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Logarithmic cohomology of the complement of a plane curve

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Abstract. Let D, x be a plane curve germ. We prove that the complex $\Omega^\bullet(\log D)_x$ computes the cohomology of the complement of D, x *only if* D is quasihomogeneous. This is a partial converse to a theorem of [5], which asserts that this complex does compute the cohomology of the complement, whenever D is a locally weighted homogeneous free divisor (and so in particular when D is a quasihomogeneous plane curve germ). We also give an example of a free divisor $D \subset \mathbb{C}^3$ which is not locally weighted homogeneous, but for which this (second) assertion continues to hold.

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1. Introduction

In [5] the last three authors showed that if D is a locally quasi-homogeneous free divisor in the complex manifold X then locally the complex $\Omega^\bullet(\log D)$ of holomorphic differential forms with logarithmic poles along D calculates the cohomology of the complement of D in X. More precisely, the following two equivalent statements hold:

Theorem 1.1. *With D as above,*

1. *If $V \subset X$ is a Stein open set then the de Rham map (integration of forms over cycles) gives rise to an isomorphism*
$$h^k(\Gamma(V, \Omega^\bullet(\log D))) \xrightarrow{\sim} H^k(V \setminus D; \mathbb{C}).$$
2. *Denoting by U the complement of D in X and by $j : U \hookrightarrow X$ the inclusion, the de Rham morphism gives rise to an isomorphism*
$$\Omega^\bullet(\log D) \xrightarrow{\sim} \mathbf{R}j_*(\mathbb{C}_U).$$ □

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By analogy with Grothendieck's Comparison Theorem [8], in which the complex $\Omega^\bullet(\log D)$ is replaced in these two statements by $\Omega^\bullet(*D)$, but which holds for an arbitrary divisor, we summarise this with a slogan: if $D \hookrightarrow X$ is a locally quasi-homogeneous free divisor then the *logarithmic comparison theorem* holds.

The definition of local quasi-homogeneity, (called *strong* quasi-homogeneity in [5]), is as follows:

Definition 1.2.

1. The polynomial $h(z_1, \cdots, z_n) = \sum a_{i_1,\cdots,i_n} z_1^{i_1} \cdots z_n^{i_n} \in \mathcal{O}_{\mathbb{C}^n}$ is *weighted homogeneous* if there exist positive integer weights $w_1, \cdots, w_n$ such that $h(z_1^{w_1}, \cdots, z_n^{w_n})$ is homogeneous.
2. The divisor $D \subset X$ is *locally quasi-homogeneous* if for all $x \in D$ there are local coordinates on X, centered at x, with respect to which D has a weighted homogeneous defining equation.

Every plane curve is a free divisor, since the module of logarithmic vector fields $\mathcal{D}\mathrm{er}(\log D)$ is reflexive and thus has depth at least 2. In [4, Cor. 4.2.2] the first author showed that if D is a plane curve then the logarithmic de Rham complex $\Omega^\bullet(\log D)$ is perverse, a necessary condition for the logarithmic comparison theorem.

In [6] the logarithmic comparison theorem has been tested for the following non locally quasi-homogeneous plane curve (cf. [9]): $D = \{f = x_1^4 + x_2^5 + x_2^4 x_1 = 0\} \subset X = \mathbb{C}^2$. A basis for $\mathcal{D}\mathrm{er}(\log D)$ is given by:

$$\begin{aligned}
\delta_1 &= (16x_1^2 + 20x_1x_2)\frac{\partial}{\partial x_1} + (12x_1x_2 + 16x_2^2)\frac{\partial}{\partial x_2} \\
\delta_2 &= (16x_1x_2^2 + 4x_2^3 - 125x_1x_2)\frac{\partial}{\partial x_1} + (12x_2^3 - 4x_1^2 + 5x_1x_2 - 100x_2^2)\frac{\partial}{\partial x_2}.
\end{aligned}$$

Let $\mathcal{D}_X$ be the sheaf of linear differential operators with holomorphic coefficients on X and I the left $\mathcal{D}_X$-ideal generated by δ_1, δ_2. By [4, Th. 4.2.1], we have a (canonical) isomorphism (in the derived category)

$$\Omega^\bullet(\log D) \simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/I, \mathcal{O}_X),$$

and so we can compute the characteristic cycle $CC(\Omega^\bullet(\log D))$ as the cycle Z in T^*X determined by the ideal $J = \sigma(I)$ generated by the principal symbols of elements in I. The symbols $\sigma_1 = \sigma(\delta_1), \sigma_2 = \sigma(\delta_2)$ form a regular sequence in $\mathcal{O}_{T^*X}$ and so, by [4, Prop. 4.1.2], the ideal J is generated by σ_1, σ_2. An easy computation shows that the multiplicity of the conormal at 0 in Z is 4. On the other hand, the multiplicity of the conormal at 0 in $CC(\mathbf{R}j_*(\mathbb{C}_U))$ is equal to $\mathrm{mult}_0(D) - 1 = 3$ (cf. [3]), and so the logarithmic comparison theorem does not hold for D.

For the family of non locally quasi-homogeneous plane curves (cf. [9])

$$x_1^q + x_2^p + x_2^{p-1}x_1 = 0, \quad p \geq q + 1 \geq 5,$$

the multiplicities of the conormal at 0 in $CC(\Omega^\bullet(\log D))$ and in $CC(\mathbf{R}j_*(\mathbb{C}_U))$ are $2(q-2)$ and $q-1$ respectively, and so these curves also do not satisfy the logarithmic comparison theorem.

A natural question is therefore whether or not the logarithmic comparison theorem holds for a given free divisor.

The purpose of this paper is to prove a partial converse to Theorem 1.1. We prove:

Theorem 1.3. *Let D be a reduced plane curve. If the logarithmic comparison theorem holds for D, then D is locally quasi-homogeneous.*

Our proof shows that if h is a local equation of D, and the logarithmic comparison theorem holds, then there is a vector field germ χ such that $\chi \cdot h = h$. As a reduced curve has isolated singularities, we can then apply the theorem of K. Saito [10]: if $h \in \mathcal{O}_{\mathbf{C}^n,0}$ has isolated singularity and h belongs to its Jacobian ideal J_h then in suitable coordinates h is weighted homogeneous.

We conjecture that in higher dimensions the following version of our Theorem 1.3 holds:

Conjecture 1.4. *If $D \hookrightarrow X$ is a free divisor and if the logarithmic comparison theorem holds, then for all $x \in D$ there is a local equation h for D around x, and a germ of vector field χ vanishing at x such that $\chi \cdot h = h$.*

A singular free divisor of dimension greater than 1 has non-isolated singularities, so even if this conjecture is true, Saito's theorem cannot be used to deduce local quasi-homogeneity. Indeed, it is *not* true in higher dimensions that if the logarithmic comparison theorem holds for a free divisor D then D is necessarily locally quasi-homogeneous. This is shown by an example in Section 4 below: the logarithmic comparison theorem holds for the free divisor

$$D = \{(x, y, z) : xy(x + y)(zx + y) = 0\}$$

(the total space of a family of four lines in the plane with varying cross-ratio, cf. [4]), in the neighbourhood of $(0, 0, \lambda)$, with $\lambda \in \mathbb{C} \setminus \{0, 1\}$; however it is well known that this divisor is not locally quasi-homogeneous. On the other hand, it does satisfy Conjecture 1.4.

Adrian Langer has indicated to us that he has subsequently found a shorter proof of Theorem 1.3, using globalisation and a comparison of Chern classes[1].

[1] Added on November 2001.

2. Preliminary results

In this section we recall the spectral sequence argument used in [5] to compare the cohomology of the logarithmic complex $\Omega^\bullet(\log D)$ with the cohomology of $X \setminus D$. Except for referring to "local" rather than "strong" quasi-homogeneity, we will use the same notation as [5].

Without loss of generality we assume $X = \mathbb{C}^n$ with coordinates z_i and $x_0 = 0$. Let V be a Stein neighbourhood (sufficiently small) of 0, let $\mathcal{U}$ be the open cover of $V \setminus \{0\}$ consisting of the sets $U_i = V \cap \{z_i \neq 0\}$, and let $\mathcal{U}'$ be the open cover of $V \setminus D$ consisting of the open sets $U_i' = (V \setminus D) \cap \{z_i \neq 0\} = U_i \setminus D$.

We consider the two double complexes

$$K^{p,q} = \check{C}^q(\mathcal{U}, \Omega^p(\log D))$$

and

$$\tilde{K}^{p,q} = \check{C}^q(\mathcal{U}', \Omega^p),$$

equipped with the exterior derivative d (the horizontal differential) and the Čech differential δ (the vertical differential). There is an obvious restriction morphism $\rho_{p,q} : K^{p,q} \to \tilde{K}^{p,q}$ which commutes with both differentials, and thus gives rise to morphisms of the two spectral sequences arising from each double complex. These spectral sequences have E_1 terms

$$''E_1^{p,q} = \check{H}^q(\mathcal{U}, \Omega^p(\log D))$$

$$''\tilde{E}_1^{p,q} = \check{H}^q(\mathcal{U}', \Omega^p)$$

$$'E_1^{p,q} = \oplus_{1 \le i_1 < \cdots < i_{q+1} \le n} h^p\Big(\Gamma\Big(\bigcap_j U_{i_j}, \Omega^\bullet(\log D)\Big)\Big)$$

$$'\tilde{E}_1^{p,q} = \oplus_{1 \le i_1 < \cdots < i_{q+1} \le n} h^p\Big(\Gamma\Big(\bigcap_j U_{i_j}', \Omega^\bullet\Big)\Big).$$

As both $\mathcal{U}$ and $\mathcal{U}'$ are Stein covers,

$$\check{H}^q(\mathcal{U}, \Omega^p(\log D)) = \check{H}^q(V \setminus \{0\}, \Omega^p(\log D))$$

and

$$\check{H}^q(\mathcal{U}', \Omega^p)) = \check{H}^q(V \setminus D, \Omega^p)).$$

As $V \setminus D$ is Stein, $\check{H}^q(V \setminus D, \Omega^p) = 0$ if $q > 0$. It follows that

$$''\tilde{E}_2^{p,q} = \begin{cases} H^p(V \setminus D; \mathbb{C}) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases},$$

and in particular the spectral sequence $''\tilde{E}$ converges to the cohomology of $V \setminus D$.

Now assume that outside 0, D is locally quasi-homogeneous, so that by 1.1 $\mathbf{R}j_*(\mathbb{C}_U) \simeq \Omega^\bullet(\log D)$, again outside 0. As $\mathcal{U}$ and $\mathcal{U}'$ are Stein covers, by 1.1 the quotient of the restriction $\rho_{p,q}$ defines an isomorphism $'\rho_{p,q}$: $'E_1^{p,q} \to {'\tilde{E}_1^{p,q}}$ for all p, q. This isomorphism persists to give an isomorphism of the cohomology of the total complexes K^{tot} and $\tilde{K}^{\mathrm{tot}}$ as calculated by the spectral sequences. It follows that the spectral sequence $''E$, like $''\tilde{E}$, also converges to the cohomology of $V \setminus D$:

$$H^k(V \setminus D; \mathbb{C}) \simeq \oplus_{p+q=k} \, ''E_\infty^{p,q}.$$

As D is a free divisor, $\check{H}^q(V \setminus \{0\}, \Omega^p(\log D)) = 0$ for $q \neq 0, n-1$, so $''E_1$ has only two non-null rows; writing for the moment $\Omega^p(D)$ and V^* in place of $\Omega^p(\log D)$ and $V \setminus \{0\}$, $''E_1$ thus looks like

$$\begin{array}{ccccc}
\check{H}^{n-1}(V^*, \Omega^0(D)) \stackrel{d_1}{\to} & \cdots \stackrel{d_1}{\to} & \check{H}^{n-1}(V^*, \Omega^p(D)) \stackrel{d_1}{\to} & \cdots \stackrel{d_1}{\to} & \check{H}^{n-1}(V^*, \Omega^n(D)) \\
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
\Gamma(V, \Omega^0(D)) \quad \stackrel{d_1}{\to} & \cdots \stackrel{d_1}{\to} & \Gamma(V, \Omega^p(\log D)) \quad \stackrel{d_1}{\to} & \cdots \stackrel{d_1}{\to} & \Gamma(V, \Omega^n(\log D)).
\end{array}$$

(Note that as $n \geq 2$ and as the $\Omega^p(\log D)$ are free modules, we have $\Gamma(V^*, \Omega^p(D)) = \Gamma(V, \Omega^p(D))$.)

As this spectral sequence converges to the cohomology of $V \setminus D$, we have

$$H^{n-1}(V \setminus D; \mathbb{C}) \simeq E_\infty^{0,n-1} \oplus \cdots \oplus E_\infty^{n-1,0} = E_{n+2}^{0,n-1} \oplus h^{n-1}(\Gamma(V, \Omega^\bullet(\log D)))$$

$$H^n(V \setminus D; \mathbb{C}) = E_\infty^{0,n} \oplus \cdots \oplus E_\infty^{0,n} = E_{n+2}^{1,n-1} \oplus \frac{h^n(\Gamma(V, \Omega^\bullet(\log D)))}{d_{n+1}(E_{n+2}^{0,n-1})},$$

where

$$E_{n+2}^{0,n-1} = \mathrm{Ker}\ d_1 : \check{H}^{n-1}(V^*, \Omega^0(D)) \to \check{H}^{n-1}(V^*, \Omega^1(D)).$$

In [5], the main theorem was proved by showing that if D is locally quasi-homogeneous then the complex

$$(\check{H}^{n-1}(V \setminus \{0\}, \Omega^\bullet(\log D)), d_1)$$

is exact.

3. Proof of the Theorem

We continue with the discussion of the last paragraph. If the natural morphism $\Omega^\bullet(\log D) \to \mathbf{R}j_*(\mathbb{C}_U)$ is a quasi-isomorphism (i.e. if the logarithmic comparison theorem holds for D) then by the formulae of the last section, $d_1 : \check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) \to \check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D))$ is injective.

Let $\{\omega_1, \cdots, \omega_n\}$ be a free basis of $\Omega^1(\log D)$ as $\mathcal{O}_V$-module, and let $\delta_1, \cdots, \delta_n$ be the dual basis of $\mathcal{D}\mathrm{er}(\log D)$. Then $\check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) = \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$ and $\check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D)) \simeq \oplus_1^n \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$. The morphism $d_1 : \check{H}^{n-1}(V \setminus \{0\}, \Omega^0(\log D)) \to \check{H}^{n-1}(V \setminus \{0\}, \Omega^1(\log D))$ now becomes

$$\begin{array}{ccc}
\check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}) & \stackrel{d_1}{\to} & \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})^n \\
[g] & \mapsto & ([\delta_1 \cdot g], \cdots, [\delta_n \cdot g]).
\end{array}$$

where $g \in \Gamma(V \setminus \cup_i\{z_i = 0\}, \mathcal{O}_{\mathbb{C}^n}) = \Gamma(\mathbb{C}^n \setminus \cup_i\{z_i = 0\}, \mathcal{O}_{\mathbb{C}^n})$ represents the class $[g]$ in $\check{H}^{n-1}(\mathbb{C}^n \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$.

For $\delta \in \mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$, we denote by d_δ the homomorphism

$$d_\delta : \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}) \to \check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n}), \quad d_\delta([g]) = [\delta \cdot g].$$

Proposition 3.1. *Let $\mathbf{m}_{\mathbb{C}^n,0}$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,0}$ and let $\delta \in \mathbf{m}_{\mathbb{C}^n,0}\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$,*

$$\delta = (x_1, \cdots, x_n)\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,n}\end{pmatrix}\begin{pmatrix}\partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n\end{pmatrix} + \delta_{\geq 1}$$

with the $a_{i,j} \in \mathbb{C}$ and $\delta_{\geq 1} \in \mathbf{m}^2_{\mathbb{C}^n,0}\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n})$. If d_δ is injective, then the eigenvalues of A do not satisfy any relation with positive integer coefficients (*in this case, we will say that δ satisfies condition* (I)).

Proof. By a coordinate change we can make A lower triangular. Its eigenvalues $a_1, \cdots, a_n$ are then the elements of the diagonal. The group $\check{H}^{n-1}(V \setminus \{0\}, \mathcal{O}_{\mathbb{C}^n})$ is isomorphic to the space of Laurent series, convergent for all $\underline{x} = (x_1, \cdots, x_n)$ with $\underline{x} \neq 0$, whose non-zero coefficients are those with strictly negative indices in all variables, i.e.

$$\sum_{i_1,\cdots,i_n<0} a_{i_1,\cdots,i_n} x_1^{i_1} \cdots x_n^{i_n}.$$

For $p \geq n$, we set

$$G^p = \left\{ \sum_{\substack{i_1, \cdots, i_n < 0 \\ i_1 + \cdots + i_n = -p}} c_{\mathbf{i}} x_1^{i_1} \cdots x_n^{i_n} \right\},$$

$$F^p = \left\{ \sum_{\substack{i_1, \cdots, i_n < 0 \\ i_1 + \cdots + i_n \geq -p}} c_{\mathbf{i}} x_1^{i_1} \cdots x_n^{i_n} \right\}.$$

Then $F^p = G^p \oplus G^{p-1} \oplus \cdots \oplus G^n$. Each G^p is a finite-dimensional $\mathbb{C}$-vector space, whose dimension we denote by r_p, and d_δ restricts to morphisms of vector spaces

$$d_{\delta\ |F^p} : F^p \to F^p$$

and

$$d_{\delta\ |G^p} : G^p \to F^p.$$

Let us denote by $d^p_{\delta,p}$ the component of this second restriction lying in G^p. Then $d^p_{\delta,p}$ depends only on the weight 0 part δ_0 of δ. We claim that with respect to a suitable ordered basis of G^p, its matrix $[d^p_{\delta,p}]$ is lower triangular.

As basis for G^p we take the monomials

$$\frac{1}{x_1^{i_1}\cdots x_n^{i_n}}$$

with $i_1+\cdots+i_n=p$.

We have

$$d_\delta(x_1^{-i_1}\cdots x_n^{-i_n}) = -\sum_{j,k} i_k\ a_{j,k}\ x_1^{-i_1}\cdots x_k^{-(i_k-1)}\cdots x_j^{-(i_j+1)}\cdots x_n^{-i_n}. \qquad (1)$$

Thus, if we give our basis of G^p the lexicographic order corresponding to the order of the coordinates $x_1,\cdots,x_n$, then since $a_{j,k}=0$ if $j<k$ (recall that we have chosen our coordinates so that A is lower triangular), the matrix $[d^p_{\delta,p}]$ is lower triangular.

Let $q\le p$. Then $d_\delta(G^q)\subset G^q+G^{q-1}+\cdots+G^n$. Thus, it follows from the above that if we give F^p the ordered basis consisting of the ordered bases for each $G^q, n\le q\le p$ that we have chosen, and order these by descending value of q, then the matrix of $d_{\delta\ |F^p}$ is also lower triangular.

What are its diagonal elements? In the right-hand side of equation (1), the coefficient of $x_1^{-i_1}\cdots x_n^{-i_n}$ is equal to

$$i_1a_{1,1}+\cdots+i_na_{n,n};$$

this is the diagonal element in the matrix of $d_{\delta\ |F^p}$ in the row and column corresponding to the basis element $x_1^{-i_1}\cdots x_n^{-i_n}$. Note that the diagonal elements of A are its eigenvalues; thus, the diagonal elements in the matrix of $d_{\delta\ |F^p}$ with respect to the chosen basis are all linear combinations $i_1\lambda_1+\cdots+i_n\lambda_n$ of the eigenvalues $\lambda_1,\cdots,\lambda_n$ of A, with the i_j positive integers and $i_1+\cdots+i_n\le p$. As this matrix is lower triangular, $d_{\delta\ |F^p}$ is injective only if the product of these diagonal elements is non-zero. □

Remark 3.2. We have used in the proof of this lemma the fact that if d_δ is injective then so is its restriction to each F^p. We do not know if the opposite implication holds. It seems likely that an argument involving faithful flatness would prove it. However, we do not need it in what follows.

Let D be a plane curve. We suppose as above that 0 is the singular point of D. In this case the upper non-zero row in the E_2 page of the spectral sequence $'\tilde{E}$ begins

$$d_1:\check{H}^1(\mathbb{C}^2\setminus\{0\},\mathcal{O}_{\mathbb{C}^2})\to\oplus_1^2\check{H}^1(\mathbb{C}^2\setminus\{0\},\mathcal{O}_{\mathbb{C}^2}).$$

Theorem 3.3. *Let D be a plane curve, singular at 0. If d_1 is injective, then there is a local equation h for D around 0, and a germ of vector field χ at 0 such that $\chi \cdot h = h$.*

Proof. Any reduced plane curve whose equation has non-zero quadratic part is quasihomogeneous, by the classification of singularities of functions of two variables: such a curve is equivalent to A_k, $x^2 + y^{k+1} = 0$, for some k. For a quasihomogeneous curve, the conclusion of the theorem of course holds. Thus, we may assume that the equation h of D lies in $\mathbf{m}^3_{\mathbb{C}^2,0}$. As the determinant of the coefficients of a free basis of $\mathcal{D}\mathrm{er}(\log D)$ is a local defining equation for D ([11]), we may therefore choose a free basis δ, γ for $\mathcal{D}\mathrm{er}(\log D)$ such that γ has zero linear part. In fact the supposition that d_1 is injective implies that at least one member of the basis has non-zero linear part, as otherwise $d_1([1/xy]) = ([\delta \cdot 1/xy], [\gamma \cdot 1/xy]) = 0$.

We may thus take

$$\delta = \delta_0 + \delta_1 + \delta_2 + \cdots = \sum_{k \geq 0} \sum_{i+j=k+1} \Big(\alpha_{ij} x^i y^j \frac{\partial}{\partial x} + \beta_{ij} x^i y^j \frac{\partial}{\partial y} \Big)$$

where $\delta_0 = \underline{x} A \underline{\partial_x}^t$, with $A \neq 0$ and in Jordan normal form, i.e.

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \lambda_1 & 0 \\ 1 & \lambda_1 \end{pmatrix}.$$

Let h be the reduced equation of D:

$$h = h_n + h_{n+1} + h_{n+2} + \cdots = \sum_{k \geq n} h_k = \sum_{k \geq n} \sum_{i+j=k} a_{ij} x^i y^j,$$

where the polynomials h_i are homogeneous of degree i.

Let us now suppose that δ is not an Euler vector field for h, we will see that (up to multiplication by a non-zero constant) the only possibility for h and δ is

$$h_1 = \cdots = h_{n-1} = 0, h_n = x^a y^b \quad \text{and} \quad \delta_0 = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}.$$

First case: $h_n = \sum_{i+j=n} a_{ij} x^i y^j$ and $\delta_0 = \lambda_1 x \frac{\partial}{\partial x} + \lambda_2 y \frac{\partial}{\partial y}$. Then

$$0 = \delta_0(h_n) = \sum_{i+j=n} (i\lambda_1 + j\lambda_2) a_{ij} x^i y^j.$$

So, $a_{ij} = 0$ if $i\lambda_1 + j\lambda_2 \neq 0$; thus, since by assumption $h_n \neq 0$, we have $q\lambda_1 = -p\lambda_2$ and $p + q = n$ ($p, q \in \mathbb{N}$). In this case,

$$h_n = x^p y^q, \quad \delta_0 = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}.$$

Second case: $h_n = \sum_{i+j=n} a_{ij} x^i y^j$ and $\delta_0 = (\lambda_1 x + y)\frac{\partial}{\partial x} + \lambda_1 y \frac{\partial}{\partial y}$. Then

$$0 = \delta_0(h_n) = n\lambda_1 a_{n0} x^n + \sum_{i+j=n, j \geq 1} (n\lambda_1 a_{ij} + i a_{i+1,j-1}) x^i y^j.$$

So, if $\lambda_1 \neq 0$, then we must have $a_{n0} = 0$, then $a_{n-1,1} = 0, \cdots, a_{1,n-1} = 0, a_{0n} = 0$, so that $h_n = 0$. This is absurd, by hypothesis.

If $\lambda_1 = 0$, then d_1 is not injective, because

$$d_1([1/xy]) = (d_\delta([1/xy]), d_\gamma([1/xy])) = (0,0).$$

Then, we have

$$h = x^p y^q + h_{n+1} + h_{n+2} + \cdots, \quad \delta_0 = qx\frac{\partial}{\partial x} - py\frac{\partial}{\partial y}.$$

We will prove that, in this case, after a coordinate change h can be reduced to $h = x^p y^q$ with $p + q = n \geq 3$. This contradicts our supposition that h is reduced. Then our initial supposition about δ is false, and δ is an Euler vector field for h.

Inductively, for all $k \geq 0$, we construct coordinates $(x_{(k)}, y_{(k)})$ and functions $h^{(k)}$ such that

$$h(x,y) = h^{(k)}(x_{(k)}, y_{(k)}) = x^p_{(k)} y^q_{(k)} + \sum_{s \geq n+k} h^{(k)}_s(x_{(k)}, y_{(k)}) \equiv x^p_{(k)} y^q_{(k)} (\mathbf{m}^{n+k}_{\mathbb{C}^2,0}),$$

where $h^{(k)}_i$ is homogeneous of degree i. Then, by Artin approximation [1, Theorem 1.2], there exist coordinates z_1, z_2 solving the equation

$$h(x,y) - z_1^p z_2^q = 0.$$

Let us construct the $x_{(k)}, y_{(k)}$, $h^{(k)}$. We suppose that we have $x_{(k)}, y_{(k)}$ and $h^{(k)} \in \mathbb{C}\{x_{(k)}, y_{(k)}\}$, such that

$$h(x,y) = h^{(k)}(x_{(k)}, y_{(k)}) = x^p_{(k)} y^q_{(k)} + \sum_{s \geq n+k} h^{(k)}_s,$$

$$\delta^{(k)}_0 = qx_{(k)}\frac{\partial}{\partial x_{(k)}} - py_{(k)}\frac{\partial}{\partial y_{(k)}}.$$

We define $x_{(k+1)}, y_{(k+1)}$ and $h^{(k+1)} \in \mathbb{C}\{x_{(k+1)}, y_{(k+1)}\}$, such that

$$h(x,y) = h^{(k+1)}(x_{(k+1)}, y_{(k+1)}) = x^p_{(k+1)} y^q_{(k+1)} + \sum_{s \geq n+k+1} h^{(k+1)}_s,$$

$$\delta^{(k+1)}_0 = qx_{(k+1)}\frac{\partial}{\partial x_{(k+1)}} - py_{(k+1)}\frac{\partial}{\partial y_{(k+1)}}.$$

Let $h^{(k)}_{n+k} = \sum_{i+j=n+k} a^{(k)}_{i,j} x^i_{(k)} y^j_{(k)}$, then

$$\delta^{(k)}_0(h_{n+k}) = \sum_{i+j=n+k} (iq - jp) a^{(k)}_{i,j} x^i_{(k)} y^j_{(k)}.$$

As the part of $h^{(k)}$ of degree less than $n+k$ is $x^p_{(k)} y^q_{(k)}$, it follows that the part of degree $n+k$ of $\delta^{(k)}(h^{(k)}) \in \mathbf{m}_{\mathbb{C}^2,0} h^{(k)}$ belongs to $(x^p_{(k)} y^q_{(k)})$:

$$[\delta^{(k)}(h^{(k)})]_{n+k} = \delta^{(k)}_0(h^{(k)}_{n+k}) + \delta^{(k)}_k(x^p_{(k)} y^q_{(k)}) \in (x^p_{(k)} y^q_{(k)}),$$

but

$$\delta_k^{(k)}(x_{(k)}^p y_{(k)}^q) \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1}),$$

then

$$\delta_0^{(k)}(h_{n+k}^{(k)}) \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1}),$$

so

$$(iq - jp)a_{i,j}^{(k)} = 0 \ (i + j = n + k) \text{ if } \ i < p - 1 \text{ or } j < q - 1,$$

but if $iq - jp = 0$, then $(i, j) = \frac{n+k}{n}(p, q)$, and $i > p$, $j > q$. So $h_{n+k}^{(k)} \in (x_{(k)}^{p-1} y_{(k)}^q, x_{(k)}^p y_{(k)}^{q-1})$:

$$h_{n+k}^{(k)} = x_{(k)}^{p-1} y_{(k)}^q f_{k+1}(x_{(k)}, y_{(k)}) + x_{(k)}^p y_{(k)}^{q-1} g_{k+1}(x_{(k)}, y_{(k)}).$$

Let

$$x_{(k+1)} = x_{(k)} + \frac{1}{p} f_{k+1}(x_{(k)}, y_{(k)}) \qquad y_{(k+1)} = y_{(k)} + \frac{1}{q} g_{k+1}(x_{(k)}, y_{(k)}).$$

We have

$$h(x, y) = x_{(k+1)}^p y_{(k+1)}^q + \sum_{r \geq k+1} \sum_{i+j=n+r} a_{i,j}^{(k+1)} x_{(k+1)}^i y_{(k+1)}^j.$$

We define $h^{(k+1)}$ by the equation $h(x, y) = h^{(k+1)}(x_{(k+1)}, y_{(k+1)})$, where

$$h^{(k+1)} = x_{(k+1)}^p y_{(k+1)}^q + \sum_{s \geq n+k+1} h_s^{(k+1)},$$

with $h_s^{(k+1)} = \sum_{i+j=s} a_{i,j}^{(k+1)} x_{(k+1)}^i y_{(k+1)}^j$ homegeneous polynomials of degree $s \geq n + k + 1$. Moreover, as

$$x_{(k+1)} = x_{(k)}; \quad y_{(k+1)} = y_{(k)} \ (\text{mod } \mathbf{m}_{\mathbb{C}^2,0}^2),$$

we have $\delta = \sum_{q \geq 0} \delta_q^{(k+1)}$, where each $\delta_q^{(k+1)}$ is homogeneous of degree q, and

$$\delta_0^{(k+1)} = q x_{(k+1)} \frac{\partial}{\partial x_{(k+1)}} - p y_{(k+1)} \frac{\partial}{\partial y_{(k+1)}}. \qquad \square$$

Proposition 3.4. *Let D a plane curve, singular at 0. If there exists $\delta \in \mathcal{D}\mathrm{er}(\log D)$ satisfying condition* (I)*, then there exists a unit α such that $\alpha\delta \cdot h = h$, and so D is Euler homogeneous.*

Proof. The proof is similar to the proof of Theorem 3.3. There, we consider the case where $h_n = x^p y^q$ and $\delta_0 = qx\partial/\partial x - py\partial/\partial y$, with $p, q \in \mathbb{N}$. Condition (I) forces one of p and q to be 0. The proof now proceeds as before, with this additional hypothesis.

Theorem 3.5. *Let $(D,0) \subset (\mathbb{C}^2,0)$ be a plane curve. The following conditions are equivalent:*

a) *There exists $\delta \in \mathcal{D}\mathrm{er}(\log D)_0$ such that d_δ is injective.*

b) *There exists $\delta \in \mathcal{D}\mathrm{er}(\log D)_0$ satisfying condition* (I).

c) *d_1 is injective.*

d) *$(D,0)$ is Euler homogeneous.*

e) *$(D,0)$ is quasi-homogeneous.*

f) *The logarithmic comparison theorem holds for $(D,0)$ on a neighbourhood of* 0.

Proof. By Theorem 3.3, if d_1 is injective, then $(D,0)$ is Euler homogeneous. By Saito's theorem [10] (for a function h with isolated singularity, $h \in J_h$ is equivalent to the quasihomogeneity of h) to be Euler homogeneous or quasi-homogeneous is the same. Theorem 1.1 proves that if $(D,0)$ is quasi-homogeneous, the logarithmic comparison theorem holds for $(D,0)$ on a neighborough of 0. From the results of section 2 we can easily deduce that logarithmic comparison theorem implies the injectivity of d_1. Then, the last four conditions are equivalent. If $\chi = w_1\frac{\partial}{\partial x} + w_2\frac{\partial}{\partial y}$ is the Euler vector field then d_χ is injective. Proposition 3.1 shows that if d_δ is injective, then δ satisfies (I) and, finally, by proposition 3.4, $\delta \in \mathcal{D}\mathrm{er}(\log D)$ implies that D is Euler homogeneous.

4. Example

In this section we give an example of a free divisor $D \subset \mathbb{C}^3$ which is Euler homogeneous but not locally quasi-homogeneous, and for which the logarithmic comparison theorem does hold. This example is studied in [4], where the perversity of $\Omega^\bullet(\log D)$ is proved. We remark that D is the total space of an equisingular one-parameter deformation of a plane curve singularity. In [7], Damon shows that under mild additional hypotheses, all surfaces obtained in this way are free divisors.

D is defined by the equation

$$h(x,y,z) = xy(x+y)((z-\lambda)x+y) = h_1h_2h_3h_4, \quad \lambda \in \mathbb{C}\setminus\{0,1\}.$$

$\mathcal{D}\mathrm{er}(\log D)$ has free basis $\{\delta_1,\delta_2,\delta_3\}$

$$\begin{aligned}
\delta_1 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\\
\delta_2 &= ((z-\lambda)x+y)\frac{\partial}{\partial z}\\
\delta_3 &= x^2\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} - (z-\lambda)(x+y)\frac{\partial}{\partial z}.
\end{aligned}$$

Note that $\delta_1 \cdot h = 4h$, so that h is Euler homogeneous. Note also that it is easy to check that each of these vector fields is logarithmic, and that the determinant of their coefficients is a reduced equation for D. From this it follows by a theorem of K. Saito ([11]) that they really do form a basis for $\mathcal{D}\mathrm{er}(\log D)$; as no linear combination of them has non-singular linear part, it follows that D cannot be quasihomogeneous.

This example of free divisor is interesting also as it provides a counterexample to the "logarithmic Sard's theorem": every point of $\mathbb{C} = z$-axis is a logarithmic critical value with respect to the projection $(x, y, z) \mapsto z$.

The basis of $\Omega^1(\log D)$ dual to $\{\delta_1, \delta_2, \delta_3\}$ is

$$\begin{aligned}
\omega_1 &= \frac{y^2\ dx + x^2\ dy}{xy(x+y)}\\
\omega_2 &= \frac{y(z-\lambda)\ dx - x(z-\lambda)\ dy + xy\ dz}{xy(x(z-\lambda)+y)}\\
\omega_3 &= \frac{y\ dx - x\ dy}{xy(x+y)}\,.
\end{aligned}$$

We have to calculate homology groups of the stalk at 0 of the logarithmic de Rham complex

$$0 \to \Omega^0(\log D) \stackrel{d_0}{\to} \Omega^1(\log D) \stackrel{d_1}{\to} \Omega^2(\log D) \stackrel{d_2}{\to} \Omega^3(\log D) \stackrel{d_3}{\to} 0.$$

Although D is not weighted homogeneous in the strict sense, it is homogeneous if we assign weights $1, 1, 0$ to the variables x, y, z. The Lie derivative with respect to the vector field δ_1,

$$L_{\delta_1}(\omega) = \iota_{\delta_1}(d\omega) + d(\iota_{\delta_1}(\omega)),$$

then defines a contracting homotopy from $\Omega^\bullet(\log D)$ to its weight-zero part $\Omega^\bullet_0(\log D)$. For if $\omega \in \Omega^k(\log D)$ is a sum of homogenenous parts ω_i, and if $d\omega = 0$, then $d\omega_i = 0$ for all i. Since $L_{\delta_1}(\omega_i) = i\omega_i$, each ω_i, for $i \neq 0$, is then exact, and ω is cohomologous to $\omega - \iota_{\delta_1}(\sum_{i\neq 0}(1/i)\omega_i)$.

Thus we consider only the weight 0 subcomplex

$$0 \to \Omega^0_0(\log D) \stackrel{d^0_0}{\to} \Omega^1_0(\log D) \stackrel{d^0_1}{\to} \Omega^2_0(\log D) \stackrel{d^0_2}{\to} \Omega^3_0(\log D) \stackrel{d^0_3}{\to} 0.$$

- We have $\Omega^0_0(\log D) = \mathbb{C}\{z\}$, and $d_0(z^k) = kz^{k-1}[((z-\lambda)x + y)\omega_2 - (z-\lambda)(x+y)\omega_3]$ $(k \geq 0)$, so

$$\mathrm{Im}(d^0_0) = \mathbb{C}\{z\}dz = \mathbb{C}\{z\}\left\langle ((z-\lambda)x + y)\omega_2 - (z-\lambda)(x+y)\omega_3\right\rangle.$$

- $\Omega^1_0(\log D) = \mathbb{C}\{z\}\left\langle \omega_1, x\omega_2, y\omega_2, x\omega_3, y\omega_3\right\rangle$, and we find

$$\begin{aligned}
d_1(\omega_1) &= d_1(x\omega_2) = d_1(x\omega_3) = d_1(y\omega_3) = 0\\
d_1(z^k\omega_1) &= kz^{k-1}((x(\lambda - z) - y)\omega_1 \wedge \omega_2 + (z-\lambda)(x+y)\omega_1 \wedge \omega_3)\\
d_1(y\omega_2) &= (xy + y^2)\omega_2 \wedge \omega_3\\
d_1(z^k x\omega_2) &= kz^{k-1}((z-\lambda)(x+y)x\omega_2 \wedge \omega_3)\\
d_1(z^k y\omega_2) &= ((k+1)z^k - k\lambda z^{k-1})(x+y)y\omega_2 \wedge \omega_3\\
d_1(z^k x\omega_3) &= kz^{k-1}x(x(z-\lambda)+y)\omega_2 \wedge \omega_3\\
d_1(z^k y\omega_3) &= kz^{k-1}y(x(z-\lambda)+y)\omega_2 \wedge \omega_3.
\end{aligned}$$

It follows that $\mathrm{Ker}(d_1^0) = \mathbb{C}\langle \omega_1, x\omega_2, x\omega_3, y\omega_3\rangle \oplus \mathrm{Im}(d_0^0)$, so

$$h^1(\Omega^\bullet(\log D)_0) = \mathbb{C}\langle \omega_1, x\omega_2, x\omega_3, y\omega_3\rangle$$

is 4-dimensional. Also we have

$$\mathrm{Im}(d_1^0) = \mathbb{C}\{z\}\langle ((\lambda - z)x - y)\omega_1 \wedge \omega_2 + (z-\lambda)(x+y)\omega_1 \wedge \omega_3)\rangle \oplus \mathbb{C}\{z\}\left\langle x^2, xy, y^2\right\rangle \omega_2 \wedge \omega_3.$$

- $\Omega_0^2(\log D)$ is generated over $\mathbb{C}\{z\}$ by

$$x\omega_1 \wedge \omega_2, y\omega_1 \wedge \omega_2, x\omega_3 \wedge \omega_1, y\omega_3 \wedge \omega_1, x^2\omega_2 \wedge \omega_3, xy\omega_2 \wedge \omega_3, y^2\omega_2 \wedge \omega_3.$$

We find

$$\begin{array}{rclclcl} d_2(x\omega_1 \wedge \omega_2) &=& d_2(x\omega_1 \wedge \omega_3) &=& d_2(y\omega_1 \wedge \omega_3) &=& 0 \\ d_2(z^k x^2\omega_2 \wedge \omega_3) &=& d_2(z^k xy\omega_2 \wedge \omega_3) &=& d_2(z^k y^2\omega_2 \wedge \omega_3) &=& 0. \end{array}$$

$$\begin{array}{rcl} d_2(z^k x\omega_1 \wedge \omega_2) &=& kz^{k-1}(\lambda - z)(x+y)x\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(y\omega_1 \wedge \omega_2) &=& (xy + y^2)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(z^k y\omega_1 \wedge \omega_2) &=& z^{k-1}(x+y)(ky(\lambda - z) - zy)\omega_1 \wedge \omega_2 \wedge \omega_3) \\ d_2(z^k x\omega_1 \wedge \omega_3) &=& -kz^{k-1}x((z-\lambda)x + y)\omega_1 \wedge \omega_2 \wedge \omega_3 \\ d_2(z^k y\omega_1 \wedge \omega_3) &=& -kz^{k-1}y((z-\lambda)x + y)\omega_1 \wedge \omega_2 \wedge \omega_3. \end{array}$$

We deduce that $\mathrm{Ker}(d_2^0) = \mathbb{C}\langle x\omega_1 \wedge \omega_2, x\omega_1 \wedge \omega_3, y\omega_1 \wedge \omega_3\rangle \oplus \mathrm{Im}(d_1^0)$, and thus that

$$h^2(\Omega^\bullet(\log D)_0) = \mathbb{C}\langle x\omega_1 \wedge \omega_2, x\omega_1 \wedge \omega_3, y\omega_1 \wedge \omega_3\rangle$$

is 3-dimensional.

- Finally,

$$\mathrm{Im}(d_2^0) = \mathbb{C}\{z\}\left\langle x^2, xy, y^2\right\rangle \omega_1 \wedge \omega_2 \wedge \omega_3 = \Omega_0^3(\log D),$$

and, consequently,

$$h^3(\Omega^\bullet(\log D)_0) = 0.$$

Now consider the intersection $D_0 = D \cap \{z = 0\}$, which has equation

$$h^0 = h_1^0 h_2^0 h_3^0 h_4^0 = xy(x+y)(-\lambda x + y).$$

It is a line arrangement, and the cohomology of its complement is therefore given by the Brieskorn complex, the exterior algebra generated over $\mathbb{C}$ by the forms dh_i^0/h_i^0, with trivial differential ([2]). This is of course a subcomplex of $\Omega^\bullet(\log D_0)$. Let $V \subset \mathbb{C}^3$ be a neighbourhood of 0. Restriction from $\mathbb{C}^3$ to $\mathbb{C}^2 = \{z = 0\}$ gives rise to a commutative diagram

$$\begin{array}{ccccc} \wedge^p \sum\limits_{1\le i\le 4} \mathbb{C}\left\langle \dfrac{dh_i}{h_i}\right\rangle & \xrightarrow{a} & h^p(\Omega^\bullet(\log D)(V)) & \xrightarrow{b} & H^p(V \setminus D; \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow\cong \\ \wedge^p \sum\limits_{1\le i\le 4} \mathbb{C}\left\langle \dfrac{dh_i^0}{h_i^0}\right\rangle & \xrightarrow{\cong} & h^p(\Omega^\bullet(\log D_0)(V_0)) & \xrightarrow{\cong} & H^p(V_0 \setminus D_0; \mathbb{C}). \end{array}$$

in which the left-hand horizontal morphisms are induced by the inclusion of the Brieskorn complex in the logarithmic complex, and the right-hand horizontal morphisms are de Rham maps. The lower horizontal morphisms are isomorphisms by the theorem of Brieskorn and by 1.1. The right-hand vertical morphism is an isomorphism because D is a topologically trivial deformation of D_0, so inclusion induces an isomorphism of the homology groups of the complements. The left-hand vertical morphism is evidently surjective, and thus the de Rham map $h^p(\Omega^\bullet(\log D)(V)) \to H^p(V \setminus D;\mathbb{C})$ is surjective. As $h^p(\Omega^\bullet(\log D)_0) = \lim_{U\ni 0} h^p(\Omega^\bullet(\log D)(V))$ and $\lim_{U\ni 0} H^p(V\setminus D;\mathbb{C}) = H^p(\mathbb{C}^3\setminus D;\mathbb{C})$, then the de Rham map $h^p(\Omega^\bullet(\log D)) \to H^p(\mathbb{C}^3\setminus D;\mathbb{C})$ is surjective. To see that it is an isomorphism we compare dimensions. A calculation (for example, using the Brieskorn complex) gives

$$\begin{aligned}\dim_{\mathbb{C}} H^1(\mathbb{C}^2\setminus D_0;\mathbb{C}) &= 4\\ \dim_{\mathbb{C}} H^2(\mathbb{C}^2\setminus D_0;\mathbb{C}) &= 3\\ \dim_{\mathbb{C}} H^3(\mathbb{C}^2\setminus D_0;\mathbb{C}) &= 0.\end{aligned}$$

As these are the same as the dimension of $h^p(\Omega^\bullet(\log D)_0)$, this completes the proof that the logarithmic comparison theorem holds for D. □

Remark 4.1. The calculations whose results we summarise here are not so simple as might be supposed. We have presented each image $d_i^0(\Omega_0^i(\log D))$ as a module over $\mathbb{C}\{z\}$ with algebraic generators, obscuring the fact that because D is not quasihomogeneous, the anti-derivatives of an algebraic exact logarithmic form are in general transcendental. For example,

$$\begin{aligned}z^k(x^2+xy)\omega_1\wedge\omega_2\wedge\omega_3 &= d\Big(\sum_{s=1}^{\infty}(z^{k+s}/\lambda^s(k+s))x\omega_1\wedge\omega_2\Big)\\ &= d\Big(-\Big(\log\Big(1-\frac{z}{\lambda}\Big)+\sum_{s=1}^{k}(z^s/\lambda^s s)\Big)\lambda^k x\omega_1\omega_2\Big)\end{aligned}$$

and

$$\begin{aligned}z^k xy\omega_1\wedge\omega_2\wedge\omega_3 &= d\Big(\sum_{s=1}^{\infty}(z^{k+s}/(\lambda+1)^s(k+s))x(\omega_1\wedge\omega_2+\omega_1\wedge\omega_3)\Big)\\ &= d\Big(-\Big((\lambda+1)^k\log(1-(z/(\lambda+1)))\\ &\qquad +\sum_{s=1}^{k}(z^s(\lambda+1)^{k-s}s)\Big)x(\omega_1\wedge\omega_2+\omega_1\wedge\omega_3)\Big).\end{aligned}$$

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