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On the Haefliger–Hirsch–Wu invariants for embeddings and immersions

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Abstract. We prove beyond the metastable dimension the PL cases of the classical theorems due to Haefliger, Harris, Hirsch and Weber) on the deleted product criteria for embeddings and immersions. The isotopy and regular homotopy versions of the above theorems are also improved. We show by examples that they cannot be improved further. These results have many interesting corollaries, e.g.

  1) Any closed homologically 2-connected smooth 7-manifold smoothly embeds in $\mathbb{R}^{11}$.

  2) If $p \leq q$ and $m \geq \frac{3q}{2} + p + 2$ then the set of PL embeddings $S^p \times S^q \to \mathbb{R}^m$ up to PL isotopy is in 1–1 correspondence with $\pi_q(V_{m-q,p+1}) \oplus \pi_p(V_{m-p,q+1})$.

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1. Introduction and main results

Denote CAT = DIFF or PL (we omit CAT if a statement holds in both categories). For $m \geq n + 3$ let $\mathrm{Emb}^m_{CAT}(N)$ be the set of CAT embeddings $N \to \mathbb{R}^m$ up to CAT isotopy. Let

$$\widetilde{N} = \{(x,y) \in N \times N \mid x \neq y\}$$

be the *deleted product* of N. Let $\mathbb{Z}_2$ act on $\widetilde{N}$ and on S^{m-1} by exchanging factors and antipodes, respectively. For an embedding $f : N \to \mathbb{R}^m$ define a map

$$\widetilde{f} : \widetilde{N} \to S^{m-1} \qquad \text{by} \qquad \widetilde{f}(x,y) = \frac{fx - fy}{|fx - fy|}.$$

The equivariant homotopy class $\alpha(f)$ of $\widetilde{f}$ in the set $\pi^{m-1}_{eq}(\widetilde{N})$ of equivariant maps $\widetilde{N} \to S^{m-1}$ is clearly an isotopy invariant. Thus is defined the *Haefliger–Wu*

(deleted product) invariant

$$\alpha = \alpha^m_{CAT}(N) : \mathrm{Emb}\,^m_{CAT}(N) \to \pi^{m-1}_{eq}(\widetilde{N})$$

[Wu59, Hae61, Gro86, 2.1.E]. The main purpose of this paper is to obtain new results on surjectivity and injectivity of α, i.e. on the classical problems on embeddability and isotopy in $\mathbb{R}^m$. The set $\pi^{m-1}_{eq}(\widetilde{N})$ can be effectively calculated [CoFl60, beginning of §2, Hae62B, Hae63, 1.7.1, Bau75, Ada93, 7.1], see also §§5,6 below. Our starting point was the following classical result.

Theorem 1.1α [Hae63, Th. 1', Web67, Th. 1 and 1']. *If N is either a polyhedron or a closed smooth n-manifold, then*

$\alpha^m(N)$ *is bijective for* $2m \geq 3n + 4$ *and surjective for* $2m \geq 3n + 3$.

The 'metastable' dimension restrictions as in Theorem 1.1α were present also in the PL cases of the classical theorems on embeddings of highly-connected (generalized) manifolds and of Poincaré complexes [PWZ61, Irw65, Hud67, Hud69, §11, BoHa70, 1.6, Wal70, §11, Boe71, 4.2, BrMi99, BrMi00]. In these results the dimension restrictions can be weakened to $m \geq n+3$. As for Theorem 1.1α, these restrictions were known to be sharp not only in the smooth case, but also for polyhedra.

Example 1.2. i) The maps $\alpha^{3l}(S^{2l-1} \sqcup S^{2l-1})$ and $\alpha^{3l}_{DIFF}(S^{2l-1})$ are not injective [Hae62A, Hae62T, Zee62].

ii) The map $\alpha^m_{PL}(N)$ is not injective for $n + 2 \leq m \leq \frac{3n+3}{2}$ and $N = (S^n \vee S^n) \sqcup S^{2m-2n-3}$.

s) The map $\alpha^m(S^n \sqcup S^n)$ is not surjective, if $m \geq n+3$ and $\Sigma^\infty : \pi_n(S^{m-n-1}) \to \pi^S_{2n+1-m}$ is not epimorphic [Hae62T, Zee62].

ss) The map $\alpha^{6k+1}_{DIFF}(S^{2k} \times S^{2k})$ is not surjective [follows from BoHa70, Boe71].

e) The map $\alpha^{n+3}_{DIFF}(N)$ is not surjective for $n \in \{8, 9, 10, 16\}$ and some homotopy n-sphere N [HLS65, Lev65, cf. Ree90, §2, MaTh95, pp. 407–408].

ee) The map $\alpha^m_{PL}(N)$ is not surjective for $\max\{4, n\} \leq m \leq \frac{3n+2}{2}$ and some n-polyhedron N [MaSe67, SeSp92, FKT94, SSS98].

Example 1.2 (and Examples 1.4 and 1.10 below) are true for *each* set of the parameters m, n, k, l satisfying to the conditions in the statement. Examples 1.2.ii and 1.2.ss are proved in §7. Examples 1.2.e and 1.2.ee have a stronger property:

(∗) *there exists an equivariant map* $\widetilde{N} \to S^{m-1}$ *but N does not embed into* $\mathbb{R}^m$.

Links and knots give many other examples of non-injectivity and non-surjectivity of α, e.g. any CAT non-trivial knot $S^n \to \mathbb{R}^m$ demonstrates non-injectivity of $\alpha^m_{CAT}(S^n)$ (note that from a link example by gluing an arc joining connected components we can obtain highly connected polyhedral example). The more surprising

is our first main result, which asserts that the metastable dimension restrictions can be weakened to $m \geq n + 3$ for the PL case of Theorem 1.1α and closed highly-connected *PL n-manifolds*:

Theorem 1.3α. *If N is a closed d-connected (for $d = 1$, just homologically 1-connected) PL n-manifold and $m \geq n + 3$, then*

$\alpha_{PL}^m(N)$ *is bijective for $2m \geq 3n + 3 - d$ and surjective for $2m \geq 3n + 2 - d$.*

(Let us made some remarks not used in the sequel. In Theorem 1.3α the surjectivity is not interesting for $m < \frac{5n+6}{4}$. Indeed, $\frac{5n+6}{4} > \frac{3n+2-d}{2}$ implies that $d > \frac{n}{2} - 1$ and $n \geq 6$, hence N is a homotopy sphere, so $N \cong S^n$, and the surjectivity in Theorem 1.3α is trivial. But our proof is not simplified for $m \geq \frac{5n+6}{4}$, and it can also be considered as a step towards the analogue of Theorem 1.3α for embeddings into *manifolds*. Analogous remark should be made for the injectivity in Theorem 1.3α. Note that our proof of Theorem 1.3α does not give relative and approximative versions, which are true for Theorem 1.1α [Hae63, 1.7.2, Web67, Theorems 3 and 7, ReSk98].)

For $d \leq 2$ Theorem 1.3α was proved in [Sko97]. Although we use some ideas of [Sko97], the proof of Theorem 1.3α in the present paper follows a distinct route via immersions. Our second main result is an extension beyond the metastable range of the Harris PL version of the Haefliger–Hirsch classification of immersions (the precise statements are given below).

By [Hae68, Hud70T], the same $(3n - 2m + 2)$-connectedness assumption as in the surjectivity part of Theorem 1.3α ($2m \geq 3n + 2 - d \Leftrightarrow d \geq 3n - 2m + 2$) is unnecessary in [Hud67] (where, roughly speaking, it was proved that a homotopy equivalence between PL manifolds is homotopic to a PL embedding). So it was natural to expect that the connectedness assumption is unnecessary in Theorem 1.3α. However, our third main result is that this connectedness assumption is *essential*.

Example 1.4. i) $\alpha^{6k}(S^p \times S^{4k-1})$ is not injective for $p < k$.

s) $\alpha^m(S^1 \times S^{n-1})$ is not surjective, if $m - n$ is odd ≥ 3 and $\Sigma^\infty : \pi_{n-1}(S^{m-n}) \to \pi_{2n-m-1}^S$ is not epimorphic, e.g.

n	7	10	13	14	15	22
m	$10 = \frac{3n-1}{2}$	$13 = \frac{3n-4}{2}$	$18 = \frac{3n-3}{2}$	$19 = \frac{3n-4}{2}$	$22 = \frac{3n-1}{2}$	$31 = \frac{3n-5}{2}$

Examples 1.4.i and 1.4.s show that $\alpha_{PL}^m(N)$ can fail to be injective for $2m = 3n - 3d$ and to be surjective for $d = 0$, $2m \leq 3n - 1$. They are constructed in §7 using linking coefficient (in fact, stronger results are proved there). Using more

advanced *the Hudson–Habegger invariant* we prove in [Sko] that the dimension restriction in Theorem 1.3α is even *sharp*: $\alpha^{6k+1}(S^1 \times S^{4k-1})$ is not injective, which shows that α^m_{PL} can fail to be injective for $2m = 3n - d + 2$.

Now we list some new corollaries of Theorem 1.3α. Throughout this paper the equality between sets denotes a 1-1 correspondence.

Corollary 1.5. *If $p \le q$ and $m \ge \frac{3q}{2} + p + 2$ ($m \ge \frac{3(p+q)}{2} + 2$ for CAT=DIFF), then*

$$\mathrm{Emb}^m(S^p \times S^q) = \pi_q(V_{m-q,p+1}) \oplus \pi_p(V_{m-p,q+1}).$$

If $s \ge 3$, $n = p_1 + \cdots + p_s$, $p_1 \le \cdots \le p_s$ and $m \ge 2n - p_1 - p_2 + 3$ (for $s = 3$ and CAT=DIFF assume also that $m \ge \frac{3n}{2} + 2$), then

$$\mathrm{Emb}^m(S^{p_1} \times \cdots \times S^{p_s}) = \oplus^s_{i=1}\pi_{n-p_i}(V_{m-n+p_i,p_i+1}).$$

Note that $\pi_p(V_{m-p,q+1}) = 0$ for $m \ge 2p + q + 2$ (which is automatic for $m \ge \frac{3(p+q)}{2} + 2$). Also note that in Corollary 1.5 $m \ge \frac{3n}{2} + 2$ is automatic for $s \ge 4$. For calculations of $\pi_q(V_{ab})$ see [Pae54]. Corollary 7.2 gives the following table of values of $|\mathrm{Emb}^m(S^1 \times S^q)|$ for $m \ge \frac{3q+6}{2}$ ($m \ge \frac{3q+7}{2}$ for CAT=DIFF).

m	$\ge 2q+3$	$2q+2$	$2q+1$	$2q$	$2q-1$	$2q-2$	$2q-3$
$\|\mathrm{Emb}\|$, q even	1	∞	2	4	4	24	1
$\|\mathrm{Emb}\|$, q odd	1	2	∞	4	48	2	1

Corollary 1.5 is known for either $p = 0$ or $p_1 = 0$ [Hae62T, Zee62] and for either $m \ge 2q + p + 1$ or $m \ge 2n - p_1 + 1$ (in particular, for $p_2 \le 2$) [Hud63, HaHi63, Hud69, §11]. Otherwise it is new even for either $m \ge \frac{3(p+q)}{2} + 2$ or $m \ge \frac{3n}{2} + 2$. See also [Zee63, Kat69, Lev69, BoHa70, Boe71, MiRe71]. Corollary 1.5 follows from Torus Lemmas 6.1 and 6.2 (because if $s \ge 3$ and $p_1 \le \cdots \le p_s$, then $m \ge 2n - p_1 - p_2 + 3$ implies that $m \ge \frac{3n+4-p_1}{2}$).

From Theorem 1.3α and the smoothing theory [Hae67, 1.6, Hae, 11.1] it follows:

Corollary 1.6. *Let N be a closed d-connected (for $d = 1$, just homologically 1-connected) smooth n-manifold and $m \ge n + 3$.*

s) If $m \ge \frac{3n+2-d}{2}$, then for each $\Phi \in \pi^{m-1}_{eq}(\widetilde{N})$ there is a PL embedding $f : N \to \mathbb{R}^m$ smooth outside a point and such that $\alpha(f) = \Phi$. A complete obstruction to smoothing of f lies in C^{m-n}_{n-1}.

i) If $m \ge \frac{3n+3-d}{2}$, then each two smooth embeddings $f, g : N \to \mathbb{R}^m$ such that $\alpha(f) = \alpha(g)$ can be joined by a PL isotopy, which is smooth outside a point. A complete obstruction to smoothing of such a PL isotopy lies in C^{m-n}_n.

Corollary 1.7. *If N is a homologically 1-connected closed smooth n-manifold, then $\alpha_{DIFF}^m(N)$ is injective for $(m,n) = (12q-2, 8q-2)$ and surjective for $(m,n) = (12q-1, 8q-1)$.*

Recall that a closed manifold N (or a pair $(N, \partial N)$) is called *homologically d-connected*, if N is connected and $H_i(N) = 0$ for each $i = 1, \ldots, d$ (or $H_i(N, \partial N) = 0$ for each $i = 0, \ldots, k$). Throughout this paper we omit $\mathbb{Z}$-coefficients from the notation of (co)homology groups. We use the agreement that 0-connectedness is equivalent to homological 0-connectedness and to connectedness, and that (homological) k-connectedness for $k < 0$ is an empty condition. Recall that $C_n^k = \mathrm{Emb}_{DIFF}^{n+k}(S^n)$.

Corollary 1.7 follows from Corollary 1.6 and $C_{8q-2}^{4q} = 0$ [Hae66A, 8.15] (there is a misprint in [Hae66A, 8.15]: $C_{4k-2}^{3k} = 0$ should be $C_{8k-2}^{4k} = 0$).

The case $d = 1$ of Theorem 1.3α can be applied to replace 'connectedness' by 'homological connectedness' in [BoHa70, 1.6, Boe71, 4.2, Hud69, §11, Sko97, 1.3 and 1.4]:

The results [Sko97, Corollaries 1.5 and 1.6] were proved in the paper J. Vrabec, Knotting a k-connected closed PL m-manifold in R^{2m-k}, Trans. Amer. Math. Soc., 233 (1977), 137–165.

Corollary 1.8. s) *A homologically $(l-2)$-connected closed PL (for $l = 4q$, also smooth) $(2l-1)$-manifold N embeds PL (for $l = 4q$, also smoothly) in $\mathbb{R}^{3l-1}$ if and only if $\bar{W}_l(N) = 0$.*

i) $\mathrm{Emb}_{PL}^{3l+1}(N) = H_l(N, \mathbb{Z}_{(l)})$ *for $l > 1$ and a homologically $(l-1)$-connected closed PL $2l$-manifold N.*

Note that $|\mathrm{Emb}_{PL}^m(N)| = 1$ for a homology n-sphere N and $m \geq n+3$ [Sch77].

Corollary 1.9. 1) *Every closed homologically 2-connected smooth 7-manifold smoothly embeds in $\mathbb{R}^{11}$.*

2) *Every closed non-orientable PL 6-manifold N such that $\bar{w}_2(N) = \bar{w}_3(N) = 0$ PL embeds into $\mathbb{R}^{10}$.*

Corollary 1.9.1 follows from Corollary 1.8.s and [Mas62, Sko97, discussion after Theorem 1.1]. Corollary 1.9.2 follows from Theorem 1.3α and [Bau75, Theorem 45, case $m = 7$].

Now we state main results on immersions and embeddings of compact manifolds with (non-empty) boundary. Recall some definitions from [LiSi69, §3]. A PL map $h : N \to \mathbb{R}^m$ from a polyhedron N is called a *PL immersion*, if it is locally injective, i.e. there is an $\varepsilon > 0$ such that $hx \neq hy$ when $\mathrm{dist}(x, y) < \varepsilon$. For $m \geq n+3$, every embedding $D^n \to \mathbb{R}^m$ is flat, so this definition coincides with the usual one for PL manifolds [HaPo64]. Two immersions $h_0, h_1 : N \to \mathbb{R}^m$ are called *regular concordant*, if there is an immersion $H : N \times I \to \mathbb{R}^m \times I$ such that $H(x, 0) = h_0(x)$

and $H(x, 1) = h_1(x)$. Denote by $\mathrm{Imm}^m_{CAT}(N)$ the set of CAT immersions $N \to \mathbb{R}^m$ up to CAT regular concordance. Two immersions $h_0, h_1 : N \to \mathbb{R}^m$ are *regular homotopic*, if there is an homotopy $H : N \times I \to \mathbb{R}^m \times I$ between $h_0(x)$ and $h_1(x)$ which is an immersion itself. For $m \geq n + 3$ regular concordance implies regular homotopy [Hae66A, 4.1, Hae, 9.2, Hud70A, cf. Mel] (this remark is not used in this paper). Two embeddings $f_0, f_1 : N \to \mathbb{R}^m$ are *concordant*, if there is an embedding $F : N \times I \to \mathbb{R}^m \times I$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. Denote by $\mathrm{Emb}^m_{CAT}(N)$ the set of CAT embeddings $N \to \mathbb{R}^m$ up to CAT concordance. For $m \geq n + 3$ concordance implies isotopy [Lic65, Hud70A], so this new definition agrees with the previous one.

For a sufficiently small neighborhood $O\Delta$ of the diagonal Δ in $N \times N$, let $SN = O\Delta - \Delta$. The reason for using such a notation is that for a smooth manifold N, the space SN has the same equivariant homotopy type as the space of unit tangent vectors. When N is a polyhedron, the equivariant homotopy type of SN does not depend on $O\Delta$, provided that it is sufficiently small. For an immersion $h : N \to \mathbb{R}^m$, the map $\widetilde{h}$ is well-defined on SN. Define the *Haefliger–Hirsch invariant*

$$\beta = \beta^m_{CAT}(N) : \mathrm{Imm}^m_{CAT}(N) \to \pi^{m-1}_{eq}(SN) \qquad \text{by} \qquad \beta(h) = [\widetilde{h}] \in \pi^{m-1}_{eq}(SN).$$

By Theorem 5.2, α and β are indeed concordance invariants for $m \geq n + 2$.

Theorem 1.1 (continuation). β) [HaHi62, §4] *If N is a closed smooth n-manifold, then*

$\beta^m_{DIFF}(N)$ *is bijective for $2m \geq 3n + 2$ and surjective for $2m \geq 3n + 1$.*

$\beta\partial$) *[HaHi62, Rem. in §5] If N is a compact smooth n-manifold with boundary and the pair $(N, \partial N)$ is homologically d-connected, then*

$\beta^m_{DIFF}(N)$ *is bijective for $2m \geq 3n + 1 - d$ and surjective for $2m \geq 3n - d$.*

$\alpha\partial$) *[Hae63, 6.4] If N is a compact smooth n-manifold with boundary and an $(n - d - 1)$-dimensional spine, then*

$\alpha^m_{DIFF}(N)$ *is bijective for $2m \geq 3n + 1 - d$ and surjective for $2m \geq 3n - d$.*

For a compact connected n-manifold with boundary, the property of having an $(n - d - 1)$-dimensional spine is close to d-connectedness. Indeed, such a manifold with this property has boundary and is homologically d-connected. On the other hand, every such manifold N for which $(N, \partial N)$ is d-connected, $\pi_1(\partial N) = 0$, $d + 3 \leq n$ and $(n, d) \notin \{(5, 2), (4, 1)\}$, has an $(n - d - 1)$-dimensional spine [Wal64, Theorem 5.5, Hor69, Lemma 5.1 and Remark 5.2].

The following is our second main result.

Theorem 1.3 (continuation). β) *If N is an n-polyhedron, then*

$\beta^m_{PL}(N)$ *is bijective for $2m \geq 3n + 3$ and surjective for $2m \geq 3n + 2$.*

$\beta\partial$) *If $m \geq n + 2$ and N is a compact PL n-manifold with boundary and an $(n - d - 1)$-dimensional spine, then*

$\beta^m_{PL}(N)$ *is bijective for $2m \geq 3n + 2 - d$ and surjective for $2m \geq 3n + 1 - d$.*

$\alpha\partial$) *The same as $(\beta\partial)$ with β replaced by α.*

Example 1.10. i) The map $\beta^m_{DIFF}(S^{4k-1})$ is not injective for $4k+2 \leq m \leq 6k-1$.
   ii) The map $\beta^{3l+1}_{PL}(S^{2l})$ is not injective.
   s) The map $\beta^m_{PL}(N)$ is not surjective for $\max\{5, n\} \leq m \leq \frac{3n+1}{2}$ and some n-polyhedron N.

The dimension restrictions are sharp in the injectivity of Theorem 1.1β, in the injectivity of Theorem 1.3β and in the surjectivity of Theorem 1.3β by Example 1.10 (which is deduced from known examples in §4). Theorems 1.3$\beta\partial$ and 1.3$\alpha\partial$ were conjectured in [Har69, discussion after Corollary 5]. Theorem 1.3β under a bit stronger dimension restrictions was proved in [Har69, Corollary 1, Theorem 2, footnote on p. 3] and essentially proved in [Web67, §6]. But Theorem 1.3β is interesting not only because it improves known results, but also for its proof (§4). This proof, in contrast to that of Theorem 1.3α, does give the approximative relative version: *if N is an n-polyhedron, $2m \geq 3n+1$, $g : N \to \mathbb{R}^m$ is a PL map, $A \subset N$ a subpolyhedron such that $g|_A$ is a PL immersion and $\Phi : SN \to S^{m-1}$ an equivariant extension of $\widetilde{g} : SA \to S^{m-1}$, then there is a PL immersion $h : N \to \mathbb{R}^m$, arbitrarily close to g and such that $h = g$ on A and $\widetilde{h} \simeq_{eq} \Phi$ on SN rel SA.* See also Corollary 5.3 and 5.4.

A possible candidate to an obstruction for embeddability and isotopy in the case when the deleted product obstruction fails to be complete is the deleted G-product obstruction. For a subgroup $G \subset S_r$ let

$$\widetilde{N}_G = \{(x_1, \ldots, x_r) \in N^r \mid x_i \neq x_{\sigma(i)} \text{ for each } \sigma \in G, \ i = 1, \ldots, r\}$$

be the *deleted G-product* of N. The group G obviously acts on $\widetilde{N}_G$. For an embedding $f : N \to \mathbb{R}^m$ the map $\widetilde{f}_G : \widetilde{N}_G \to \widetilde{\mathbb{R}^m}_G$ is defined by $\widetilde{f}_G(x_1, \ldots, x_r) = (fx_1, \ldots, fx_r)$. Clearly, the map $\widetilde{f}_G$ is G-equivariant. Thus the existence of a G-equivariant map $\Phi : \widetilde{N}_G \to \widetilde{\mathbb{R}^m}_G$ is the *deleted G-product necessary condition for embeddability $N \to \mathbb{R}^m$*. Similarly one can define the *deleted G-product necessary condition for immersability*, the *G-Haefliger–Wu* and the *G-Haefliger–Hirsch* isotopy invariants α_G and β_G.

This approach worked well in the link theory (the simplest example is classification of 'higher-dimensional Borromean rings' [Hae62T, §3, Mas90, Proposition

8.3] demonstrating non-injectivity of α^m by means of $\alpha^m_{S_3}$). In contrast to that, Examples 1.2.i, 1.2.ii, 1.10.i and 1.10.ii (and the example of [Sko]) demonstrate non-injectivity of α_G and β_G (for each G): in their formulations α or β can be replaced by α_G or β_G (for each G). This follows by the construction of these examples (§4, §7). Clearly, if α or β is not surjective, then so is α_G or β_G, respectively (for each G). Under the conditions of Examples 1.2.e and 1.2.ee, $(*)$ is true even if we replace $\mathbb{Z}_2$-equivariant map $\widetilde{N} \to S^{m-1}$ by G-equivariant map $\widetilde{N}_G \to \widetilde{\mathbb{R}^m}_G$.

These results were announced in [Sko98', Sko98", ReSk99, §4, ReSk99',§4, Sko99]. Previous versions of the present paper were entitled as [Sko98']. I would like to acknowledge A. Haefliger for sending me a copy of [Hae], S. Melikhov and I. Izmestiev for many remarks, A. Melnichenko for a nice surprise, M. M. Postnikov for useful discussions, the referee for his comments and A. Kuligin for preparing computer versions of most figures.

2. Plan of the proof and related results

The proof of Theorems 1.3α and 1.3$\alpha\partial$ consists of two steps: construction of an immersion (Theorem 1.3$\beta\partial$) and modification of the immersion to an embedding [see also ReSk99, §11]. The first step could be replaced by reference to Theorem 1.1$\beta\partial$ for the case when N is smooth outside a point (and, in the injectivity part, when given embeddings f_0 and f_1 such that $\alpha(f_0) = \alpha(f_1)$ are smooth outside a point). This section is devoted to the plan of the second step. We also present the plan of the whole paper and notation which is used throughout the paper.

We state several classical results and their generalizations, which are of independent interest and which imply Theorems 1.3α and 1.3$\alpha\partial$. These results also imply Theorems 1.1α and 1.1$\alpha\partial$ in the PL case, thus providing new short proofs. First let us introduce some definitions. We say that $\alpha/\beta^m(N)$ *is surjective*, if for each immersion $h : N \to \mathbb{R}^m$ and an equivariant map $\Phi : \widetilde{N} \to S^{m-1}$ such that $\Phi \simeq_{eq} \widetilde{h}$ on SN there is a regular homotopy from h to an embedding $f : N \to \mathbb{R}^m$ such that $\widetilde{f} \simeq_{eq} \Phi$. We say that $\alpha/\beta^m(N)$ *is injective*, if for each embeddings $f_0, f_1 : N \to \mathbb{R}^m$, an equivariant homotopy $\varphi : \widetilde{N} \times I \to S^{m-1}$ from $\widetilde{f}_0$ to $\widetilde{f}_1$ and a regular homotopy

$H : N \times I \to \mathbb{R}^m$ from f_0 to f_1 such that $\widetilde{H}|_{SN \times \mathrm{diag}\, I} \simeq_{eq} \varphi|_{SN \times I}$ rel $SN \times \{0,1\}$,

there is a regular homotopy from H to a concordance between f_0 and f_1. Note that neither the surjectivity of α/β nor the injectivity of α/β are properties of maps α or β, but no confusion would arise.

Theorem 2.1. α/β) [Hae63, Th. 2', Web67, Th. 8] *If N is either an n-polyhedron or a smooth n-manifold, then*

$\alpha/\beta^m(N)$ *is injective for $2m \geq 3n + 4$ and is surjective for $2m \geq 3n + 3$.*

$\alpha/\beta\partial$) [Hae63, 6.4, Har69, Cor. 5.(ii) and Rem. after Cor. 5]. If N is a non-closed n-manifold, possessing an $(n-d-1)$-dimensional spine, then

$$\alpha/\beta^m(N) \text{ is injective for } 2m \geq 3n+1-3d \text{ and is surjective for } 2m \geq 3n-3d.$$

Clearly, the surjectivity of α/β and the surjectivity of β imply the surjectivity of α. Thus Theorem 1.1α (Theorem 1.3$\alpha\partial$) essentially follows from Theorems 1.1β and 2.1.α/β (from Theorems 1.3$\beta\partial$ and 2.1.$\alpha/\beta\partial$). Some extra remarks are required for the injectivity [Hae63, cf. Web67, Sko97], but we omit details.

Note that $\alpha/\beta_{PL}^m(N)$ is not surjective for $2m = 3n+2$ and some n-polyhedron N. Indeed, $\alpha_{PL}^m(N)$ is not surjective by Example 1.2.ee, but $\beta_{PL}^m(N)$ is by Theorem 1.3β. Note that the 'almost-embedding' $g : N \to \mathbb{R}^m$ from [SSS98] is an immersion, but it does not necessarily have the property $\widetilde{g} \simeq_{eq} \Phi$ on SN.

Let us introduce some more definitions, which would allow us to give formal proof of Theorem 1.3α. We identify $\mathring{B}^m$ and $\mathbb{R}^m$. For a map $f : N \to B^m$ denote

$$\Sigma(f) = \mathrm{Cl}\{x \in N : |f^{-1}fx| > 1\}.$$

A map $f : N \to B^m$ of a PL manifold (a polyhedron) N is a *quasi-embedding*, if $\Sigma(f)$ is contained in a n-ball $B^n \subset N$ (in a regular neighborhood of a point, respectively) and $f\Sigma(f) \subset \mathring{B}^m$. Two embeddings $f_0, f_1 : N \to B^m$ are PL *quasi-concordant*, if there is a PL quasi-embedding $F : N \times I \to B^m \times I = B^{m+1}$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$.

For a triangulation T of a polyhedron N let

$$\widetilde{T} = \cup\{\sigma \times \tau \in T \times T \mid \sigma \cap \tau = \emptyset\}$$

be the *simplicial deleted product* of T. By [Hu60], $\widetilde{T}$ is an equivariant deformation retract of $\widetilde{N}$ (but we will not confuse $\widetilde{N}$ and $\widetilde{T}$, because e.g. for $B \subset N$, the space $\widetilde{T} - \widetilde{B}$ is not necessarily an equivariant deformation retract of $\widetilde{N} - \widetilde{B}$). Therefore if $f : N \to B^m$ is a quasi-embedding, then there exists an equivariant map $\widetilde{N} \to S^{m-1}$. Analogous result holds for $\widetilde{N}_G$ and α_G (for each G). See also Theorem 5.2.

We say that the map $\alpha^m(N)$ is *quasi-surjective*, if for each equivariant map $\Phi : \widetilde{N} \to S^{m-1}$ there is a quasi-embedding $f : N \to \mathbb{R}^m$ such that $\widetilde{f} \simeq_{eq} \Phi$ on $\widetilde{N} - \widetilde{B^n}$. We say that the map $\alpha^m(N)$ is *quasi-injective*, if each embeddings $f_0, f_1 : N \to \mathbb{R}^m$ such that $\widetilde{f_0} \simeq_{eq} \widetilde{f_1}$ are quasi-concordant. Note that neither quasi-surjectivity nor quasi-injectivity is a property of α, but no confusion could arise. The definition of *quasi-surjectivity (quasi-injectivity) of $\alpha/\beta^m(N)$* is obtained from the definition of surjectivity (injectivity) of $\alpha/\beta^m(N)$ by replacing 'embedding' by 'quasi-embedding' and '$\widetilde{f} \simeq_{eq} \Phi$' by '$\widetilde{f} \simeq_{eq} \Phi$ on $\widetilde{N} - \widetilde{B^n}$' (by replacing 'concordant' by 'quasi-concordant'). Note that neither the quasi-surjectivity of α (α/β) nor the

quasi-injectivity of α (α/β) are properties of maps α or β, but no confusion would arise.

Theorem 2.2. *Let N be a closed PL n-manifold and $d \geq 0$.*
  q) *If N is d-connected and $m \geq n + 2$, then*

$$\alpha_{PL}^m(N) \text{ is quasi-injective for } 2m \geq 3n + 2 - d$$
$$\text{and quasi-surjective for } 2m \geq 3n + 1 - d.$$

  q/β) *If N is d-connected, then*

$$\alpha/\beta_{PL}^m(N) \text{ is quasi-injective for } 2m \geq 3n + 2 - 2d$$
$$\text{and quasi-surjective for } 2m \geq 3n + 1 - 2d.$$

  αq) *Suppose that N is homologically d-connected and $m \geq n + 3$. If $2m \geq 3n + 3 - d$ and $\alpha_{PL}^m(N)$ is quasi-injective, then it is injective. If $2m \geq 3n + 2 - d$ and $\alpha_{PL}^m(N)$ is quasi-surjective, then it is surjective.*

Theorem 1.3α follows from Theorems 2.2.q and 2.2.αq. Note that Theorem 1.3α does not follow merely from Theorems 1.3$\alpha\partial$ and 2.2.αq: the given embedding $f : N - B^n \to \mathbb{R}^m$ can be extended over N, but the extension is not necessarily a quasi-embedding. Note that Theorems 2.2.q and 2.2.αq give an improvement of the PL case of Theorem 2.1.α/β beyond the metastable range, but only in a weaker form, i.e. the homotopy from f to h is not regular (this homotopy is through immersions, but is an immersion itself only outside a point). For $2m \geq 3n + 2 - d$ ($2m \geq 3n + 3 - d$) the quasi-surjectivity (quasi-injectivity) in Theorem 2.2.q is covered by the surjectivity (injectivity) in Theorem 1.3α.

Theorem 2.3. *Let N be a closed PL n-manifold, smooth outside a point.*
  q) *If N is d-connected, $m \geq n + 2$ and $d \geq 1$, then $\alpha_{PL}^m(N)$ is quasi-surjective for $2m \geq 3n - d$ and is quasi-injective on the subset of smooth quasi-embeddings for $2m \geq 3n + 1 - d$.*
  α) *If N is homologically 2-connected and $m \geq n + 3$, then $\alpha_{PL}^m(N)$ is surjective for $2m \geq 3n$ and is injective on the subset of smooth embeddings for $2m \geq 3n + 1$.*

Theorem 2.3.q is proved analogously to Theorem 2.2.q (see §3), using Theorem 1.1$\beta\partial$ on smooth immersions instead of Theorem 1.3$\beta\partial$ on PL immersions, which for $d \geq 1$ allows to relax the dimension assumption. Theorem 2.3.α follows from Theorems 2.3.q and 2.2.αq.

Theorem 2.2.αq follows from Theorem 2.4 below, which was essentially proved in [Hir65, cf. Sko97, Theorem 2.1.2] (in order to prove the surjectivity of Theorem 2.2.αq, before applying Theorem 2.4 we take a triangulation T of N such that B^n

is contained in an only simplex of T and recall that $\widetilde{T}$ is an equivariant deformation retract of $\widetilde{N}$).

Theorem 2.4. *Let N be a homologically d-connected PL n-manifold (either closed or non-closed) and $m \geq \max(\frac{3n+2-d}{2}, n+3)$. If $g : N \to B^m$ is a proper quasi-embedding, then there is an embedding $f : N \to B^m$ such that $f = g$ on $N - \mathring{B}^n$.*

For polyhedra $N \supset Z \supset Y$ the notation $R(Z, Y)$ means 'a sufficiently small regular neighborhood of Z rel Y in N', when first appears, and 'the regular neighborhood of Z rel Y in N', after the first appearance. Also, $R(Z) = R(Z, \emptyset)$. The notation $R_M(Z, Y)$, $R_M(Z)$ has the same meaning, only N is replaced to M.

Proof of Theorem 2.4. We may assume that N is connected, because we shall apply Theorem 2.4 only for $d \geq 0$ or because we can take a connected component of N, containing $\Sigma(g)$. Let $M = B^m - \text{Int } R_{B^m}(g(N - \mathring{B}^n), g\partial B^n)$. Since N is homologically $(3n - 2m + 2)$-connected, by Alexander and Poincaré duality theorems we have

$$H_i(M) \cong H^{m-1-i}(B^m - M, \partial B^m - M) \cong H^{m-1-i}(N - \mathring{B}^n, \partial N) \cong H_{n-m+1+i}(N) = 0$$

for $i \leq 2n - m + 1$. Since $m - n \geq 3$, it follows that M is simply connected. Therefore by the Hurewicz Isomorphism Theorem we have that M is $(2n - m + 1)$-connected. Hence by [Irw65], the embedding $g : \partial B^n \to \partial M$ extends to an embedding $f : B^n \to M$. Extending f as g outside B^n we complete the proof. $\square$

The plan of the paper is as follows. In §3 we prove the surjectivity of Theorems 2.2.q, 2.2.q/β, 2.1.α/β and 2.1.α/$\beta\partial$ for PL case using Theorem 1.3$\beta\partial$ (or Theorem 1.1$\beta\partial$, if N is smooth outside a point) and Disjunction Theorem 3.1. We also prove there Disjunction Theorem 3.1, which is one of our main tools. In §4 we prove Immersion Theorem 4.1 (using Disjunction Theorem 3.1) and deduce from it Theorems 1.3β and 1.3$\beta\partial$. We also deduce there from known results Example 1.10. In §5 we prove Cylinder Lemma 5.1 and use it to prove the injectivity in Theorems 2.2.q, 2.2.q/β, 2.1.α/β and 2.1.α/$\beta\partial$. We also deduce there Corollaries 5.2, 5.3, 5.4 and Pseudo-Isotopy Theorem 5.5, which are of independent interest. In §6 we prove Torus Lemmas 6.1 and 6.2, which imply Corollary 1.5 and are used in construction of Example 1.4. In §7 we prove Decomposition Lemma 7.1, Example 1.4 and Example 7.4, and deduce from known results Examples 1.2.ii and 1.2.ss.

Let us introduce some definitions (Figure 1). Throughout this paper we work in the PL category and follow the notation of [RoSa72], unless the contrary is explicitly stated. For $A \subset N$ denote $A^* = A \times N \cup N \times A$. For a map $f : N \to B^m$ let

$$\widetilde{\Delta}(f) = \text{Cl}_{N \times N}\{(x, y) \in N \times N \mid x \neq y, \; fx = fy\}.$$

Clearly, the map $\widetilde{f} : \widetilde{N} - \widetilde{\Delta}(f) \to S^{m-1}$ is well-defined by the formula $\widetilde{f}(x,y) = \frac{fx-fy}{|fx-fy|}$. Suppose that N is an n-polyhedron with a fixed triangulation T (or with a fixed cell subdivision T in the sence of [RoSa72]). We denote by small Greek letters simplices of T, unless otherwise indicated. We fix the product cell subdivision of $T \times T$ and $T \times I$. Denote $TN = \cup\{\sigma \times \tau \mid \sigma, \tau \in T, \ \sigma \cap \tau \neq \emptyset\}$. Clearly, for each immersion $h : N \to \mathbb{R}^m$ there is a triangulation T of N such that $\widetilde{\Delta}(h) \cap TN = \emptyset$.
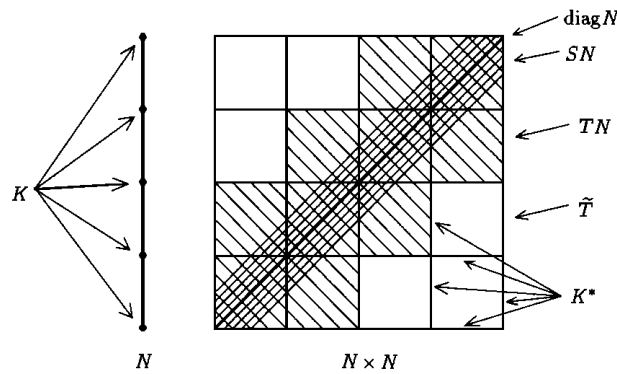

Figure 1

We often use the same notation for an element and its equivalence class, but no confusion should arise. By $|x,y|$ we denote the distance between points x and y. We consider the antipodal involution a_q on S^q (unless the opposite is explicitly stated). Two maps $\varphi, \psi : X \to S^{m-1}$ are *close*, if $\varphi(x) \neq -\psi(x)$ for each $x \in X$. Clearly, close equivariant maps are equivariantly homotopic. The phrase $\varphi \simeq_{eq} \psi$, and hence by the equivariant Borsuk Homotopy Extension Theorem we can modify ψ by a homotopy and assume further that $\varphi = \psi'$ is abbreviated to '$\varphi \simeq_{eq} \psi$, even $\varphi = \psi'$. We shall use the following essentially known result.

Theorem 2.5. *For a polyhedral pair (Y, Θ) with an involution and an equivariant map $\psi : \Theta \to S^m$ denote by $\pi_{eq}^m(Y, \Theta, \psi)$ the set of equivariant extensions $Y \to S^m$ of ψ, up to equivariant homotopy rel Θ. Then*

$$\Sigma : \pi_{eq}^{m-1}(Y, \Theta, \psi) \to \pi_{eq}^m(\Sigma Y, \Sigma \Theta, \Sigma \psi)$$

is a bijection for $\dim Y \leq 2m - 4$ and a surjection for $\dim Y \leq 2m - 3$.

　If $p : X \to Y$ is an equivariant onto map and the union of non-trivial preimages of p is homotopy equivalent to an l-polyhedron, then

$$p^* : \pi_{eq}^m(Y, \Theta, \psi) \to \pi_{eq}^m(X, p^{-1}\Theta, \psi \circ p)$$

is a bijection for $l \leq m - 2$ and a surjection for $l \leq m - 1$.

The proof of the first part (relative equivariant Suspension Theorem) is analogous to [CoFl60, Theorem 2.5, cf. Spa66, Chapter 8, §5, Theorem 11]. The second part is proved using general position and the Borsuk Homotopy Extension Theorem.

3. Modification of the immersion to a (quasi-)embedding

Disjunction Theorem 3.1. *Let N be a polyhedron with fixed triangulation T,*
   A a subcomplex of T,
   $A^* \subset E_0 \subset E_1$ equivariant subcomplexes of $T \times T$ such that $E_1 - E_0 \subset \widetilde{T}$,
   $\Phi : E_1 \cap \widetilde{T} \to S^{m-1}$ an equivariant map,
   *$h_0 : N \to B^m$ a PL map such that $h_0^{-1} \partial B^m = A$, $\widetilde{\Delta}(h_0) \cap E_0 = \emptyset$ and $\widetilde{h_0} = \Phi$
on $E_0 \cap \widetilde{T}$.*
   Suppose that for each simplices $\sigma^p, \tau^q, \nu^n \in T$ such that $p \leq q$, $\sigma \times \tau \subset \mathrm{Cl}(E_1 - E_0)$ and $\nu \times \tau \subset E_1$ we have $p + q + n \leq 2m - 3$ and $q \leq m - 2$.
   *Then there is a PL homotopy $h_t : N \to B^m$ rel A such that $h_t(N - A) \subset \mathring{B}^m$
for each t,*
   (3.1.1) *$\widetilde{\Delta}(h_t) \cap E_0 = \emptyset$ for each t and $\widetilde{\Delta}(h_1) \cap E_1 = \emptyset$,*
   (3.1.2) *the homotopy $\widetilde{h_t}$ from $\Phi = \widetilde{h_0}$ to $\widetilde{h_1}$ on $E_0 \cap \widetilde{T}$ extends to a homotopy
from Φ to $\widetilde{h_1}$ on $E_1 \cap \widetilde{T}$.*

Disjunction Theorem 3.1 generalizes the surjectivity of the PL case of Theorem 2.1.α/β, [Sko97, Theorems 2.1.1 and 3.1] and the pre-limit version of [ReSk98, Theorem 1.1, see also Mel]. Our proof of Disjunction Theorem 3.1 (below) extends the method of [Web67, §5] as it was exposed in [ReSk99, §9] (the extension of [SpTo91, Sko00] is in a different direction).

For many results of this paper we need the following simpler particular cases of Disjunction Theorem 3.1. In the proof of the injectivity (the surjectivity, respectively) in Theorems 2.1.α/β and 2.1.$\alpha/\beta\partial$ of the properties 3.1.1 and 3.1.2 we need only that $\widetilde{\Delta}(h_1) \cap E_1 = \emptyset$ (that $\widetilde{\Delta}(h_1) \cap E_1 = \emptyset$ and $\widetilde{h_1} \simeq \Phi$ on $E_1 \cap \widetilde{T}$, respectively). However, in the proof of Theorems 2.2.q, 2.2.q/β, 1.3β and 1.3$\beta\partial$ we use the complete strength of 3.1.2 (in the proof of Theorem 1.3$\beta\partial$ we apply Disjunction Theorem 3.1 for $N \cong S^n$ and $E_1 = \widetilde{S^n}$, then $\widetilde{h_1} \simeq \Phi$ on $\widetilde{S^n}$ for *any* embedding $h_1 : S^n \to \mathbb{R}^m$). Note that in Disjunction Theorem 3.1 we cannot enlarge E_0 to contain TN, even if $\widetilde{\Delta}(h_0) \cap TN = \emptyset$, because the dimension restrictions need not be preserved under this enlargement.

In this paragraph assume that $\bigcup_{\tau \in T} \tau \times \tau \subset E_0$. This case of Theorem 3.1 is sufficient to prove the surjectivity (the injectivity) in Theorem 2.2.q for $2m \geq 3n + 2 - d$ ($2m \geq 3n + 3 - d$) and $m \geq n + 3$, which in turn is sufficient to prove

Theorem 1.3α without the improvement for $d = 1$. The condition $E_1 - E_0 \subset \widetilde{T}$ can be dropped from Theorem 3.1 [Web67, §6, Sko98] (a minor mistake in [Sko98] is corrected in [ReSk99, §10]). In Theorem 3.1 we may assume that h_t is a $C \cdot \varepsilon$-homotopy, where $\varepsilon = \max\{\text{diam}(f\sigma \cup f\tau) \mid \sigma \times \tau \subset \text{Cl}(E_1 - E_0)\}$ and C depends only on dim N. This is proved analogously to [ReSk98, §3] (since the required inequality $2(p + q) \leq 3m - 5$ holds).

Proof of the quasi-surjectivity in Theorem 2.2.q. Take a map $\Phi : \widetilde{N} \to S^{m-1}$. Let K be the $(n - d - 1)$-skeleton of the dual cell-subdivision to a triangulation T_1 of N. Apply Theorem 1.3$\beta\partial$ (or Theorem 1.1$\beta\partial$, if N is smooth outside a point) to get a PL immersion $h_0 : R(K) \to \mathbb{R}^m$ such that $\widetilde{h_0} \simeq_{eq} \Phi$ on $S(R(K))$. Extend h_0 over N to obtain a map $h_0 : N \to \mathbb{R}^m$. Take a new triangulation T of N in which K is a subcomplex and such that for $E_0 = TN \cap K^*$ we have

$$\widetilde{\Delta}(h_0) \cap E_0 = \emptyset \quad \text{and} \quad \widetilde{h_0} \simeq_{eq} \Phi, \quad \text{even} \quad \widetilde{h_0} = \Phi \quad \text{on} \quad E_0 \cap \widetilde{N}.$$

Since $n + 2(n - 1 - d) \leq 2m - 3$, we can apply Disjunction Theorem 3.1 to $A = \emptyset$, $E_0 = TN \cap K^*$ and $E_1 = K^*$. Since $\widetilde{\Delta}(h_1)$ is closed, by 3.1.1 it follows that there exists a new regular neighborhood $R(K) \subset N - T_1^{(d)}$ such that $\widetilde{\Delta}(h_1) \cap R(K)^* = \emptyset$. Hence

$$\text{Cl}(N - R(K)) = R(T_1^{(d)}) \quad \text{and} \quad \Sigma(h_1) \cap R(K) = \emptyset.$$

Since N is d-connected, by the Engulfing Lemma [Irw65, Theorems 2.1 and 2.3] it follows that $T_1^{(d)}$ is contained in some PL n-ball in N. Therefore by the Uniqueness of Regular Neighborhood Theorem, $N - R(K)$ is also contained in some (possibly, another) PL n-ball $B^n \subset N$. We have $\Sigma(h_1) \subset N - R(K) \subset B^n$, hence h_1 is a quasi-embedding.

By 3.1.1 there is $U = R_{N \times N}(K^*, \widetilde{T}) \subset R(K)^*$ such that $\widetilde{\Delta}(h_t) \cap U \cap TN = \emptyset$ for each t (Figure 1). Since $\text{Cl}(E_1 - E_0) \cap U \subset E_0$, by 3.1.2 it follows that the homotopy $\widetilde{h_t}$ on $(U \cap TN) - \text{diag}\, N$ extends to a homotopy from Φ to $\widetilde{h_1}$ on $U - \text{diag}\, N$. By the equivariant version of [Coh69, Theorem 3.1 and Addendum 3.4], $U - \text{diag}\, N$ is a strong equivariant deformation retract of $R(K)^* - \text{diag}\, N$. Therefore

$$\widetilde{h_1} \simeq_{eq} \Phi \quad \text{on} \quad R(K)^* - \text{diag}\, N \supset \widetilde{N} - \widetilde{N - R(K)} \supset \widetilde{N} - \widetilde{B^n}. \qquad \square$$

Proof of the (quasi-)surjectivity in Theorems 2.2.q/β, 2.1.α/β and 2.1.α/$\beta\partial$ for the PL case. Take an immersion $h_0 : N \to \mathbb{R}^m$ and an equivariant map $\Phi : \widetilde{N} \to S^{m-1}$ such that $\widetilde{h_0} \simeq_{eq} \Phi$ on SN. Fix a triangulation T of N such that

$$\widetilde{\Delta}(h_0) \cap TN = \emptyset \quad \text{and} \quad \widetilde{h_0} \simeq_{eq} \Phi, \quad \text{even} \quad \widetilde{h_0} = \Phi \quad \text{on} \quad TN - \text{diag}\, N.$$

The quasi-surjectivity in Theorem 2.2.q/β is proved as in Theorem 2.2.q ($K = T^{(n-1-d)}$ and T_1 is the cell-subdivision dual to T).

The surjectivity in Theorem 2.1.α/β for the PL case follows by applying Disjunction Theorem 3.1 to $A = \emptyset$, $E_0 = TN$ and $E_1 = N \times N$.

In order to prove the surjectivity in Theorem 2.1.$\alpha/\beta\partial$ for the PL case, assume that the $(n-d-1)$-spine K is a subcomplex of T. Since $3(n-1-d) \leq 2m-3$, we can apply Disjunction Theorem 3.1 for $A = \emptyset$, $E_0 = TK$ and $E_1 = K \times K$. Since $\widetilde{\Delta}(h_1)$ is closed, it follows that there exists $R(K)$ such that $\widetilde{\Delta}(h_1) \cap (R(K) \times R(K)) = \emptyset$. Hence $h_1|_{R(K)}$ is an embedding. Analogously to the proof of Theorem 2.2.q, replacing U by $R_{N \times N}(K \times K, \widetilde{T}) \subset R(K) \times R(K)$, we have that $U - \operatorname{diag} N$ is an equivariant deformation retract of $\widetilde{R(K)}$, hence $\widetilde{h_1} \simeq_{eq} \Phi$ on $\widetilde{R(K)}$. By [Coh69, Theorem 3.1] there exists an isotopy $g_t : N \to N$ between $g_0 = \operatorname{id}$ and a homeomorphism $g_1 : N \to R(K)$. Therefore $h_1 \circ g_1 : N \to \mathbb{R}^m$ is an embedding and

$$\widetilde{h_1 \circ g_1} = \widetilde{h_1} \circ (g_1 \times g_1) \simeq_{eq} \Phi \circ (g_1 \times g_1) \simeq_{eq} \Phi \circ (g_0 \times g_0) = \Phi. \qquad \square$$

Although Theorem 2.2.q does not follow from Theorem 1.3$\beta\partial$, the former can hardly be improved without improving the latter. Indeed, even if there are no f-intersections of distant simplices, the intersections of close simplices are just as bad and may form a $(d+1)$-cycle.

Proof of Disjunction Theorem 3.1. Consider the case $A = \emptyset$ (the general case is proved analogously). We may assume that h_0 is in general position. It suffices to prove Theorem 3.1 for $\operatorname{Cl}(E_1 - E_0) = \sigma \times \tau \cup \tau \times \sigma \subset \widetilde{T}$. Denote $p = \dim \sigma \leq \dim \tau = q$. By the dimension hypothesis it follows that $2p + q \leq 2m - 3$.

Construction of a homotopy h_t satisfying to 3.1.1. We may assume that $p + q \geq m$ (otherwise 3.1.1 holds for $h_t = h_0$ by general position). Therefore $2p + q \leq 2m - 3$ implies that $p \leq m - 3$. Let $h = h_0$ and $x = 2m - p - q - 3$.

We begin with construction of certain balls D^m, D^p and D^q. Since $h\sigma \cap h\partial\tau = h\partial\sigma \cap h\tau = \emptyset$, it follows that $\sigma \cap h^{-1}h\tau \subset \mathring{\sigma}$ and $\tau \cap h^{-1}h\sigma \subset \mathring{\tau}$. By general position, $\dim(h\sigma \cap h\tau) \leq p + q - m$. Let $C_\sigma \subset \mathring{\sigma}$ be a general position trace of $\sigma \cap h^{-1}h\tau$ under a sequence of collapses $\sigma \searrow$ (a point in $\mathring{\sigma}$). Define similarly $C_\tau \subset \mathring{\tau}$. The polyhedra C_σ, C_τ are collapsible,

$$h\sigma \cap h\tau \subset hC_\sigma \cup hC_\tau \quad \text{and} \quad \dim C_\sigma, \dim C_\tau \leq p + q - m + 1.$$

Since $(p+q-m+1) + (2p-m) < p$, by general position it follows that $C_\sigma \cap \Sigma(h|_\sigma) = \emptyset$.

Let $C \subset \mathring{B}^m$ be a general position trace of $hC_\sigma \cup hC_\tau$ under a sequence of collapses $B^m \searrow$ (a point in $\mathring{B}^m$). The polyhedron C is collapsible, $h\sigma \cap h\tau \subset C$ and $\dim C \leq p + q - m + 2$. Hence by general position

$$C \cap hN^{(x)} = hC_\sigma \cup hC_\tau \quad \text{for} \quad x \geq q \quad \text{and} \quad C \cap hN^{(x)} = hC_\sigma \quad \text{for} \quad x < q.$$

Therefore we can take PL balls $D^m = R_{\mathring{B}^m}(C)$, $D^p = R_{\mathring{\sigma}}(C_\sigma)$ and $D^q = R_{\mathring{\tau}}(C_\tau)$ such that

(a) $D^p = \sigma \cap h^{-1}D^m$ and $h|_{D^p}$ is a proper unknotted embedding into D^m.

(b) $N^{(x)} \cap h^{-1}D^m = R_{N^{(x)}}(D^p, \partial D^p) \sqcup R_{N^{(x)}}(D^q, \partial D^q)$. Note that

$$R_{N^{(x)}}(D^q, \partial D^q) = \emptyset \text{ for } x < q.$$

(c) $\sigma \cap h^{-1}h\tau \subset \mathring{D}^p$, $\tau \cap h^{-1}h\sigma \subset \mathring{D}^q$ and $hD^q \subset D^m$.

Now we construct a map $h^+ : D^q \to D^m - hD^p$ which is 'the first approximation' to $h_1|_{D^q}$. By (a), $D^m - hD^p \simeq S^{m-p-1}$. By (c), $h\partial D^p \cap hD^q = h\partial D^q \cap hD^p = \emptyset$, so $\widetilde{h}$ is well-defined on $\partial(D^p \times D^q)$. We have

$$\pm\Sigma^p[h|_{\partial D^q} : \partial D^q \to D^m - hD^p] = [\widetilde{h}|_{\partial(D^p \times D^q)}] = [\Phi|_{\partial(D^p \times D^q)}] = 0 \in \pi_{p+q-1}(S^{m-1}).$$

Here the first equality holds by [Web67, Proposition 1]. The second equality holds since $\partial(\sigma \times \tau) \subset E_0 \cap \widetilde{T}$. The third equality holds since Φ is defined over $E_1 \cap \widetilde{T} \supset D^p \times D^q$. Since $q - 1 \leq 2(m - p - 1) - 2$, by the Freudenthal Suspension Theorem it follows that $h|_{\partial D^q}$ is null-homotopic in $D^m - hD^p$, i.e. the embedding $h|_{\partial D^q}$ extends to a map $h^+ : D^q \to D^m - hD^p$.

Now we construct the homotopy h_t in the case $\tau \times \tau \not\subset E_0$. Take $U = R(D^q, \partial D^q) \subset h^{-1}D^m$. Set h_1 to be h^+ on D^q and h outside U. Then extend h_1 over U to an arbitrary map whose image is in D^m. Join h_0 and h_1 by a homotopy h_t fixed outside U and mapping U to D^m.

Now we construct the homotopy h_t in the case $\tau \times \tau \subset E_0$. Then $p + 2q \leq 2m - 3$ and (since $p + q \geq m$) $q \leq m - 3$. Hence (a) holds also for D^q. Therefore by [PWZ 61, Irw65] from the map h^+ we can obtain an *embedding* $h_1 : D^q \to D^m - hD^p$ such that $h_1 = h$ on ∂D^q. Since $q \leq m - 3$, by [Zee63-6, Corollary 1 to Theorem 9] it follows that there is an ambient isotopy $g_t : D^m \to D^m$ rel ∂D^m carrying $h|_{D^q}$ to h_1. If $N^{(x)} = N$ (this case suffices to prove Theorem 1.3α), then by (b) we can define the homotopy $h_t : N \to B^m$ by setting

$$h_t = g_t \circ h \quad \text{on} \quad V = U := R_{N^{(x)}}(D^q, \partial D^q) \quad \text{and} \quad h_t = h \quad \text{outside} \quad U.$$

In general (i.e. without the assumption $N^{(x)} = N$, when D^p and D^q can lie in the same connected component of $h^{-1}D^m$) take

$$V = R(R_{N^{(x)}}(D^q, \partial D^q)) \cap h^{-1}D^m \quad \text{and} \quad U = R(V, h^{-1}(B^m - \mathring{D}^m))$$

(Figure 2). Then $\text{Cl}(U - V) \cap N^{(x)} = \emptyset$. Set $h_t = g_t \circ h$ on V and $h_t = h$ outside U. Since $g_t = \text{id}$ on ∂D^m, the map h_t is thus well-defined and could be extended over $U - V$ so that $h_t(h^{-1}D^m) \subset D^m$.

Now we check 3.1.1. By (c),

$$h_1\sigma \cap h_1\tau \subset h_1 D^p \cap h_1 D^q = \emptyset, \quad \text{so} \quad \widetilde{\Delta}(h_1) \cap \sigma \times \tau = \emptyset.$$

If, to the contrary, $\widetilde{\Delta}(h_t) \cap E_0 \neq \emptyset$ for some t, then there is a pair $\nu \times \mu \subset E_0$ such that $\nu \times \mu \cap (\widetilde{\Delta}(h_t) - \widetilde{\Delta}(h_0)) \neq \emptyset$. Recall that in both cases $\tau \times \tau \subset E_0$ and $\tau \times \tau \not\subset E_0$ above h_t is fixed outside U and $h_t U \subset D^m$. Therefore we may assume that $\mu \cap U \neq \emptyset$ and $\nu \cap h^{-1} D^m \neq \emptyset$. Since $\mu \cap U \neq \emptyset$, it follows that $\mu \supset \tau$, hence $\nu \times \tau \subset E_0$, therefore $\nu \subset N^{(x)}$ by our dimension assumption. Then by (b) and $\nu \cap h^{-1} D^m \neq \emptyset$ it follows that either $\nu \supset \sigma$ or $\nu \supset \tau$. Since $\mu \supset \tau$, $\nu \times \mu \subset E_0$ but $\sigma \times \tau \not\subset E_0$, it follows that in fact $\nu \supset \tau$. Hence $\tau \times \tau \subset E_0$ (in the first case this is already a contradiction) and $\mu \subset N^{(x)}$. Since h_t moves V isotopically and $\mathrm{Cl}(V - U) \cap N^{(x)} = \emptyset$, it follows that h_t moves $R(\nu \cup \mu)$ isotopically. So $\widetilde{\Delta}(h_t) \cap \nu \times \mu = \emptyset$. This contradiction shows that $\widetilde{\Delta}(h_t) \cap E_0 = \emptyset$.

Let
$$E = (E_0 \cap \widetilde{T}) \cup (\sigma \times \tau - \mathring{D}^p \times \mathring{D}^q) \cup (\tau \times \sigma - \mathring{D}^q \times \mathring{D}^p).$$

By the Borsuk Homotopy Extension Theorem, there is an extension $\Phi' : E_1 \to S^{m-1}$ of $\widetilde{h}|_E$ such that the homotopy $\widetilde{h}_t$ from $\Phi = \widetilde{h}_0$ to $\widetilde{h}_1$ on E extends to a homotopy from Φ to Φ' on $E_1 \cap \widetilde{T}$. Therefore now it suffices to prove the Theorem assuming that $\Delta(h_0) \cap E_1 = \emptyset$.
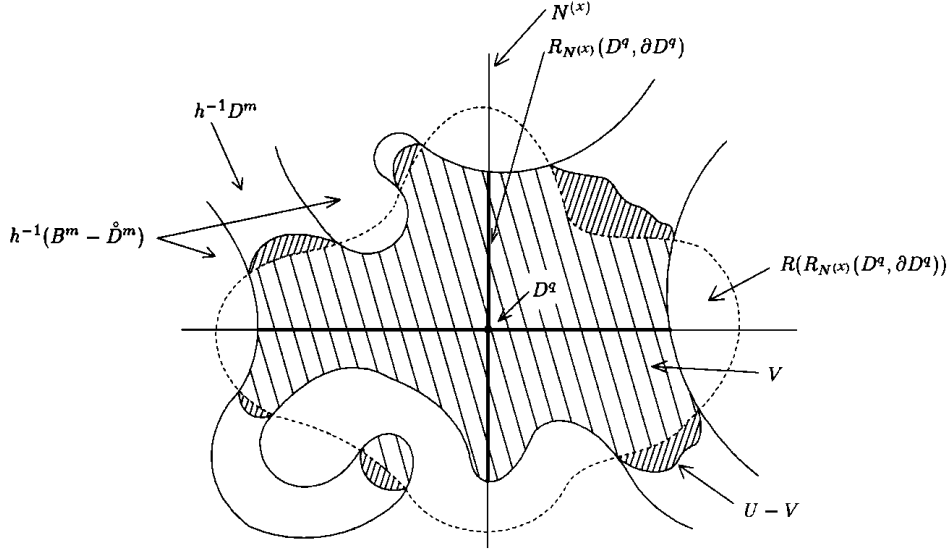

Figure 2

Proof of Theorem 3.1 assuming that $\Delta(h_0) \cap E_1 = \emptyset$. We use the notation and assumptions from the first paragraph of the proof. Let $h = h_0$ and $x = m - 2$.

We begin with a construction of certain balls D^m, D^p and D^q. By general position, we can take points $C_\sigma \in \mathring{\sigma}$ and $C_\tau \in \mathring{\tau}$ so that the restrictions of h to $R_\sigma(C_\sigma)$ and $R_\tau(C_\tau)$ are flat embeddings. Join points hC_σ and hC_τ by an arc $C \subset \mathring{B}^m$ such that $C \cap hN^{(x)} = \{hC_\sigma, hC_\tau\}$. We have $p \leq q \leq m-2$, and the balls D^m, D^p and D^q are constructed as above and have the properties (a) and (b), whereas (a) holds even for D^p replaced by D^q (note that $x \geq q$).

Now we construct the homotopy h_t as in the first part depending on some element $y \in \pi_q(S^{m-p-1})$. Suppose that $h^+ : D^p \sqcup D^q \to D^m$ is a map such that $h^+ = h$ on $D^p \sqcup \partial D^q$ and $h^+ D^p \cap h^+ D^q = \emptyset$. Define a map

$$d_{hh^+} : S^q = D^q \bigcup_{\partial D^q = \partial D^q_+} D^q_+ \to D^m - hD^p \text{ by } d_{hh^+}|_{D^q} = h \text{ and } d_{hh^+}|_{D^q_+} = h^+.$$

By (a), $D^m - hD^p \simeq S^{m-p-1}$, hence $[d_{hh^+}] \in \pi_q(S^{m-p-1})$. For each $y \in \pi_q(S^{m-p-1})$ we can construct a map h^+ such that $d_{hh^+} = y$. Then analogously to the first part we can construct a homotopy h_t such that $h_1|_{D^q}$ is homotopic to $h^+|_{D^q}$ in $D^m - h_1 D^p$ and 3.1.1 holds. (The only difference is construction of h_t in the case $\tau \times \tau \subset E_0$ and $q = m-2$, which is though not used in the proof of Theorem 1.3α. In this case we may assume that $p + q \geq m-1$, otherwise 3.1.2 holds by general position. Since $p + 2q \leq 2m-3$, it follows that $p = 1$. Then analogously to [MaRo86, Proposition 7.1] we may assume that h^+ is an *embedding, ambiently isotopic to the standard one*, so the required isotopy g_t is constructed without use of [PWZ61, Irw65].)

Now we choose y so that 3.1.2 holds. For any maps $\varphi_0, \varphi_1 : D^p \times D^q \to S^{m-1}$ and a homotopy

$$\varphi_t : \partial(D^p \times D^q) \to S^{m-1} \quad \text{from} \quad \varphi_0|_{\partial(D^p \times D^q)} \quad \text{to} \quad \varphi_1|_{\partial(D^p \times D^q)}$$

define a map $H_{\varphi_0 \varphi_t \varphi_1} : \partial(D^p \times D^q \times I) \to S^{m-1}$ by

$$H_{\varphi_0 \varphi_t \varphi_1}|_{D^p \times D^q \times 0} = \varphi_0, \quad H_{\varphi_0 \varphi_t \varphi_1}|_{\partial(D^p \times D^q) \times t} = \varphi_t \text{ and } H_{\varphi_0 \varphi_t \varphi_1}|_{D^p \times D^q \times 1} = \varphi_1.$$

Then

$$[H_{\Phi \widetilde{h}_t \widetilde{h}_1}] = [H_{\Phi \Phi_t \widetilde{h}}] + [H_{\widetilde{h} \widetilde{h}_t \widetilde{h}_1}] = [H_{\Phi \Phi_t \widetilde{h}}] \pm \Sigma^p [d_{hh_1}] \in \pi_{p+q}(S^{m-1}).$$

Here by Φ, $\widetilde{h}$ and $\widetilde{h}_1$ are denoted the restrictions of these maps onto $D^p \times D^q$; by $\Phi_t = \Phi$ and $\widetilde{h}_t$ are denoted the restrictions of these maps to $\partial(D^p \times D^q)$. The second equality holds by [Web67, lemme 1] (the hypothesis $m - p \geq 3$ in [Web67] can clearly be replaced to (a)). Since $q \leq 2(m-p-1) - 1$, by the Freudenthal Suspension Theorem it follows that we can take $y \in \pi_q(S^{m-p-1})$ so that $[H_{\Phi \Phi_t \widetilde{h}}] \pm \Sigma^p y = 0$. Then $[H_{\Phi \widetilde{h}_t \widetilde{h}_1}] = 0$, i.e the homotopy $\widetilde{h}_t$ on $\partial(D^p \times D^q)$ extends to a homotopy from Φ to $\widetilde{h}_1$ on $D^p \times D^q$. Then 3.1.1 and 3.1.2 hold.　□

4. Construction of an immersion

A possible approach to the proof of Theorem $1.3\beta\partial$ is to mimick the proof of the smooth case [HaHi62], see [ReSk99, §11] for the details. But this idea does not work because the *PL Stiefel manifold* V_{mn}^{PL} (which is the space of PL embeddings $S^{n-1} \to S^{m-1}$) is not a subset of the *equivariant Stiefel manifold* V_{mn}^{eq} (which is the space of equivariant maps $S^{n-1} \to S^{m-1}$). Neither it is obvious (and is possibly false) that $V_{mn}^{PL} \cap V_{mn}^{eq}$ is a deformation retract of V_{mn}^{PL}. Even if this problem can be fixed, the proof of Theorem $1.3\beta\partial$ in the present paper (based on a different idea) is shorter even than the proof of the results on homotopy groups of V_{mn}^{PL} [Hae67, Hae, Mil75], necessary to carry out the above approach.

Immersion Theorem 4.1. *Let N be an n-polyhedron with a triangulation T, $m \geq n+2$ and $\Phi : TN - \operatorname{diag} N \to S^{m-1}$ an equivariant map. Then there exist a PL map $h : N \to \mathbb{R}^m$ and a collection $\{\nu^+ = R(\nu, \partial\nu)\}_{\nu \in T}$ such that*

h is nondegenerate (i.e. $h|_\sigma$ is an embedding for each $\sigma \in T$),

$h|_{\nu^+}$ is an embedding whenever $\dim \nu \geq 3n - 2m + 2$ and

$\widetilde{h} \simeq_{eq} \Phi$ on $\bigcup\limits_{\dim \nu \geq 3n-2m+2} \widetilde{\nu^+}$.

Proof of Theorems 1.3β and $1.3\beta\partial$. In order to prove the surjectivity, take $\Phi \in \pi_{eq}^{m-1}(SN)$ and a triangulation T of N. First we prove the surjectivity in Theorem 1.3β. Since $2m \geq 3n+2$, we have $m \geq n+2$, so we can apply Immersion Theorem 4.1 to get a map h and a collection $\{\nu^+\}$. Since $h|_{\nu^+}$ is an embedding for each ν, it follows that h is an immersion. Since $SN \simeq_{eq} \bigcup\limits_{\nu \in T} \widetilde{\nu^+}$, it follows that $\beta(h) = \Phi$.

Now we prove the surjectivity in Theorem $1.3\beta\partial$. Take a triangulation T of N. We may assume that the $(n-d-1)$-spine K is contained in the dual skeleton of $T^{(d)}$. Apply Immersion Theorem 4.1 to get a map h and a collection $\{\nu^+\}$. Since $3n - 2m + 2 \leq d+1$, it follows that $R(K) \subset \bigcup\limits_{\dim \nu > d} \nu^+$, hence $h|_{R(K)}$ is an immersion. Since $S(R(K)) \subset \bigcup\limits_{\dim \nu > d} \widetilde{\nu^+}$, it follows that $\widetilde{h} \simeq_{eq} \Phi$ on $S(R(K))$. If $g_t : N \to N$ is an isotopy between $g_0 = \operatorname{id}$ and a homeomorphism $g_1 : N \to R(K)$, then $h \circ g_1 : N \to \mathbb{R}^m$ is an immersion and

$$\widetilde{h \circ g_1} = \widetilde{h} \circ (g_1 \times g_1) \simeq_{eq} \Phi \circ (g_1 \times g_1) \simeq_{eq} \Phi \circ (g_0 \times g_0) = \Phi \quad \text{on} \quad SN.$$

The injectivity is reduced to the boundary version of the surjectivity (which is proved analogously) in the same way as in §5 (using Cylinder Lemma $5.1.\beta$). $\quad\square$

Our proof of Immersion Theorem 4.1 uses Disjunction Theorem 3.1 and induction on simplices. Our idea is in a sense similar both to [Hir59] and to [Web67, §6, Har69, proof of Theorem 2]. The method of [Web67, Har69] was actually designed for the proof of Theorem 1.1α and thus is not optimal: the result of [Web67,

Har69] concerning the existence of an immersion has the dimension restriction $2m \geq 3n + 3$ instead of $2m \geq 3n + 2$ as in Theorem 1.3β. The proofs [Web67, Har69] can be perhaps modified to work for $2m \geq 3n + 2$ (at least to obtain *some* immersion h, i.e. without the property $\widetilde{h} \simeq_{eq} \Phi$ on SN).

Beginning of the proof of Immersion Theorem 4.1. Order simplices of T with respect to increasing of dimension. By '$<$' we denote the corresponding order relation and by '$\subset$' a proper face (thus $\nu \supset \eta \Rightarrow \dim \nu > \dim \eta \Rightarrow \nu > \eta$). By downward induction on η we may assume that for an r-simplex $\eta \in T$ such that $r \geq 3n - 2m + 2$ there exists a nondegenerate PL map $H : N \to \mathbb{R}^m$ and a collection $\{\nu^+ = R(\nu, \partial\nu)\}_{\nu \in T}$ such that

(a) $H|_{\nu^+}$ is an embedding for each $\nu > \eta$;

(b) $\widetilde{H} \simeq_{eq} \Phi$ on $E := \bigcup\limits_{\nu > \eta} \widetilde{\nu^+}$.

The induction base $\eta = \{\text{the last simplex of } T\}$ is proved by taking H to be a map, linear on simplices of T, and $\nu^+ = \mathrm{st}_T \nu$ for each ν. Thus we may additionally assume by induction that

(c) for each k-simplex $\nu \leq \eta$ there exist homeomorphisms (compatible with each other for distinct ν)

$$\mathrm{lk}\,\nu * \nu \cong \nu^+ \quad \text{and} \quad S^{m-k-1} * H\nu \cong R_{\mathbb{R}^m}(H\nu, H\partial\nu)$$

such that $H(\mathrm{lk}\,\nu) \subset S^{m-k-1}$ and $H|_{\mathrm{lk}\,\nu * \nu} = H|_{\mathrm{lk}\,\nu} * H|_\nu$.

The above make sense because H is nondegenerate and so $H|_\nu$ is an embedding. In this proof we identify X and Y with the subsets $X * \emptyset$ and $\emptyset * Y$ of $X * Y$. We also identify $\mathrm{lk}\,\eta$ with the image of $\mathrm{lk}\,\eta$ under the above homeomorphism. We recommend the reader to read this proof for the particular case when N a PL manifold (this case is sufficient for the proof of Theorem 1.3β and hence of 1.3α). For this case $\mathrm{lk}\,\eta$ below is homeomorphic to S^{n-r-1}.

By (c) we can take a PL ball $B^{m-r-1} \subset S^{m-r-1}$ containing $H\,\mathrm{lk}\,\eta$ and denote $h_0 = H|_{\mathrm{lk}\,\eta} : \mathrm{lk}\,\eta \to B^{m-r-1}$ (Figure 3). We have $\dim \mathrm{lk}\,\eta \leq n - r - 1$. Denote

$$L_\nu = \nu^+ \cap \mathrm{lk}\,\eta \quad \text{and} \quad E_0 = \bigcup\limits_{\nu \supset \eta} L_\nu \times L_\nu.$$

By (a) and (b)

$$\widetilde{\Delta}(h_0) \cap E_0 = \emptyset \quad \text{and} \quad \Phi \simeq_{eq} \widetilde{H}, \quad \text{even} \quad \Phi = \widetilde{H} \quad \text{on} \quad E.$$

Cone Lemma 4.2. *Let X be a polyhedron. Denote $cX = \frac{X \times [0,1]}{X \times 1}$ and identify $X = X \times 0 \subset cX$. Fix a triangulation T of X and let cT be the cone triangulation of cX.*
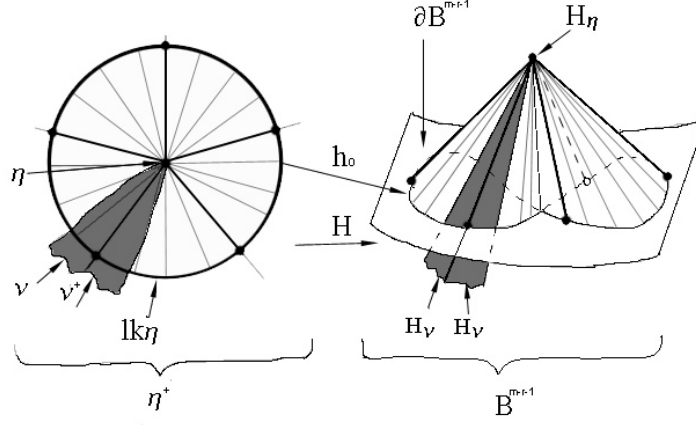
Figure 3

(4.2.1) *There is an equivariant surjective map* $p : \widetilde{cT} \to \Sigma\widetilde{T}$ *whose only non-trivial preimages are those of the vertices of the suspension and are* $c \times X$ *and* $X \times c$. *This map is natural on the inclusion, i.e. for a subcomplex* T_1 *of* T *we have* $p_{T_1} = p_T|_{\widetilde{cT_1}}$.

(4.2.2) *If* $h : X \to B^{m-1}$ *is an embedding and* $ch : cX \to cB^{m-1} \subset B^m$ *is the cone map, then the map* $\widetilde{ch} : \widetilde{cT} \to S^{m-1}$ *is well-defined and close to* $(\Sigma\widetilde{h}) \circ p$.

(4.2.3) $H_{eq}^i(\widetilde{cX}, \widetilde{cT} \cup \theta) = 0$ *for each* i, *where* L_ν *are subcomplexes of* T *and* $\theta = \cup_\nu \widetilde{cL_\nu}$.

Proof of 4.2.1 and 4.2.2. For each $([x,s],[y,t]) \in \widetilde{cT}$ either $s = 0$ or $t = 0$ (Figure 4). Let $p([x,s],[y,t]) = [(x,y), s - t]$. It is easy to check that p is well-defined, surjective, equivariant and natural on the inclusion. The p-preimage of a point $[(x,y),s] \in \Sigma\widetilde{T}$ is not a unique point if and only if $s = \pm 1$. Such non-trivial preimages $p^{-1}[\widetilde{T} \times 1]$ and $p^{-1}[\widetilde{T} \times \{-1\}]$ are $c \times X$ and $X \times c$, respectively.

If $([x,s],[y,t]) \in \widetilde{cT}$, then $(x,y) \in \widetilde{T}$, hence $hx \neq hy$, so $ch[x,s] \neq ch[y,t]$. Therefore $\widetilde{ch}$ is well-defined on $p^{-1}\Sigma\widetilde{T}|_L$. Let $\overline{u,v} = \frac{u-v}{|u-v|}$. We have

$$\Sigma\widetilde{h} \circ p([x,s],[y,t]) = \Sigma\widetilde{h}[(x,y), s - t] = [\widetilde{h}(x,y), s - t] = [\overline{hx, hy}, s - t]$$

and $\widetilde{ch}([x,s],[y,t]) = \overline{[hx,s],[hy,t]}$. For $s > t$ ($s < t$) these two points both lie in the open northern (southern) hemisphere, for $s = t$ they even coincide. So they are indeed not antipodal. $\qquad\square$

The proof of 4.2.2 shows that in 4.2.2 we can replace the map ch by any map $H : cX \to cB^{m-1}$ such that $Hc = c$ and $HX \subset B^{m-1}$.
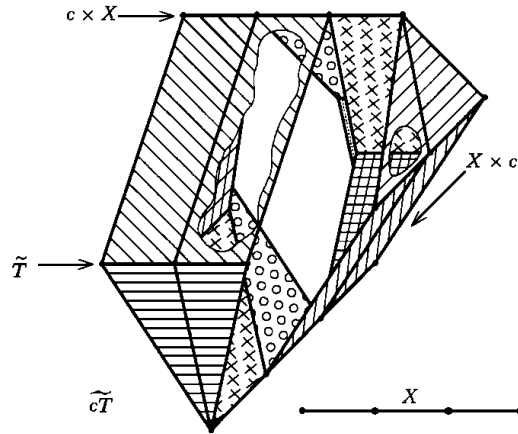
Figure 4

Proof of 4.2.3. We tacitly consider *equivariant* cohomology groups. Consider two restriction homomorphisms $H^i(\widetilde{cX}) \xrightarrow{r} H^i(\widetilde{cT} \cup \theta) \xrightarrow{r_1} H^i(\widetilde{cT})$. From the exact sequence of the pair $(\widetilde{cX}, \widetilde{cT} \cup \theta)$ we obtain that it suffices to prove that r is an isomorphism. Since $\widetilde{cT}$ is an equivariant deformation retract of $\widetilde{cX}$, it follows that $r_1 \circ r = \mathrm{id}$. Thus it suffices to prove that r_1 is an isomorphism. Consider the Mayer–Vietoris sequence

$$\ldots \xrightarrow{s_1 \oplus s_2} H^{i-1}(\widetilde{cT} \cap \theta) \xrightarrow{\delta} H^i(\widetilde{cT} \cup \theta) \xrightarrow{r_1 \oplus r_2} H^i(\widetilde{cT}) \oplus H^i(\theta) \xrightarrow{s_1 \oplus s_2} H^i(\widetilde{cT} \cap \theta) \rightarrow \ldots.$$

Since $\widetilde{cL_\nu}$ equivariantly deformationally retracts to $\widetilde{cT|_{L_\nu}}$ and these retractions are natural on the inclusion, they agree on the intersections, so θ equivariantly deformationally retracts to $\widetilde{cT} \cap \theta$. Therefore s_2 is an isomorphism. Hence $\delta = 0$ and the projection $\ker(s_1 \oplus s_2) = \mathrm{im}(r_1 \oplus r_2) \rightarrow H^i(\widetilde{cT})$ is an isomorphism. Thus r_1 is an epimorphism. Since $\delta = 0$, it follows that $r_1 \oplus r_2$ is a monomorphism, so r_1 is an isomorphism. $\square$

Continuation of the proof of Immersion Theorem 4.1: construction of H_1 and $\{\nu_1^+\}$. We recommend the reader to read the proof first for the particular case $r = 0$. Take a triangulation T_η of lk η in which all $\nu \cap$ lk η and L_ν are subcomplexes. Denote by T_η^+ the join triangulation of η^+. By an isotopy of $\mathbb{R}^m$ we can modify B^{m-r-1} to a *standard* ball. Since the join $\eta^+ = \eta * \mathrm{lk}\,\eta$ is homeomorphic to the iteration of $r + 1$ cones, by (c) we can apply Cone Lemma 4.2.1 $r + 1$ times to

obtain an equivariant surjective map $p : \widetilde{T_\eta^+} \to \Sigma^{r+1} \widetilde{T_\eta}$. By applying 4.2.2 $r + 1$ times we have that

$$\widetilde{H} \simeq_{eq} p^* \Sigma^{r+1} \widetilde{h_0} \quad \text{on} \quad \Theta := p^{-1} \Sigma^{r+1} \bigcup_{\nu \supset \eta} \widetilde{T_\eta|_{L_\nu}}.$$

For $r = 0$ we have

$$\Theta = \bigcup_{\nu \supset \eta} (\widetilde{cT_\eta|_{L_\nu}} \cup c \times L_\nu \cup L_\nu \times c) = \bigcup_{\nu \supset \eta} \widetilde{cT_\eta|_{L_\nu}} = \widetilde{T_\eta^+} \cap E$$

because $\cup_\nu L_\nu = \operatorname{lk} \eta$. Analogously $\Theta = \widetilde{T_\eta^+} \cap E$ for arbitrary r.

By the equivariant Borsuk Homotopy Extension Theorem Φ is homotopic to an equivariant extension

$$\Phi' : \widetilde{T_\eta^+} \to S^{m-1} \quad \text{of} \quad p^* \Sigma^{r+1} \widetilde{h_0} : \Theta \to S^{m-1}.$$

Since $n \le m - 2$, by applying $r + 1$ times Theorem 2.5 we obtain that the map

$$p^* \Sigma^{r+1} : \pi_{eq}^{m-r-2}(\widetilde{T_\eta}, E_0 \cap \widetilde{T_\eta}, \widetilde{h_0}) \to \pi_{eq}^{m-1}(\widetilde{T_\eta^+}, \Theta, \Phi'|_\Theta)$$

is surjective. The preimage of Φ' under this map is an equivariant extension

$$\varphi : \widetilde{T_\eta} \to S^{m-r-2} \quad \text{of} \quad \widetilde{h_0} : E_0 \cap \widetilde{T_\eta} \to S^{m-r-2}.$$

Since $r \ge 3n - 2m + 2$, we have $3(n - r - 1) \le 2(m - r - 1) - 3$, hence we can apply Disjunction Theorem 3.1 to $\operatorname{lk} \eta, T_\eta, A = \emptyset, E_0, E_1 = \operatorname{lk} \eta \times \operatorname{lk} \eta$ and $\Phi = \varphi$. We obtain a homotopy $h_t : \operatorname{lk} \eta \to B^{m-r-1}$ from h_0 to an embedding h_1 such that $\widetilde{\Delta}(h_t) \cap E_0 = \emptyset$ for each t.

By (c) we can identify

$$(\eta^+, \eta, \partial \eta) \quad \text{with} \quad (c \operatorname{lk} \eta * \partial \eta, c * \partial \eta, \partial \eta)$$

and $(R_{\mathbb{R}^m}(H\eta, H\partial \eta), H\eta, H\partial \eta) \quad \text{with} \quad (cS^{m-r-1} * H\partial \eta, c * H\partial \eta, H\partial \eta).$

Let $\quad \nu_1^+ = \nu^+ \quad$ for $\quad \nu \ne \eta \quad$ and $\quad \eta_1^+ = \{[x, t] \subset c \operatorname{lk} \eta \mid \frac{1}{2} \le t \le 1\} * \partial \eta.$

Define a map

$$H_t : c \operatorname{lk} \eta \to cS^{m-r-1} \quad \text{by} \quad H_t[x, s] = \begin{cases} [h_{2s}(x), s] & 0 \le s \le \frac{t}{2} \\ [h_t(x), s] & \frac{t}{2} \le s \le 1 \end{cases}$$

(Figure 5). Extend H_t to $c \operatorname{lk} \eta * \partial \eta$ as a join with $H|_{\partial \eta}$ and then to the rest of N by H to obtain a map $H_t : N \to \mathbb{R}^m$.
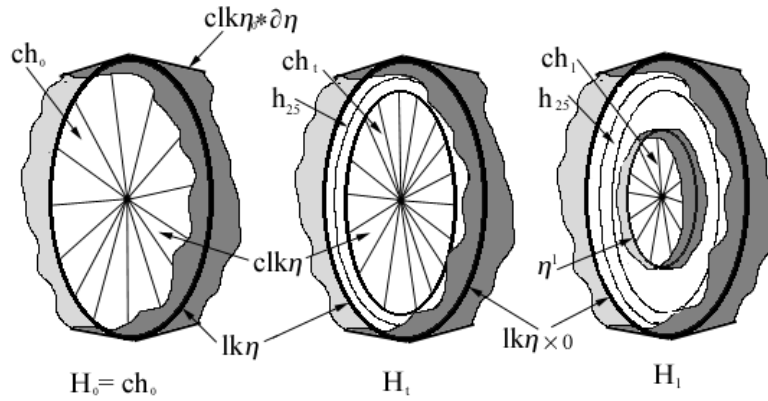
Figure 5

Completion of the proof of Immersion Theorem 4.1: verification of the properties of H_1 and $\{\nu_1^+\}$. The map $H_1|_{\eta_1^+}$ is an embedding as a join with an embedding h_1. If $\nu > \eta$ and $\nu \not\supset \eta$, then $H_1 = H$ on ν^+, hence $H_1|_{\nu_1^+}$ is an embedding. Since $\widetilde{\Delta}(h_t) \cap E_0 = \emptyset$, it follows that $H_1|_{\nu_1^+}$ is an embedding for each $\nu \supset \eta$. Thus $H_1|_{\nu_1^+}$ is an embedding for each $\nu \geq \eta$.

If $\nu < \eta$, then either $\nu \subset \eta$ or $\nu \cap \mathring{\eta}^+ = \emptyset$, so $H_1|_\nu$ is an embedding. Thus H_1 is nondegenerate.

If $\nu \leq \eta$, then $H_1|_{\nu^+}$ is a join both in the intersection with η^+ by the construction of H_1, and outside this interection by the inductive hypothesis, and these join structures are compatible. Thus (c) holds for H and ν^+ replaced by H_1 and ν_1^+.

It remains to prove that $\widetilde{H_1} \simeq_{eq} \Phi$ on $E \cup \widetilde{\eta_1^+}$. The triangulation T_η^+ induces a join triangulation T_1 of η_1^+. By Cone Lemma 4.3.3 it suffices to prove that $\widetilde{H_1} \simeq_{eq} \Phi$ on $E \cup \widetilde{T_1}$. Clearly, $\widetilde{H_t}$ is well-defined on E. We shall construct a homotopy $\Phi_t : \widetilde{T_1} \to S^{m-1}$ from $\Phi_0 = \Phi$ to $\Phi_1 = \widetilde{H_1}$ and then sew $\widetilde{H_t}$ with Φ_t to obtain the required homotopy Ψ_t between $\Psi_0 = \Phi$ and $\Psi_1 = \widetilde{H_1}$. Note that $H_0 = H$ and $H_t = ch_t|_{c\,\mathrm{lk}\,\eta} * H|_{\partial\eta} = h_t|_{\mathrm{lk}\,\eta} * H|_\eta$ on η_1^+. Denote by $p : \widetilde{T_1} \to \Sigma^{r+1}\widetilde{T_\eta}$ the *new* map given by application of Cone Lemma 4.2.1 $r+1$ times and define the *new* Θ as above. It is easy to check that the properties of p and Θ used below (analogous to and using the properties of *old* p and Θ).

By the construction of φ we have $p^*\Sigma^{r+1}\varphi \simeq \Phi'$ rel Θ. Therefore the linear homotopy between close maps $\Phi = \widetilde{H_0}$ and $\Phi' = p^*\Sigma^{r+1}\widetilde{h_0}$ on Θ extends to a homotopy Φ_t, $t \in [0, \frac{1}{3}]$, between $\Phi_0 = \Phi$ and $\Phi_{1/3} = p^*\Sigma^{r+1}\varphi$ on $\widetilde{T_1}$.

By the construction of h_t using 3.1.2 we obtain that there exists a homotopy $\varphi_t : \widetilde{T_\eta} \to S^{m-r-2}$ between $\varphi_0 = \varphi$ and $\varphi_1 = \widetilde{h_1}$ extending the homotopy $\widetilde{h_t}$:

$E_0 \cap \widetilde{T_\eta} \to S^{m-r-2}$. For $t \in [\frac{1}{3}, \frac{2}{3}]$ let $\Phi_t = p^* \Sigma^{r+1} \varphi_{3t-1}$. Then by Cone Lemma 4.2.2 $\Phi_t(x)$ is close to $\widetilde{H_{3t-1}}(x)$ for $t \in [\frac{1}{3}, \frac{2}{3}]$ and $x \in \Theta$.

Finally, for $t \in [\frac{2}{3}, 1]$ let Φ_t be the linear homotopy between close (by Cone Lemma 4.2.2) maps $\Phi_{2/3} = p^* \Sigma^{r+1} \widetilde{h_1}$ and $\Phi_1 = \widetilde{H_1}$.

In order to sew $\widetilde{H_t}$ and Φ_t take an $\varepsilon \geq 0$ such that $\widetilde{H_t}$ is defined on the ε-neighborhood $O_\varepsilon E$ of E (or just extend $\widetilde{H_t}$ to $O_\varepsilon E$). By compactness we may assume that ε is so small that the maps $\widetilde{H_0}$ and Φ_t for $t \in [0, \frac{1}{3}]$, $\widetilde{H_{3t-1}}$ and Φ_t for $t \in [\frac{1}{3}, \frac{2}{3}]$, $\widetilde{H_1}$ and Φ_t for $t \in [\frac{2}{3}, 1]$ are close on $\widetilde{T_1} \cap O_\varepsilon E$. Let $\tau(x) = \frac{|x,E|}{3\varepsilon}$. Define a map

$$\Psi'_t : \widetilde{T_1} \cap O_\varepsilon E \to \mathbb{R}^m - \{0\} \quad \text{by} \quad \Psi'_t = \begin{cases} 3\tau \Phi_{\frac{t}{3\tau}} + (1 - 3\tau)\widetilde{H_0} & t \leq \tau \\ 3\tau \Phi_q + (1 - 3\tau)\widetilde{H_{3q-1}} & \tau \leq t \leq 1 - \tau \\ 3\tau \Phi_{1 - \frac{1-t}{3\tau}} + (1 - 3\tau)\widetilde{H_1} & t \geq 1 - \tau \end{cases},$$

where $q = \frac{1+t-3\tau}{3(1-2\tau)}$ (Figure 6). Note that in the above formula $\tau \leq \frac{1}{3}$ and we use the agreement that $\Psi'_t = \widetilde{H_t}$ for $\tau = 0$ and $t \in \{0, 1\}$. We omit x which is the argument of $\Psi'_t, \Phi_t, \widetilde{H_t}$ and τ. It is easy to check that Ψ'_t is continuous (even at points $x \in E$, for all of which $\tau(x) = 0$). Then we can define the required homotopy Ψ_t by

$$\Psi_t = \frac{\Psi'_t}{|\Psi'_t|} \text{ on } \widetilde{T_1} \cap O_\varepsilon E, \quad \Psi_t = \widetilde{H_t} \text{ on } E \quad \text{and} \quad \Psi_t = \Phi_t \text{ on } \widetilde{T_1} - O_\varepsilon E. \quad \square$$

For the approximative relative version of Theorems 1.3β and 1.3$\beta\partial$ stated in §1, in the above proof we take sufficiently fine T and use the corresponding approximative relative version of Immersion Theorem 4.1, which is proved analogously (we start with taking H very close to g and ν^+ so small that (c) holds).

Proof of Example 1.10. Example 1.10.i follows because there exists a smooth embedding $S^{4l-1} \to \mathbb{R}^m$, which is not smoothly regular homotopic to the standard inclusion [Hae66A, Remark 6.8].

Example 1.10.ii follows because $\beta_{DIFF}^{3l+1}(S^{2l})$ is surjective (by Theorem 1.1β) but there exists a piecewise-smooth immersion $S^{2l} \to \mathbb{R}^{3l+1}$, which is not piecewise-smoothly regularly homotopic to a smooth immersion [Hae67, 5.4, Theorem 5.6].

In order to construct Example 1.10.s take an $(n-1)$-polyhedron K from Example 1.2.ee with the property $(*)$ of §1. Then $N = cK$ does not PL immerse in $\mathbb{R}^m$, but $\pi_{eq}^{m-1}(SN) \neq \emptyset$, even $\pi_{eq}^{m-1}(\widetilde{N}) \neq \emptyset$, by Cone Lemma 4.2.1. Example 1.10.s is true also for $m = n = 4$, which follows from [MaSe67, Theorem 2]. $\quad \square$

We conjecture that $\alpha/\beta_{PL}^{\frac{3n}{2}+1}(S^n)$ is not surjective. An evidence for this conjecture is as follows. By Example 1.10.ii, the restriction $2m \geq 3n + 3$ is sharp in the
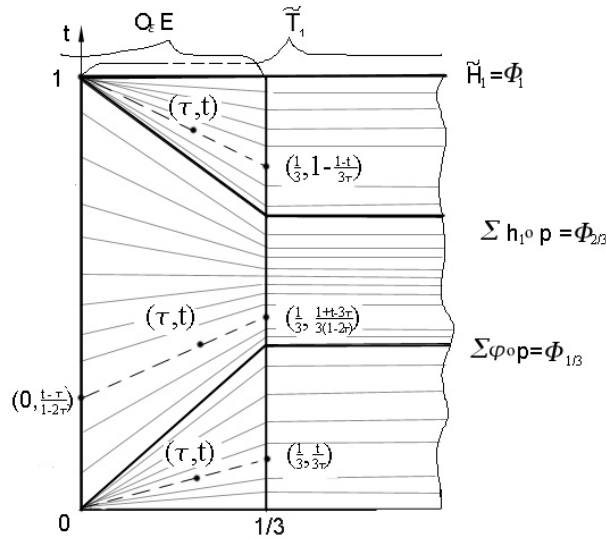
Figure 6

injectivity part of Theorem 1.3β even for closed PL manifolds. The only point in the proof of the injectivity in Theorem 1.3β for closed PL manifolds, which would not go for $2m = 3n+2$, is the application of a partial case of the Disjunction Theorem 3.1. Therefore this partial case is false, and this partial case is very similar to the surjectivity of $\alpha/\beta_{PL}^{\frac{3n}{2}+1}(S^n)$.

We conjecture that if N is an n-polyhedron and $m > n$, then the map $f \mapsto \Delta f$ defines a 1–1 correspondence between the set of PL immersions $N \to \mathbb{R}^m$ up to PL regular homotopy and the set of PL bundle monomorphisms $\Phi : TN \to \mathbb{R}^m$ up to homotopy through PL bundle monomorphisms, cf. [Lee69]. Here TN is sufficiently small neighborhood of the diagonal in $N \times N$ (not to be confused with the previous notation for TN). For a map $f : N \to \mathbb{R}^m$ the map $\Delta f : N \times N \to \mathbb{R}^m$ can be defined by the formula $\Delta f(x,y) = fx - fy$. The reason for using such a notation is that Δf is a 'finite difference, approximating df'. The map $\Phi : TN \to \mathbb{R}^m$ is a *PL bundle monomorphism*, if the restriction of Φ onto each fiber (of the projection onto the first factor) is a PL embedding. A similar conjecture can be stated for the classification of TOP immersions of locally contractible compacta.

We conjecture that in the proof of Immersion Theorem 4.1 we may require that (c) holds for *each ν*, not only for $\nu \le \eta$. To check this improved property (c') for H_1 and $\{\nu_1^+\}$, constructed in the proof, observe that (c') for $m \ge n + 3$ follows from (c) and (a) by the Unknotting Cones Theorem [Lic65, Corollary on p. 71]. For $m = n + 2$ we have $3n - 2m + 2 = n - 2$. So for $\dim \nu = n$ (c') holds since $\nu^+ = \nu$, for $\dim \nu = n - 2$ (c') is not violated during the constructions of

H_1 for distinct $(n-2)$-simplices ν (since the corresponding ν^+ may be chosen to have disjoint interiors), and for $\dim \nu = 1$ (c') could follow from a version of the Unknotting Cones Theorem. Or else for $m = n + 2$ preservation of (c') for $\nu > \eta$ could follow from corresponding improvement of Disjunction Theorem 3.1.

5. Deleted product of cylinder and applications

Denote $\Sigma X = \frac{X \times [-1,1]}{X \times 1, X \times (-1)}$. Embed X as $X \times 0$ into ΣX.

Cylinder Lemma 5.1. *(5.1.α) For a polyhedron N there is an equivariant surjective map $p : \widetilde{N \times I} \to \Sigma(\widetilde{N} \times I)$ whose only non-trivial preimages are those of the vertices of the suspension and are homotopy equivalent to $N \times N$.*

Moreover, if $H : N \times I \to \mathbb{R}^m \times I \subset \mathbb{R}^{m+1}$ is a level-preserving map, $E \subset_{eq} N \times I \times N \times I$ and $\varphi : \widetilde{N} \times \operatorname{diag} I \to S^{m-1}$ is an equivariant homotopy between $\widetilde{H}|_{N \times 0}$ and $\widetilde{H}|_{N \times 1}$ such that $\widetilde{\Delta}(H) \cap E = \emptyset$ and $\varphi = \widetilde{H}$ on $E \cap (\widetilde{N} \times \operatorname{diag} I)$, then $\Phi := \Sigma\varphi \circ p \simeq_{eq} \widetilde{H}$ on $E \cap \widetilde{N \times I}$.

(5.1.β) The same as above replacing $\widetilde{N}$ by SN, $\widetilde{N \times I}$ by $S(N \times I)$ and $N \times N$ by N.

Proof of 5.1.α [cf. Web67, 7.1, Sko97, 3.3, Sko00, 3.1]. Take a metric on N such that $\operatorname{diam} N < 1$. Define the map p by

$$ p(x, s, y, t) = \left[\left(x, y, \frac{s+t}{2} \right), \frac{s-t}{\max\{|x,y|, |s-t|\}} \right]. $$

See [Sko97, Figure 1 and idea of proof of Lemma 3.3], cf. Figure 7.a. It is easy to see that p is well-defined, equivariant and surjective. Clearly, the non-trivial preimages of p are those of the vertices of the suspension. They retract deformationally to $N \times 0 \times N \times 1$ and to $N \times 1 \times N \times 0$, so they are homotopy equivalent to $N \times N$. Observe that $\Phi(x, t, y, t) = \varphi(x, y, t)$, hence $\Phi = \widetilde{H}$ on $E \cap (\widetilde{N} \times \operatorname{diag} I)$. Since H preserves levels, it follows that for $s < t$ ($s > t$) and $(x, s, y, t) \in E$, both $\Phi(x, s, y, t)$ and $\widetilde{H}(x, s, y, t)$ are in the northern (southern) open hemisphere. Therefore Φ and $\widetilde{H}$ are close on $E \cap \widetilde{N \times I}$. Hence $\Phi \simeq_{eq} \widetilde{H}$ on $E \cap \widetilde{N \times I}$. $\square$

Proof of 5.1.β. Take a small $\varepsilon > 0$ such that

$$ S(N \times I) \simeq_{eq} \{(x, s, y, t) \in \widetilde{N \times I} : |x, y| + |s - t| = \varepsilon\}. $$

Define the map

$$ p : S(N \times I) \to \Sigma(SN \times I) \quad \text{by} \quad p(x, s, y, t) = \left[\left(x, y, \frac{s+t}{2} \right), \frac{s-t}{\varepsilon} \right] $$
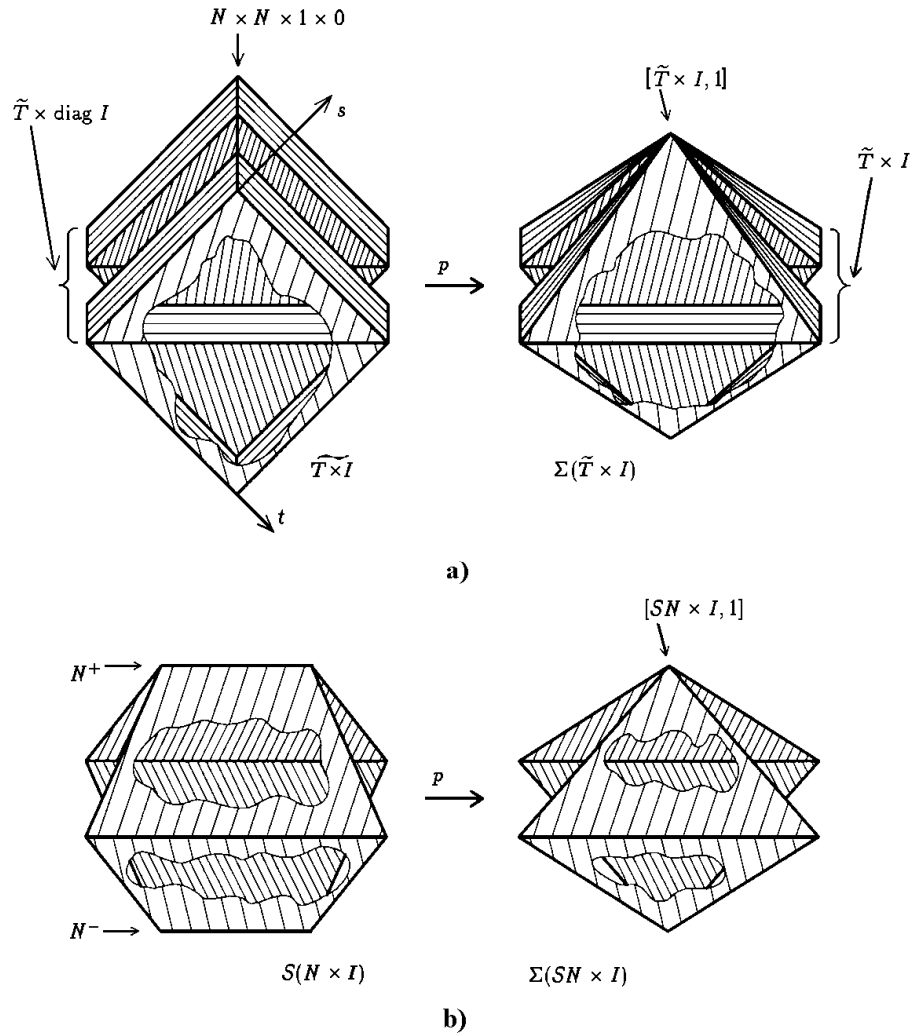
Figure 7

(Figure 7.b). It is easy to see that p well-defined and to verify the properties of p (the non-trivial preimages are $N^{\pm} = \{(x, s, x, s \pm \varepsilon) \in S(N \times I)\} \cong N \times I \simeq N)$.

□

Proof of the quasi-injectivity in Theorem 2.2.q. Take PL embeddings $f_0, f_1 : N \to \mathbb{R}^m$ and an equivariant homotopy $\varphi : \widetilde{N} \times I \to S^{m-1}$ from $\widetilde{f}_0$ to $\widetilde{f}_1$. Take the

linear homotopy $G : N \times I \to \mathbb{R}^m \times I$ between f_0 and f_1 and a triangulation T_1 of $N \times I$. Let $E = (N \times \{0,1\})^*$. Since

$$\widetilde{G} = \varphi \quad \text{on} \quad E \cap \widetilde{N} \times \operatorname{diag} I = \widetilde{N} \times 0 \times 0 \cup \widetilde{N} \times 1 \times 1,$$

we can apply Cylinder Lemma 5.1.α to obtain an equivariant map $\Phi : \widetilde{N \times I} \to S^m$ such that $\Phi \simeq_{eq} \widetilde{G}$ on $E \cap \widetilde{N \times I}$. Let K be the union of $N \times \{0,1\}$ and the $(n-d)$-skeleton of the dual cell-subdivision to T_1. We may assume that $K \cap (T_1^{(d)} \cap N \times 0) \times \frac{1}{2} = \emptyset$. Apply boundary version of the surjectivity in Theorem 1.3$\beta\partial$ [cf. Sko97, Theorems 1.1.b and 3.1] (or Theorem 1.1$\beta\partial$, if N, f_0 and f_1 are smooth outside a point) to obtain a PL immersion $h_0 : R(K) \to \mathbb{R}^m \times I$ such that

$$h_0(x,0) = f_0(x),\ h_0(x,1) = f_1(x) \text{ and } \widetilde{h_0} \simeq_{eq} \Phi \text{ on } \widetilde{N} \times 0 \times 0 \cup \widetilde{N} \times 1 \times 1 \cup S(R(K)).$$

Extend h_0 to a map $N \times I \to \mathbb{R}^m \times I$. Take a new triangulation T of $N \times I$ such that K is a subcomplex of T and such that for $E_0 = (N \times \{0,1\})^* \cup [K^* \cap T(N \times I)]$ we have

$$\widetilde{\Delta}(h_0) \cap E_0 = \emptyset \quad \text{and} \quad \widetilde{h_0} \simeq_{eq} \Phi, \quad \text{even} \quad \widetilde{h_0} = \Phi \quad \text{on} \quad E_0 \cap \widetilde{T}.$$

Since $n + 1 + 2(n - d) \le 2(m + 1) - 3$, we can apply Disjunction Theorem 3.1 for

$$N = N \times I,\ \ A = N \times \{0,1\},\ \ E_0 = (N \times \{0,1\})^* \cup [K^* \cap T(N \times I)]\ \text{ and }\ E_1 = K^*.$$

Since $\widetilde{\Delta}(h_1)$ is closed, by 3.1.1 it follows that there exists

$$R_{N \times I}(K) \subset N \times I - (T_1^{(d)} \cap N \times 0) \times \frac{1}{2} \quad \text{such that} \quad \widetilde{\Delta}(h_1) \cap R_{N \times I}(K)^* = \emptyset.$$

Hence $V := N \times I - R_{N \times I}(K) = R_{N \times I}\Big((T_1^{(d)} \cap N \times 0) \times \frac{1}{2}\Big)$ and $\Sigma(h_1) \subset V$.

Since $N \times I$ is d-connected, by the Engulfing Lemma [Irw65, Theorems 2.1 and 2.3] it follows that $(T_1^{(d)} \cap N \times 0) \times \frac{1}{2}$ is contained in some PL n-ball in $N \times \mathring{I}$. Therefore by the Uniqueness of Regular Neighborhood Theorem, V is also contained in some (possibly, another) PL n-ball $B^n \subset N \times \mathring{I}$. We have $\Sigma(h_1) \subset V \subset B^n$, hence h_1 is a quasi-concordance. $\qquad \square$

Proof of the (quasi-)injectivity in Theorems 2.2.q/β, 2.1.α/β and 2.1.α/$\beta\partial$ for the PL case. Take PL embeddings $f_0, f_1 : N \to \mathbb{R}^m$ and an equivariant homotopy $\varphi : \widetilde{N} \times I \to S^{m-1}$ from $\widetilde{f_0}$ to $\widetilde{f_1}$. Take a PL regular homotopy $h_0 : N \times I \to \mathbb{R}^m$ from f_0 to f_1 such that

$$\widetilde{h_0} \simeq_{eq} \varphi \quad \text{on} \quad SN \times \operatorname{diag} I = SN \times I \text{ rel } SN \times \{0,1\}.$$

Fix a triangulation T of $N \times I$ such that $\widetilde{\Delta}(h_0) \cap T(N \times I) = \emptyset$. Extend the map $\varphi : SN \times \operatorname{diag} I \to S^{m-1}$ to $\widetilde{N} \times 0 \times 0$ as $\widetilde{f}_0$ and to $\widetilde{N} \times 1 \times 1$ as $\widetilde{f}_1$. Let $E = (N \times \{0,1\})^* \cup T(N \times I)$. Then

$$\widetilde{h_0} \simeq_{eq} \varphi, \quad \text{even} \quad \widetilde{h_0} = \varphi \quad \text{on} \quad E \cap (\widetilde{N} \times \operatorname{diag} I) = SN \times \operatorname{diag} I \cup \widetilde{N} \times 0 \times 0 \cup \widetilde{N} \times 1 \times 1.$$

Therefore we can apply Cylinder Lemma 5.1.α to obtain an equivariant map $\Phi : \widetilde{N \times I} \to S^m$ such that $\widetilde{h_0} \simeq_{eq} \Phi$, even $\widetilde{h_0} = \Phi$ on $E \cap \widetilde{T}$.

The quasi-injectivity in Theorem 2.2.q/β is now proved as in the proof of Theorem 2.2.q ($K = T^{(n-d)} \cup N \times \{0,1\}$ and T_1 is the cell-subdivision dual to T such that $K \cap (T_1^{(d)} \cap N \times 0) \times \frac{1}{2} = \emptyset$).

The injectivity in Theorem 2.1.α/β follows by applying Disjunction Theorem 3.1 to

$$N \times I, \quad A = N \times \{0,1\}, \quad E_0 = E \quad \text{and} \quad E_1 = N \times I \times N \times I.$$

In order to prove the injectivity in Theorem 2.1.α/$\beta\partial$, assume that the union K of $N \times \{0,1\}$ and the product of the $(n-d-1)$-spine with the interval I is a subcomplex of T. Since $3(n-d) \leq 2(m+1) - 3$, we can apply Disjunction Theorem 3.1 for

$$N \times I, \ A = N \times \{0,1\}, \ E_0 = (N \times \{0,1\})^* \cup TK \subset E \ \text{and} \ E_1 = (N \times \{0,1\})^* \cup K \times K.$$

Then the theorem follows because $h_1|_{R_{N \times I}(K)}$ is an embedding for some $R_{N \times I}(K)$ such that
$$(R_{N \times I}(K), N \times 0, N \times 1) \cong (N \times I, N \times 0, N \times 1). \qquad \square$$

Theorem 5.2 (5.2.α)**.** *If PL embeddings $f_0, f_1 : N \to \mathbb{R}^m$ of an n-polyhedron N are TOP quasi-concordant and $m \geq n + 2$, then $\alpha(f_0) = \alpha(f_1)$.*

(5.2.β) *If PL immersions $h_0, h_1 : N \to \mathbb{R}^m$ of an n-polyhedron N are TOP regular concordant and $m \geq n + 2$, then $\beta(h_0) = \beta(h_1)$.*

Proof [cf. Sko00, Lemma 1.0]. We prove only 5.2.α (the proof of 5.2.β is analogous). Take a TOP quasi-concordance $F : N \times I \to \mathbb{R}^m \times I$ between f_0 and f_1. Take a triangulation T of N such that $\Sigma(F)$ is contained in a star of some point of N. Let $T \times I$ be the product cell-subdivision of $N \times I$. Since $\widetilde{F}(N \times N \times 0 \times 1)$ and $\widetilde{F}(N \times N \times 1 \times 0)$ lie in the northern and in the southern open hemispheres of S^m, it follows that the map $\widetilde{F} : \widetilde{T \times I} \to S^m$ is homotopic rel $\widetilde{T} \times 0 \times 0 \cup \widetilde{T} \times 1 \times 1$ to some map $\Phi : \widetilde{T \times I} \to S^m$ such that $\Phi(N \times N \times 0 \times 1)$ and $\Phi(N \times N \times 1 \times 0)$ are the northern and in the southern poles of S^m, respectively.

Now we need a simplicial version of Cylinder Lemma 5.1.α. For each $(x, s, y, t) \in \widetilde{T \times I}$ we have either $x \neq y$ or $\{s, t\} = \{0, 1\}$. Define a map

$$p : \widetilde{T \times I} \to \Sigma(\widetilde{T} \times I) \quad \text{by} \quad p(x, s, y, t) = \left[\left(x, y, \frac{s+t}{2} \right), s - t \right]$$

(Figure 7.a). It is easy to check that p is equivariant surjective, that its only non-trivial preimages are those of the vertices of the suspension and they are $N \times N \times 0 \times 1$ and $N \times N \times 1 \times 0$. Hence there is a map $\Phi' : \Sigma(\widetilde{T} \times I) \to S^m$ such that $\Phi = \Phi' \circ p$. Analogously to Cylinder Lemma 5.1.α we have

$$\Sigma(\widetilde{f_0} \sqcup \widetilde{f_1}) \circ p \simeq_{eq} \widetilde{F} \quad \text{on} \quad X := (N \times \{0,1\})^* \cap \widetilde{T \times I}.$$

Moreover, we can check that $X = p^{-1}\Sigma(\widetilde{T} \times \{0,1\})$. By the construction of the homotopy $\Sigma(\widetilde{f_0} \sqcup \widetilde{f_1}) \circ p \simeq_{eq} \widetilde{F} \simeq_{eq} \Phi' \circ p$ on X we can see that

$$\Sigma(\widetilde{f_0} \sqcup \widetilde{f_1}) \simeq_{eq} \Phi', \quad \text{even} \quad \Sigma(\widetilde{f_0} \sqcup \widetilde{f_1}) = \Phi' \quad \text{on} \quad pX = \Sigma(T \times \{0,1\}.$$

Since $2n + 1 \leq 2(m-1) - 1$, by Theorem 2.5 the map

$$\Sigma : \pi_{eq}^{m-1}(\widetilde{T} \times I, \widetilde{T} \times \{0,1\}, \widetilde{f_0} \sqcup \widetilde{f_1}) \to \pi_{eq}^m(\Sigma(\widetilde{T} \times I), pX, \Phi'|_{pX})$$

is surjective. Therefore there exists an equivariant homotopy $\varphi : \widetilde{T} \times I \to S^{m-1}$ between $\widetilde{f_0}$ and $\widetilde{f_1}$ (such that $\Sigma\varphi \simeq_{eq} \Phi'$, but this property is unnecessary). $\square$

Note that the proof of Theorem 5.2.β works under the weaker assumption that there is an immersion $F : N \times I \to \mathbb{R}^{m+1}$ (not $\to \mathbb{R}^m \times I$) such that $F(x, 0) = h_0(x)$ and $F(x, 1) = h_1(x)$. Therefore the restriction $n \leq m-2$ is sharp. Indeed, for the two embeddings $h_0, h_1 : S^1 \to \mathbb{R}^2$ with different orientation, $\widetilde{h_0} \not\simeq_{eq} \widetilde{h_1}$ on SS^1, but the immersion F as above do exist.

Another motivation for Theorems 1.1β, 1.3β and 1.3$\beta\partial$ is the following simple corollary of 5.2.

Corollary 5.3. *If N is TOP immersible in $\mathbb{R}^m$ and $\beta_{PL}^m(N)$ is surjective, then N is PL immersible in $\mathbb{R}^m$. Particularly, PL immersability of N into $\mathbb{R}^m$ does not depend on the PL structure on N. If $m \geq n+2$, $\beta_{PL}^m(N)$ is injective and two PL immersions $N \to \mathbb{R}^m$ are TOP regular concordant, then they are PL regular concordant.*

Theorem 1.3 can analogously be applied to obtain smooth immersion from a PL or TOP one (as well as to the corresponding results on regular homotopies, embeddings and isotopies), but these applications (in contrast to Corollary 5.3) are covered by known results.

Corollary 5.4 [cf. Hir66, Ros93, RSS95]**.** *Suppose that N is either an n-polyhedron or a smooth n-manifold, $m \geq n+2$ and $\alpha^m(N)$ is surjective (see Theorems 1.1α, 1.1$\alpha\partial$ and 1.3α). If either cN TOP embeds into $\mathbb{R}^{m+1}$ or ($N \times I$ TOP embeds in $\mathbb{R}^{m+1}$ and $H^i(N) = 0$ for $i \geq m-n$), then N embeds into $\mathbb{R}^m$.*

Proof. Suppose that $N \times I$ TOP embeds in $\mathbb{R}^{m+1}$ and $H^i(N) = 0$ for $i \geq m - n$. Let T be a triangulation of N. Consider the maps

$$\text{Emb}^m(N) \xrightarrow{\alpha} \pi_{eq}^{m-1}(\widetilde{T}) \xrightarrow{\text{pr}^*} \pi_{eq}^{m-1}(\widetilde{T} \times I) \xrightarrow{\Sigma} \pi_{eq}^{m-1}(\Sigma(\widetilde{T} \times I)) \xrightarrow{p^*} \pi_{eq}^m(\widetilde{T \times I}),$$

where $\text{pr} : \widetilde{T} \times I \to \widetilde{T}$ is the projection and p is the map from the simplicial version of Cylinder Lemma 5.1.α (formulated in the proof of Theorem 5.2). Clearly, pr^* is an equivariant homotopy equivalence. Since $\dim \widetilde{T} \leq 2n$, by the equivariant Suspension Theorem Σ is surjective for $m \geq n + 2$. By the Künneth formula the condition $H^i(N) = 0$ for $i \geq m - n$ implies that $H^i(N \times N) = 0$ for $i \geq m$. Hence any map $N \times N \to S^m$ is null-homotopic. Therefore by the equivariant Borsuk Homotopy Extension Theorem, any equivariant map $\widetilde{N \times I} \to S^m$ is homotopic to the one which factors through the compression of non-trivial preimages of p. So p^* is surjective. Since $N \times I$ embeds into $\mathbb{R}^{m+1}$, it follows that $\pi_{eq}^m(\widetilde{T \times I}) \neq \emptyset$. Since α is surjective, this implies that $\text{Emb}^m(N) \neq \emptyset$.

The case when cN TOP embeds into $\mathbb{R}^{m+1}$ is proved analogously using Cone Lemma 4.2 instead of Cylinder Lemma 5.1.α. $\square$

The assumption on TOP embeddability in Corollary 5.4 can be relaxed to quasi-embeddability, cf. [MaSe67].

Note that if N is a polyhedron, cN TOP embeds into $\mathbb{R}^{m+1}$ and (cN, c) does not contain topologically $(B^{m+1}, 0)$, then there is an equivariant map $\Sigma\widetilde{N} \to S^m$. For $\dim N \leq m - 1$ this is proved analogously to Corollary 5.4. The below proof works without dimension restrictions. By Cone Lemma 4.2 and the equivariant Borsuk Homotopy Extension Theorem, it suffices to prove that for an embedding $f : cN \to \mathbb{R}^{m+1}$, the map $\widetilde{f}|_{(c \times N) \sqcup (N \times c)}$ is null-homotopic. There is an m-sphere in $\mathbb{R}^{m+1}$ such that the vertex fc and the base fN of the embedded cone are contained in different connected components of the complement to this sphere. Since (cN, c) does not contain topologically $(B^{m+1}, 0)$, it follows that fcN does not contain any neighborhood of fc in $\mathbb{R}^{m+1}$. Therefore we may assume that $fc \in \mathbb{R}^m \times (0, +\infty)$ and $fN \in \mathbb{R}^m \times (-\infty, 0)$. Then $\widetilde{f}(c \times N)$ and $\widetilde{f}(N \times c)$ are in the northern and in the southern hemisphere of the unit sphere S^m, respectively, and we are done.

The following Pseudo-Isotopy Theorem 5.5 confirms [ReSk98, Conjecture 1.9.c], generalizes [Web67, Theorem 1'], improves [ReSk98, Theorem 1.2] and also provides a shorter proof of [ReSk98, Theorem 1.2] for $2m \geq 3n + 4$. The introduction and motivations can be found in [ReSk96, §9, CRS98, §4, ReSk98, §1]. For a map $g : N \to \mathbb{R}^m$ denote

$$\widetilde{N}^g = \{(x, y) \in N \times N \mid gx \neq gy\} = \widetilde{N} - \widetilde{\Delta}(g) \quad \text{and} \quad \text{Map } g = \frac{N \times I}{\{g^{-1}x \times 1\}_{x \in g(N)}}.$$

Pseudo-Isotopy Theorem 5.5. *Let N be an n-polyhedron, $g : N \to \mathbb{R}^m$ a PL map and $2m \geq 3n + 4$.*

a) *If $f : N \to \mathbb{R}^m$ is a PL embedding such that $\widetilde{g} \simeq_{eq} \widetilde{f}$ on $\widetilde{N}^g$, then f is PL pseudo-isotopic to g (i.e. there is a PL homotopy $f_t : N \to \mathbb{R}^m$ from $f_0 = f$ to $f_1 = g$ such that f_t is an embedding for each $0 \le t < 1$).*

b) *[cf. Mel] The following conditions are equivalent:*

(I) *g is isotopically approximable by embeddings (i.e. there exists a pseudo-isotopy, arbitrary close to g, from an embedding f to g);*

(A) *g is approximable by embeddings;*

(Φ) *there exists an equivariant map $\Phi : \widetilde{N} \to S^{m-1}$ such that $\Phi|_{\widetilde{N}^g} \simeq_{eq} \widetilde{g}$.*

Proof. In Theorem 5.5.b $(I) \Rightarrow (A)$ is clear, $(A) \Rightarrow (\Phi)$ is simple [ReSk98, Theorem 1.2, necessity] and $(\Phi) \Rightarrow (I)$ follows from [Web67, Theorem 1] and Theorem 5.5.a. So let us prove Theorem 5.5.a. Apply Mapping Cylinder Lemma 5.6 to the linear homotopy G between f and g. Then by [Sko97, Theorem 1.1.c] — the boundary version of [Web67, Theorem 1] — it follows that there exists a PL embedding $F : \mathrm{Map}\, g \to \mathbb{R}^m \times I$ (a 'pseudo-concordance' from f to g) such that

$$F(x, 0) = f(x) \times 0, \quad F(g(x), 1) = g(x) \times 1 \quad \text{and} \quad F(N \times (0, 1)) \subset \mathbb{R}^m \times (0, 1)$$

[cf. Sko94, §4]. Since F is PL, we can find a small $\varepsilon > 0$ and modify F to achieve $F(x, t) \subset \mathbb{R}^m \times t$ for each $t \le \varepsilon$. Therefore by the Concordance Implies Isotopy Theorem [Lic65, Hud70] we can modify F to achieve $F(x, t) \subset \mathbb{R}^m \times t$ for each $\varepsilon \le t \le 1$. So F become a pseudo-isotopy from f to g. $\qquad\square$

Mapping Cylinder Lemma 5.6. *Let N be an n-polyhedron, $g, f : N \to \mathbb{R}^m$ a PL map and a PL embedding such that $\widetilde{f} \simeq_{eq} \widetilde{g}$ on $\widetilde{N}^g$ and $G : \mathrm{Map}\, g \to \mathbb{R}^m \times I$ a map such that*

$$G(x, 0) = f(x) \times 0, \quad G(N \times (0, 1)) \subset \mathbb{R}^m \times (0, 1)$$

and $G|_{g(N)}$ is a composition $g(N) \subset \mathbb{R}^m \cong \mathbb{R}^m \times 1$. Then $\widetilde{G}$ is defined over $E := \widetilde{\mathrm{Map}\, g} \cap (p \times p)(N \times N \times \partial(I \times I))$ and there exists an equivariant map $\Psi : \widetilde{\mathrm{Map}\, g} \to S^m$ such that $\Psi \simeq_{eq} \widetilde{G}$ on E.

Proof. Take a triangulation T of N in which g is simplicial, and the corresponding triangulation of $\mathrm{Map}\, g$. It suffices to prove the same result with the *simplicial* deleted product of $\mathrm{Map}\, g$, which we denote by $\widetilde{\mathrm{Map}\, g}$. Let

$$M = \widetilde{T} \times 0 \times I \cup \widetilde{T} \times I \times 0 \cup \widetilde{T}^g \times I \times I \cup N \times N \times 0 \times 1 \cup N \times N \times 1 \times 0,$$

where $\widetilde{T}^g = \bigcup_{g\sigma \cap g\tau = \emptyset} \sigma \times \tau$. Let $\pi : N \times I \to \mathrm{Map}\, g$ be the projection. Since π is a surjection, it follows that $\pi \times \pi : M \to \widetilde{\mathrm{Map}\, g}$ is a surjection. Therefore in

order to construct the required map Ψ it suffices to construct an equivariant map $\Psi' : M \to S^m$ which assume constant value on each fiber of $\pi \times \pi$ and is such that

$$\Psi' \simeq_{eq} \widetilde{G} \circ (\pi \times \pi) \quad \text{on} \quad M \cap (N \times N \times \partial(I \times I)).$$

Let $\varphi : \widetilde{T}^g \times I \to S^{m-1}$ be an equivariant homotopy between $\widetilde{f}$ and $\widetilde{g}$ on $\widetilde{T}^g$. Extend φ to $Y = \widetilde{T} \times 0 \times 0 \cup \widetilde{T}^g \times \operatorname{diag} I$ by $\widetilde{f}$. Take a metric on N such that $\operatorname{diam} N < 1$. Define a map

$$p : M \to \Sigma(\widetilde{N} \times I) \quad \text{by} \quad p(x, s, y, t) = \begin{cases} \left[\left(x, y, \frac{\min\{s,t\}}{1-|s-t|} \right), 2(s-t) \right] & 2|s-t| \leq 1 \\ \left[Y, \frac{s-t}{|s-t|} \right] & 2|s-t| \geq 1 \end{cases}.$$

Then $pM \subset \Sigma Y$, hence the map $\Psi' = \Sigma \varphi \circ p$ is well-defined. Since

$$\Psi'|_{\widetilde{T}^g \times 1 \times 1} = \widetilde{g} \quad \text{and} \quad \Psi'(x, 1, y, t) = \Sigma\varphi[(x, y, 1), 2(s-t)] = [\widetilde{g}(x, y), 2(s-t)],$$

it follows that Ψ' assume constant value on each fiber of $\pi \times \pi$. Analogously to Cylinder Lemma 5.1.α and Cone Lemma 4.2.2 we prove that

$$\Psi' = \widetilde{G} \circ (\pi \times \pi) \quad \text{on} \quad \widetilde{T} \times 0 \times 0 \cup \widetilde{T}^g \times 1 \times 1$$

and these maps are close on the rest of $M \cap (N \times N \times \partial(I \times I))$. Hence they are indeed equivariantly homotopic. $\qquad\square$

6. Knotted tori

Torus Lemma 6.1. *For $p \leq q$ and $m \geq p+q+2$ there are homomorphisms σ, γ, ρ and maps τ, ω such that the diagram below (anti)commute, the homomorphisms σ, γ, ρ and pr_1 are isomorphisms under the dimension restrictions $m \geq A$, where A is shown near the notation of a map:*

$$
\begin{array}{ccccc}
\pi_q(V_{m-q,p+1}) & \xrightarrow[\tau]{} & \operatorname{Emb}^m(S^p \times S^q) & \xrightarrow[\alpha]{} & \pi_{eq}^{m-1}(\widetilde{S^p \times S^q}) \\
\downarrow{\scriptstyle \rho \ \frac{3q}{2}+p+2} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \gamma \ p+q+3} \\
\pi_q(V_{m-q,p+1}^{eq}) & \xrightarrow[\sigma \ \frac{3q+p}{2}+2]{} & \Pi_{pq}^{m-1} & \xleftarrow[\operatorname{pr}_1 \ q+2p+2]{} & \Pi_{pq}^{m-1} \oplus \Pi_{qp}^{m-1}
\end{array}
$$

Here the *equivariant Stiefel manifold* V_{mn}^{eq} is the space of equivariant maps $S^{n-1} \to S^{m-1}$. Define $\Pi_{pq}^{m-1} := \pi_{eq}^{m-1}(S^p \times S^{2q})$, where the involution on $S^p \times S^{2q}$ is $a_p \times t_q$ and $t_q : S^{2q} \to S^{2q}$ is the symmetry with respect to $S^q \subset S^{2q}$. The group structure on Π_{pq}^{m-1} is defined as follows. For equivariant maps $\varphi, \psi : S^p \times S^{2q} \to$

S^{m-1} define the map $\varphi + \psi : S^p \times S^{2q} \to S^{m-1}$ (the unity and the inverse of φ) on $x \times S^{2q}$ to be the ordinary sum of the restrictions of φ and ψ to $x \times S^{2q}$ (the ordinary unity and the ordinary inverse of the $\varphi|_{x \times S^{2q}}$).

Torus Lemma 6.1 generalizes, in particular, the following observation [MaRo86, §3]. We have $\widetilde{S^q \sqcup S^q} \simeq_{eq} S^q \times S^q \sqcup S^q \times S^q \sqcup S^q \sqcup S^q$, where the involution on the right-hand term exchanges antipodes on each copy of S^q and also the corresponding points from the two copies of $S^q \times S^q$. Therefore

$$\pi_{eq}^{m-1}(\widetilde{S^q \sqcup S^q}) \cong \pi^{m-1}(S^q \times S^q) \overset{v_q^*}{\cong} \pi^{m-1}(S^{2q}) \cong \pi_{2q+1-m}^S$$

for $m \geq q+2$. Here the map $v_q : S^q \times S^q \to \frac{S^q \times S^q}{S^q \vee S^q} \cong S^{2q}$ is the quotient map. We can prove that the map v_q^* is an isomorphism for $m \geq q+3$ using general position and for $m = q+2$ using the cofibration exact sequence of the pair $(S^q \times S^q, S^q \vee S^q)$ and the existence of a retraction $r_q : \Sigma(S^q \times S^q) \to \Sigma(S^q \vee S^q)$.

Proof of Torus Lemma 6.1. In order to define the map τ, recall that $\pi_q(V_{m-q,p+1})$ is isomorphic to a group of CAT maps $S^q \to V_{m-q,p+1}$ up to CAT homotopy. The latter maps can be considered as CAT maps $\varphi : S^q \times S^p \to \partial D^{m-q}$. Define a CAT embedding $\tau_{CAT}(\varphi)$ as the composition

$$S^p \times S^q \overset{\varphi \times \mathrm{pr}_2}{\to} \partial D^{m-q} \times S^q \subset D^{m-q} \times S^q \subset \mathbb{R}^m.$$
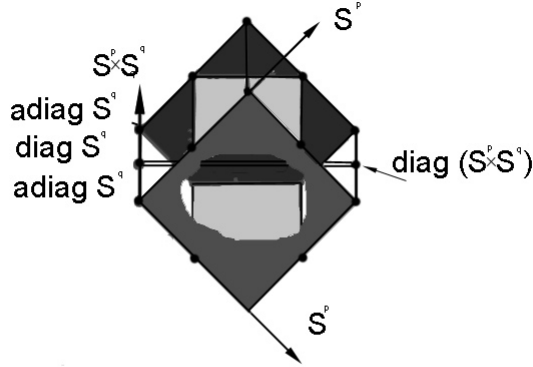

Figure 8

There is an equivariant deformation retraction

$$\widetilde{S^p \times S^q} \to \mathrm{adiag}\, S^p \times S^q \times S^q \bigcup_{\mathrm{adiag}\, S^p \times \mathrm{adiag}\, S^q} S^p \times S^p \times \mathrm{adiag}\, S^q$$

(Figure 8). Consider the involution $(s, x, y) \to (-s, y, x)$ on $S^p \times S^q \times S^q$. For $m \geq p + q + 2$ the map $\mathrm{id}_{S^p} \times v_q$ induces an isomorphism $\pi_{eq}^{m-1}(S^p \times S^q \times S^q) \cong \Pi_{pq}^{m-1}$. For $m \geq p + q + 3$ this is proved using general position and for $m = p + q + 2$ using the cofibration exact sequence of the pair $(S^p \times S^q \times S^q, S^p \times (S^q \vee S^q))$ and a retraction $\Sigma(S^p \times S^q \times S^q) \to \Sigma(S^p \times (S^q \vee S^q))$ obtained from the retraction $\mathrm{id}_{S^p} \times r_q : S^p \times \Sigma(S^q \times S^q) \to S^p \times \Sigma(S^q \vee S^q)$ by shrinking to a point the product of S^p with the vertex of the suspension. We also need to check that the involution on $S^q \times S^q$ exchanging factors corresponds to t_q.

Consider restrictions of an equivariant map $\widetilde{S^p \times S^q} \to S^{m-1}$ to $\mathrm{adiag}\, S^p \times S^q \times S^q$ and to $S^p \times S^p \times \mathrm{adiag}\, S^q$ (where adiag is antidiagonal). Define the map γ to be a direct sum of compositions of such restrictions and the isomorphisms $(\mathrm{id}_{S^p} \times v_q)^*$ and $(\mathrm{id}_{S^q} \times v_p)^*$. If $\dim(\mathrm{adiag}\, S^p \times \mathrm{adiag}\, S^q) = p + q \leq (m - 1) - 2$, then γ is an isomorphism by general position and the Borsuk Homotopy Extension Theorem.

By general position, for $2p + q \leq m - 2$ we have $\Pi_{qp}^{m-1} = 0$, hence pr_1 is an isomorphism. Let ω be the map corresponding under $(\mathrm{id}_{S^p} \times v_q)^*$ to the map

$$S^p \times S^q \times S^q \to S^{m-1} \quad \text{defined by} \quad (s, x, y) \mapsto \widetilde{f}((s, x), (-s, y)).$$

Clearly, the right-hand square of the diagram commutes.

Recall that ρ is the inclusion homomorphism. By [HaHi62], ρ is an isomorphisms for $m \geq \frac{3q}{2} + p + 2$.

Define σ as a composition

$$\pi_q(V_{m-q,p+1}^{eq}) = \pi_{eq}^{m-q-1}(S^p \times S^q) \xrightarrow{\Sigma^q} \pi_{eq}^{m-1}(\Sigma^q(S^p \times S^q)) \xrightarrow{\mathrm{pr}^*} \Pi_{pq}^{m-1}$$

Here the involution on $S^p \times S^q$ is $a_p \times \mathrm{id}_{S^q}$, the involutions on $\Sigma^q S^q$ and on $\Sigma^q(S^p \times S^q)$ are the 'suspension' involutions over id_{S^q} and $a_p \times \mathrm{id}_{S^q}$; the map

$$\mathrm{pr} : S^p \times \Sigma^q S^q = S^p \times \frac{S^q \times D^q}{S^q \times y, \ y \in \partial D^q} \to \frac{S^p \times S^q \times D^q}{S^p \times S^q \times y, \ y \in \partial D^q} = \Sigma^q(S^p \times S^q)$$

is a quotient map. The 'S^p-fiberwise' group structures on $\pi_{eq}^{m-q-1}(S^p \times S^q)$ and on $\pi_{eq}^{m-1}(\Sigma^q(S^p \times S^q))$ are defined analogously to that on Π_{pq}^{m-1}. It is easy to see that σ is an isomorphism. Analogously to [Ker59] it is proved that the left-hand square of the diagram (anti)commutes.

By the Equivariant Suspension Theorem, it follows that Σ^q is an isomorphism for $p+q \leq 2(m-q-1)-2$. The non-trivial preimages of pr are $S^p \times [S^q \times y], y \in \partial D^q$. Their union is homeomorphic to $S^p \times \partial D^q$. Since $\dim(S^p \times \partial D^q) = p + q - 1$, by general position it follows that pr^* is an isomorphism for $p + q - 1 \leq m - 3$. Therefore σ is an isomorphism for $m \geq \frac{3q+p}{2} + 2 \geq p + q + 2$. $\square$

By Theorems 1.1α and 1.3α, the maps α_{DIFF} and α_{PL} are bijective for $m \geq \frac{3(q+p)}{2} + 2$ and $m \geq \frac{3q}{2} + p + 2$, respectively. From the existence of τ and Torus

Lemma 6.1 it follows that α_{DIFF} is surjective for $m \geq \max(\frac{3q}{2} + p + 2, q + 2p + 2)$ [cf. Boe71, BoHa70]. Note that some maps of Torus Lemma 6.1 are *epimorphisms* under the weaker by one dimension restrictions than stated there.

The *PL Stiefel manifold* V_{mn}^{PL} is the space of PL embeddings $S^{n-1} \to S^{m-1}$. Replacing $V \to V^{PL}$, we can define analogously a map $\tau^{PL} : \pi_q(V_{m-q,p+1}^{PL}) \to \mathrm{Emb}_{PL}^m(S^p \times S^q)$.

It would be interesting to apply smoothing theory to prove the following conjectures:

$$\mathrm{Emb}_{DIFF}^m(S^p \times S^q) = \pi_q(V_{m-q,p+1}) \oplus C_{p+q}^{m-p-q} \text{ for } m \geq \max\left(\frac{3q}{2} + p + 2, q + 2p + 2\right),$$

where $C_{p+q}^{m-p-q} = \mathrm{Emb}_{DIFF}^m(S^{p+q})$ (note that $C_{p+q}^{m-p-q} = 0$ for $m \geq \frac{3(p+q)}{2} + 2$); and

$$\mathrm{Emb}_{DIFF}^{3k+1}(S^k \times S^k) = X \oplus C_{2k}^{k+1} \text{ for } k \geq 2,$$ where X is $\mathbb{Z} \vee \mathbb{Z}$ for k even and is either $\mathbb{Z}_2 \vee \mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for k odd (for a group G we define $G \vee G = \{(x,y) \in G \oplus G \mid \text{either } x = 0 \text{ or } y = 0\}$).

The classification of knotted tori, i.e. description of isotopy classes of embeddings $S^p \times S^q \to \mathbb{R}^m$ seems to be an interesting problem because it generalizes an important classical theory of 2-componented links (of the same dimension) [Hae66C], and just as the link theory, provides interesting examples and connections between geometric topology and homotopy theory. In particular, this classification is a natural next step (after the link theory) towards understanding isotopy classes of an *arbitrary* manifold in $\mathbb{R}^m$ (by the Handle Decomposition Theorem).

Torus Lemma 6.2. *If $s \geq 3$, $p_1 \leq \cdots \leq p_s$, $n = p_1 + \cdots + p_s$ and $N = S^{p_1} \times \cdots \times S^{p_s}$, then the same assertion as in Torus Lemma 6.1 holds for the following diagram:*

$$
\begin{array}{ccccc}
\pi_{n-p_1}(V_{m-n+p_1,p_1+1}) & \overset{\tau}{\longrightarrow} & \mathrm{Emb}^m(N) & \overset{\alpha}{\longrightarrow} & \pi_{eq}^{m-1}(\widetilde{N}) \\
\downarrow{\scriptstyle \rho\ \frac{3n-p_1}{2}+2} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \gamma\ 2n-p_1-p_2+3} \\
\pi_{n-p_1}(V_{m-n+p_1,p_1+1}^{eq}) & \underset{\sigma\ \frac{3n}{2}-p_1+2}{\longrightarrow} & \Pi_{p_1,n-p_1}^{m-1} & \underset{\mathrm{pr}_1\ 2n-p_2+2}{\longleftarrow} & \oplus_i \Pi_{p_i,n-p_i}^{m-1}
\end{array}
$$

Proof. Analogous to Torus Lemma 6.1. We shall only give definitions of τ and σ and omit the details. The map τ is defined as follows. An element $\varphi \in \pi_{n-p_1}(V_{m-n+p_1,p_1+1})$ is represented by a map $S^{n-p_1} \times S^{p_1} \to S^{m-n+p_1-1}$. Consider the projections

$$\mathrm{pr}_1 : N \to S^{p_1} \times S^{p_2+\cdots+p_s} = S^{p_1} \times S^{n-p_1} \quad \text{and} \quad \mathrm{pr}_2 : N \to S^{p_2} \times \cdots \times S^{p_s}.$$

Analogously to the case $s = 2$, define an embedding $\tau(\varphi)$ as the composition

$$S^{p_1} \times S^{p_2} \times \cdots \times S^{p_s} \overset{(\varphi \circ \mathrm{pr}_1) \times \mathrm{pr}_2}{\longrightarrow} \partial D^{m-n+p_1} \times S^{p_2} \times \cdots \times S^{p_s} \subset \mathbb{R}^m.$$

The map σ is defined analogously to the case $s = 2$ as a composition

$$\pi_{n-p_1}(V^{eq}_{m-n+p_1,p_1+1})$$
$$= \pi^{m-n+p_1-1}_{eq}(S^{p_1} \times S^{n-p_1}) \overset{\Sigma^{n-p_1}}{\to} \pi^{m-1}_{eq}(\Sigma^{n-p_1}(S^{p_1} \times S^{n-p_1})) \overset{\mathrm{pr}^*}{\to} \Pi^{m-1}_{p_1,n-p_1}.$$

Here the maps Σ^{n-p_1} and pr^* are isomorphisms for $2m \geq 3n - 2p_1 + 4$ and $m \geq n + 2$, respectively. □

We conjecture that $\alpha^4_{PL}(N)$ is surjective for a closed 2-manifold (e.g. a torus) N. Note that this is true for non-closed connected 2-manifolds. Since $\pi^3_{eq}(\widetilde{N}) \cong H_1(N, \mathbb{Z}_2)$, this conjecture for orientable surfaces is implied by the following one: If N is a closed orientable 2-surface, $\gamma \subset N$ is a circle representing an element $[\gamma] \in H_1(N, \mathbb{Z}_2)$, $e : N^2 \to \mathbb{R}^4$ is the standard embedding, $h_\gamma : N \cong N$ is the Dehn twist along γ, and $d(e, e \circ h_\gamma)$ is the difference element [Hud69, §11], then $d(e, e \circ h_\gamma) = [\gamma]$. Indeed, the latter conjecture implies that every element of $H_1(N, \mathbb{Z}_2)$ is representable by a difference element $d(e, f)$ for some embedding $f : N \to \mathbb{R}^4$. Similar conjecture can be stated for non-orientable 2-surfaces, using local coefficients.

7. Construction of examples

Proof of Example 1.2.ss. Example 1.2.ss follows because $\alpha^{6k+1}_{PL}(S^{2k} \times S^{2k})$ is bijective by Theorem 1.3α (or by [Boe71, Hae62B]) but there exists a PL embedding $S^{2k} \times S^{2k} \to \mathbb{R}^{6k+1}$, non-isotopic to a smooth embedding [Hae62A, BoHa70, p. 165, Boe71, 6.2]. □

Proof of Example 1.2.ii. Take a standard embedding $f : S^n \vee S^n \to S^m$. Then $S^m - f(S^n \vee S^n) \simeq S^{m-n-1} \vee S^{m-n-1}$. Take a map $\varphi : S^{2m-2n-3} \to S^m - f(S^n \vee S^n)$ representing the Whitehead product of generators. If $n = 1$ and $m = 3$, then φ is homotopic to an embedding by general position. If $n > 1$, then

$$2(2m - 2n - 3) - m + 1 \leq m - n - 2 \quad \text{and} \quad m - (2m - 2n - 3) \geq 3,$$

so φ is homotopic to an embedding by the Irwin Embedding Theorem. Define f on $S^{2m-2n-3}$ to be such an embedding. Since the homotopy class of φ is non-trivial, it follows that f is not isotopic to the standard embedding g. Using 'finger moves' analogously to [SSS98] we construct a map $F : N \times I \to \mathbb{R}^m \times I$ such that

$$F(x,0) = (f(x),0),\ F(x,1) = (g(x),1) \text{ and } F((S^n \vee S^n) \times I) \cap F(S^{2m-2n-3} \times I) = \emptyset.$$

Analogously to the proof of Theorem 5.2.α we obtain that $\alpha^m_G(N)f = \alpha^m_G(N)g$ (for each G). So $\alpha^m_G(N)$ is not injective. □

We conjecture that the non-trivial embedding f of Example 1.2.ii can be obtained from the Borromean rings $S^n \sqcup S^n \sqcup S^{2m-2n-3} \subset \mathbb{R}^m$ [Hae62T, Mas90, Proposition 8.3] by 'wedging' $S^n \sqcup S^n$. We also conjecture that by joining $S^n \sqcup S^n$ with a tube we obtain analogous example $S^n \sqcup S^{2m-2n-3} \to \mathbb{R}^m$ (this is though harder to prove: either we need to check that the linking coefficient of such a link is $[\iota_{m-n-1}, \iota_{m-n-1}] \neq 0$ for $m - n \notin \{2, 4, 8\}$, or we need to apply the Hudson–Habegger invariant [cf. Hae62T, §3, Sko]). Note that for $2m = 3n + 3 \geq 12$ by joining all the three components of the Borromean rings by two tubes the celebrated DIFF non-trivial but PL trivial knots are obtained [Hae62A, Hae66A].

In this section we assume that $m \geq \max\{2p + q + 2, p + q + 3\}$ (unless the opposite is stated). So by Torus Lemma 6.1 we can identify $\pi_{eq}^{m-1}(\widetilde{S^p \times S^q})$ with Π_{pq}^{m-1} and α with ω.

Decomposition Lemma 7.1. *For $m \geq 2p + q + 2$ and $p \geq 1$ there is the following (anti)commutative diagram, in which the first and the third lines are exact sequences of homomorphisms.*

$$
\begin{array}{ccccc}
\pi_q(V_{m-q-1,p}) & \xrightarrow{\mu''} & \pi_q(V_{m-q,p+1}) & \xrightarrow{\nu''} & \pi_q(V_{m-q,1}) \\
\downarrow{\scriptstyle \tau_{p-1}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle =} \\
\mathrm{Emb}_{PL}^{m-1}(S^{p-1} \times S^q) & \xrightarrow{\mu'} & \mathrm{Emb}_{PL}^{m}(S^p \times S^q) & \xrightarrow{\nu'} & \pi_q(S^{m-q-1}) \\
\downarrow{\scriptstyle \alpha_{p-1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \Sigma^\infty} \\
\Pi_{p-1,q}^{m-2} & \xrightarrow{\mu} & \Pi_{pq}^{m-1} & \xrightarrow{\nu} & \pi_{2q+1-m}^S
\end{array}.
$$

Proof. Let ν'' and μ'' be the homomorphisms induced by the mappings of the well-known bundle $V_{m-q-1,p} \to V_{m-q,p+1} \to V_{m-q,1}$. For an embedding $f : S^p \times S^q \to \mathbb{R}^m$ let $\nu'(f)$ be the linking coefficient of $f(x \times S^q)$ and $f(-x \times S^q)$ in $\mathbb{R}^m$. Define the map $\nu : \Pi_{pq}^{m-1} \to \Pi_{0q}^{m-1} \cong \pi_{2q-m+1}^S$ as 'the restriction over $* \times S^{2q}$'. Clearly, the right-upper square of the diagram commutes. The right-bottom square of the diagram (anti)commutes by [Ker59, Lemma 5.1].

By $S^p = D_+^p \underset{\partial D_+^p = S^{p-1} = \partial D_-^p}{\bigcup} D_-^p$ we denote the standard decomposition of S^p.

Analogously are defined $\mathbb{R}_\pm^m$ and $\mathbb{R}^{m-1}$. By the Irwin–Zeeman Embedding and Isotopy Theorem [Zee62, Irw65], for $m \geq 2p+q+2$ any embedding $f : S^{p-1} \times S^q \to S^{m-1}$ can be uniquely up to isotopy extended to an embedding $f_\pm : D_\pm^p \times S^q \to \mathbb{R}_\pm^m$. Two embeddings f_+ and f_- define an embedding $\mu'(f) : S^p \times S^q \to \mathbb{R}^m$ [cf. Hud63]. Clearly, the left-upper square of the diagram commutes.

Let us define the map μ first for the case $p = 1$. For a map $\varphi : S^{2q} \to S^{m-2}$ define the map $\mu\varphi$ to be the equivariant extension of the composition $D^1 \times S^{2q} \xrightarrow{\mathrm{pr}}$

$\Sigma S^{2q} \overset{\Sigma\varphi}{\to} S^{m-1}$. In order to define the map μ for arbitrary p, replace

$$\Pi^{m-2}_{p-1,q} \quad \text{and} \quad \Pi^{m-1}_{pq} \quad \text{by} \quad \pi^{m-2}_{eq}(\Sigma^q(S^{p-1} \times S^q)) \quad \text{and} \quad \pi^{m-1}_{eq}(\Sigma^q(S^p \times S^q)),$$

respectively (see the proof of Torus Lemma 6.1). For an equivariant map φ : $\Sigma^q(S^{p-1} \times S^q) \to S^{m-2}$ let $\mu\varphi$ be the composition

$$\Sigma^q(S^p \times S^q) = \Sigma^q(\Sigma S^{p-1} \times S^q) \overset{\Sigma^q \operatorname{pr}}{\to} \Sigma^{q+1}(S^{p-1} \times S^q) \overset{\Sigma\varphi}{\to} \Sigma S^{m-2},$$

where pr is the map from the proof of Torus Lemma 6.1. Clearly, the definition for arbitrary p agrees with that for $p = 1$. It is easy to check that both ν and μ are homomorphisms.

Let us prove the commutativity of the left-bottom square. We prove this for $p = 1$, for general case the proof is analogous. Take an embedding $f : S^0 \times S^q \to \mathbb{R}^{m-1}$. We have $\mu'f(D^1_\pm \times S^q) \subset \mathbb{R}^m_\pm$. In this paragraph denote by Π^{m-1}_{pq} the *space* of equivariant maps $S^p \times S^q \times S^q \to S^{m-1}$ with respect to the involution $(s, x, y) \to (-s, y, x)$ on $S^p \times S^q \times S^q$; modify accordingly the definition of ω; denote by μ', ω and ω_0 the maps of spaces (not of equivalence classes), corresponding to μ', ω and ω_0. See the proof of Torus Lemma 6.1; recall that we identify $\alpha = \omega$ and isomorphic groups $\pi^{m-2}_{eq}(\widetilde{S^0 \times S^q}) \cong \pi^S_{2q-m+2}$. Let pr be the projection from the definition of μ for the case $p = 1$. Observe that $\omega\mu'f = \omega_0 f$ on $S^0 \times S^q \times S^q$. For each $y \in S^1 \times S^q \times S^q$ the points $(\omega\mu'f)y$ and $(\Sigma\omega_0 f \circ \operatorname{pr})y$ are either both in the upper or both in the lower open hemisphere of S^{m-1}. Hence $\omega\mu'f \simeq_{eq} \Sigma\omega_0 f \circ \operatorname{pr}$, i.e. $\omega\mu' = \mu\omega_0$.

Let us prove the exactness at Π^{m-1}_{pq}. Clearly, $\nu\mu = 0$. On the other hand, if $\Phi : S^p \times S^{2q} \to S^{m-1}$ is an equivariant map such that $\Phi|_{* \times S^{2q}}$ is null-homotopic, then by the Borsuk Homotopy Extension Theorem, Φ is equivariantly homotopic to a map which maps $* \times S^{2q}$ and $a_p(*) \times S^{2q}$ to antipodal points of S^{m-1}. By the equivariant Suspension Theorem, the latter map is in $\operatorname{im}\mu$, since $p - 1 + 2q \le 2(m-2) - 1$. So $\ker\nu = \operatorname{im}\mu$. $\square$

Note that all the maps of Decomposition Lemma 7.1 except μ' are defined for $m \ge p + q + 3$.

Proof of Example 1.4. Set $q = n - 1 \le m - 4$. Now Example 1.4.s is proved looking at the right-bottom square of the diagram from Decomposition Lemma 7.1 and using the surjectivity of ν'' from Corollary 7.2 below. The specific examples can be found using [Tod64, §14] (set $l = m - n = m - q - 1$ and $k = 2q + 1 - m$).

Since $p < k$, we have $m \ge 2p + q + 2$. Now Example 1.4.i is proved looking at the right squares of the diagram from Decomposition Lemma 7.1 and using Lemma 7.3 below. $\square$

For a group G let $G_{(k)} = G_{[k]} = G$ for k even, let $G_{(k)} = G/2G$ for k odd and let $G_{[k]}$ be the subgroup of G formed by elements of order 2 for k odd. If G is finite abelian, then $G_{(k)} \cong G_{[k]} \cong G \otimes \mathbb{Z}_{(k)}$. Denote $\pi^S_l = 0$ for $l < 0$.

Corollary 7.2. *Suppose that $p = 1$ and $m \geq q + 4$. For the diagram of Decomposition Lemma 7.1 we have*

$$\operatorname{im} \nu = \pi^S_{2q-m+1,[m-q]} \quad and \quad \operatorname{coim} \mu = \pi^S_{2q-m+2,(m-q)}$$

(recall that we identify $\Pi^{m-2}_{0q} = \pi^S_{2q-m+2}$). So Π^{m-1}_{1q} is adjoint to $(\pi^S_{2q-m+2} \oplus \pi^S_{2q-m+1}) \otimes \mathbb{Z}_{(m-q)}$, unless $m = 2q+1$ and q is even, when $\Pi^{m-1}_{1q} \cong \mathbb{Z}_2$. If $m - q$ is even and $2m \geq 3q + 4$, then $\Pi^{m-1}_{1q} \cong \pi^S_{2q-m+2} \oplus \pi^S_{2q-m+1}$. If $m - q$ is even, then both ν'' and ν' are epimorphic.

Proof. Clearly, $\operatorname{im} \nu$ consists of homotopy classes $\varphi \in \Pi^{m-1}_{0q}$ extendable to a map $D^1 \times S^{2q} \to S^{m-1}$. These φ, considered as maps $\varphi : S^{2q} \to S^{m-1}$ are exactly such that $a_{m-1} \circ \varphi \circ t_q \simeq \varphi$. The latter condition is equivalent to $(-1)^m \varphi = (-1)^q \varphi$ (for m odd this follows by [Pos85, complement to lecture 6, (10), p. 264], since $h_0 : \pi_{2q}(S^{m-1}) \to \pi_{2q}(S^{2m-3})$ and $2q < 2m - 3$). So $\operatorname{im} \nu = \ker(1 - (-1)^{m-q})$.

In order to calculate $\ker \mu$ denote by M the composition

$$S^1 \times \Sigma^q S^q \overset{\operatorname{pr}}{\to} \Sigma^q (S^1 \times S^q) \cong \Sigma^q (\Sigma S^0 \times S^q) \overset{\Sigma^q \operatorname{pr}}{\to} \Sigma^{q+1}(S^0 \times S^q),$$

where pr and pr are maps from the proof of Torus Lemma 6.1 (we use the same notation pr for two distinct maps). Each map $\psi : S^{2q} \to S^{m-2}$ can be identified with an equivariant map $\psi : \Sigma^q(S^0 \times S^q) \to S^{m-2}$. For each map $\psi : S^{2q} \to S^{m-2}$ we can construct an equivariant map $h : \partial I^2 \times S^{2q} \to S^{m-1}$ such that $h|_{\partial I \times I \times S^{2q}}$ 'represents' $M \circ *$, $h|_{I \times 0 \times S^{2q}}$ 'represents' $\Sigma \psi \circ M$ and

$$h(s, 1, x, y) = -h(-s, 0, y, x) \quad \text{for} \quad (s, 1, x, y) \in I \times 1 \times S^{2q}.$$

Then $h|_{I \times 1 \times S^{2q}}$ 'represents' $((-1)^{m+q+1}\Sigma\psi) \circ M$. Hence

$$h = \Sigma((1 - (-1)^{m-q})\psi) \circ M, \quad \text{so} \quad \mu((1 - (-1)^{m-q})\pi^S_{2q-m+2}) = 0.$$

It is easy to see that the above construction describes the entire $\ker \mu$.

Recall that $V_{m-q,2} \cong TS^{m-q-1}$. For $m - q$ even we even have a section $s : S^{m-q-1} \to V_{m-q,2}$ such that $\nu'' s_* = \operatorname{id}$ (note that the map τs_* is a generalization of [Zee62, Example] for $q - r = 1$). Therefore ν'' is epimorphic, hence ν' is epimorphic. If $2m \geq 3q + 4$, then $\Sigma^\infty : \pi_q(S^{m-q-1}) \to \pi^S_{2q+1-m}$ is an isomorphism. Hence there is a section

$$\alpha \tau s_* (\Sigma^\infty)^{-1} : \pi^S_{2q+1-m} \to \Pi^{m-1}_{1q}, \quad \text{so} \quad \Pi^{m-1}_{1q} \cong \pi^S_{2q-m+2} \oplus \pi^S_{2q-m+1}. \quad \square$$

Lemma 7.3. (7.3.a) Π^{m-1}_{pq} *is finite if either $p + q + 2 \leq m \leq 2q$ or $p \geq 1$, q odd ≥ 3 and $m = 2q + p + 1$.*

(7.3.b) *The image of the restriction homomorphism $\nu''_p : \pi_{4k-1}(V_{2k+1,p+1}) \to \pi_{4k-1}(S^{2k})$ is infinite for $p < 2k$.*

Proof. Let us prove (a) by induction on p. For the case $p + q + 2 \leq m \leq 2q$ the induction base is $p = 0$, when $\Pi^{m-1}_{0q} \cong \pi_{2q}(S^{m-1})$ is indeed finite. For the case $m = 2q + p + 1$, q odd ≥ 3 and $p \geq 1$ the induction base is $p = 1$, when $\Pi^{2q+1}_{1q} \cong \mathbb{Z}_{(q+2)}$ is indeed finite by Corollary 7.2. The inductive step of (a) follows by the induction hypothesis and Decomposition Lemma 7.1.

In order to prove (b) for $p = 0$ observe that the map ν_0 is an isomorphism and $\pi_{4k-1}(S^{2k})$ is infinite. Suppose that $p \geq 1$ and there is an infinite set $\{x_i\} \in \pi_{4k-1}(V_{2k+1,p})$ with distinct ν''_{p-1}-images. Consider the Serre fibration $S^{2k+1-p} \to V_{2k+1,p+1} \xrightarrow{\psi} V_{2k+1,p}$ and the following segment of its exact sequence:

$$\pi_{4k-1}(V_{2k+1,p+1}) \xrightarrow{\psi_*} \pi_{4k-1}(V_{2k+1,p}) \to \pi_{4k-2}(S^{2k-p}).$$

Since $\pi_{4k-2}(S^{2k-p})$ is finite, by exactness it follows that the number of congruence classes of $\pi_{4k-1}(V_{2k+1,p})$ modulo $\operatorname{im}\psi_*$ is finite. Therefore an infinite number of x_i (we may assume that all x_i) lie in the same congruence class. By passing from $\{x_i\}$ to $\{x_i - x_1\}$ we may assume that this congruence class is the subgroup $\operatorname{im}\psi_*$ itself. Hence the inductive step follows from $\nu''_p = \nu''_{p-1}\psi_*$. $\qquad\square$

For k even $\pi_{k+l}(V_{k+2,2}) \cong \pi^S_l \oplus \pi^S_{l-1}$ because the tangent bundle $V_{k+2,2} \to S^{k+1}$ has a section. Corollary 7.2 and Torus Lemma 6.1 imply that for k odd and $2 \leq l \leq k$ the group $\pi_{k+l}(V_{k+2,2})$ is adjoint to $(\pi^S_l \oplus \pi^S_{l-1})\otimes\mathbb{Z}_2$ and $\pi_{k+1}(V_{k+2,2}) \cong \mathbb{Z}_2$. Note that $\pi_{2k+1}(V_{k+2,2})$ is not adjoint to $(\pi^S_{k+1}\oplus\pi^S_k)\otimes\mathbb{Z}_2$ for k odd by Lemma 7.3.b. Note that $\pi_q(V_{n,p+1})$ is finite if either $n \leq \frac{q}{2} + 1$ or $p + \frac{q+3}{2} \leq n \leq q$ or $n \geq q + p + 3$ (by induction on p using the exact sequence of the above Serre fibration). This yields another proof of Lemma 7.3.b. F. Cohen kindly informed me that these remarks are known, although not in this explicit form.

Example 7.4. If $l \neq 3, 7$ is odd and $\Sigma^3 : \pi_{2l-1}(S^{l-1}) \to \pi^S_l$ is epimorphic, then $\alpha^{3l}(S^1 \times S^{2l-1})$ is not injective.

$$
\begin{array}{ccccc}
\pi_{2l-1}(V_{l,1}) & \xrightarrow{\mu''} & \pi_{2l-1}(V_{l+1,2}) & \xrightarrow{\nu''} & \pi_{2l-1}(S^l) \\
\downarrow{=} & & \downarrow{\tau} & & \downarrow{=} \\
\pi_{2l-1}(S^{l-1}) & \xrightarrow{\tau\mu''} & \mathrm{Emb}^{3l}_{PL}(S^1 \times S^{2l-1}) & \xrightarrow{\nu'} & \pi_{2l-1}(S^l). \\
\downarrow{\Sigma^3} & & \downarrow{\omega} & & \downarrow{\Sigma} \\
\pi^S_l & \xrightarrow{\mu} & \Pi^{3l-1}_{1,2l-1} & \xrightarrow{\nu} & \pi^S_{l+1}
\end{array}
$$

Proof. The case $l = 1$ is obvious, so suppose that $l > 1$. By Decomposition Lemma 7.1 there exists (anti)commutative diagram as above. Since $3l \geq 2 + (2l - 1) + 2$, it suffices to construct embeddings $f, g : S^1 \times S^{2l-1} \to \mathbb{R}^{3l}$ such that $\nu'(f) \neq \nu'(g)$ but $\omega g = \omega f$. Let $\varphi = [\iota_l, \iota_l] \in \pi_{2l-1}(S^l)$. Recall that $\varphi \neq 0$ for $l \neq 1, 3, 7$ but $\Sigma\varphi = 0$. Since l is odd, it follows that there is a section $s : S^l \to V_{l+1,2}$ such that $\nu'' s_* = \mathrm{id}$. Let $f = \tau s_* \varphi$. We have $\omega f \in \ker \nu = \mathrm{im}\, \mu$, hence there is $y \in \pi_l^S$ such that $\mu y = \omega f$. Since Σ^3 is epimorphic, it follows that there is $y' \in \pi_{2l-1}(S^{l-1})$ such that $\Sigma^\infty y' = y$. Let $g = \tau \mu'' y'$. Now the example follows from

$$\omega f = \mu \Sigma^3 y' = \omega \tau \mu'' y' = \omega g \quad \text{and} \quad \nu' f = \varphi \neq 0 = \nu' \tau \mu'' y' = \nu' g. \qquad \square$$

We conjecture that $\Sigma^3 : \pi_{2n-1}(S^{n-1}) \to \pi_{2n+2}(S^{n+2}) = \pi_n^S$ is epimorphic for each integer $n \notin \{1, 2, 3, 7\}$ (Triple Suspension Conjecture, cf. [Jam54] and Example 7.4). This conjecture is true for $n = 4s$ and for each $n \leq 30$ by [Tod62, §14 and addition to the Russian edition]. Indeed, for $n \geq 4$, the EHP sequence [Jam54] is $\pi_{2n-1}(S^{n-1}) \xrightarrow{\Sigma^3} \pi_n^S \xrightarrow{H_3} \pi_n(V_{n+2,3}) \xrightarrow{P_3} \pi_{2n-2}(S^{n-1}) \xrightarrow{\Sigma^2} \pi_{n-1}^S$ (the last homomorphism is Σ^3 but it equals to Σ^2 by stability). Recall that $\pi_n(V_{n+2,3})$ is $0, \mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4$ according to $n = 4s, 4s + 2, 4s + 1$ and $4s - 1$ [Pae54]. Since Σ^2 above is an epimorphism for $n \neq 2, 4, 8$, it follows that Triple Suspension Conjecture is equivalent to '$P : \pi_n(V_{n+2,3}) \to \pi_{2n-2}(S^{n-1})$ is monomorphic for $n \geq 4$, $n \neq 7$' and to '$|\pi_{n-1}^S| = v_n |\pi_{2n-2}(S^{n-1})|$' for $n = 5, 6$ or $n \geq 9$' (or to the same assertions for the 2-components). Here $v_n = 1, 2, 4$ according to $n = 4s + 1$, $4s - 1$ and $2s$, respectively.

Note that $\mathrm{im}\, \nu' = \mathrm{im}\, \nu'' = \pi_{2q-m+1,[m-q]}$ for $2m \geq 3q + 4$. For $m \geq q + 4$ and $m - q$ odd we have $\ker 2 \subset \mathrm{im}\, \nu'' \subset \mathrm{im}\, \nu' \subset \ker 2\Sigma^\infty$, where by 2 is denoted the multiplication by 2. Indeed, from the exact sequence of the bundle $S^{m-q-2} \to V_{m-q,2} \to S^{m-q-1}$ it follows that

$$\ker(1 - (-1)^{m-q}) \subset \mathrm{im}\, \nu'' \subset \ker[(1 - (-1)^{m-q})\Sigma^\infty].$$

Since $\nu'' = \nu'\tau$, it follows that $\mathrm{im}\, \nu'' \subset \mathrm{im}\, \nu'$. If $f : S^1 \times S^q \to \mathbb{R}^m$ is an embedding, then the linking $f(x \times S^q \sqcup -x \times S^q)$ is PL isotopic to the linking $f(-x \times S^q \sqcup x \times S^q)$. Therefore $\Sigma^\infty (-1)^{m-q}\nu'(f) = \Sigma^\infty \nu'(f)$ i.e. $\nu'(f) \in \ker[(1 - (-1)^{m-q})\Sigma^\infty]$. Hence the above assertions on $\mathrm{im}\, \nu'$ and $\mathrm{im}\, \nu''$ follow. By Lemma 7.3.b $\mathrm{im}\, \nu'' \neq \pi_q(S^{m-q-1})_{[m-q]}$ for $2m \leq 3q + 3$.

Analogously to the proof of Example 1.4.s the results of the previous paragraph show that

$\alpha^m(S^1 \times S^q)$ *is not surjective if* $m - q$ *is odd* ≥ 3 *and* $\Sigma_{[1]}^\infty \pi_q(S^{m-q-1})_{[1]} \to \pi_{2q+1-m,[1]}^S$ *is not epimorphic.*

Specific examples can be found using [Tod64, §14]. Note that if $\mathrm{im}\, \Sigma_{[m-q]}^\infty \not\supset \pi_{2q-m+1,[m-q]}^S$, then by the Freudenthal Suspension Theorem either $(q, l) =$

$(6,3), (14,7)$ or $q \geq 2l+1$ and $m = q+l+1 \leq \frac{3q+1}{2}$ (from the $[l-1]$-version of Triple Suspension Conjecture it follows that even $q \neq 2l+1$ and $m \neq \frac{3q+1}{2}$).

We conjecture that $\alpha^m(S^p \times S^q)$ is not surjective for (at least some) integers $a \geq 0$, $b \geq 0$, $c \in \{0,1,2,3\}$, $1 \leq p \leq 2^c + 8b - 1$, $k = (2a+1)2^{4b+c} \notin \{2,4,8\}$ and $m = q+k+1$ such that $\Sigma^\infty : \pi_q(S^{m-q-1}) \to \pi^S_{2q+1-m}$ is not epimorphic. Perhaps this can be proved analogously to Example 1.4.s (note that ∂D^{m-q} would admit p linearly independent vector fields and there would be a section $s : S^{m-q-1} \to V_{m-q,p+1}$ such that $\nu'' s_* = \mathrm{id}$).

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