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Some properties of locally conformal symplectic structures

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Abstract. We show that the d_ω-cohomology is isomorphic to a conformally invariant usual de Rham cohomology of an appropriate cover. We also prove a Moser theorem for locally conformal symplectic (lcs) forms. We point out a connection between lcs geometry and contact geometry. Finally, we show the connections between first kind, second kind, essential, inessential, local, and global conformal symplectic structures through several invariants.

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1. Preliminaries

A *locally conformal symplectic* (lcs) form on a smooth manifold M is a non-degenerate 2-form Ω such that there exists an open cover $\mathcal{U} = (U_i)$ and smooth positive functions λ_i on U_i such that

$$\Omega_i = \lambda_i(\Omega_{|U_i})$$

is a symplectic form on U_i. If for all i, $\lambda_i = 1$, the form Ω is a symplectic form. Lee [15] observed that the 1-forms $\{d(\ln \lambda_i)\}$ fit together into a closed 1-form ω such that

$$d\Omega = -\omega \wedge \Omega. \tag{1}$$

Such 1-form is uniquely determined by Ω and is called the Lee form of Ω.

Conversely, if a non-degenerate 2-form Ω satisfies (1), and $\mathcal{U} = (U)_i$ is an open cover with contractible open sets, then $\omega_{|U_i} = d \ln \lambda_i$, for some positive function λ_i on U_i and $\lambda_i \Omega_{|U_i}$ is symplectic.

Two lcs forms Ω, Ω' on a smooth manifold M are said to be (conformally) equivalent if $\Omega' = f\Omega$, for some positive function f on M.

A locally conformal symplectic (lcs) structure $\mathcal{S}$ on a smooth manifold M is an equivalence class of lcs forms.

The couple $(M, \mathcal{S})$ is called a *lcs manifold.* If Ω is a representative of $\mathcal{S}$, we write $\Omega \in \mathcal{S}$. If $\omega = 0$ in the definition above, then Ω is a symplectic form. In that case the lcs structure $\mathcal{S}$ is said to be a *global conformal symplectic* (gcs) structure and we write $\mathcal{S} = \mathcal{O}$.

Let $(M, \mathcal{S})$ be a lcs manifold, and let $\Omega \in \mathcal{S}$ and ω its Lee form. If $\Omega' = \lambda\Omega$ for some positive function λ, then an immediate calculation shows that the Lee form of Ω' is $\omega' = \omega - d\ln(\lambda)$.

Hence the cohomology class $[\omega] \in H^1(M, \mathbb{R})$ is an invariant $\mathcal{L}_{\mathcal{S}}$ of $\mathcal{S}$, we call the Lee class of $\mathcal{S}$. Clearly, $\mathcal{S} = \mathcal{O}$ iff $\mathcal{L}_{\mathcal{S}} = 0$.

Locally conformal symplectic forms were introduced by Lee [15], and have been extensively studied by Vaisman [18], [19]. The first properties of their automorphism groups were established by Lefebvre [16].

We will assume that all manifolds considered are connected, but not necessarily compact, and have dimension at least 4. (In dimension 2, a lcs form is simply a volume-form, and the corresponding structure is an orientation.)

For any closed 1-form ω on a smooth manifold M, the operator d_ω which assigns to a p-form γ the $(p+1)$-form

$$d_\omega\gamma = d\gamma + \omega \wedge \gamma$$

is a coboundary operator, i.e. $d_\omega \circ d_\omega = 0$.

The cohomology of differential forms with this coboundary operator will be denoted by $H^*_\omega(M)$ and will be called the d_ω-cohomology. For more information on this cohomology, see [11] or [19].

A lcs form Ω is precisely a non-degenerate d_ω closed 2-form (where ω is the Lee form).

This cohomology is "almost" an invariant of the lcs structure $\mathcal{S} = [\Omega]$: given $\Omega' \in \mathcal{S}$, there is an isomorphism between $H_\omega(M)$ and $H_{\omega'}(M)$, (ω' the Lee form of Ω'), depending on the choice of λ such that $\omega' = \omega - d\ln\lambda$. More precisely the isomorphism is given by $\alpha \mapsto \lambda\alpha$.

In section 3, we show that the $c\mathcal{A}$ cohomology constructed in [5], [6], is isomorphic to $H_\omega(M)$. This shows that the d_ω cohomology (which is a sort of twisted de Rham cohomology of M) is a conformally invariant usual de Rham cohomology of an appropiate cover of M.

Let $\mathrm{Diff}_{\mathcal{S}}(M)$ be the group of all automorphisms of a lcs structure $\mathcal{S}$ on a smooth manifold M. It is clear that for any representative $\Omega \in \mathcal{S}$, then $\mathrm{Diff}_{\mathcal{S}}(M)$ is the set of all diffeomorphisms ϕ of M such that $\phi^*\Omega = f_\phi\Omega$, where f_ϕ is a nowhere zero (positive) smooth function on M.

We also may choose (or fix) an underlying $\Omega \in \mathcal{S}$, and consider the group $G_\Omega(M)$ of diffeomorphisms of M which preserve the form Ω. This is a non-invariant subgroup of $\mathrm{Diff}_{\mathcal{S}}(M)$.

The Lie algebra $\mathcal{X}_{\mathcal{S}}(M)$ of infinitesimal automorphisms of $\mathcal{S}$, consists of vector fields X on M such that $L_X\Omega = (u_\Omega(X))\Omega$, where $u_\Omega(X)$ is a smooth function on M. Here L_X stands for the Lie derivative in the direction X. We denote $\mathcal{X}_{\mathcal{S}}(M)_c$

the subalgebra of compact supported automorphisms. We will also consider the subalgebra $\mathcal{X}_\Omega(M)$ of $\mathcal{X}_\mathcal{S}(M)$ consisting of vector fields X such that $L_X\Omega = 0$.

Definition. *A lcs form Ω on M is said to be of the first kind if there exists $X \in \mathcal{X}_\Omega(M)$, with $\omega(X) \neq 0$, where ω is the corresponding Lee form. Otherwise it is said to be of the second kind* [18].

A lcs structure $\mathcal{S}$ on M is said to be of the first kind if there is a representative $\Omega \in \mathcal{S}$ of the first kind. The lcs structure $\mathcal{S}$ is said to be of the second kind otherwise.

Warning. Vaisman [18] observed that a first kind lcs structure admits representatives which are second kind lcs forms.

For $X \in \mathcal{X}_\Omega(M)$, and M connected, $\omega(X)$, is a constant number since:

$$0 = dL_X\Omega = L_X d\Omega = L_X(-\omega \wedge \Omega) = -((L_X\omega) \wedge \Omega + \omega \wedge L_X\Omega) = -(di(X)\omega) \wedge \Omega$$

and Ω is non-degenerate.

Hence if Ω is a first kind lcs form with Lee form ω, the condition:

$$\text{There is } X \in \mathcal{X}_\Omega(M), \text{ with } \omega(X) \neq 0$$

is equivalent to saying that there a 1-form θ such that

$$\Omega = d\theta + \omega \wedge \theta$$

Indeed just normalize X as above so that $\omega(X) = 1$ and set $\theta = i(X)\Omega$. First kind lcs forms are d_ω exact.

2. Examples

We describe here a few examples of lcs forms. The reader can consult the book [9] for more examples.

2.1. Examples connected with Contact Geometry

A contact form α on a (2n+1) dimensional manifold N is a 1-form α such that $\alpha \wedge (d\alpha)^n$ is everywhere non-zero. Two contact forms α and α' are equivalent if there is a smooth positive function f on N such that $\alpha' = f\alpha$. The contact structure $\mathcal{C}(\alpha)$, determined by α is the equivalence class of α.

Consider the cartesian product $M = N \times S^1$, and the projections $p_1 : M \to N$, $p_2 : M \to S^1$. Let β be the canonical 1-form on S^1 with integral 1. If we set $\theta = p_1^*\alpha$ and $\omega = p_2^*\beta$, then

$$\Omega = d\theta + \omega \wedge \theta$$

is non-degenerate and $d\Omega = -\omega\wedge d\theta = -\omega\wedge(\Omega - \omega\wedge\theta) = -\omega\wedge\Omega + \omega\wedge\omega\wedge\theta = -\omega\wedge\Omega$. Hence the conformal class of Ω is a lcs structure on M, we denote $\mathcal{S}(\alpha)$. This structure is of the first kind.

The following result will be proved in section 4.

Theorem 1. *The lcs structure $\mathcal{S}(\alpha)$ depends only on the contact structure $\mathcal{C}(\alpha)$. In fact there is a well defined mapping from the group $\mathrm{Diff}_{\mathcal{C}(\alpha)}(M)$ of automorphisms of the contact structure $\mathcal{C}(\alpha)$ (the group of contact diffeomorphisms of (M, α)) to the group $\mathrm{Diff}_{\mathcal{S}(\alpha)}(M \times S^1)$.*

2.2. Deformations of lcs structures

If we add a 2-form η_ϵ C^0 close to 0 to a lcs form Ω, the resulting form $\Omega_\epsilon = \Omega + \eta_\epsilon$ is again non-degenerate. An immediate calculation gives:

$$d\Omega_\epsilon = -\omega \wedge \Omega_\epsilon + (d\eta_\epsilon + \omega \wedge \eta_\epsilon) = -\omega \wedge \Omega_\epsilon + d_\omega \eta_\epsilon.$$

Hence if η_ϵ is d_ω closed, then Ω_ϵ is a lcs form with ω as Lee form. For instance take $\eta_\epsilon = d_\omega \gamma_\epsilon$ where γ_ϵ is C^1 close to zero.

To construct general deformations of a lcs form Ω, with Lee form ω, we may look for 2-forms η_ϵ C^0 closed to zero, and closed 1-forms ρ (not necessarily small) such that $d\Omega_\epsilon = -(\omega + \rho) \wedge \Omega_\epsilon$. In that connection, we note that if $\mathcal{L}_{cs}(M)$ is the set of all lcs forms on a smooth manifold M, and $\mathcal{F}^*(M)$ the space of differential forms, both with the C^∞ topology, $\mathcal{L}_{cs}(M)$ is not an open subset of $\mathcal{F}^*(M)$.

Note that if the lcs form Ω is of first kind and we add to it a non-d_ω-exact form, the resulting lcs form is not d_ω-exact, hence of the second kind.

We have the following fact:

Theorem 2. *Let $(M, \mathcal{S})$ be a compact lcs manifold, and let $\Omega \in \mathcal{S}$ be a representative, with Lee form ω. Then for any d_ω exact 2-form η_ϵ, C^0 close to zero, the lcs form $\Omega_\epsilon = \Omega + \eta_\epsilon$ represents a lcs structure equivalent to $\mathcal{S}$.*

Hence the non-trivial deformations of lcs structures are parametrized by elements of the second cohomology group $H^2_\omega(M)$.

2.3. Lcs on cotangent bundles [12]

Let $M = T^*(N)$ be the total space of the cotangent bundle $\pi : T^*(N) \to N$ over a smooth manifold N. Let Λ_N be the Liouville 1-form on M and α a closed 1-form on N, then

$$\Omega_\alpha = d_\omega \Lambda_N$$

where $\omega = \pi^*\alpha$, is a lcs form on M. The conformal structure defined by this lcs form depends only on the cohomology class of α.

3. The $c\mathcal{A}$ -cohomology and the d_ω-cohomology

For any closed 1-form ω on a smooth manifold M, the operator d_ω which assigns to a p-form γ the $(p+1)$-form

$$d_\omega \gamma = d\gamma + \omega \wedge \gamma$$

is a coboundary operator, i.e. $d_\omega \circ d_\omega = 0$.

The cohomology of differential forms with this coboundary operator will be denoted by $H^*_\omega(M)$ and will be called the d_ω-cohomology. For more information on this cohomology, see [11] or [19]. For instance, it was proved in [19] that the groups $H^p_\omega(M)$ are isomorphic to the cohomology groups of M with coefficients in the sheaf $\mathcal{F}_\omega(M)$ of germs of smooth functions f on M such that $d_\omega f = 0$.

In this section, we give another interpretation of the d_ω cohomology.

One associates with a closed 1-form ω on a smooth manifold M the minimum regular cover $\pi : \tilde{M} \to M$ over which the 1-form ω pulls back to an exact 1-form. The manifold $\tilde{M}$ is a connected component of the sheaf of germs of smooth functions f on M such that $\omega = df$ [10].

Let $\lambda : \tilde{M} \to \mathbb{R}$ be a positive function on $\tilde{M}$ such that

$$\pi^* \omega = d(\ln \lambda).$$

It is well known that the group $\mathcal{A}$ of automorphisms of the covering $\tilde{M}$, is isomorphic to the group of periods of ω [10]. We will need the following:

Lemma 1 [6]**.** *For any $\tau \in \mathcal{A}$, the function*

$$(\lambda \circ \tau)/\lambda$$

is a constant, we denote c_τ, independent of the choice of λ and

$$\tau \mapsto c_\tau$$

is a group homomorphism c from $\mathcal{A}$ to the multiplicative group $\mathbb{R}^+$ of positive real numbers.

For the convenience of the reader, we give here the proof [6].

Proof. Clearly if $\lambda' = a\lambda$ for some constant a, $\lambda' \circ \tau/\lambda' = \lambda \circ \tau/\lambda$.

For any $\tau \in \mathcal{A}$, we have:

$$d(\ln(\lambda \circ \tau) - \ln \lambda)) = \tau^* \pi^* \omega - \pi^* \omega = (\pi\tau)^* \omega - \pi^* \omega = \pi^* \omega - \pi^* \omega = 0.$$

Hence $\ln(\lambda \circ \tau/\lambda) = K$, a constant and $\lambda \circ \tau/\lambda = e^K = c_\tau$.

If $\tau, \tau' \in \mathcal{A}$:

$$c_{\tau\tau'} = (\lambda \circ \tau\tau')/\lambda = ((\lambda \circ (\tau\tau'))/(\lambda \circ \tau')).(\lambda \circ \tau')/\lambda$$

$$= ((\lambda \circ \tau)/\lambda) \circ \tau').((\lambda \circ \tau')/\lambda) = ((\lambda \circ \tau)/\lambda).((\lambda \circ \tau')/\lambda) = c_\tau.c_{\tau'}. \qquad \square$$

The set $\mathcal{F}^*_{c\mathcal{A}}(M)$ of all differential forms α on $\tilde{M}$ such that $\tau^*\alpha = c_\tau \alpha$ for all $\tau \in \mathcal{A}$, is a subcomplex of the de Rham complex of $\tilde{M}$. We denote its cohomology by $H^*_{c\mathcal{A}}(M)$ and call it the conformally $\mathcal{A}$-invariant cohomology of M. Clearly, if the cohomology class of ω is trivial, then $H^*_{c\mathcal{A}}(M)$ coincides with the de Rham cohomology of M.

Remark 1. For any differential form α on M, then $U_\alpha = \lambda\pi^*\alpha \in \mathcal{F}^*_{c\mathcal{A}}(M)$
Indeed, for any $\tau \in \mathcal{A}$,

$$\tau^* U_\alpha = \lambda \circ \tau \cdot \tau^*\pi^*\alpha = \frac{\lambda \circ \tau}{\lambda} \cdot \lambda \cdot (\pi \circ \tau)^*\alpha = c_\tau(\lambda\pi^*\alpha) = c_\tau U_\alpha. \qquad \square$$

Lemma 2. *For any differential form, α, $d_\omega\alpha = 0$ if and only if $d(\lambda\pi^*\alpha) = 0$.*

Proof. Suppose $d_\omega\alpha = 0$. Then: $d(\lambda\pi^*\alpha) = d\lambda \wedge \pi^*\alpha + \lambda\pi^*(-\omega \wedge \alpha) = d\lambda \wedge \pi^*\alpha - \lambda d(\ln \lambda) \wedge \pi^*\alpha = 0$.

Suppose now $d(\lambda\pi^*\alpha) = 0$, and compute:
$\lambda\pi^*(d_\omega\alpha) = \lambda\pi^* d\alpha + \lambda\pi^*\omega \wedge \pi^*\alpha = \lambda\pi^* d\alpha + \lambda d(\ln\lambda) \wedge \pi^*\alpha = d(\lambda\pi^*\alpha) = 0$.
Since λ is a positive function and π is a local diffeomorphism, $d_\omega\alpha = 0$. $\square$

Theorem 3. *$H^*_{c\mathcal{A}}(M)$ is (non-canonically) isomorphic with $H^*_\omega(M)$*

Proof. The natural homomorphism

$$H^*_\omega(M) \to H^*_{c\mathcal{A}}(M) \quad [\alpha] \mapsto [\lambda\pi^*\alpha]$$

is onto: indeed, let β be a form such that $d\beta = 0$ and $\tau^*\beta = c_\tau\beta$ for all $\tau \in \mathcal{A}$. Then:

$$\tau^*(\beta/\lambda) = \tau^*\beta/\lambda \circ \tau = (c_\tau.\beta/\lambda).(\lambda/\lambda \circ \tau) = \beta/\lambda$$

for all $\tau \in \mathcal{A}$. Hence β/λ is basic, i.e. there is a form α on M such that $\beta/\lambda = \pi^*\alpha$. Since $\beta = \lambda\pi^*\alpha$ is closed, α is d_ω closed, by Lemma 2.

It is also one-to-one: suppose $d_\omega\alpha = 0$ and $\lambda\pi^*\alpha = d\rho$ with $\tau^*\rho = c_\tau\rho$ for all $\tau \in \mathcal{A}$. Then: rewriting the equations above with β replaced by ρ, we see that ρ/λ is basic, i.e. there is a formγ on M such that $\rho/\lambda = \pi^*\gamma$.

Let us now compute: $\pi^*(d_\omega\gamma) = \pi^*(d\gamma + \omega \wedge \gamma) = d(\rho/\lambda) + d\ln\lambda \wedge \rho/\lambda = d\rho/\lambda - d\lambda/(\lambda)^2 \wedge \rho + (d\lambda/\lambda) \wedge \rho/\lambda = d\rho/\lambda = \pi^*\alpha$.

Since π is a covering map, $\alpha = d_\omega \gamma$. □

In [5], [6], we had already observed that $H_{c\mathcal{A}}(M)$ is a quotient of $H_\omega(M)$. We deduce the following well known fact ([11])

Corollary. *If ω is a non-exact 1-form on a smooth manifold M, $H^0_\omega(M) = 0$.*

Proof. An element of $H^0_\omega(M) \approx H^0_{c\mathcal{A}}(M)$ is represented by a constant K such that $K \circ \tau = K = c_\tau K$ for all $\tau \in \mathcal{A}$. Since ω is not exact, there is a $\tau \in \mathcal{A}$ with $c_\tau \neq 1$. Hence $K = 0$. □

Let $(M, \mathcal{S})$ be a lcs manifold, $\Omega \in \mathcal{S}$ a representative, with Lee form ω. Let $\pi : \tilde{M} \to M$ be the minimum regular covering of M associated with the 1-form ω and let $\lambda : \tilde{M} \to \mathbb{R}$ be a positive function on $\tilde{M}$ such that

$$\pi^*\omega = d(\ln \lambda).$$

Then $\tilde{\Omega} = \lambda(\pi^*\Omega)$ is a symplectic form on $\tilde{M}$ and its conformal class $\tilde{\mathcal{S}}$ is independent of the choice of $\Omega \in \mathcal{S}$ and of λ.

Note that given a lcs $\Omega \in \mathcal{S}$, with Lee form ω, the cohomology classes $[\Omega] \in H^2_\omega(M)$ and $[\lambda\pi^*\Omega] \in H^2_{c\mathcal{A}}(M)$ are not invariants of the lcs structure $\mathcal{S}$.

The cohomology groups $H^*_{c\mathcal{A}}(M)$ and the d_ω cohomology are "almost" invariants of the lcs structure: since if ω and $\omega' = \omega - d\ln\lambda$ are two Lee forms, then $H_\omega(M)$ is isomorphic to $H_{\omega'}(M)$, by the isomorphism $\alpha \to \lambda\alpha$, which unfortunately depends on the choice of λ. Two such λ's differ by a constant.

4. Equivalence of lcs structures

We have the following Moser type result:

Theorem 4. *Let Ω_t be a smooth family of lcs forms on a compact manifold M. Suppose that for all t, the Lee form of Ω_t is the same 1-form ω and that $\Lambda_t = \Omega_t - \Omega_0$ is d_ω- exact, then there exist a smooth family of diffeomorphisms ϕ_t with $\phi_0 = id$ and a smooth family of functions f_t such that $\phi_t^*\Omega_t = f_t\Omega_0$.*

Remark 2. If the smooth family of lcs forms Ω_t has a smooth family ω_t of corresponding Lee forms, and we write $\omega_t = \omega_0 + d\ln u_t$ for some positive functions u_t (see the beginning of the proof of Theorem 5), then $\Omega'_t = u_t\Omega_t$ has ω_0 as Lee form for all t. Hence assuming $\Lambda'_t = \Omega'_t - \Omega'_0$ to be d_{ω_0}-exact, yields that Ω_t represent equivalent lcs structures for all t.

Proof. By assumption, $\partial/\partial t(\Omega_t)$ is d_ω exact for all t. A result of [12], (Lemma 1.9) asserts that there exists a smooth family of 1-forms η_t such that

$$\partial/\partial t(\Omega_t) = d_\omega \eta_t.$$

The argument used to find a smooth lifting of d_ω-coboundaries is the same as in [1], (Lemma II.2.2), which is an application of Grothendieck's theory of nuclear topological vector spaces. This replaces the Hodge–de Rham theorem in Moser's theorem for symplectic forms [17].

Let $\tilde{\Omega}_t = \lambda\pi^*\Omega_t$, where $\pi : \tilde{M} \to M$ is the minimum regular cover and λ is such that $\pi^*\omega = d\ln\lambda$. We define a smooth family of vector fields X_t on $\tilde{M}$ by:

$$i(X_t)\tilde{\Omega}_t = -\lambda\pi^*\eta_t$$

Since $d(\lambda\pi^*\eta_t) = \lambda\pi^* d_\omega\eta_t$, we have:

$$L_{X_t}\tilde{\Omega}_t + \partial/\partial t(\tilde{\Omega}_t) = 0.$$

We claim that X_t is complete. Hence it defines a smooth family of diffeomorphisms ψ_t of $\tilde{M}$ such that $\psi_t^*\tilde{\Omega}_t = \tilde{\Omega}_0$.

This argument is Moser's standard path method [17].

To prove that X_t is complete, it is enough to show that it is basic, i.e., there is a family of vector fields Y_t on M such that $\pi_* X_t = Y_t$. Since M is compact, Y_t is integrable, and so will be X_t.

For any $\tau \in \mathcal{A}$, we easily see that:

$$\tau^*\tilde{\Omega}_t = c_\tau\tilde{\Omega}_t,$$

and

$$\tau^*(\lambda\pi^*\eta_t) = c_\tau(\lambda\pi^*\eta_t).$$

We therefore have:

$$-c_\tau i(X_t)\tilde{\Omega}_t) = \tau^*(\lambda\pi^*\eta_t) = -\tau^*(i(X_t)\tilde{\Omega}_t) = -i((\tau)^{-1})_* X_t)(\tau^*\tilde{\Omega}_t)$$
$$= -i((\tau)^{-1})_* X_t)(c_\tau\tilde{\Omega}_t) = -c_\tau i((\tau)^{-1})_* X_t)(\tilde{\Omega}_t).$$

Hence

$$c_\tau i((\tau)^{-1})_* X_t)(\tilde{\Omega}_t) = c_\tau i(X_t)\tilde{\Omega}_t).$$

Since $c_\tau \neq 0$, we have: $i((\tau)^{-1})_* X_t)(\tilde{\Omega}_t) = i(X_t)\tilde{\Omega}_t$. Therefore $((\tau)^{-1})_* X_t) = X_t$.

Let now ϕ_t be the family of diffeomorphisms of M covered by ψ_t, i.e. $\pi \circ \psi_t = \phi_t \circ \pi$, then $\psi_t^*\tilde{\Omega}_t = (\lambda_t \circ \psi_t).\pi^*(\phi_t^*\Omega_t) = \lambda_0\pi^*\Omega_0$. Hence $\pi^*(\phi_t^*\Omega_t) = (\lambda_0/(\lambda_t \circ \phi_t))\pi^*\Omega_0$. For all $\tau \in \mathcal{A}$, we have:

$$(\lambda_0/(\lambda_t \circ \phi_t))\pi^*\Omega_0 = \pi^*(\phi_t^*\Omega_t) = \tau^*\pi^*(\phi_t^*\Omega_t) = ((\lambda_0/(\lambda_t \circ \phi_t) \circ \tau)\pi^*\Omega_0.$$

Therefore, $(\lambda_0/\lambda_t \circ \phi_t)$ is invariant by all $\tau \in \mathcal{A}$, hence $(\lambda_0/\lambda_t \circ \phi_t) = f_t \circ \pi$ for some function f_t on M. We thus get that $\pi^*(\phi_t^*\Omega_t) = \pi^*(f_t\Omega_0)$, and hence $\phi_t^*\Omega_t = f_t\Omega_0$.

This finishes the proof of Theorem 4. □

Exactly like in Moser's theorem in Symplectic Geometry [17], there are examples in which we get smooth liftings of the coboundaries Λ_t without using the deep lemma (which is an application of Grothendieck's theory of topological vector spaces). The most trivial example is provided by Theorem 2: if $\eta_\epsilon = d_\omega \gamma_\epsilon$, then $\Lambda_t = d_\omega(t\gamma_\epsilon)$

In the following situation, we also have an immediate smooth lifting of the coboundaries Λ_t.

Theorem 5. *Let Ω_t be a smooth family of lcs forms on a compact manifold M, with a smooth family ω_t of Lee forms having a fixed de Rham cohomology, i.e. $[\omega_0] = [\omega_t], \forall t$, and such that there exists a smooth family θ_t, with $\Omega_t = d\theta_t + \omega_t \wedge \theta_t$, then the lcs forms Ω_t define equivalent lcs structures.*

Proof. There is a smooth family of positive functions u_t on M with $\omega_t = \omega_0 + d\ln(u_t)$ and $u_0 = 1$. Indeed, since $(\partial/\partial t)(\omega_t)$ is exact, there is a smooth family of positive functions v_t such that $(\partial/\partial t)(\omega_t) = d\ln(v_t)$. Use for instance the Hodge–de Rham decomposition theorem. Now integrate both side and set $u_t = \int_0^t (v_s) ds$.

Let $\pi : \tilde{M} \to M$ be the minimum cover associated with ω_0, and let $\lambda_0 : \tilde{M} \to \mathbb{R}$ be a positive function such that $\pi^*\omega_0 = d\ln\lambda_0$. Then $\pi^*\omega_t = d\ln\lambda_0 + d\ln(u_t \circ \pi) = d\ln\lambda_t$ with $\lambda_t = \lambda_0.(u_t \circ \pi)$. We have:

$$\tilde{\Omega}_t = \lambda_t \pi^* \Omega_t = \lambda_t \pi^*(d\theta_t) + \lambda_t d\ln\lambda_t \wedge \pi^*\theta_t = d(\lambda_t \pi^*\theta_t).$$

Setting $\partial/\partial t(\lambda_t \pi^* \theta_t) = \rho_t$, we define a smooth family of vector fields X_t on $\tilde{M}$ by:

$$i(X_t)\tilde{\Omega}_t = -\rho_t.$$

We have:

$$L_{X_t}\tilde{\Omega}_t + \partial/\partial t(\tilde{\Omega}_t) = 0.$$

We claim that X_t is complete. Hence it defines a smooth family of diffeomorphisms ψ_t of $\tilde{M}$ such that $\psi_t^* \tilde{\Omega}_t = \tilde{\Omega}_0$.

From here proceed like in the proof of Theorem 3. □

Remark 3. Let u_t be a smooth family of positive functions such that $\omega_t = \omega_0 + d\ln u_t$. Then $\Omega'_t = u_t \Omega_t$ has ω_0 as Lee form for all t. Moreover setting $\theta'_t = u_t \theta_t$, we have:

$$d_{\omega_0}(\theta'_t) = u_t d\theta_t + \frac{du_t}{u_t} \wedge (u_t\theta_t) + \omega_0 \wedge u_t\theta_t = u_t(d\theta_t + (d\ln u_t + \omega_0) \wedge \theta_t) = u_t\Omega_t = \Omega'_t.$$

Hence $\Omega'_t = d_{\omega_0}(\theta'_t)$. The coboundary $\Lambda'_t = \Omega'_t - \Omega'_0$ has the smooth lifting $d_{\omega_0}(\theta'_t - \theta'_0)$.

Proof of Theorem 1. Theorem 1 is a consequence of Theorem 5 since two contact forms α, α' define the same contact structure if $\alpha' = w\alpha$, with w a smooth positive function. Now set $\alpha_t = \exp(t\ln(w))\alpha$. The family of lcs forms is $\Omega_t = d\theta_t + \omega \wedge \theta_t$ with $\theta_t = p_1^*\alpha_t$.

The mapping $\rho : \mathrm{Diff}_{\mathcal{C}(\alpha)}(M) \to \mathrm{Diff}_{\mathcal{S}(\alpha)}(M \times S^1)$ comes from the proof. For $h \in \mathrm{Diff}_{\mathcal{C}(\alpha)}(M)$, $h^*\alpha = w.\alpha$, then the diffeomorphism ϕ_1 above obtained using $\Omega_t = d\theta_t + \omega \wedge \theta_t$, with $\theta_t = p_1^*\alpha_t$ and $\alpha_t = \exp(t.\ln(w))\alpha$, takes Ω_1 to $a\Omega_0$. Taking a path from $h\alpha$ to α, which does not reverse the first one, for instance $\alpha'_t = (t + (1-t)h)\alpha$, $\theta'_t = p_1^*\alpha'_t$ and $\Omega'_t = d\theta'_t + \omega \wedge \theta'_t$, get a diffeomorphism ϕ_1 taking Ω_0 back to a multiple of Ω_1. Now set $\rho(h) = \phi_1 \circ \psi_1$. □

5. Invariants of lcs structures

Given a lcs manifold $(M, \mathcal{S})$, we have considered the following objects attached to $\mathcal{S}$:

1. The cohomology class of the Lee form ω of any representative lcs form $\Omega \in \mathcal{S}$. We saw that this is an invariant $\mathcal{L}_{\mathcal{S}}$, we called the Lee class of $\mathcal{S}$. The group $\mathcal{A}$ of periods of ω is an object depending only on the conformal class $\mathcal{S}$.

2. We considered the minimum cover of M which has a group of deck transformations isomorphic with the group $\mathcal{A}$ of periods of ω as group of automorphisms, and the $c\mathcal{A}$ cohomology.

In Proposition 1, we gather other invariants built using the automorphisms of the lcs structure.

If $\mathcal{G}$ is a Lie algebra and K is a $\mathcal{G}$-module, we denote by $H^*(\mathcal{G}, K)$, the cohomology of $\mathcal{G}$ with coefficients in K [14]. This is the cohomology of the complex $(C^*(\mathcal{G}, K), \delta)$ where p-cochains are p-linear alternating mappings on $\mathcal{G}$ with values in K and the coboundary operator is given by:

$$\begin{aligned}\partial f(X_1, \ldots, X_{p+1}) = &\sum_i (-1)^{i+1} X_i \cdot f(X_1, \ldots, \hat{X}_i, \ldots X_{p+1}) \\ &+ \sum_{i \leq j} (-1)^{i+j} f([X_i, X_j], \ldots, \hat{X}_i, \ldots \hat{X}_j, \ldots).\end{aligned}$$

We also consider the cohomology $H^*(G, K)$ of an (abstract) group G into a G-module K [13]. The p-cochains now are mappings from G^p to K and the coboundary operator δ is given by

$$\begin{aligned}\delta g(a_0, \ldots, a_p) = a_0 \cdot c(a_1, \ldots, a_p) - \Big(\sum_i (-1)^i c(a_0, \ldots, a_i a_{i+1}, \ldots a_p) \Big) \\ + (-1)^{p+1} c(a_0, \ldots, a_{p-1}).\end{aligned}$$

$H^1(G, K)$ is the quotient of derivations (1-cocycles) by inner derivations (coboundaries). Recall that derivations are maps $d : G \to K$ such that $d(gh) = g.d(h) + dg$ and an inner derivation is a map $v : G \to K$ such that there exists $k \in K$ such that $v(g) = g.k - k$.

$H^1(\mathcal{G}, K)$ is the quotient of the space of linear maps $v : \mathcal{G} \to K$ such that $u([X,Y]) = X.u(Y) - Y.u(X)$ (1-cocycles), modulo (the coboundaries) consisting of linear maps v such that there exists $k \in K$ with $v(X) = X.k$, for all $X, Y \in \mathcal{G}$.

Proposition 1. *Let $\mathcal{S}$ be a lcs structure on M, and $\Omega \in \mathcal{S}$ with Lee form ω.*

1. *The map $D_\Omega : \mathrm{Diff}_{\mathcal{S}}(M) \to C^\infty(M)$, $\phi \mapsto \ln(f_{\phi^{-1}})$, if $\phi^*\Omega = f_\phi \Omega$ is a 1-cocycle on $\mathrm{Diff}_{\mathcal{S}}(M)$ whose cohomology class $a_{\mathcal{S}} \in H^1(\mathrm{Diff}_{\mathcal{S}}(M), C^\infty(M))$ is independent of the choice of $\Omega \in \mathcal{S}$, i.e. an invariant of $\mathcal{S}$.*
2. *The map $d_\Omega : \mathcal{X}_{\mathcal{S}}(M) \to C^\infty(M)$, $X \mapsto u_\Omega(X)$, where $L_X\Omega = (u_\Omega(X))\Omega$, is a 1-cocycle, whose cohomology class $b_{\mathcal{S}} \in H^1(\mathcal{X}_{\mathcal{S}}(M), C^\infty(M))$ is independent of the choice of $\Omega \in \mathcal{S}$, i.e., an invariant of $\mathcal{S}$.*
3. *The map $\hat{\omega} : \mathcal{X}_{\mathcal{S}}(M) \to C^\infty(M)$, $X \mapsto \omega(X)$ is a 1-cocycle, whose cohomology class $c_{\mathcal{S}} \in H^1(\mathcal{X}_{\mathcal{S}}(M), C^\infty(M))$ is independent of the choice of $\Omega \in \mathcal{S}$, i.e. an invariant of $\mathcal{S}$.*
4. *The sum $d_\Omega + \hat{\omega}$ is a 1-cocycle on $\mathcal{X}_{\mathcal{S}}(M)$ with values in $\mathbb{R}$, hence a homomorphism l, called the extended Lee homomorphism, an invariant of $\mathcal{S}$.*
5. *Suppose M is compact and fix a riemannian metric. For each $h \in \mathrm{Diff}_{\mathcal{S}}(M)$ (not even homotopic to the identity) $h^*\omega - \omega$ is an exact 1-form. Let u_h be the unique function provided by the Hodge decomposition of $h^*\omega - \omega$ such that $h^*\omega - \omega = du_h$.*

For $h, h' \in \mathrm{Diff}_{\mathcal{S}}(M)$:

$$(h, h') \mapsto u_h \circ h' + u_{h'} - u_{hh'}$$

is a 2-cocycle K_ω with values in $\mathbb{R}$. Its cohomology class in $H^2(\mathrm{Diff}_{\mathcal{S}}(M), \mathbb{R})$ is an invariant $\mathcal{K}_{\mathcal{S}}$ of $\mathcal{S}$.

Statements 1, and 2 have been observed in [2]. The statement 3 is obvious, since the coboundary operator in the Gelfand–Fucks cohomology (cohomology on Lie algebras of vector fields) is the same as in the de Rham cohomology.

The class $c_{\mathcal{S}}$ may be called the Gelfand–Fucks class of $\mathcal{S}$.

Statement 4 was proved by Vaisman [18]. See also [6].

Statement 5 was proved in [8]. The Hodge–de Rham theory gives a smooth lifting of de Rham coboundaries: i.e. any exact p-form θ determines uniquely a $(p-1)$-form α such that $\theta = d\alpha$ as follows: let δ be the codifferential, and G the Green operator defined by a riemannian metric, then $\alpha = \delta G(\theta)$. Here the function u_h is $u_h = \delta(G(h^*\omega - \omega))$. See for instance [3].

Remark 4. We can define similar invariants using objects with compact support, and denote them by $a^c_{\mathcal{S}}$, $b^c_{\mathcal{S}}$, $c^c_{\mathcal{S}}$.

Definition. *The structure $\mathcal{S}$ is called* **inessential** *if there exists $\Omega_* \in \mathcal{S}$ such that $G_{\Omega_*}(M) = \mathrm{Diff}_{\mathcal{S}}(M)$. The structure $\mathcal{S}$ is called* **essential** *otherwise.*

The following fact was observed in [4]:

Proposition 2. *Let $(M, \mathcal{S})$ be a lcs manifold. Then $\mathcal{S}$ is inessential iff $a_{\mathcal{S}} = 0$.*

The connection between these invariants, and the problem of essentiality, and globality of locally conformal structure is given by the following:

Theorem 6. *Let $(M, \mathcal{S})$ be a lcs manifold.*

1. *If $a_{\mathcal{S}} = 0$, then $\mathcal{S} = \mathcal{O}$. Furthermore, the Lee homomorphism is trivial, and the structure $\mathcal{S}$ is of the second kind. Thus inessential structures are of the second kind. This also says that if $\mathcal{S}$ is of the first kind, then $a_{\mathcal{S}} \neq 0$.*
2. *If M is compact, then $\mathcal{S} = \mathcal{O}$ implies that $a_{\mathcal{S}} = 0$.*
3. *The Gelfand–Fucks class $c_{\mathcal{S}}$ vanishes iff the Lee class $\mathcal{L}_{\mathcal{S}}$ does.*
4. *If M is compact, the vanishing of one of the four classes $a_{\mathcal{S}}$, $b_{\mathcal{S}}$, $c_{\mathcal{S}}$, $\mathcal{L}_{\mathcal{S}}$, implies the vanishing of the remaining three classes.*

We will need the following "local transitivity" result. Lefebvre's [16] proved it away from the zeros of the Lee form. Since for any point, the lcs structure can be represented by a lcs form with Lee form not vanishing at that point, Lefebvre's argument applies. For the convenience of the reader, we rewrote it in our style.

Theorem 7. *Let $(M, \mathcal{S})$ be a lcs manifold of dimension $2n$. For each $x \in M$, there exist $2n$ vector fields $V_j^x \in \mathcal{X}_{\mathcal{S}}(M)$ with arbitrarily small compact support in an open neighborhood of x and such that $\{V_j^x(x)\}_{j=1,\dots,2n}$ form a basis of the tangent space T_xM.*

Proof. 1. For each point $x \in M$, there is $\Omega \in \mathcal{S}$, with Lee form ω such that $\omega(x) \neq 0$. Indeed, if the Lee form ω of $\Omega \in \mathcal{S}$ vanishes at x, consider a contractible neighborhood U of x at which $\omega|_U = d\ln(\lambda)$, and choose a smooth positive function ρ, constant outside of U with $d\rho(x) \neq 0$ and $d\ln\lambda \neq d\ln\rho$ on a neighborhood of x. The form $\rho\Omega \in \mathcal{S}$ and has Lee form $\omega' = \omega - d\ln(\rho)$. The new Lee form does not vanish at x (and in a neighborhood).

2. Any function u on an open set U where $f\Omega_{|U}$ is symplectic defines a vector field X_u on U by the equation:

$$i(X_u)f\Omega_|U) = d(fu).$$

A direct calculation shows that $L_{X_u}\Omega_|U) = (-X_u \cdot \ln f)\Omega$ [18].

3. The form $\Omega \in \mathcal{S}$ above has a Lee form ω not vanishing on an open neighborhood $V \subset U$ of x. Hence, there are local coordinates $(x_1, \dots x_n, y_1, \dots, y_n)$ defined on a smaller neighborhood V_1 of x such that $y_1 \neq 0$, and

$$\Omega|_{U_1} = y_1\Big(\sum_{k=1}^{n} dx_k \wedge dy_k\Big).$$

Let μ be a smooth function, supported in V_2 and which is equal to 1 on a closed neighborhood F of x, where $F \subset V_2 \subset V_1$.

We define 2n vector fields by:

$$i(Y_1)\Big(\frac{1}{y_1}\Omega_{|V_1}\Big) = d\Big(\mu\frac{y_1^2}{y_1}\Big) = d(\mu y_1)$$

and for $j = 2, \ldots, n$,

$$i(Y_j)\Big(\frac{1}{y_1}\Omega_{|V_1}\Big) = d\Big(\mu\frac{y_j}{y_1}\Big).$$

For $j = 1, \ldots, n$ define X_j by:

$$i(X_j)\Big(\frac{1}{y_1}\Omega_{|V_1}\Big) = d\Big(\mu\frac{x_j}{y_1}\Big).$$

Then X_i, Y_i are smooth vector fields on M with compact support in V_1, which all belong to $\mathcal{X}_{\mathcal{S}}(M)_c$.

Let us note $e_j = \partial/\partial x_j$ and $e'_j = \partial/\partial y_j$, then on F, we have

$$Y_1 = e_1, \quad Y_j = \frac{1}{y_1}e_j - \frac{y_j}{y_1^2}e_1, \quad j = 2, \ldots, n$$

$$X_j = -\frac{1}{y_1}e'_j - \frac{x_j}{y_1^2}e_1, \quad j = 1, \ldots, n.$$

Writing that $\sum_{i=1}^n (a_i X_i + b_i Y_i) = 0$, gives immediately that $b_i = 0$ and $a_i = 0$, i.e. these vector fields are linearly independent near x. □

Proof of Theorem 6. 1. Suppose that $a_{\mathcal{S}} = 0$, that is $\mathcal{S}$ is inessential (Proposition 2). Let $\Omega_* \in \mathcal{S}$ with $\mathrm{Diff}_{\mathcal{S}}(M) = G_{\Omega_*}(M)$, and let ω_* be the corresponding Lee form. It follows that

$$\mathcal{X}_{\mathcal{S}}(M)_c = \mathcal{X}_{\Omega_*}(M)_c.$$

Let us now show that $\omega_* = 0$.

For each $x \in M$, and any tangent vector $\xi \in T_x M$, we want to show that $\omega_*(x)(\xi) = 0$. By Theorem 7, $\xi = \sum_{j=1}^{2n} c_j(x) V_j^x(x)$. Extend now the coefficients $c_j(x)$ into smooth functions c_j with compact support near x. We get a smooth vector field with compact support $V = \sum_{j=1}^{2n} c_j V_j^x$, which coincides with ξ at $x \in M$. Therefore,

$$\omega_*(x)(\xi) = \omega_*(x)(V(x)) = (\omega_*(V))(x) = \sum_{j=1}^{2n}(c_j \omega_*(V_j^x))(x).$$

Since $V_j^x \in \mathcal{X}_{\mathcal{S}}(M)_c = \mathcal{X}_{\Omega_*}(M)_c$, $\omega_*(V_j^x))$ is a constant function (see Remark 5.3) with compact support, and hence identically zero. This proves that $\omega_*(x) = 0$.

This implies that $\mathcal{S} = \mathcal{O}$.

Since the Lee homomorphism can be computed using Ω_* and ω_*, we see that

$$l = \hat{\omega}_* = 0.$$

This implies that the structure is of the second kind. Indeed, if Ω is any representative of $\mathcal{S}$ with Lee form ω and $X \in \mathcal{X}_\Omega(M)$, then $l(X) = \omega(X) = 0$.

2. If $\mathcal{S} = \mathcal{O}$, there is a symplectic form $\Omega \in \mathcal{S}$. If $\phi \in \mathrm{Diff}_\mathcal{S}(M)$, then $\phi^*\Omega = f\Omega$. By the classical theorem of Libermann (see [6]), f is a constant, provided that the dimension of M is at least 4, (which is assumed here) and if M is compact, this constant must be 1. This follows from the fact that $\int_M \phi^*\Omega^n = f^n \int_M \Omega^n$ and by the formula of change of variable, we have equality with $\int_M \Omega^n$. Hence $f = 1$ and therefore $a_\mathcal{S} = 0$.

3. It is clear that $[\omega] = 0$ implies that $[\hat{\omega}] = 0$. Conversely, suppose there exists a smooth function u such that $\omega(X) = X.u = du(X)$ for all $X \in \mathcal{X}_\mathcal{S}(M)$. We show that indeed $\omega(\xi) = du(\xi)$ for all vector fields ξ, i.e that $\omega = du$. For each point $x \in M$, we need to show that $\omega(\xi)(x) = (du(\xi)(x))$.

As above, we consider the vector field $V = \sum_{j=1}^{2n} c_j V_j^x$, which is equal to ξ at x. Then, like above: $\omega(\xi)(x) = \sum_{j=1}^{2n}(c_j\omega(V_j^x))(x) = \sum_{j=1}^{2n}(c_j du(x)(V_j^x)) = du(x)(\sum_{j=1}^{2n} c_j V_j^x) = du(x)(V) = du(x)(\xi)$. Therefore the de Rham class of ω is trivial.

4. In the compact case $(a_\mathcal{S} = 0) \Leftrightarrow (\mathcal{S} = \mathcal{O})$ and $(a_\mathcal{S} = 0) \Leftrightarrow (b_\mathcal{S} = 0)$.

We also have that in general, $(\mathcal{S} = \mathcal{O} \Leftrightarrow (\mathcal{L}_\mathcal{S} = 0)$ and $(c_\mathcal{S} = 0) \Leftrightarrow (\mathcal{L}_\mathcal{S} = 0)$

Putting these facts together, yields the last assertion of Theorem 5. □

Remarks. 1. If M is not compact, $\mathcal{S} = 0$ does not imply that $a_\mathcal{S} = 0$. Take for instance the global conformal symplectic structure defined by the standard symplectic form on $\mathbb{R}^{2n}$, and more generally non-compact manifolds with complete Liouville vector fields, like Stein manifolds [4].

2. The vanishing of the compactly supported invariant $a^c_\mathcal{S}$ also implies that $\mathcal{S} = 0$. This was proved in [12].

6. Concluding remarks and questions

1. The mapping $L : \mathcal{L}_{cs}(M) \to \mathcal{F}^1(M)$ assigning to a lcs form its Lee form is not continuous in the C^0 topology. Indeed if u is a smooth function which is C^0 close to 1 and C^1 far from 0, then the Lee forms of $u\Omega$ and Ω, are far apart. How about the continuity for the C^∞ topology?

If M has a complex structure J and a hermitian metric g such that the lcs form Ω is given by $\Omega(X, Y) = g(X, JY)$ (M is said to be a locally conformal Kaehler manifold), then L is continuous for the C^∞ topology. Indeed in that case we have an explicit formula for $L(\Omega)$ [9]:

$$L(\Omega) = \frac{1}{n-1}(\delta\Omega \circ J).$$

Here δ is the codifferential with respect to the metric g, and $2n$ is the dimension of M.

2. The Lee homomorphism $l : \mathcal{X}_{\mathcal{S}}(M) \to \mathbb{R}$ can be integrated into a homomorphism $\mathcal{L} : \mathrm{Diff}_{\mathcal{S}}(M)_+ \to \mathbb{R}/\Delta$ (where Δ is some countable subgroup of $\mathbb{R}$), and $\mathrm{Diff}_{\mathcal{S}}(M)_+$ is the group of automorphisms of $\mathcal{S}$ which admit a lift to the minimal regular cover $\tilde{M}$ [6].

If α is a contact form on a compact manifold M, we constructed in Theorem 1 a map $\rho : \mathrm{Diff}_{\mathcal{C}(\alpha)}(M) \to \mathrm{Diff}_{\mathcal{S}(\alpha)}(M \times S^1)_+$. Composing ρ with the extended global Lee homomorphism, we get a map:

$$\mu = \mathcal{L} \circ \rho : \mathrm{Diff}_{\mathcal{C}(\alpha)}(M) \to \mathbb{R}/\Delta.$$

This map is not a group homomorphism. This allows us to define a 2-cocycle η on the the group $\mathrm{Diff}_{\mathcal{C}(\alpha)}(M)$:

$$\eta(\phi, \psi) = \rho(\phi).\rho(\psi).(\rho(\phi\psi))^{-1}$$

for all $\phi, \psi \in \mathrm{Diff}_{\mathcal{C}(\alpha)}(M)$.

What is the meaning of that cocycle?

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