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A gap theorem for hypersurfaces of the sphere with constant scalar curvature one

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Abstract. We consider closed hypersurfaces of the sphere with scalar curvature one, prove a gap theorem for a modified second fundamental form and determine the hypersurfaces that are at the end points of the gap. As an application we characterize the closed, two-sided index one hypersurfaces with scalar curvature one in the real projective space.

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1. Introduction

To state our main result we need some notation.

$x : M^n \to S^{n+1}(1)$ will be a closed (compact without boundary) hypersurface of the unit sphere $S^{n+1}(1)$. We denote by A the linear map associated to the second fundamental form and by $k_1, \ldots, k_n$ its eigenvalues (principal curvatures of M). We will use the first two elementary symmetric function of the principal curvatures:

$$S_1 = \sum_{i=1}^{n} k_i, \; S_2 = \sum_{i<j=1}^{n} k_i k_j.$$

We will also use the normalized means: *the mean curvature* $H = \frac{1}{n} S_1$ and the *scalar curvature* R, given by $n(n-1)(R-1) = S_2$. Finally, we introduce the first two *Newton tensors* by

$$P_0 = Id, \; P_1 = S_1 Id - A.$$

Clearly P_1 commutes with A and it is also a self-adjoint operator. We will show later (see Remark 2.1) that if $R = 1$ and $S_1 \geq 0$, then all eigenvalues of P_1 are nonnegative, hence we can consider $\sqrt{P_1}$.

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We can now state our gap theorem.

Theorem 1. *Let $x : M^n \to S^{n+1}(1)$ be a closed orientable hypersurface with scalar curvature $R = 1$ (equivalently, $S_2 = 0$). Assume that S_1 does not change sign and choose the orientation such that $S_1 \geq 0$. Assume further that*

$$\|\sqrt{P_1}A\|^2 \leq \mathrm{trace}P_1.$$

Then:

(i) $\|\sqrt{P_1}A\|^2 = \mathrm{trace}P_1$.

(ii) *M^n is either a totally geodesic submanifold or $M^n = S^{n_1}(r_1) \times S^{n_2}(r_2) \subset S^{n+1}(1)$, where $n_1 + n_2 = n$, $r_1^2 + r_2^2 = 1$ and $\left(\frac{r_2}{r_1}\right)^2 = \beta$ satisfies the quadratic equation:*

$$n_1(n_1 - 1)\beta^2 - 2n_1n_2\beta + n_2(n_2 - 1) = 0.$$

Our theorem was inspired by a similar theorem on minimal submanifolds of the sphere first proved by J. Simons [S] (part (i)) and latter completed (part (ii)) by S. S. Chern, M. do Carmo and Kobayashi [CdCK] and, independently, by H. B. Lawson [L].

Remark. The condition on the modified second fundamental form in above theorem can not be dropped, as can be seen by the following example: Let $M^6 \to S^7(1)$ be an isoparametric hypersurface with principal curvatures given by

$$\lambda_1 = \lambda_2 = \theta,\ \lambda_3 = \frac{\theta + 1}{1 - \theta},\ \lambda_4 = \lambda_5 = -\frac{1}{\theta} \text{ and } \lambda_6 = -\frac{1 - \theta}{1 + \theta},$$

where θ is given by $\theta = \sqrt{\frac{13 + \sqrt{165}}{2}}$ (see [M]). It is easy to see that M^6 has $R = 1$ and $S_1 > 0$. We would like to thank Luiz Amancio de Sousa Junior for showing us this example.

As an application of Theorem 1, we will present a characterization of index one closed hypersurfaces with constant scalar curvature one of the real projective space $\mathbb{P}(\mathbb{R})^{n+1}$. For minimal submanifolds this result was obtained recently by M. do Carmo, M. Ritoré and A. Ros [dCRR].

Before giving a formal statement we need some considerations. Hypersurfaces of a curvature one space form with constant scalar curvature one are solutions to a variational problem (see [Re], [Ro], [BC]) whose *Jacobi equation* is

$$T_1 f = L_1 f + \{\|\sqrt{P_1}A\|^2 + \mathrm{trace}P_1\} f = 0.$$

Here $f \in C^\infty(M)$ and L_1 is a second order differential operator given by

$$L_1 f = \mathrm{div}(P_1 \nabla f),$$

where ∇f is the gradient of f. Notice that L_1 generalizes the Laplacian. However, differently from the Laplacian, L_1 is not always elliptic. J. Hounie and M. L. Leite [HL] have proved that if $S_3 \neq 0$ everywhere, then L_1 is elliptic. Of course, from the definition of L_1, it follows that L_1 is elliptic if and only if P_1 is positive definite (or negative definite). For the next theorem we will assume that L_1 is elliptic and P_1 is positive definite. Denote by $Ind(M)$ the *Morse index* of M, i.e., the number of negative eigenvalues of T_1.

Theorem 2. *Let $x : M^n \to \mathbb{P}(\mathbb{R})^{n+1}(1)$ be a closed two-sided hypersurface with scalar curvature one. Then $Ind(M) \geq 1$ and if $Ind(M) = 1$, M is the Clifford hypersurfaces obtained by the projection of the Clifford torus of Theorem 1.*

2. Preliminaries

In this section we will present some properties of the r^{th} Newton tensors in M and describe the Clifford hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$.

2.1. The r^{th} Newton tensors

We introduce the r^{th} Newton tensors, $P_r : T_pM \to T_pM$, which are defined inductively by

$$P_0 = I,$$
$$P_r = S_r I - AP_{r-1},\ r > 1,$$

where $S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \ldots k_{i_r}$ is the r^{th} symmetric function of the principal curvatures $k_1, \ldots, k_n$.

It is easy to see that each P_r commutes with A and if e_i an eigenvector of A associated to principal curvature k_i, then

$$P_1(e_i) = \mu_i e_i = (S_1 - k_i)e_i.$$

In [Re], Reilly showed that the P_r's satisfy the following

Proposition 2.1 ([Re], see also [BC] – Lemma 2.1)**.** *Let $x : M^n \to N^{n+1}$ be an isometric immersion between two Riemannian manifolds and let A be its second fundamental form. The r'th Newton tensor P_r associated to A satisfies:*

1. trace$(P_r) = (n - r)S_r$,
2. trace$(AP_r) = (r + 1)S_{r+1}$,
3. trace$(A^2 P_r) = S_1 S_{r+1} - (r + 2)S_{r+2}$.

It follows from (3) that if $S_2 = 0$, trace$(A^2 P_1) = -3S_3$.

Remark 2.1. Observe that if $S_2 = 0$, we have that

$$S_1^2 = |A|^2 + 2S_2 \geq k_i^2, \text{ for all } i.$$

Thus, $0 \leq (S_1^2 - k_i^2) = (S_1 - k_i)(S_1 + k_i)$, what implies that all eigenvalues of P_1 are nonnegative if $S_1 \geq 0$, that is, P_1 is a nonnegative operator. We also remark that if $S_2 = 0$ and P_1 has one eigenvalue equal to zero, then

$$P_1 A \equiv 0. \tag{1}$$

In fact, if $\mu_{i_0} = 0$, then $k_{i_0} = S_1$. As $S_1^2 = |A|^2$, we get

$$\sum_{i \neq i_0} k_{i_0}^2 = 0.$$

So $k_i = 0$, for all $i \neq i_0$, hence $P_1 A \equiv 0$.

Associated to each Newton tensor P_r, we define a second order differential operator

$$L_r(f) = \text{trace}(P_r \text{Hess } f).$$

If N^{n+1} has constant sectional curvature, it follows from Codazzi equation (see Rosenberg [Ro], p. 225) that L_r is

$$L_r(f) = \text{div}_M(P_r \nabla f).$$

Hence L_r is a self-adjoint operator and for any differentiable functions f and g on M^n,

$$\int_M f L_r g dM = \int_M g L_r f dM \tag{2}$$

We observe that for $r = 0$, L_0 is the Laplacian which is always an elliptic operator. For $r > 0$ we have to add some extra condition in order to ensure that L_r is elliptic. For hypersurfaces of $\mathbb{R}^{n+1}$ with $S_r = 0$, Hounie and Leite, [HL], were able to give a geometric condition that is equivalent to L_r being elliptic. In fact their proof can be generalized to hypersurfaces of the sphere and we have that

Theorem 2.1 ([HL] – Proposition 1.5)**.** *Let M be a hypersurface in $\mathbb{R}^{n+1}$ or S^{n+1} with $S_r = 0$, $2 \leq r < n$. Then the operator $L_{r-1}(f) = \text{div}(P_{r-1}\nabla f)$ is elliptic at $p \in M$ if and only if $S_{r+1}(p) \neq 0$.*

Thus, for hypersurfaces with $S_2 = 0$, L_1 is an elliptic operator if and only if $S_3 \neq 0$. Since $L_1(f) = \text{div}_M(P_1 \nabla f)$, it follows that the ellipticity of L_1 implies that P_1 is definite, hence then $S_1 \neq 0$.

Let $a \in \mathbb{R}^{n+2}$ be a fixed vector. Let $x : M \to S^{n+1}(1) \subset \mathbb{R}^{n+2}$ be an isometric immersion with $S_2 = 0$ and let N be its unit normal vector. The functions $f = \langle N, a \rangle$ and $g = \langle x, a \rangle$ satisfy (see [BC], lemma 5.2)

$$L_1(g) = -(n-1)S_1 g \tag{3}$$

and

$$L_1(f) = 3S_3 f. \tag{4}$$

2.2. Clifford hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$

We are now going to describe some properties of the *Clifford hypersurface* in $\mathbb{P}(\mathbb{R})^{n+1}$. A Clifford torus in $S^{n+1}(1)$ is given by the product immersion of $M = S^{n_1}(r_1) \times S^{n_2}(r_2)$, with $n_1 + n_2 = n$ and $r_1^2 + r_2^2 = 1$, which is a closed hypersurface of $S^{n+1}(1)$. It is easy to see that this immersion is invariant under the antipodal map, hence it induces an immersion of M into $\mathbb{P}(\mathbb{R})^{n+1}$. This hypersurface will be called *Clifford hypersurface.* If $x : S^{n_1}(r_1) \times S^{n_2}(r_2) \to S^{n+1}(1)$ is a Clifford torus, then the unit normal vector at a point $p = (p_1, p_2) \in S^{n_1}(r_1) \times S^{n_2}(r_2)$ is given by

$$N = \left(-\frac{r_2}{r_1} p_1, \frac{r_1}{r_2} p_2 \right).$$

Thus, the principal curvatures of M are $\frac{r_2}{r_1}$ with multiplicity n_1 and $-\frac{r_1}{r_2}$ with multiplicity n_2. It is easily checked that the scalar curvature of M is equal to one ($S_2 = 0$) if and only if $\left(\frac{r_2}{r_1}\right)^2 = \beta$ satisfies the quadratic equation:

$$n_1(n_1 - 1)\beta^2 - 2n_1 n_2 \beta + n_2(n_2 - 1) = 0. \tag{5}$$

We will show in a while that only one of the torus given by (5) yields $S_1 > 0$. Notice that L_1 is an elliptic operator and in order to calculate the index of M, we first observe that in a principal basis, P_1 is a diagonal matrix whose elements are

$$\left\{ (n_1 - 1)\frac{r_2}{r_1} - n_2 \frac{r_1}{r_2} \right\} \text{ with multiplicity } n_1$$

and

$$\left\{ n_1 \frac{r_2}{r_1} - (n_2 - 1)\frac{r_1}{r_2} \right\} \text{ with multiplicity } n_2.$$

Thus,

$$\mathrm{trace} P_1 = (n-1)S_1 = (n-1)\left(n_1 \frac{r_2}{r_1} - n_2 \frac{r_1}{r_2} \right).$$

We will need the following relation:

$$\|\sqrt{P_1} A\|^2 = -3S_3 = (n-1)S_1.$$

The first equality is a general fact that follows from Proposition 2.1, part 3, by setting $r = 1$ and $S_2 = 0$. The second equality is specific for Clifford tori with

$S_2 = 0$ and can be proved as follows. Write:

$$\begin{aligned} S_1 &= n_1\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}, \\ S_2 &= \frac{n_1(n_1-1)}{2}\left(\frac{r_2}{r_1}\right)^2 + \frac{n_2(n_2-1)}{2}\left(\frac{r_1}{r_2}\right)^2 - n_1n_2, \\ S_3 &= \frac{n_1(n_1-1)(n_1-2)}{6}\left(\frac{r_2}{r_1}\right)^3 - \frac{n_2(n_2-1)(n_2-2)}{6}\left(\frac{r_1}{r_2}\right)^3 \\ &\quad + \frac{n_1n_2(n_2-1)}{2}\left(\frac{r_1}{r_2}\right)^2\frac{r_2}{r_1} - \frac{n_1n_2(n_1-1)}{2}\left(\frac{r_2}{r_1}\right)^2\frac{r_1}{r_2}. \end{aligned}$$

By introducing the condition $S_2 = 0$ into S_3, we obtain, after a long but straightforward computation, that

$$3S_3 = \frac{1}{2}\left[-2(n-1)n_1\frac{r_2}{r_1} + 2(n-1)n_2\frac{r_1}{r_2}\right] = -(n-1)S_1,$$

and this proves our claim. Thus the *Jacobi operator* reduces to

$$T_1(f) = L_1(f) + \{\|\sqrt{P_1}A\|^2 + \mathrm{trace}P_1\}f = L_1(f) + 2(n-1)S_1f.$$

If $\varphi = \text{const.}$, $L_1(\varphi) = 0$ and

$$T_1(\varphi) + 2(n-1)S_1\varphi = 0.$$

Thus the first eigenvalue of T_1 is negative, hence $Ind(M)$ is at least 1. Now let us look at the second eigenvalue of T_1. By using the expression of the eigenvalues of P_1 given above, we have that

$$\begin{aligned} L_1(f) &= \mathrm{div}(P_1\nabla f) \\ &= \left\{(n_1-1)\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}\right\}\Delta^{n_1}(f) + \left\{n_1\frac{r_2}{r_1} - (n_2-1)\frac{r_1}{r_2}\right\}\Delta^{n_2}(f), \end{aligned}$$

where Δ^{n_i} is the Laplacian in $S^{n_i}(r_i),\ i = 1, 2$. Thus the second eigenvalue of L_1 is given by

$$\lambda_2 = -\left\{(n_1-1)\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}\right\}\nu_2^{\Delta^{n_1}} + \left\{n_1\frac{r_2}{r_1} - (n_2-1)\frac{r_1}{r_2}\right\}\nu_2^{\Delta^{n_2}},$$

where $\nu_2^{\Delta^{n_i}}$ is the first nonzero eigenvalue of Δ^{n_i} that corresponds to an eigenfunction which is invariant by the antipodal map (see [BGM] chap III, CII). Thus

$$\begin{aligned} \lambda_2 &= -\left[\left\{(n_1-1)\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}\right\}\frac{n_1}{r_1^2} + \left\{n_1\frac{r_2}{r_1} - (n_2-1)\frac{r_1}{r_2}\right\}\frac{n_2}{r_2^2}\right] \\ &= \frac{-1}{r_1^3r_2^3}\left\{[n_1(n_1-1) - n_1(n-1)r_1^2]r_2^2 + [n_2(n-1)r_2^2 - n_2(n_2-1)]r_1^2\right\}. \end{aligned} \tag{6}$$

Observe that

$$S_1 = n_1 \frac{r_2}{r_1} - n_2 \frac{r_1}{r_2} = \frac{n_1 r_2^2 - n_2 r_1^2}{r_1 r_2}. \tag{7}$$

The fact that $S_2 = 0$ is equivalent to

$$n(n-1)r_1^4 - 2n_1(n-1)r_1^2 + n_1(n_1-1) = n(n-1)r_2^4 - 2n_2(n-1)r_1 2^2 + n_2(n_2-1) = 0. \tag{8}$$

By using (7) and (8), we have that

$$[n_1(n_1-1) - n_1(n-1)r_1^2]r_2^2 = (n-1)S_1 r_1^3 r_2^3$$

and

$$[n_2(n-1)r_2^2 - n_2(n_2-1)]r_1^2 = (n-1)S_1 r_1^3 r_2^3.$$

Thus,

$$\lambda_2 = -2(n-1)S_1.$$

Since the second eigenvalue of T_1 is given by $\lambda_2 + 2(n-1)S_1$, it is equal to zero. This shows then that the Clifford hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$ have index one.

Remark. Observe that, by equation (7), the condition $S_1 \geq 0$ means that

$$n_1 r_2^2 - n_2 r_1^2 \geq 0.$$

On the other hand, since $\beta = \left(\frac{r_2}{r_1}\right)^2$, the above inequality implies that

$$n_1 \beta \geq n_2. \tag{9}$$

The condition $S_2 = 0$ is equivalent to

$$n_1(n_1-1)\beta^2 - 2n_1 n_2 \beta + n_2(n_2-1) = 0, \tag{10}$$

and one can easily see that only one solution of (10) is compatible with (9).

3. A gap theorem for hypersurfaces of the sphere with $R = 1$

In this section we prove a gap theorem for hypersurfaces of the sphere with $R = 1$.

Theorem 3.1 (Theorem 1 of the Introduction)**.** *Let $x : M^n \to S^{n+1}(1)$ be a closed orientable hypersurface with scalar curvature $R = 1$ (equivalently, $S_2 = 0$). Assume that S_1 does not change sign and choose the orientation such that $S_1 \geq 0$. Assume further that*

$$\|\sqrt{P_1}A\|^2 \leq \text{trace} P_1.$$

Then:

(i) $\|\sqrt{P_1}A\|^2 = \text{trace} P_1$.

(ii) *M^n is either a totally geodesic submanifold or $M^n = S^{n_1}(r_1) \times S^{n_2}(r_2) \subset S^{n+1}(1)$, where $n_1 + n_2 = n$, $r_1^2 + r_2^2 = 1$ and $\left(\frac{r_2}{r_1}\right)^2 = \beta$ satisfies the quadratic equation:*

$$n_1(n_1-1)\beta^2 - 2n_1n_2\beta + n_2(n_2-1) = 0.$$

Proof. Let us calculate $\frac{1}{2}L_1\|A\|^2$. Since $R=1$, $S_2 = n(n-1)(R-1) = 0$, by the Gauss' formula. Thus $\|A\|^2 = (nH)^2 = S_1^2$. Hence,

$$\frac{1}{2}L_1\|A\|^2 = \frac{1}{2}L_1S_1^2 = S_1L_1S_1 + \langle P_1\nabla S_1, \nabla S_1\rangle.$$

From [AdCC](Lemma 3.7), by using that $2S_2 = n(n-1)(R-1) = 0$, we have

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + n\|A\|^2 - S_1^2 + 3S_1S_3.$$

Therefore,

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 + 3S_1S_3. \tag{11}$$

Now, by using Proposition 2.1 (3), we obtain that

$$\|\sqrt{P_1}A\|^2 = \mathrm{trace}P_1A^2 = -3S_3.$$

Then, equation (11) becomes

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 - S_1\|\sqrt{P_1}A\|^2.$$

Thus,

$$\begin{aligned}\frac{1}{2}L_1\|A\|^2 &= S_1L_1S_1 + \langle P_1\nabla S_1, \nabla S_1\rangle \\ &= S_1(|\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 - 3S_1\|\sqrt{P_1}A\|^2) + \langle P_1\nabla S_1, \nabla S_1\rangle \\ &= S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1}A\|^2) + \langle P_1\nabla S_1, \nabla S_1\rangle.\end{aligned}$$

Since M is compact, we obtain

$$\begin{aligned}0 &= \frac{1}{2}\int_M L_1\|A\|^2 dM \\ &= \int_M \{S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1}A\|^2) + \langle P_1\nabla S_1, \nabla S_1\rangle\}dM.\end{aligned} \tag{12}$$

We recall the following result (see [AdCC] – Lemma 4.1):

Lemma 3.1 ([AdCC]). *Let M be an n-dimensional compact hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. If the normalized scalar curvature R is constant and $R-1 \geq 0$, then*

$$|\nabla A|^2 - |\nabla S_1|^2 \geq 0. \tag{13}$$

Since $S_1 \geq 0$ and P_1 is positive, we have that

$$\langle P_1 \nabla S_1, \nabla S_1 \rangle = \|\sqrt{P_1} \nabla S_1\|^2 \geq 0. \tag{14}$$

Our hypothesis and inequalities (13) and (14) implies that the right-hand side of (12) is non-negative. Thus we conclude that

$$S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1}A\|^2) + \langle P_1 \nabla S_1, \nabla S_1 \rangle = 0. \tag{15}$$

Since each term in above equation is non-negative, we have

$$S_1((n-1)S_1 - \|\sqrt{P_1}A\|^2) = 0.$$

Observe that when $S_1 = 0$, $\|A\|^2 = 0$ and $\|\sqrt{P_1}A\|^2 = 0$. Since by Lemma 2.1, $trace P_1 = (n-1)S_1$, the first part of the theorem is proved.

Now, let us assume that $\|\sqrt{P_1}A(p)\|^2 = (n-1)S_1(p)$, for all $p \in M$. If $S_1(p) = 0$ for all $p \in M$, since $S_2 = 0$, $\|A\|^2 = 0$ and M is totally geodesic. Let us suppose that there exists a point p_0 in M such that $S_1(p_0) > 0$. So the set $\mathcal{A} \subset M$ where $S_1(p) > 0$ is an open and non-void set of M. We claim that P_1 is positive definite in $\mathcal{A}$. In fact, if P_1 has one eigenvalue equal to zero, then by Remark 2.1, $P_1 A \equiv 0$ and since $\|\sqrt{P_1}A(p)\|^2 = (n-1)S_1(p)$, we conclude that $S_1 = 0$, which is a contradiction. On each connected component of $\mathcal{A}$, we have that

$$\langle P_1 \nabla S_1, \nabla S_1 \rangle = 0$$

and

$$|\nabla A|^2 - |\nabla S_1|^2 = 0.$$

Since P_1 is positive definite, the first equation implies that $\nabla S_1 = 0$. This implies that $|\nabla A|^2 = 0$, by the second equation, i.e., the second fundamental form of M is covariant constant. It follows that the component $\mathcal{A}$ is a piece of a Clifford torus, by using the following theorem of H. B. Lawson ([L] – Theorem 4, see also [CdCK] Lemma 3).

Theorem 3.2 [L]**.** *Let M^n be an isometrically immersed hypersurface of S^{n+1}, over which the second fundamental form is covariant constant. Then, up to isometries of S^{n+1}, M^n is an open set of $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$.*

Finally, since along the boundary of $\mathcal{A}$, $\|A\|^2 = S_1^2 = 0$, we conclude that $\partial\mathcal{A} = \emptyset$ and M is a Clifford torus. □

4. Characterization of index one closed hypersurfaces with $R = 1$ in the real projective space

In this section we will assume that the operator L_1 is elliptic and will describe the index of closed hypersurfaces in the real projective space $\mathbb{P}(\mathbb{R})^{n+1}$. In order to do that we are going to use the covering map of S^{n+1} onto $\mathbb{P}(\mathbb{R})^{n+1}$. The following result will be needed.

Lemma 4.1. *Let $M^n \to S^{n+1}$ is a closed orientable hypersurface with $R = 1$. Then the index of the quadratic form*

$$I(f,f) = -\int_M fT_1 f dM$$
$$= -\int_M fL_1 f + ((n-1)S_1 - 3S_3)f^2 dM$$

is greater than one.

Proof. First of all observe that for constant functions $f = \text{const.}$, we have that

$$I(f,f) = -\int_M fL_1 f + ((n-1)S_1 - 3S_3)f^2 dM$$
$$= -\int_M ((n-1)S_1 - 3S_3)f^2 dM < 0.$$

Thus $ind(M) \geq 1$.

Suppose that this index is equal to one. Let $\{e_1, \dots, e_{n+2}\}$ be an orthonormal basis of $\mathbb{R}^{n+2}$. If we write the normal vector field of the immersion as $N = \sum_{i=1}^{n+2} n_i e_i$, we obtain that

$$L_1(n_i) = 3S_3 n_i, \text{ for all } i = 1, \dots, n+2.$$

Thus

$$I(n_i, n_i) = -\int_M ((n-1)S_1)n_i^2 dM \leq 0.$$

Since the functions n_i are linearly independent, the index one hypothesis implies that $(n-1)$ of the $n_i's$ have to be null and since $|N| = 1$, after reordering if necessary, we have $n_1 = 1$ and $n_i = 0$ for $i = 2, \dots, n+2$. Thus the normal vector field $N = e_1$. This implies that M^n is totally geodesic. On the other hand, since L_1 is elliptic, we have that $S_1 > 0$, and this contradicts the fact that M^n is totally geodesic. We conclude then that $ind(M) > 1$.

The main result of this section is the following characterization of index one closed hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$.

Theorem 4.1 (Theorem 2 of the introduction)**.** *Let $x : M^n \to \mathbb{P}(\mathbb{R})^{n+1}(1)$ be a closed two-sided hypersurface with scalar curvature one. Then $Ind(M) \geq 1$ and if $Ind(M) = 1$, M is the Clifford hypersurfaces obtained by the projection of the Clifford torus of Theorem 3.1.*

Proof. The proof is inspired by the proof of the minimal case in [dCRR]. Observe that the index one hypothesis implies that M must be connected. Since, by lemma 4.1, S^{n+1} does not have an index one hypersurface with $R = 1$, x cannot lift to an immersion of M into S^{n+1}. Thus we obtain that there exists a connected twofold covering $\widetilde{M} \to M$ and an isometric immersion $\widetilde{x} : \widetilde{M} \to S^{n+1}$ which is locally congruent to the immersion of M in $\mathbb{P}(\mathbb{R})^{n+1}$. An object in $\widetilde{M}$ that corresponds to an object in M will be denoted by the same notation as in M. If we denote by $\pi : \widetilde{M} \to \widetilde{M}$ the isometric involution induced by the covering, then $\widetilde{x}$ must satisfy

$$\widetilde{x} \circ \pi = -\widetilde{x}$$

and, since $\widetilde{x}(M)$ is two-sided, $\widetilde{M}$ is orientable, and

$$N \circ \pi = -N,$$

where N is the unit normal vector field of the immersion. We have that the immersion $\widetilde{x}$ is such that $R = 1$ and $S_3 \neq 0$. By ellipticity we can choose the orientation of $\widetilde{M}$ in such way that $S_1 > 0$.

Let λ_1 be the first eigenvalue of the operator

$$T_1(\varphi) = L_1(\varphi) + ((n-1)S_1 + 3S_3)\varphi.$$

We know that its first eigenspace is one-dimensional and generated by a function φ that does not change sign on $\widetilde{M}$. Now, let $\varphi_1 = \varphi \circ \pi$. Since π is an isometry, we obtain that $T_1(\varphi_1) = \lambda_1 \varphi_1$. This implies that $\varphi = \pm\varphi \circ \pi$. Observe that if $\varphi = -\varphi \circ \pi$, φ has to change sign on $\widetilde{M}$. Thus $\varphi = \varphi \circ \pi$.

From the fact that $Ind(M) = 1$, we obtain that any function $u : \widetilde{M} \to \mathbb{R}$ such that $u \circ \pi = u$ and $\int_{\widetilde{M}} u\varphi d\widetilde{M} = 0$ satisfies

$$I(u,u) = -\int_{\widetilde{M}} \{uL_1u + ((n-1)S_1 + 3S_3)u^2\} d\widetilde{M} \geq 0.$$

Moreover, if such a function u satisfies $I(u,u) = 0$, then u is a Jacobi function, that is,

$$L_1u + ((n-1)S_1 + 3S_3)u = 0.$$

Given $a, b \in \mathbb{R}^{n+2}$, let $\phi_{a,b} : \widetilde{M} \to \mathbb{R}^{n+2}$ be defined by

$$\phi_{a,b} = \langle \widetilde{x}, a\rangle \widetilde{x} + \langle N, a\rangle N + \langle \widetilde{x}, b\rangle N.$$

By doing the calculation coordinatewise and using equations (3) and (4) we have that

$$L_1(\widetilde{x}) = -(n-1)S_1\widetilde{x}$$

and

$$L_1(N) = 3S_3N.$$

Thus,

$$L_1(\langle \widetilde{x}, a\rangle \widetilde{x}) = -2(n-1)S_1\langle \widetilde{x}, a\rangle \widetilde{x} - P_1A(a^t),$$
$$L_1(\langle N, a\rangle N) = 6S_3\langle N, a\rangle N - P_1A^2(a^t)$$

and

$$L_1(\langle \widetilde{x}, b\rangle N) = [-(n-1)S_1 + 3S_3]\langle \widetilde{x}, b\rangle N - P_1A(b^t),$$

where a^t, b^t are the tangent projection of a and b. This implies that

$$T_1(\phi_{a,b}) = -[(n-1)S_1 + 3S_3][\langle \widetilde{x}, a\rangle \widetilde{x} - \langle N, a\rangle N] + X_{a,b}, \tag{16}$$

where $X_{a,b}$ is a tangent vector field. Then,

$$-\int_{\widetilde{M}} \langle T_1(\phi_{a,b}), \phi_{a,b}\rangle d\widetilde{M}$$

$$= \int_{\widetilde{M}} [(n-1)S_1 + 3S_3][\langle \widetilde{x}, a\rangle^2 - \langle N, a\rangle^2 - \langle \widetilde{x}, b\rangle\langle N, a\rangle] d\widetilde{M}.$$

Now, by (2), we have

$$\int_{\widetilde{M}} [(n-1)S_1 + 3S_3]\langle \widetilde{x}, b\rangle\langle N, a\rangle d\widetilde{M}$$

$$= -\int_{\widetilde{M}} \{\langle N, a\rangle L_1(\langle \widetilde{x}, b\rangle) - \langle \widetilde{x}, b\rangle L_1(\langle N, a\rangle)\} d\widetilde{M} = 0.$$

Thus,

$$-\int_{\widetilde{M}} \langle T_1(\phi_{a,b}), \phi_{a,b}\rangle d\widetilde{M} = \int_{\widetilde{M}} [(n-1)S_1 + 3S_3][\langle \widetilde{x}, a\rangle^2 - \langle N, a\rangle^2] d\widetilde{M}. \tag{17}$$

Observe that the above expression does not depend on b. We are going to show that for any $a \in \mathbb{R}^{n+2}$, it is possible to choose $b \in \mathbb{R}^{n+2}$ such that $\int_{\widetilde{M}} \varphi\phi_{a,b} d\widetilde{M} = 0$. To do this, consider a linear map $F : \mathbb{R}^{n+2} \to \mathbb{R}^{n+2}$ given by

$$F(b) = \int_{\widetilde{M}} \varphi\langle \widetilde{x}, b\rangle N d\widetilde{M}.$$

We claim that F is injective (thus a linear isomorphism). In fact, if $b \neq 0$ is such that $F(b) = 0$, one has that (17), with $\phi = \phi_{0,b} = \langle \widetilde{x}, b\rangle N$, implies that

$$I(\phi, \phi) = 0.$$

Then, $T_1(\phi) = 0$. On the other hand, for $a = 0$,

$$T_1(\phi) = X_{0,b} = -P_1 A(b^t) = 0, \tag{18}$$

where b^t is the tangent projection of b along $\widetilde{M}$. Since P_1 is positive definite, (18) says that $A(b^t) = 0$ on $\widetilde{M}$, which is the same that $\langle N, b\rangle$ is constant along $\widetilde{M}$. As we have that $N \circ \pi = -N$, we get that $\langle N, b\rangle = 0$. This implies that the function $u = \langle \widetilde{x}, b\rangle$ satisfies that $\mathrm{Hess}u(X, Y) = \langle X, Y\rangle u$. We need the following result of M. Obata.

Theorem 4.2 ([O] – Theorem A)**.** *In order that a complete Riemannian manifold of dimension $n \geq 2$ admit a non-constant function ϕ with* $\mathrm{Hess}\phi(X, Y) = c^2\phi\langle X, Y\rangle$*, it is necessary and sufficient that the manifold be isometric to a sphere $S^n(c)$ of radius $\frac{1}{c}$ in the $(n+1)$ Euclidean space.*

Thus, if u is non-constant, then $\widetilde{M}$ is isometric to a unit sphere and since $\widetilde{M}$ is isometrically immersed in $S^{n+1}(1)$, this implies that $\widetilde{M}$ is totally geodesic. On the other hand, if u is constant, $\widetilde{M}$is totally umbilic. Since $S_2 = 0$, $\widetilde{M}$ is again totally geodesic. In both cases, $S_1^2 = |A|^2 = 0$, which is a contradiction to the fact that $S_1 > 0$. Thus the claim is proved.

Take an orthonormal basis $\{a_1, \ldots, a_{n+2}\}$ of $\mathbb{R}^{n+2}$. By using the isomorphism F, for any $i = 1, \ldots, n+2$, it is possible to find $b_i \in \mathbb{R}^{n+2}$ such that $\int_{\widetilde{M}} \varphi \phi_{a_i,b_i} d\widetilde{M} = 0$. Thus each coordinate ϕ_{ij} of ϕ_{a_i,b_i} is such that $\int_{\widetilde{M}} \varphi \phi_{ij} d\widetilde{M} = 0$. Then, $I(\phi_{ij}, \phi_{ij}) \geq 0$. From equation (17), we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n+2} \int_{\widetilde{M}} [(n-1)S_1 + 3S_3][\langle \widetilde{x}, a_i \rangle^2 - \langle N, a_i \rangle^2] d\widetilde{M} \\ &= \sum_{i=1}^{n+2} \int_{\widetilde{M}} [(n-1)S_1 + 3S_3](|\widetilde{x}|^2 - |N|^2) d\widetilde{M} = 0. \end{aligned}$$

This implies that $T_1(\phi_{a_i,b_i}) = 0,\ i = 1, \ldots, n+2$. Hence, $\langle T_1(\phi_{a_i,b_i}), \widetilde{x} \rangle = 0$ and, by equation (16), we obtain that

$$[(n-1)S_1 + 3S_3]\langle \widetilde{x}, a_i \rangle = 0,\ i = 1, \ldots, n+2.$$

But this is only possible if $(n-1)S_1 + 3S_3 = 0$. Since $\|\sqrt{P_1}A\|^2 = -3S_3 = (n-1)S_1$, theorem (3) implies that $\widetilde{M}$ is a Clifford torus. □

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