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On the dynamics of certain actions of free groups on closed real analytic manifolds

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Abstract. Let M be a closed connected real analytic manifold; let Γ be a free group on two generators. The set of analytic actions of Γ on M endowed with Taken's topology contains a nonempty open subset whose corresponding actions share three properties: (a) they have every orbit dense, (b) they leave invariant no geometric structure on M, (c) any homeomorphism conjugating two of them is analytic.

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1. Introduction

Let M be a closed connected real analytic manifold. Various topologies can be defined on the group $\mathrm{Diff}^\omega(M)$ of its analytic diffeomorphisms; we choose to work with the so-called "C^ω" topology introduced by Takens. See Section 3 for a definition of it.

Let Γ be a finitely generated group; let $\mathrm{Act}(\Gamma, M)$ denote the set of actions of Γ on M. When studying the dynamical properties of Γ-actions on M it is not compulsory, but customary, to grant special attention to those that are *open* properties, whether they are open by definition or as the result of a theorem. Here we show that when Γ is a free group on two generators, there exists a nonempty open subset $\mathcal{O}$ of $\mathrm{Act}(\Gamma, M)$ whose elements share two important properties: firstly, they are minimal (every orbit dense) and even remain so when lifted to the bundle of jets of coframes of any finite order on M; secondly, they are *locally rigid* (a homeomorphism of M which conjugates two close enough points of $\mathcal{O}$ is (a) unique, (b) analytic and (c) tends to the identity when the two conjugate actions tend to one another).

Let us now provide more precise statements: first recall that a *Morse–Smale* diffeomorphism of M is an $f \in \mathrm{Diff}^\omega(M)$ whose nonwandering set consists of finitely many periodic points, each of them hyperbolic; it is also required that the invariant manifolds of these periodic points intersect transversely: if W^s_α is the stable manifold of the periodic point α, if W^u_β is the unstable manifold of the periodic point β, then $W^s_\alpha \cap W^u_\beta$ has dimension $\dim(W^s_\alpha) + \dim(W^u_\beta) - \dim(M)$.

For any Morse–Smale diffeomorphism f there is a least integer $k > 0$ such that every periodic point of f^k is a fixed point, and the map $f \to k$ is locally constant; for the sake of simplicity, we will call a periodic point of f a *sink*, a *source* or a *saddle* if it is such a fixed point of f^k (what we call "saddles" are the hyperbolic fixed points whose stable and unstable manifolds both have strictly positive dimension). Then, if s is a "source" or "sink" of f in our sense, its *basin* (of repulsion or attraction, respectively) is the set of points x such that $f^{-kn}(x)$ (respectively $f^{kn}(x)$) tends to s when n tends to $+\infty$.

A Morse–Smale diffeomorphism f is called *special* if any Morse–Smale diffeomorphism g close enough to f is linearizable in a neighborhood of each of its sources and sinks. This property is of course devised to be stable; due to Poincaré's theorem ([2] p. 99) it amounts to saying that for any source or sink s of f, the jacobian matrix $df^k(s)$ belongs to the Poincaré domain and is nonresonant, so that special Morse–Smale diffeomorphisms are in fact generic: they form an open and dense set in the set of all Morse–Smale diffeomorphisms (Sternberg, [38]) which in turn is open and nonempty in $\mathrm{Diff}^\omega(M)$. It is readily seen that the maximal domain on which the linearization of f can be performed near a source or sink s is the basin of s; moreover the linearization is unique up to linear coordinate changes.

A *linear contraction* of $\mathbb{R}^n$ is a linear map A such that for any vector v, the sequence $A^n(v)$ tends to zero; by elementary linear algebra this means that the spectrum of A lies in the open unit disc in the complex plane, and as a consequence some power A^k of A with k large enough will send the closed unit ball of $\mathbb{R}^n$ inside its interior. In Section 6 we define a conjugacy-invariant, dense and open subset $\mathcal{X}$ in the space of linear contractions of $\mathbb{R}^n$; such a subset provides us with yet another dense and open subset in the space of Morse–Smale diffeomorphisms:

Definition 1.1. An *admissible* diffeomorphism h of M is a special Morse–Smale diffeomorphism whose jacobian matrix $dh(s)$ at sinks belongs to $\mathcal{X}$, and whose jacobian matrix at sources has inverse in $\mathcal{X}$.

Definition 1.2. Two admissible maps f and g are in *general position* if:

- For any source or sink s of f the image $g(s)$ of s by g lies in the basin of s, and the 3-jet of g at s is in "general position" in the linearizing chart around s (i.e. it belongs to some Zariski-open subset invariant by linear conjugacy (which remains to be described) in the space of all 3-jets).
- For any saddle β of f (if any), $g(\beta)$ is outside the closed subset of M which is the union of all fixed points of f and invariant manifolds W^s_i, W^u_i of its saddle-points.

This notion is clearly stable under C^3 perturbations (being admissible is a C^1-

stable property, transversality of $g(\beta)$ to the W_i^s and W_i^u also is, and the genericity condition on the 3-jet of g is by essence C^3-stable).

In the following theorems, we let Γ denote the free group on two letters a, b. An element of $\mathrm{Act}(\Gamma, M)$ is defined by the images $\rho(a) = f$ and $\rho(b) = g$ of a and b; we assume that f and g are admissible, close enough to one another and in general position.

Theorem A. *Let D be a domain in M and let ϕ be an analytic map from D to M. Let m be any point of D. Then there is a neighborhood U of m in D and a sequence $\gamma_n \in \Gamma$ such that $\rho(\gamma_n)$ tends to ϕ on U in C^ω topology.*

Corollary B (minimality)**.** *Let $J^k(M)$ ($k \in \mathbb{N}$) be the bundle of k-jets of coframes on M (see Section 4); then the natural lift of ρ to $J^k(M)$ is, for any k, a minimal action (in particular ρ is a minimal action on $J^0(M) = M$). As a consequence there exists no nontrivial $\rho(\Gamma)$-invariant geometric structure on M (see Section 4).*

Theorem C (rigidity)**.** *Let $\rho' \in \mathrm{Act}(\Gamma, M)$ be close enough ρ. Then, the set $\mathrm{Hom}(\rho', \rho)$ of homeomorphisms of M conjugating ρ' to ρ is either empty or a singleton, and in the latter case its only element belongs to $\mathrm{Diff}^\omega(M)$.*

Let us now consider the meaning of these theorems. The part of dynamical systems theory which studies actions of finitely presented groups on closed manifolds is of course very wide, and abundant in interesting results; certain classes of groups naturally arise: first, the abelian ones (starting with $\mathbb{Z}$, whose actions are "classical" dynamical systems with discrete time), first studied as a whole by N. Koppel in her thesis [26] and the object of many works since then; the class of abelian groups admits the successive classical generalizations nilpotent, solvable, and amenable. Opposite to these are the lattices of semi-simple Lie groups, which always contain free nonabelian groups and have also been the object of a wide research activity. Finally, it seems to us that nonabelian free groups form a very natural class to look at. Now, amenable groups preserve measures and thus yield to the apparatus of ergodic theory, while lattices in semisimple Lie groups of higher rank seem, metaphorically, to "remember" their noble origin by not acting willingly on manifolds that are too low (in dimension, e.g.) for them (see [10] or [11], amongst others). Our philosophy is then that free groups, again metaphorically speaking, may not be bound to obey any dynamic rule (theorem A, corollary B) nor may they bend easily (theorem C). The moral of these results is in our mind that free groups are up to anything, and their dynamical study is a desperate one.

Let us have a closer look at theorem A and its two corollaries. As one knows, at a time when "dynamical system" meant "hamiltonian dynamical system", the theory of those was mostly concerned with their *integration* – this meaning, roughly, the quest for sufficiently many first integrals for a given system, as in the construction of canonical action-angle coordinates for the so-called completely integrable dynamical systems. As Poincaré taught us, this procedure will fail for generic systems, not only because their first integrals cannot be explicited (in the sense of differential Galois theory) but because there are not, in general, enough first integrals. The notion of dynamical system then slowly begun to encompass actions of groups other than $\mathbb{R}$, or $\mathbb{Z}$, on manifolds that were not as a rule equipped with a riemannian, symplectic or contact structure, and the theory of foliations and pseudogroups on these as well. Many famous researchers, amongst whom Stephen Smale ([37], Section 4), pointed out the necessity of systematically studying the dynamics of Lie group actions on abstract manifolds; in that wide context, first integrals are scarce and we have to look for something else: indeed, most interesting systems are topologically transitive (one orbit is dense) or even minimal (every orbit dense) ! Here are some classical examples of transitive systems: Anosov diffeomorphisms of tori, some fuchsian actions on the circle, linear flows on tori with generic angles, geodesic flows on compact negatively curved surfaces... Notice that all these examples but the third are structurally stable. Here are less classical interesting examples of minimal dynamics, our list surely being by no means complete: first, Hector showed that if a subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^1)$ acts with no global fixed point it has dense orbits unless it is cyclic (unpublished); before that he had described in [19] minimal C^∞ foliations of $\mathbb{R}^3$ with all leaves transverse to the verticals, but these cannot be C^ω by a theorem of Haefliger. Then there is a theorem of Duminy stating that there is a neighborhood of the identity in $\mathrm{Diff}^2(\mathbb{S}^1)$ such that the group generated by any part of it either has a finite orbit or every orbit dense; since any two generic elements of $\mathrm{Diff}^2(\mathbb{S}^1)$ are Morse–Smale with disjoint nonwandering sets, this gives the minimality of a generic free subgroup of $\mathrm{Diff}^2(\mathbb{S}^1)$ having at least two generators. The theorem of Duminy is also unpublished, but has been improved in the analytic case by Ghys, [11]. Finally, in the neighbouring field of pseudogroup actions, Nakai [32] showed that a generic nonsolvable pseudosubgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ had all orbits dense near the origin, thus improving an older result by Scherbakhov on the same lines; in [4] it was shown that this density also held on the bundle of k-jets of invertible holomorphic functions, for any k (a fact that was probably known to Nakai). In the neighbouring field of foliations, one can but think of Il'yashenko's conjecture, which in dimension $n = 2$ becomes Il'yashenko's theorem, stating that on $\mathbb{C}^2$ a polynomial ordinary differential equation with degree at least two which leaves invariant the line at infinity has every regular integral curve dense (for a still concise but more precise survey of these matters we immodestly refer the reader to the introduction of our previous work [5], which deals with a multidimensional version of [4]).

Let ρ be an action of Γ on M. A *first integral* for ρ is a map from M to $\mathbb{R}$ which is invariant under ρ; in applications to physics this corresponds to scalar invariants. If we wish to take other types of invariants into account, e.g. tensor fields, we should replace the notion of a map $M \to \mathbb{R}$ by that of a section $M \to E$, where $E \twoheadrightarrow M$ is some fibre bundle whose nature depends on the problem under consideration; then for this section to be invariant (or, rather, for the notion of an invariant section to mean something at all) it is necessary that the action of Γ on M lifts to E in some more or less canonical way. Consider then the case when E has the bundle of k-jets of coframes as its associated principal bundle (see Section 3 for a lengthier explanation): our requirement of canonical extensibility of the action is met, and we are endowed with a notion of generalized first integral of the uttermost generality. But in view of the second assertion of corollary B, this notion is still insufficient to apply to general two-generators actions. Now if this part of corollary B was our sole result, it wouldn't come as a big surprise, since a generic Morse–Smale diffeomorphism already preserves no locally homogeneous geometric structure; but what makes it worth its salt is that usually, the rigidity property of theorem C is linked with the existence of one type or other of invariant geometric structure; let us provide a small list of examples: diffeomorphisms of the circle having an irrational diophantine rotation number preserve a smooth metric by the famous theorem [20] of Herman; fuchsian groups by definition preserve a projective structure (see [12]); also notice that in the neighboring context of foliations, the Elkacimi–Nicolau examples of C^∞-stable foliations (cf. [8]) are transversely affine, and the so-called "compact foliations" of [9] are transversely riemannian by [36].

Talking of rigidity leads us to having a closer look at theorem C. It's what we call a *rigidity* result: any conjugacy between two of the objects under consideration (here, the dynamics of a free group) is analytic; moreover in many important cases the space of such conjugacies is a homogeneous space of a Lie group (here, a trivial Lie group). Such results abound in the literature; we can but think to the following one due to Nakai, [33] which can be seen as a geometrico-differential generalization of the so-called *fundamental theorem of affine geometry*:

Theorem. *Let $\mathcal{F}_i^k$ $(i = 1, 2; k = 1, \dots, n+1)$ be two families of $n+1$ transverse analytic foliations of codimension one in $\mathbb{C}^n$ (this is what one calls an $n+1$-web in $\mathbb{C}^n$). If some homeomorphism c sends $\mathcal{F}_1^k$ to $\mathcal{F}_2^k$ for all k, then c is real analytic; if nonempty, the space of such homeomorphisms identifies in a natural way with a homogeneous space of some well-defined solvable Lie group of dimension $\leq n+1$. Finally if for any triple j, k, l of distinct indexes the 3-web induced by $\mathcal{F}_1^j$, $\mathcal{F}_1^k$ and $\mathcal{F}_1^l$ on the intersection of the remaining $\mathcal{F}_1^m$ $(m \notin \{j, k, l\})$ is non-hexagonal then c is holomorphic or antiholomorphic as well.*

Readers interested in this kind of results should consult Nakai's survey in [40]. In our case rigidity comes from the strong interaction between the dynamics of f and g, which themselves are far from rigid: each of them is structurally stable, but their space of analytic deformations may be expected to have infinite dimension due to the existence of Mather's invariant, as happens for Morse diffeomorphisms of the circle.

Recently, several papers appeared on the subject of rigid actions, in our sense of this word (the word "rigid" is quite fashionable, and has received many different definitions, some of them quite formal ("rigid geometric structure" (Gromov)) and some quite not; many being definitely remote from the common meaning of the word: "deficient in or devoid of flexibility", according to [39]). We will now provide a short list. First, the papers of Nakai and Il'yashenko, already mentioned, do not only prove the minimality of the pseudogroup or foliation under consideration, but its rigidity as well: a conjugacy between two pseudosubgroups of $\mathrm{Diff}(\mathbb{C}, 0)$ is holomorphic as soon as those pseudosubgroups are nonsolvable; a conjugacy between generic holomorphic polynomial foliations of $\mathbb{C}^2$ which leave the line at infinity invariant is transversely holomorphic. These results have led to various investigations; in [5], the rigidity of generic pseudogroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$ generated by two generators close to the identity is shown, while in [29] it is shown that the set of rigid foliations of $\mathbb{CP}^n$ has nonempty interior. Also inspired by Nakai's methods is the theorem of Rebelo, [34] stating that a generic group of analytic transformations of the circle is rigid:

Theorem. *Let Γ be a nonsolvable subgroup of $\mathit{Diff}^\omega(\mathbb{S}^1)$ with a finite number of generators close enough to the identity. Then any topological conjugacy of Γ to another such group is analytic except perhaps at the points of a finite Γ-invariant set.*

The assumption of proximity made in this theorem, ours and various others (as Duminy's, quoted above) is a rather natural one when studying actions of free groups (were it not made, groups of Schottky type would arise, and these do not behave rigidly). On the contrary, if the groups studied possess many relators (e.g. many commuting elements, as is the case for $SL(n, \mathbb{Z})$ when $n > 2$) then rigidity may arise without any proximity-to-the-identity assumption: we can quote for instance [22], [24] or [12], which respectively show the rigidity of $SL(n, \mathbb{Z})$-actions on spheres or on tori and that of certain fuchsian actions on the circle. All in all, actions generated by small generators and actions of semisimple Lie group lattices cover most of the rigidity results for countable groups.

Then, some rigidity results have been obtained for actions of connected groups; being not interested with them in this paper, we content ourselves with quoting the two main references [13] and [15], the latter containing extensive bibliography on the subject. Finally, we should mention results of rigidity for real codimension one foliations; [31] more or less extends [32] to the real analytic context, while in [14] it is shown that a C^1 conjugacy between C^r foliations, $r \geq 2$, is automatically transversely C^r along the noncompact leaves. The foliations of [9] and [8], already quoted, are not only rigid but C^∞-stable (= smoothly conjugated to all their neighbours). This allows statements in the shape of implicit functions theorems (and, indeed, the use of such theorems in the proof, *via* K.A.M. theory): "there exists a neighborhood U of the identity map in $\mathrm{Diff}(M)$, a neighborhood V of $\mathcal{F}$ and a continuous map ϕ from V to U such that for all $\mathcal{F}' \in V$, the diffeomorphism $\phi(\mathcal{F}')$ conjugates $\mathcal{F}'$ to $\mathcal{F}$". The theorem of Gomez–Mont [16] may also be viewed as a rigidity theorem: it states that if a holomorphic deformation of a generic polynomial differential equation on $\mathbb{C}^n$ is transversely holomorphically trivial then it will also be globally holomorphically trivial, which is rather the spirit of our notion of rigidity (conjugacies are smoother than expected).

Finally, notice that the distinction between the discrete group actions, connected group actions and foliations described above is formal: one has e.g.

Theorem ([12]). *Let Γ be the fundamental group of a compact surface with genus $g \geq 2$, let Φ be a C^∞ action of Γ on the circle, and assume its Euler number is $2g-2$. Then there is a C^∞ conjugacy from $\Phi(\Gamma)$ to a group of homographies.*

Through the well-known suspension procedure, one can associate to Φ a foliation of a certain circle-bundle on the surface whose fundamental group is Γ, and this foliation is rigid as a consequence of Ghys's theorem. Then, it so happens that the affine group of the real line acts along this foliation's leaves, and this action is also rigid. So, the above result by E. Ghys competes in all three categories.

A motivation for our theorems lies in [5] and [11]: in the spirit of the latter, and with the methods of the first of these papers, we wished to show that the dynamics generated by two generic diffeomorphisms of any manifold was rigid; however we had to introduce certain additional assumptions, the strongest being that one of the diffeomorphisms was Morse–Smale. Then, the existence of things like Schottky groups induced us, as in [5], to use "small" generators. The minimality of the dynamics on every bundle of jets of coframes and the subsequent lack of an interesting invariant geometric structure came as byproducts of the method of proof, as in [4] and [5].

The whole purpose of this paper is to show theorems A and C; the fact that theorem A implies corollary B is obvious from the definitions given in Section 4. To conclude this introduction, we remark that we work with a free group on two generators only for the sake of simplicity, because our theorem holds in fact for any finitely generated group Γ possessing such a free group as a quotient, as the reader will surely grant after reading the proofs; another remark is that as already said the technique of *suspension* would allow us to translate our definitions and theorems in the language of foliation theory; we haven't wished to do so because this paper is long enough as it is and – contrarily to what happens in [12] – this translation procedure yields nothing really interesting.

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2. Contents

The oncoming notation will serve throughout this paper. Let ρ be an element of $\mathrm{Act}(\Gamma, M)$; let Λ denote the pseudogroup generated by $\rho(\Gamma)$ on M and, following P. Libermann, associate to Λ its *infinitesimal pseudogroup* $\mathfrak{g}_\Lambda$, which is a sheaf of Lie algebras on M (see Section 5 for the definitions). We let f and g be the images in $\mathrm{Diff}^\omega(M)$ of the generators of Γ. The rest of this paper consists of 6 paragraphs. In Section 3 we recall the definition of the C^ω topology; in Section 4 we recall what a geometric structure is. In Section 5 we define $\mathfrak{g}_\Lambda$ and recall a few of its classical properties. The original part of this paper is contained in Sections 6-7. In Section 6 we give sufficient conditions for $\mathfrak{g}_\Lambda$ not to vanish at special points. Then in Section 7 we give other conditions under which the germ of $\mathfrak{g}_\Lambda$ at special points must either vanish or contain every germ of analytic vector field. There is some redundance between Sections 6 and 7 because we wished to make them as independent as possible; Section 6 relies mostly on dynamical arguments whereas Section 7 is of a more analytic nature. In Section 8 we prove theorems A and C, then we ask ourselves a few natural questions on possible extensions of our results.

3. The analytic topology

Let M be a closed connected real analytic manifold. We identify M with the zero section of both its tangent space TM and the complexified tangent space $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$; then we first consider an analytic embedding π of M in some $\mathbb{R}^N$ (this will exist if N is large enough as a consequence of Morrey's analytic version of Whitney's theorem; see [17] for a proof). Since M is compact the following construction will not depend essentially on π: thanks to an auxiliary riemannian metric, we construct the exponential map $\exp : TM \to M$ where TM is the tangent bundle of M; then we compose with π and extend the resulting map holomorphically along the fibers of $TM \to M$ to get a map ϕ from a neighborhood W of M in $T_{\mathbb{C}}M$ to $\mathbb{C}^N$ which is holomorphic along the fibers of $W \to M$. By construction the real tangent space to $\phi(W)$ is in fact a complex space at every point (this is because the exponential is locally surjective). For the same reason $\pi(M) \subset \phi(W)$. So, if W is small enough the image of W is a smooth complex submanifold $M_{\mathbb{C}}$ of $\mathbb{C}^N$ having complex dimension $\dim(M)$ and admitting M as a Lagrangian submanifold. We then notice that any analytic diffeomorphism of M extends to some neighborhood of M in $M_{\mathbb{C}}$ as a map with values in some other such neighborhood; we define W_ε as the tubular neighborhood of M in $M_{\mathbb{C}}$ having diameter ε, we let U_ε be the subset of $\phi \in \mathrm{Diff}^\omega(M)$ which extend at least to W_ε and endow U_ε with the topology of uniform convergence on compact subsets of W_ε. Since $\mathrm{Diff}^\omega(M)$ is the union of all U_ε and the inclusion $U_\varepsilon \subset U_\delta$ for $\varepsilon > \delta$ is obviously continuous we can endow $\mathrm{Diff}^\omega(M)$ with the union of the topologies defined on each U_ε. It is not too difficult to check that this topology is compatible with the group structure. It is strictly contained in the C^∞ topology, is not metrizable but has the Baire property; it was recently put to good use in dynamical system theory by several authors (see e.g. [35], [11] or [6] and compare [20], p. 14, footnote).

4. Invariant geometric structures

A subject of outstanding interest in the literature is that of a *geometric structure* invariant by a given group of transformations. Already present, at least implicitly, in early work of Lie and Cartan, this notion was thoroughly formalized by Ehresman in the middle of the twentieth century, then systematically studied in the subsequent years by various authors; one may consult [3], chap. 6. Recently, Gromov achieved to prove astounding results in this domain; *e.g.* his celebrated "dense-orbit" theorem [18]. Let us again recall a few basic definitions. First, a *k-jet of coframes* on M with base-point $m \in M$ is the k-jet of a germ of invertible analytic transformation between pointed analytic spaces with source (M, m) and target $(\mathbb{R}^n, 0)$, with $n = \dim(M)$. The collection $J^k(M, m)$ of all k-jets of coframes with the same base-point m is an analytic manifold; indeed, let G^k denote the set of vector-valued polynomial functions of degree $\leq k$ in n variables with vanishing constant parts and invertible Jacobian matrix at 0: then the rule of composition for k^{th}-order Taylor polynomials makes G^k into a real algebraic group, this group acts on $J^k(M, m)$ by composition on the right, and the resulting action is free and transitive; thus one sees that the collection $J^k(M)$ of all the $J^k(M, m)$ is endowed with the structure of an analytic G^k-principal fiber bundle.

Definition 4.1. *Let X be a Hausdorff space and let σ be any continuous G^k-action on X. Then, a geometric structure of type (X, σ) is a section of the X-bundle with base M and group G^k associated to $J^k(M)$.*

For instance, a vector field on M is a geometric structure of type $(\mathbb{R}^n, \sigma)$ with σ the usual action of $G^1 = GL(n, \mathbb{R})$ on $\mathbb{R}^n$. More generally, any tensor field is a geometric structure. A geometric structure obtained by taking for (X, σ) a homogeneous space G^k/H of G^k with its natural G^k-action by left translations is called an *H-structure*, see [25]; e.g. a riemannian metric is an $O(n)$-structure.

Since G^{k+1} is, for any k, an extension of G^k, a geometric structure with group G^k will also be a geometric structure with group G^{k+l} for any integer l. This calls for another definition: the *order* of a geometric structure s is the least integer k such that σ factors through G^k when restricted to the image of s in X.

The action of $\mathrm{Diff}^\omega(M)$ on M lifts naturally to $J^k(M)$ for any k. Therefore, given any G^k-space (X, σ) and any subgroup Λ of $\mathrm{Diff}^\omega(M)$ there will be a natural action of Λ on $X \times J^k(M)$, namely the product of the trivial action on X with the lift of Λ. Assume there is a geometric structure s of type (X, σ) on M: then due to definition 4.1, s can be seen as a G^k-equivariant morphism from $J^k(M)$ to X, and may or may not be invariant by the aforementioned lifted Λ-action. We can therefore speak of Λ-*invariant geometric structures of type* (X, σ) for any given Λ, X and σ. When studying the dynamics of Λ it is a very natural question to search for such invariant structures; one should then note that when σ is the trivial morphism, geometric structures of type (X, σ) (either Λ-invariant or not) will necessarily exist on M, have order zero and amount to sections of the trivial bundle $X \times M$; amongst those, the invariant ones may be thought of as *first integrals* of ρ with values in X. By analogy, a geometric structure of order k may be thought of as a G^k-equivariant first integral with values in X of the lifted action of Λ to $J^k(M)$. Constant first integrals of dynamical systems are of no interest at all in the study of these systems, so we define *trivial geometric structures* to be those which consist of a constant map. These definitions make it clear what corollary B means, and at the same time show it to be a direct consequence of theorem A.

5. Infinitesimal pseudogroups

We let $\mathfrak{G}$ denote the pseudogroup of all analytic diffeomorphisms from an open set of M to another; if $f \in \mathfrak{G}$ has U as its domain we write $f \in \mathfrak{G}(U)$, thus keeping in mind the natural structure of presheaf on $\mathfrak{G}$. Similarly we let $\mathfrak{g}(U)$ denote the Lie algebra of all analytic vector fields on U, and write $\mathfrak{g}$ for the corresponding sheaf of Lie algebras on M. We endow both $\mathfrak{G}(U)$ and $\mathfrak{g}(U)$ with the C^ω topology (notice that we thus use the sheaf structure on $\mathfrak{g}(U)$ but not the sheaf topology, for which fibers would be discrete instead of contractible). Given a pseudogroup Λ on M, we let $\mathfrak{G}_\Lambda$ denote the "closure" of Λ in $\mathfrak{G}$; thus an element g of $\mathfrak{G}(U)$ belongs to $\mathfrak{G}_\Lambda(U)$ if for any $x \in U$ there is a neighborhood V of x in U on which g is the limit, in $\mathfrak{G}(V)$ and for the C^ω topology,[1] of some sequence of elements of $\Lambda(V)$.

We now quickly review some well known facts about the link between pseudogroup subpresheaves of $\mathfrak{G}$ and Lie algebras subsheaves of $\mathfrak{g}$. Consider first some Lie algebras subsheaf $\mathfrak{h}$ of $\mathfrak{g}$. Given any nonempty open subset U of M and any section X of $\mathfrak{h}$ over U, we can define the pseudoflow ϕ_X^t of X in the usual way: precisely, if $K \subset U$ is compact there is a maximal $T = T(X, K) \in]0, +\infty]$ such that any point $k \in K$ has its X-trajectory well defined for all times $t \in [0, T[$; in this case, the point of this trajectory corresponding to time t is denoted by $\phi_X^t(k)$, just as in the case where $X \in \mathfrak{g}(M)$ is a global field and $T(M) = +\infty$. One may then define the *pseudogroup* $\mathfrak{G}_\mathfrak{h}$ *of* $\mathfrak{h}$ as the closure in $\mathfrak{G}$ of the pseudosubgroup generated by all the elements of all the pseodoflows ϕ_X^t with $X \in \mathfrak{h}$.

Conversely, consider some pseudosubgroup Λ of $\mathfrak{G}$; given an open set U, define $\mathfrak{g}_\Lambda^0(U)$ as the set of all vector fields $X \in \mathfrak{g}(U)$ such that for any compact $K \subset U$ and

[1] From now on, we will generally omit the precision "in C^ω topology" when referring to $\mathfrak{G}$ or $\mathfrak{g}$.

any $t < T(X, K)$ there exists some neighborhood U_K of K in U and some sequence $g_k \in \Lambda(U_k)$ which converges to ϕ_X^t on K. This provides us with a presheaf that needs not yet be a sheaf, because if X is in $\mathfrak{g}_\Lambda^0(U)$ as the "limit" of a sequence g_n, if X' is in $\mathfrak{g}_\Lambda^0(U')$ as the "limit" of another sequence g'_n, one may be able to glue X to X' on $U \cup U'$ without managing to glue the g_n to the g'_n. Anyway, we make $\mathfrak{g}_\Lambda^0$ into a sheaf $\mathfrak{g}_\Lambda$ in the usual way and, following P. Libermann [28], call $\mathfrak{g}_\Lambda$ the *infinitesimal pseudogroup of* Λ.

A priori, $\mathfrak{g}_\Lambda$ is no more than a sheaf of subsets of $\mathfrak{g}$; it turns out that $\mathfrak{g}_\Lambda$ is, in fact, a subsheaf of Lie algebras of $\mathfrak{g}$. Moreover, $\mathfrak{g}_\Lambda(U)$ is closed in $\mathfrak{g}(U)$ for all U. These facts are fairly standard; one can find their proof in [4], proposition 3 (it is given there in the holomorphic case, but extends nicely to our situation due to our use of the convenient C^ω topology). The link between $\mathfrak{g}_\Lambda$ and the geometric structures Λ can preserve is classical matter; see [25], chap. I, Section 8 and the references therein. The following very useful facts are obvious from the definitions.

Proposition 5.1. *One has (of course !)* $\mathfrak{G}_\mathfrak{g} = \mathfrak{G}$ *and* $\mathfrak{g}_\mathfrak{G} = \mathfrak{g}$.

Proposition 5.2. (1). *Let* γ *belong to* $\mathfrak{G}_\Lambda(U)$ *and* X *belong to* $\mathfrak{g}_\Lambda(U)$*: then,* $\gamma_*(X)$ *belongs to* $\mathfrak{g}_\Lambda(\gamma(U))$. (2). *Let* X *belong to* $\mathfrak{g}_\Lambda(U)$*, let* V *be an open subset of* U *on which* ϕ_X^1 *is defined: then* ϕ_X^1 *belongs to* $\mathfrak{G}_\Lambda(V)$.

The following proposition expresses rigorously the intuitive fact that the larger $\mathfrak{g}_\Lambda$ is, the larger the Λ-orbits and therefore the smaller the set of Λ-invariant geometric structures must be (see [25] or [4]):

Proposition 5.3. *Let* U *be open in* M *and let* X *belong to* $\mathfrak{g}_\Lambda(U)$*. Then* (1). *The trace on* U *of the closure of any* Λ*-orbit meeting* U *is saturated by the pseudoflow of* X. (2). *The restriction to* U *of a* Λ*-invariant geometric structure must also be* X*-invariant.*

Let m belong to M. Consider the natural *evaluation morphism* which, to any germ $X \in \mathfrak{g}_\Lambda(m)$, associates the value $X(m)$ of this germ at m. This is a linear map from $\mathfrak{g}_\Lambda(m)$ to the tangent space $T_m M$ of M at m; its image is therefore generated by the image of finitely many elements of $\mathfrak{g}_\Lambda$ (and, in fact, at most n of them). These elements have general position at m, and therefore have general position as well on all of some neighborhood of m. This shows that the *dimension* $d_\Lambda(m)$ of $\mathfrak{g}_\Lambda(m)$ (defined as that of its evaluation at m) can only increase locally. Therefore the level-sets of $d_\Lambda(m)$ induce a stratification of M:

$$M_0 \subset \cdots \subset M_k = M \quad (k \in \mathbb{N},\ 0 \le k \le n) \tag{1}$$

with each M_{i-1} (if any) closed in M_i. Recall Nagano's theorem [30]:

Theorem 5.4. *Let* m *be a point in* M*. Then there is a real analytic manifold*

$L(m)$ and an analytic immersion from $L(m)$ to M which is g_Λ-invariant and has dimension $d_\Lambda(m)$. This manifold is characterized as the set of points m' of M that are linked to m by a piecewise analytic curve whose segments are integral arcs of elements of $\mathfrak{g}_\Lambda$.

We can think of the collection $\mathcal{F}$ of manifolds $L(m)$ as a singular foliation on M. There are two drawbacks to this approach: first, in many interesting cases $\mathcal{F}$ will turn out to be the trivial zero-dimensional foliation; and secondly, if $\mathcal{F}$ is not a regular foliation, the stratification (1) of M by the dimensions of leaves might be awful. To see that, pick a nonempty open subset U of M and let Λ be the pseudogroup generated on M by the collection of analytic maps whose domain and target both lie within U: then in (1) one has $k = 1$, $M_0 = M - U$, $\mathfrak{g}_\Lambda = \{0\}$ on M_0 and $\mathfrak{g}_\Lambda = \mathfrak{g}$ on $M - M_0$. Worse examples can be readily constructed, e.g. by taking products of copies of this one. Still, not everything is possible:

Proposition 5.5. *Let m be a point in M. If its orbit $\Lambda.m$ is closed then m belongs to M_0 and $d_\Lambda(m) = 0$. If it is dense, then either $k = 0$ in (1) or m lies outside M_{k-1}. In particular, if Λ acts minimally on M then $k = 0$ and $\mathcal{F}$ is a regular foliation.*

Proof. The set $\Lambda.m$ is both countable and dense in the orbit of m. So, if $\Lambda.m$ is closed then $d_\Lambda(m) = 0$, which forces m to lie in M_0. If $k > 0$ and m is in M_{k-1} then the orbit of m belongs to a proper closed subset of M so it cannot be dense. □

To end this paragraph we describe two interesting examples of foliations $\mathcal{F}$ corresponding to minimal actions on the torus $M = \mathbb{R}^2/\mathbb{Z}^2$.

Example 5.6. Identify M with the product of two copies P_1, P_2 of $\mathbb{RP}^1$. Let G_i $(i = 1, 2)$ be the projective group of P_i and $G = G_1 \times G_2$ act on M by the product action. Finally let $\Lambda = \Gamma_1 \times \Gamma_2$ with Γ_1 a lattice in G_1 and Γ_2 a dense subgroup in G_2. Then, $\mathcal{F}$ is the vertical foliation given by $dx_1 = 0$ and $\mathfrak{g}_\Lambda$ is the sheaf of local sections of the Lie algebra of G_2. This $\mathcal{F}$ has all leaves closed.

Example 5.7. Let X be a nowhere vanishing field with dense orbits on M. Let f and g be respectively ϕ_X^a and ϕ_X^b with $a, b \in \mathbb{R}-\{0\}$ and $\frac{a}{b} \notin \mathbb{Q}$. Then obviously, X is a global section of $\mathfrak{g}_\Lambda$. On the other hand if some sequence λ_n in Λ tends to some field Y on the domain U then since $f_*X = X$, $g_*X = X$ and λ_n is a word in f and g we have $[X, Y] = 0$; moreover, $\lambda_n \circ \phi_X^t = \phi_X^t \circ \lambda_n$ for all $t \in \mathbb{R}$, which shows that λ_n converges on an X-invariant domain (therefore on all of M). Now according to Arnol'd, [1] there is a diffeomorphism ϕ of $\mathbb{R}/\mathbb{Z}$ whose centralizer in $\mathrm{Diff}^\omega(\mathbb{R}/\mathbb{Z})$ is the group ϕ spans. Consider the constant field $X' = (0, 1)$ on $M' = \mathbb{R}/\mathbb{Z} \times [0, 1]$ and glue the boundary components of M' by identifying $(x, 0)$ with $(\phi(x), 1)$; the obtained manifold is diffeomorphic to M. In the previous discussion take for X the vector field induced by X' on M. Then the symmetry group of X is ϕ_X^t by construction, so that $d_\Lambda = 1$ and $\mathcal{F}$ is a codimension-one minimal foliation.

6. Construction of nontrivial elements of $\mathfrak{g}_\Lambda$

To construct elements of $\mathfrak{g}_\Lambda$, one may rely on Euler's method of polygonal approximations. Endow M with some analytic riemannian metric. As is well known, since M is compact there is some $\varepsilon > 0$ such that the exponential map is a local diffeomorphism when restricted to the bundle of tangent disks of radii ε on M. This means that if f is an element of $\mathrm{Diff}^\omega(M)$ which is uniformly close enough to the identity, then there will exist an analytic vector field X_f on M uniformly close to the zero field, depending analytically on f, uniquely defined by the requirement of its proximity to zero and the fact that for every m, $f(m)$ is the time-one point of the parametrized geodesic curve defined by $\gamma(0) = m$, $\dot\gamma(0) = X(m)$ and $\ddot\gamma(0) = 0$. We might write informally $X_f(m) = f(m) - m$ or $f(m) = m + X_f(m)$.

Proposition 6.1. *Let λ_k be a sequence of elements in Λ. Assume λ_k tends to the identity in $\mathfrak{G}_\Lambda(U)$ for some nonempty open subset U of M. Write $X_k(m) = \lambda_k(m) - m$ and assume that there is some sequence p_k of integers which tends to $+\infty$ and is such that $p_k X_k$ tends to some vector field $X \in \mathfrak{g}(U)$. Then, X belongs to $\mathfrak{g}_\Lambda(U)$. In addition, for any real number r such that $\exp(rX)$ is defined at least somewhere in U, this map will coincide on its whole domain of definition with the limit of the sequence $\lambda_k^{[p_k r]}$ where $[p_k r]$ represents the integral part of $p_k r$.*

This result is just theorem 5 of chapter II of [23], in a slightly different context ([23] deals with locally compact groups without small subgroups, whereas proposition 5.1 above deals with pseudogroups of holomorphic maps). The proof of [23] still extends *mutatis mutandis* (for details see [5], proposition 3.3.1).

We will apply proposition 6.1 in a local context. Recall the following classical definitions and facts ([2] chap. 6 Section 2). A matrix A belongs to the *Poincaré domain* if its eigenvalues lie either all within the open unit disk $\mathbb{D}$ of $\mathbb{C}$, or all outside the closure of $\mathbb{D}$; otherwise, A belongs to the *Siegel domain.* If h is a germ of holomorphic map fixing 0 with jacobian A at this point, and if A is nonresonant, then h is formally linearizable; that is, there is an invertible formal series ϕ fixing 0 such that $\phi \circ h = A \circ \phi$. In case h leaves the germ of $\mathbb{R}^n$ invariant in $\mathbb{C}^n$, ϕ may be chosen to do so. The convergence of ϕ will strongly depend on the geometry of the spectrum of A. It has been shown by Poincaré that if A belongs to the Poincaré domain, then ϕ necessarily converges, whereas in the Siegel domain the situation is much richer and involves the diophantine properties of the eigenvalues of A. Also recall that a matrix $A \in GL(n, \mathbb{C})$ is called *nonresonant* if its eigenvalues $\lambda_1, \dots, \lambda_n$ satisfy no equality of the form $\lambda_i = \lambda_1^{\zeta_1} \dots \lambda_n^{\zeta_n}$ with each ζ_i a nonnegative integer and $\sum \zeta_i \geq 2$. We now consider a nonresonant linear contraction $A \in GL(n, \mathbb{C})$ and an invertible holomorphic map $h_0(x) = x + \alpha_0(x)$ from the open unit ball $\mathbb{B}$ of $\mathbb{C}^n$ to some bounded domain of $\mathbb{C}^n$; we let $\mathfrak{h}$ denote the infinitesimal pseudogroup generated on $\mathbb{C}^n$ by A and h_0. It is to be understood that $\mathfrak{h}$ will turn out to be the image of $\mathfrak{g}_\Lambda$ in some local chart of M. Let $P(X) = a_d X^d + \cdots + a_1 X$ be a polynomial with zero constant term and integer coefficients. As long as this makes sense, we recursively define

$$h_{k+1} = A^{-d} \circ (h_k^{a_d} \circ A) \circ (h_k^{a_{d-1}} \circ A) \circ \cdots \circ (h_k^{a_1} \circ A) \tag{3}$$

(depending on A, h_0 and P this sequence may or may not be infinite). Finally we let $\mathfrak{B}$ denote the Banach space of invertible holomorphic bounded maps from $\mathbb{B}$ to $\mathbb{C}^n$ with its usual norm $\|\alpha\| = \sup_{z \in \mathbb{B}} \|\alpha(z)\|$. The purpose of this paragraph will now be to show the following result:

Theorem 6.2. *There is a dense and open subset $\mathcal{X}$ in the set of linear contractions of $\mathbb{R}^n$ such that if A_0 belongs to $\mathcal{X}$ then there is an $\varepsilon > 0$, a neighborhood U of A_0 and a polynomial P with the property that if $A \in U$, $\alpha_0 \in \mathfrak{B}$, $\alpha(0) \neq 0$ and $\|\alpha_0\| < \varepsilon$ then the sequence h_k constructed above tends in the sense of proposition 6.1 to a constant nonzero vector field on $\mathbb{B}$.*

As the proof is rather intricate, we give its plan: (1) show that h_1 is well defined for α_0 small enough, (2) now that the map $T : h_0 \to h_1$ is well defined near Id, show that it's C^1 and compute its differential, (3a) find a suitable P for $dT(Id)$ to be a contraction, so that $h_k = Id + \alpha_k$ will exist for any k and α_k will tend to 0; (3b) moreover, arrange for $dT(Id)$ to contract the translation subspace strictly less than its complementary subspace of maps vanishing at 0, so that we'll have $\alpha_k = \alpha_k(0) + o(\alpha_k(0))$ as planned. While steps (1) and (2) are nearly straightforward, step (3) is rather painstaking to establish and requires some technique.

Before we get started on step (1) we make a technical remark. We can always define $\mathcal{X}$ as the set of contractions A_0 meeting the requirements of theorem 6.2; then $\mathcal{X}$ is obviously open, and what remains to be done is just to show its density in the set of contractions. Now let A_0 be any contraction, and suppose that for some $k \in \mathbb{N}$ the contraction A_0^k belongs to $\mathcal{X}$; *then we claim that A_0 does, too.* This is because since $A_0^k \in \mathcal{X}$, there exists a polynomial P_1, a neighborhood U_1 of A_0^k and an $\varepsilon > 0$ such that if A_1 belongs to U, $\alpha_0(0) \neq 0$ and $\|\alpha_0\| < \varepsilon$ then the sequence h_k^1 given by (3) with first term $h_0^1 = h_0$ and A_1 and P_1 replacing A and P will tend to some constant nonzero vector field Z. But then, the set U of those contractions A such that $A^k \in U_1$ is a neighborhood of A_0; and if we let $P(X)$ and A_1 respectively denote $P_1(X^k)$ and A^k, we see that the sequence h_k given by (3) is exactly h_k^1.

As a consequence of this remark, it suffices to show that $\mathcal{X}$ contains every contraction A_0 such that $A_0(\mathbb{B})$ is relatively compact in $\mathbb{B}$ (since for any contraction A_0 there exists a $k \in \mathbb{N}$ such that $A_0^k(\mathbb{B})$ lies in, say, $\frac{1}{2}\mathbb{B}$). From now on let A_0 be such a contraction; fix $P(X) \in \mathbb{Z}[X]$ with $P(0) = 0$.

Lemma 6.3. *There is a neighborhood U of A_0 and an $\varepsilon > 0$ such that if $A \in U$ and $\|\alpha_0\| < \varepsilon$ then h_1 exists and belongs to $\mathfrak{B}$.*

Proof. If $\|\alpha_0\| < \varepsilon$ then $h(\mathbb{B})$ contains $(1-\varepsilon)\mathbb{B}$ and h sends $(1-\varepsilon)\mathbb{B}$ to a domain contained in $\mathbb{B}$. From the last of these facts it follows that $h \circ h$ is defined on $(1-\varepsilon)\mathbb{B}$ and sends $(1-2\varepsilon)\mathbb{B}$ to within $\mathbb{B}$, while the first fact implies that h^{-1} is defined on $(1-\varepsilon)\mathbb{B}$. By induction one then shows easily that h^k is defined on at least $(1-(|k|+1)\varepsilon)\mathbb{B}$ and sends this ball to within $\mathbb{B}$ for $|k| < \frac{1}{\varepsilon} - 1$. Now let a be $\max(|a_i|+1)$ and ρ be $\|A\|$: for ε small enough, one has $1 - a\varepsilon > \rho$, so that $h_0^{a_i} A$ is for all i a map with source at least $\mathbb{B}$ and range at most $\mathbb{B}$. Therefore, the composition h_1 of A^{-d} with all these maps is well defined and bounded on $\mathbb{B}$, as asserted. □

We have in fact shown a bit more than the existence of h_1. To explain that, we will need some extra notation. Consider the right term of (3) as a word W in a free alphabet $\langle A_0, h_k\rangle$ (and drop the indices $0, k$ for simplicity). Read W from right to left: $W = L_\ell L_{\ell-1} \dots L_1$ where $L_1 = A, \dots, L_\ell = A^{-1}$ are the "letters" of W, and $\ell = 2d + \sum |a_i|$ is its length; *e.g.* if $P(X) = X^3 - 2X$ then $W = A^{-1}A^{-1}A^{-1}hAAh^{-1}h^{-1}A$, $\ell = 9$ and $L_5 = A$. Finally, define the i-th left and right parts of W by $W = W_i^l L_i W_i^r$ (so that $W_\ell^l = W_1^r$ =empty word). What lemma 6.3 means is that each right subword W_i^r, $2 \le i \le \ell$, is well defined on $\mathbb{B}$, which allows us to construct W recursively; but the proof we gave of it implies moreover that for $i \le \ell - d$ each of these words sends $\mathbb{B}$ to some relatively compact subset of it. In particular, there exists $\delta > 0$ such that $W_i^r(\mathbb{B}) \subset (1-\delta)\mathbb{B}$ for every $i \in \{2, \dots, \ell - d\}$. Now, recall that by Cauchy's estimates, there is a constant C_δ depending only on δ and such that if ϕ belongs to $\mathfrak{B}$ then the supremum of $\|d^2\phi\|(z)$ on $(1-\delta)\mathbb{B}$ is at most $C_\delta\|\phi\|$, where $\|d^2\phi\|$ is seen as a map from $\mathbb{B}$ to the space of bilinear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ and endowed with the obvious norm. From this we get:

Lemma 6.4. *For A_0, P and U as in lemma 6.3 the transformation T that sends h_0 to h_1 is C^1 near Id; moreover its differential is given by the formula*

$$dT(h).\delta h = \sum_{k=1}^{\ell} s(L_k) dW_k^l.\delta h \circ W_k^r \tag{4}$$

where $s(A) = s(A^{-1}) = 0$, $s(h) = 1$, $s(h^{-1}) = -1$; in particular,

$$\alpha_1 = \sum_{k=1}^{d} a_k A^{-k} \circ \alpha_0 \circ A^k + o(\|\alpha_0\|) = P(Ad(A)).\alpha_0 + o(\|\alpha_0\|) \tag{5}$$

where $Ad(A)$ stands for the map $\alpha_0 \to A^{-1} \circ \alpha_0 \circ A$.

Proof. According to Lagrange's theorem, if ϕ and ψ are two elements of $\mathfrak{B}$ with $\psi(\mathbb{B})$ relatively compact in $\mathbb{B}$ (so that $\phi \circ (\psi + \delta\psi) \in \mathfrak{B}$ as well for any variation $\delta\psi$ of ψ that is small enough) then

$$\phi \circ (\psi + \delta\psi) = \phi \circ \psi + d\phi(\psi).\delta\psi + O(\sup_{\mathcal{D}} \|d^2\phi\|.\|\delta\psi\|^2) \ ;$$

the supremum norm of $d^2\phi$ in the right-hand side of this formula is to be taken on a sufficiently large domain $\mathcal{D}$, depending on ψ and $\delta\psi$. Now if $\|\psi\| + \|\delta\psi\| \leq 1 - \delta$ then $(1-\delta)\mathbb{B}$ can be taken for $\mathcal{D}$; in that case Cauchy's estimate yields:

$$\phi \circ (\psi + \delta\psi) = \phi \circ \psi + d\phi(\psi).\delta\psi + O(\|\phi\|.\|\delta\psi\|^2).$$

Using this, we obtain formula (4) by a straightforward recursion, which consists of computing one after the other the differentials of the "partial" operators

$$T_1(h_0) = A_0, T_2(h_0) = h_0 \circ A_0, \ldots, T_{\ell-1}(h_0) = AT(h_0), T_\ell(h_0) = T(h_0),$$

using the fact that each W_i^r sends $\mathbb{B}$ within $(1-\delta)\mathbb{B}$ to replace the quadratic differentials of h_0 and δh_0 by h_0 and δh_0 themselves in the estimates. In turn, formula (5) is a straightforward consequence of (4). □

We now show that for suitable P, the map T is a contraction of $\mathfrak{B}$. Choose some positive integer k. Any element α of $\mathfrak{B}$ has the form $\alpha(z) = Q(z) + \phi(z)$ where Q is a degree-k polynomial map and $(*)$ $\phi(z) \leq C|z|^{k+1}$ for some constant C; this is an $Ad(A)$-invariant decomposition of $\mathfrak{B}$ into a direct sum of closed vector subspaces. One readily computes the spectrum of $Ad(A)$ on the finite dimensional space of polynomials with degree $\leq k$; on the complementary space of functions ϕ, one has the majoration $\|Ad(A)\| \leq (\sup|\lambda_i^{-1}|)(\sup|\lambda_i|)^{k+1}$ which is a direct consequence of $(*)$. So, applying this to any k, we see that the spectrum of $Ad(A)$ on $\mathfrak{B}$ is the set of numbers

$$\lambda_i^{-1} \times \lambda_1^{a_1} \times \cdots \times \lambda_n^{a_n} \quad (i \in \{1, \ldots, n\},\ a_1, \ldots, a_n \in \mathbb{N}). \tag{6}$$

By the same argument, the spectrum of $P(Ad(A))$ is the set of values taken by P on the spectrum of $Ad(A)$. We now proceed to construct some P such that $P(Ad(A))$ is a contraction. In the following statement, recall that the a-stable space of a bounded operator T on a Banach space means the vector subspace of all vectors x such that the sequence $a^{-n}T^n(x)$ tends to zero.

Proposition 6.5. *There exists in $GL(n,\mathbb{R})$ an open and conjugacy-invariant subset $\mathcal{X}$ which is dense in the space of linear contractions and has the following property: if A belongs to $\mathcal{X}$ then there is some polynomial P such that $P(Ad(A))$ is a contraction of $\mathfrak{B}$; moreover there is an $a > 0$ such that the a-stable space of $P(Ad(A))$ in $\mathfrak{B}$ is the closed subspace $\mathfrak{B}_0$ of maps fixing 0, and $P(Ad(A))$ leaves invariant the complementary subspace of constant maps as well.*

Proof. In $\mathbb{C}^n$, we define the real-semialgebraic and conjugacy-invariant subset $\mathcal{C}$ by the conditions $(\lambda_1, \ldots, \lambda_n) \in \mathcal{C}$ iff $(\forall i)$ $|\lambda_i| < 1$ and $\{\lambda_1, \ldots, \lambda_n\} = \{\overline{\lambda_1}, \ldots, \overline{\lambda_n}\}$.

So, the quotient of $\mathcal{C}$ by the obvious action of the symmetric group $\mathfrak{S}_n$ on it identifies naturally with the space of conjugacy classes of diagonalizable contractions of $\mathbb{R}^n$. We now enunciate an obvious fact (we omit the proof):

Lemma 6.6. *The subset* $\mathcal{C}_0 = \{(\forall i)\ \lambda_i \in \mathbb{Q}[\sqrt{-1}]\}$ *of* $\mathcal{C}$ *is dense in* $\mathcal{C}$.

Observe that this lemma would still hold true if we replaced $\mathbb{Q}[\sqrt{-1}]$ with any non-real field number (or even with some nonalgebraic extension of $\mathbb{Q}$ as long as it is not contained in $\mathbb{R}$ and is conjugacy-invariant). Let us now show proposition 6.5. It clearly amounts to the statement that there exists a dense and open subset $\mathcal{Y}$ of $\mathcal{C}$ such that if $(\lambda_1,\dots,\lambda_n)$ belongs to $\mathcal{Y}$ then there exists a polynomial $P(X)\in\mathbb{Z}[X]$ with the property that

$$1 > \sup_{1\le i\le n} |P(\lambda_i^{-1})| \quad \text{and} \quad \inf_{1\le i\le n} |P(\lambda_i^{-1})| > \sup_{\lambda\in\mathcal{S}} |P(\lambda)|.$$

Since this property is clearly an open one, it suffices to construct a dense subset $\mathcal{Y}$ with the desired property. This will be done in three steps. We start with an arbitrary $(\lambda_1,\dots,\lambda_n)\in\mathcal{C}$ and make it satisfy the above property by only applying arbitrarily small perturbations to it.

First step: $1 > \sup_{1\le i\le n} |R(\lambda_i^{-1})|$ *and* $1 > \sup_{x\in\mathcal{S}} |R(x)|$ *for some* $R\in\mathbb{Z}[X]$. Start by noticing that $\overline{\{0\}\cup\mathcal{S}}$ is a compact subset K of $\mathbb{C}$. So, only a finite number of elements of $\mathcal{S}$ may have modulus greater or equal to one; call them $\mu_1,\dots,\mu_N$. Arrange the remaining elements of $\mathcal{S}$ in a sequence $\nu_1,\nu_2,\dots$; choose $a\in]0,1[$ such that $|\nu_k| < a$ for any $k\in\mathbb{N}$; finally, call F the finite set whose elements are the λ_i^{-1} and the μ_j.

Start by assuming that $(\lambda_1,\dots,\lambda_n)\in\mathcal{C}_0$: then, any element of $\mathcal{S}$ is in $\mathbb{Q}[\sqrt{-1}]$, and there exists a polynomial $Q(X)\in\mathbb{Z}[X]$ such that each λ_i^{-1} and each μ_j is a root of Q. For k large enough one also has $\sup_{|z|\le a} |z^kQ(z)| < a$, so that the polynomial $R(X) = X^kQ(X)$ sends each λ_i^{-1} and all of $\mathcal{S}$ within the disk $|z|\le a$. It can be seen that the property "$R(F\cup S\cup\{0\})\subset\{|z|<1\}$" is stable by perturbations of $(\lambda_1,\dots,\lambda_n)$. So, there is a dense and open subset $\mathcal{Z}$ of $\mathcal{C}$ such that if $(\lambda_1,\dots,\lambda_n)\in\mathcal{Z}$ then there exists some $R\in\mathbb{Z}[X]$ such that

$$1 > \sup_{1\le i\le n} |R(\lambda_i^{-1})| \quad \text{and} \quad 1 > \sup_{\lambda\in\mathcal{S}} |R(\lambda)|.$$

Second step: *no* $R(\lambda_i^{-1})$ *vanishes, or has the same modulus as a* $R(x)$ *with* $x\in\mathcal{S}$. We observe that the same R works for every element of a sufficiently small neighborhood of $(\lambda_1^{-1},\dots,\lambda_n^{-1})$. We now assume $(\lambda_1^{-1},\dots,\lambda_n^{-1})$ to belong to $\mathcal{Z}$ and let ϕ_i denote $R(\lambda_i^{-1})$ for short; similarly, the sequence ψ_j is the sequence of all values taken by R over $\mathcal{S}$. Now if we multiply each λ_i by the same number $1+\varepsilon$ for some small $\varepsilon\in\mathbb{R}$ (which does not take $(\lambda_1,\dots,\lambda_n)$ out of $\mathcal{C}$ since $\mathcal{C}$ is a semi-algebraic subset of a real cone) then we can manage to have every λ_i transcendent at the same time. In that case no ϕ_i will vanish; using lemma 6.6

we can next assume that each λ_i is in $\mathbb{Q}[\sqrt{-1}]$ again (so, we've put $(\lambda_1,\dots,\lambda_n)$ outside of $\mathbb{Q}[\sqrt{-1}]$ and back again; the benefit is that now no ϕ_i vanishes).

Since $\inf_{i=1}^n |\phi_i| > 0$, there is only a finite number ℓ of the ψ_j which are outside the open disk $|z| < \inf_{1\le i\le n}|\lambda_i|$. Let $(\theta_k)_{k\in\mathbb{N}}$ denote the rest of the ψ_j, so that for all $i \le n$ and $k \in \mathbb{N}$ one has $|\phi_i| > |\theta_k|$. Remark, again, that $\{0\}\cup\{\theta_k\}$ is a compact set, so that it remains in the open disk $\mathbb{D} = \{z/|z| < \inf_{1\le i\le n}|\phi_i|\}$ as long as the λ_i do not move too much. Next multiply the λ_i by some $r \in \mathbb{Q}$ (this leaves them in $\mathbb{Q}[\sqrt{-1}]$). We claim that for generic r the ψ_j (which are outside $\mathbb{D}$) will all be outside the closure of $\mathbb{D}$ as well. Indeed, one has $\phi_i = R(\lambda_i^{-1})$ and $\psi_j = R(\lambda_j^{-1}\lambda_1^{a_1}\dots\lambda_n^{a_n})$ with $\sum a_i > 0$, so that the rational functions $|\phi_i|^2$ and $|\psi_j|^2$ of the variables $\lambda_1,\dots,\lambda_n$ cannot coincide. Therefore the ψ_j will have to move away from the boundary of $\mathbb{D}$ while r varies in $\mathbb{Q}$, either to get inside $\mathbb{D}$ (in which case ℓ diminishes and the ψ_i involved become "new members" of the family θ_k) or they will get outside the closure of $\mathbb{D}$. Then, the same argument yields more: by conveniently choosing r we may assume that no $|\phi_i|$ is equal to some $|\psi_j|$.

Third step: $\inf_{1\le i\le n}|P(\lambda_i^{-1})| > \sup_{\lambda\in\mathcal{S}}|P(\lambda)|$ *for some* $P = Q\circ R$.

Note that since the λ_i are in $\mathbb{Q}[\sqrt{-1}]$, the ψ_i also are. Call $\alpha_1,\dots,\alpha_p$ the rational ones, and $\beta_1,\overline{\beta_1},\dots,\beta_q,\overline{\beta_q}$ the quadratic ones. Consider the following polynomial

$$Q(X) = aX^b\prod_1^p(X-\alpha_i)\prod_1^q(X^2 - 2\Re(\beta_i) + |\beta_i^2|)$$

with $a,b\in\mathbb{N}$. For suitable $a\neq 0$, Q lies in $\mathbb{Z}[X]$; let such an a be given. For any b, Q sends the ψ_i to zero; finally for $b=0$, Q will send the θ_k within some ball $|z| < R$ and the ϕ_i inside a certain annulus $\rho_- < |z| < \rho_+$. There exist moreover numbers κ_-,κ_0 and κ_+ such that $\theta_k < \kappa_- < \kappa_0 < \phi_i < \kappa_+ < 1$ for any k and i as a result of steps 1 and 2; so, for given b, Q will send θ_k and ϕ_i respectively in the disk of radius $R\kappa_-^b$ and in the annulus of inner radius $\kappa_0^b\rho_-$ and outer radius $\kappa_+^b\rho_-$. So if b is large enough the polynomial $P = Q\circ R$ satisfies the desired property. □

Finally, recall the so-called *stable manifold theorem* of Hadamard and Perron. This theorem possesses many versions;[2] ours will be:

Stable manifold theorem. *If $\mathfrak{B}$ is a Banach space and T is a linear, continuous endomorphism of $\mathfrak{B}$ whose spectrum does not meet the circle $|z| = a$ $(a>0)$ in the complex plane then for any Lipschitz germ f from $(\mathfrak{B},0)$ to itself with small Lipschitz constant, there exists an ε such that in the ball $\|x\| < \varepsilon$, the points x such that $a^{-n}(T+f)^{\circ n}(x)$ tends to 0 form a (germ of a) continuous manifold M_a^s, usually called the* a-stable manifold *of $T+f$. Moreover, if the spectrum of T does*

[2] As Anosov once cynically said, every odd year someone comes up with a "new" proof of the Hadamard–Perron theorem which is essentially the proof of Perron or that of Hadamard. The inventive proof in [21] is an exception to this rule.

not meet the annulus with inner *radius a and* outer *radius a^r (a condition which is tautologic if $a < 1$) then M is of class C^r.*

We now show theorem 6.2. Let A belong to $\mathcal{X}$ in proposition 6.5; choose P so that T is a contraction and fix a such that the a-stable manifold of T is a complement in $\mathfrak{B}$ of the space of translations (this can be done by proposition 6.5). Since T is a contraction there is an $\varepsilon > 0$ such that for $\|\alpha_0\| < \varepsilon$ the sequence h_k tends to Id uniformly on $\mathbb{B}$ but does so slowlier than a^k does because $\alpha(0) \neq 0$. Then by the assumption on M_a^s, this sequence is equivalent to that of constant maps $h_k(0)$, so that $h_k(0)$ does not vanish and if one takes $p_k = [\|h_k(0)\|^{-1}]$ in proposition 6.1, then h_k tends uniformly on $\mathbb{B}$ to a constant map with unit norm. So, theorem 6.2 holds true.

7. From one field to many more

In this paragraph we show the following assertion:

Theorem 7.1. *Assume that in some local chart (U, ϕ) of M with origin $m = \phi(0)$, the diffeomorphisms f and g respectively write $f(x) = Ax$ with $A \in GL(n, \mathbb{R})$ a nonresonant diagonalizable contraction, and $g(x_1, \dots, x_n) = (x'_1, \dots, x'_n)$ with*

$$x'_i = a_i + \sum_{j=1}^{n} b_i^j x_j + \sum_{j,k=1}^{n} c_i^{jk} x_j x_k + o(|x|^2) \tag{7}$$

and each of the tensors a_i, b_i^j, c_i^{jk} generic in the sense of Zariski.[3] *Then either $\mathfrak{g}_\Lambda(m) = \{0\}$ or $\mathfrak{g}_\Lambda(m) = \mathfrak{g}(m)$.*

First recall some useful properties of nonresonant linear maps:

Proposition 7.2. *Let $A \in GL(n, \mathbb{R})$ be nonresonant and diagonalizable. Then any germ of a smooth real analytic manifold through 0 which is A-invariant is a germ of a linear subspace.*

Proof. A germ Σ_0 of a smooth analytic manifold through $0 \in \mathbb{R}^n$ is the intersection of $\mathbb{R}^n$ with a germ Σ of a smooth holomorphic manifold through $0 \in \mathbb{C}^n$; if Σ_0 is A-invariant then so will be Σ. In some basis of $\mathbb{C}^n$, A is diagonal; if the basis is well chosen the tangent space to Σ at 0 is spanned by its p first vectors so that Σ has a parametrization $x_{p+k} = \phi_k(x_1, \dots, x_p)$ with $1 \leq k \leq n-p$ and each ϕ_k holomorphic. Expand ϕ_k as a power series:

$$\phi_k = \sum_{i \geq 1}^{j_1 + \cdots + j_p = i} a(i)_k^{j_1 \dots j_p} x^{j_1} \dots x^{j_p}$$

[3] By writing this, we implicitly assume that $g(0)$ belongs to the domain of U.

and write the condition for Σ to be invariant,

$$\lambda_{p+k}\phi_k = \sum_{i\geq 1}^{j_1+\cdots+j_p=i} a(i)_k^{j_1\ldots j_p}\lambda_1^{j_1}\ldots\lambda_p^{j_p}x^{j_1}\ldots x^{j_p}.$$

Identifying coefficients and using the fact that λ_{p+k} is equal to none of the $\lambda_1^{j_1}...\lambda_p^{j_p}$, we obtain the desired result: each ϕ_k is zero, so that Σ coincides with its tangent space at 0. □

Let A be as above, and fix some A-invariant decomposition $\mathbb{R}^n = E \oplus F$. Let $\mathcal{F}$ be a germ of an analytic regular A-invariant foliation at $0 \in \mathbb{R}^n$; by proposition 7.2 the leaf of $\mathcal{F}$ through the origin must be an A-invariant subspace. Assume it's E. Then locally, tangent spaces to leaves of $\mathcal{F}$ remain transverse to the verticals $\{e\} + F$ ($e \in E$), so that one may project E onto F parallelly to them at each point, thus giving rise to an analytic field ω of 1-forms on E with values in F which is A-invariant and vanishes along E.

Proposition 7.3. *In fact $\omega = 0$, so that each leaf of $\mathcal{F}$ is an affine space parallel to E.*

The proof works on the same lines as the previous one (extend to $\mathbb{C}^n$, expand ω as a power series, compare with $A^*\omega$ and conclude). Here is a third consequence of nonresonance when A is a contracting map:

Proposition 7.4. *Let $X_1, \ldots, X_n$ be n germs of vector fields defined at $0 \in \mathbb{R}^n$ and linearly independent at this point. Let $\mathfrak{h}$ be the closed Lie algebra they span; assume that $\mathfrak{h}$ is A-invariant and that $\|A\| < 1$: then, $\mathfrak{h}$ contains every constant field.*

Proof. For the sake of simplicity, we let A be diagonal (we hope the reader will grant that pairs of conjugate complex eigenvalues, while they make the proof more abstruse, bring no real additional difficulty). The effect of this assumption is that for every $d \in \mathbb{N}$, A is diagonal on the finite dimensional vector space E_0 of polynomial vector fields with degree d. We let δ be equal to $\inf_i |\lambda_i^{-1}|$ and d be the smallest integer such that $\lambda_i^{-1}\lambda_1^{a_1}\ldots\lambda_n^{a_n} < \frac{\delta}{2}$ for all i and $a_1, \ldots, a_n$ such that $\sum a_i > d$. By assumption, after replacing the X_i by linear combinations, we can assume that $A.X_i(0) = \lambda_i X_i(0)$. Endow E_0 with a scalar product and let X_i^{0k} denote $c_i^k A^k X_i$ where the constant c_i^k is chosen so that the projection P_i^{0k} of X_i^{0k} on E_0 has unit norm: then because of our definition of d, the remainder $X_i^{0k} - P_i^{0k}$ can be neglected w.r.t. P_i^{0k}, so that X_i^{0k} converges uniformly near 0 to a nontrivial polynomial vector field which we call Y_i^0. Let F_1 be the vector subspace of E_0 spanned by $Y_1^1, \ldots, Y_n^1$. Since A is diagonal on E_0, there is an A-invariant subspace E_1 complementary to F_1. By induction, once we have defined an A-invariant subspace F_p we consider its complementary subspace E_p in E and let

X_i^{p0} be the projection on E_p of X_i parallel to F_p (more precisely, of the polynomial part of degree d of X_i). As long as at least one of the X_i^{p0} is nonzero, we use it to define a sequence $c_i^k A^k X_{ip} = X_i^{pk}$ and define c_i^k by the condition that $\|X_i^{pk}\| = 1$. Then X_i^{pk} converges to Y_i^p with $Y_i^p \in E_p$ and $\|Y_i^p\| = 1$; the set of the Y_i^p thus obtained spans together with F_p a vector subspace of E which we call F_{p+1} and is strictly larger than F_p. So, E_0 being finite dimensional, there must be a p such that no more Y_i^p can be constructed, i.e. every X_i projects trivially on E_p parallel to F_p. Then, F_p must contain all constant fields since the $X_i(0)$ span $\mathbb{R}^n$. But F_p is a subset of $\mathfrak{h}$, so the proof is done. □

Proof of theorem 7.1. We assume that $\mathfrak{g}_\Lambda(m)$ is nontrivial. Then it is spanned by a finite number of analytic germs of vector fields at m, whose images in the chart we denote by $X_1, \dots, X_k$. These fields are linearly independent as germs, but *a priori* their evaluation at some point must not necessarily be so. Our assumptions will first serve to rule out the possibility that the X_i be nontransverse somewhere. We reduce this possibility to the absurd by letting Σ be the germ of analytic manifold defined on U as the locus where $X_1, \dots, X_k$ are not transverse: by construction of $\mathfrak{g}_\Lambda$ this germ is invariant by the restriction of Λ to the image of U, and since A contracts U, one sees that Σ passes through zero. Then there is an analytic stratification $\Sigma_0 \subset \cdots \subset \Sigma_p = \Sigma$ by A-invariant germs through 0, uniquely defined by the property that Σ_0 is smooth and Σ_k is the locus of singular points of Σ_{k+1}. This stratification must also be g-invariant, so that Σ_0 contains both 0 and $g(0)$ which, since the a_i are generic, does not lie on any proper A-invariant subspace. The same can be said of every point $A^n g(0)$, and this sequence tends to 0 nontrivially, so that Σ_0 must have strictly positive dimension. Then, it is an A-invariant subspace by proposition 7.2, and it is not contained in any proper A-invariant one so it's all of U, which is absurd since the X_i have been chosen so as to be independent as global sections of $\mathfrak{g}(U)$.

By what precedes, the X_i are locally independent at every point, so that they span a germ of a regular foliation $\mathcal{F}$ near 0, which is A-invariant. Then by proposition 7.3 this foliation is parallel to some A-invariant subspace E of $\mathbb{R}^n$ which is nonzero. So, $dg(0)$ must send E to within itself, but since b_i^j is generic, this implies that E is all of $\mathbb{R}^n$ because the matrices A and $B = (b_i^j)$, being in general position, share no proper invariant subspace of $\mathbb{R}^n$. Therefore g_Λ contains all constant vector fields by proposition 7.4. Next, one computes the image of those through g^*,

$$g^*\Big(\frac{\partial}{\partial x_j}\Big) = \sum_{i=1}^{n} (b_i^j + (1+\delta_i^j)c_i^{jk} x_k + \dots)\frac{\partial}{\partial x_i} = Y_j$$

with δ_i^j the Kronecker symbol. We know that $g_\Lambda(0)$ contains every constant field, so it contains $Z_j = Y_j - \sum b_i^j \frac{\partial}{\partial x_i}$ which vanishes at 0. Since c_i^{jk} is a generic tensor, the linear part of the Z_j span the Lie algebra of all linear fields, because two generic linear fields do so (the condition is that at least one of them has nonzero trace and no vector subspace of $\mathfrak{sl}(n,\mathbb{R})$ is both $\mathrm{ad}(X)$- and $\mathrm{ad}(Y)$-invariant, see [5] Section 2).

Now, $\mathfrak{g}_\Lambda(m)$ contains every affine field up to second order terms, so we can pick in it a field which writes $X = \sum x_i \frac{\partial}{\partial x_i} + \ldots$ where the dots represent the higher-order terms. Such a field is linearizable in a neighborhood of the origin. Fix a local chart where X is linear and defined on the unit ball $\mathbb{B}$; let Y be any field in $\mathfrak{g}_\lambda(m)$ and let Y_k be the k-th order homogeneous term in the Taylor series for Y: $Y = Y_{-1} + Y_0 + \ldots$. We notice that the flow ϕ^t of $-X$ is defined on $\mathbb{B}$ for all $t \geq 0$ and sends $\mathbb{B}$ within smaller and smaller subballs, so that the field $Y_t = (\phi^t)^*(Y)$ lies in $\mathfrak{g}_\Lambda(\mathbb{B})$ for t large enough by item (1) of proposition 5.2. We can next compute $Y_t = \sum_{k=-1}^{\infty} e^{-kt} Y_k$. We then see that $e^{-t} Y_t$ tends to Y_{-1} on $\mathbb{B}$, so that $\mathfrak{g}_\Lambda(\mathbb{B})$ (which contains both Y and Y_{-1}) contains their difference $Y_0 + Y_1 + \ldots$ Replacing Y by $Y - Y_{-1}$ and taking this time the limit of Y_t we see that Y_0 lies in $\mathfrak{g}_\Lambda(\mathbb{B})$ too; by an easy induction we also see that every Y_k is in $\mathfrak{g}_\Lambda(\mathbb{B})$ as well. So, for any germ Y of $\mathfrak{g}_\Lambda(m)$ and any $k \geq 1$ the k-th polynomial term in the Taylor series for Y lies in $\mathfrak{g}_\Lambda(\mathbb{B})$. This means that $\mathfrak{g}_\Lambda(\mathbb{B})$ is a *graded Lie algebra*. Moreover it is an *irreducible and transitive* one (see the definitions in the survey [7]); the list of those was written down by E. Cartan: in view of the list, there are only two possibilities; $\mathfrak{g}_\Lambda(\mathbb{B})$ may be all of $\mathfrak{g}(\mathbb{B})$, or a subalgebra of the Lie algebra of $PGL(n+1,\mathbb{R})$ (two subcases). But the last case has finite dimension at most n^2+2n, whereas for the choice of g in (7) there are $n^2+n+\frac{n^2(n+1)}{2}$ degrees of freedom: so for generic g, this last case is excluded. $\square$

8. Proofs, and final remarks

We start this paragraph with an easy fact:

Proposition 8.1. *The closure of any Λ-orbit contains a source or a sink of f.*

Proof. The basins of attraction/repulsion of sinks/sources of f are cells whose complementary set is a closed subset Σ of M with empty interior, consisting of those points who go from a saddle to another along the intersection of their stable and unstable manifolds. If some $m \in M$ lies on Σ then the closure of its orbit contains a saddle β of f. This saddle is sent by g outside Σ, necessarily within the basin of a source or sink s of f. Then the closure of $\Lambda.m$ contains β and that of $\Lambda.\beta$ contains s. $\square$

The proof of theorem A is now a rather straightforward consequence of what has already been obtained. By proposition 8.1 the ρ-orbit of any $m \in M$ meets the basin of some source or sink s of f. Near s, define a linearizing map ϕ which turns f into a diagonal matrix A, then apply theorem 6.2 to A and $h = \phi \circ f \circ g^{-1} \circ \phi^{-1}$:

we see that $\mathfrak{g}_\Lambda$ is nontrivial at s, and contains the image $\phi(X)$ of a nonzero constant field. By theorem 7.1, it must then coincide with $\mathfrak{g}$ in some neighborhood U of s. Thus $\mathfrak{g}_\Lambda(m) = \mathfrak{g}(m)$ by proposition 5.2, so that $\mathfrak{G}_\Lambda = \mathfrak{G}$ by proposition 5.1; this is another way of enunciating theorem A.

Next we show theorem C. For that, we consider another action ρ' near ρ with generators f' and g' and assume that some homeomorphism c exists such that $c \circ f = f' \circ c$ and $c \circ g = g' \circ c$. Just above, starting with $A = \phi \circ f \circ \phi^{-1}$ and $h = \phi \circ f \circ g^{-1} \circ \phi^{-1}$ we constructed a vector field X defined near s; the same construction done with f', g' and a linearizing map ϕ' for f' will yield a field X' defined near s' (the source of f' which is near f: remember that f is structurally stable !) Then, since c sends by assumption f to f' and g to g' it must conjugate the flow of X with that of X'. Now consider some $\gamma \in \Gamma$ and apply the same construction to the action ρ^γ such that $\rho^\gamma(\gamma') = \rho(\gamma \circ \gamma' \circ \gamma^{-1})$: we see that c sends $\rho(\gamma)_*(X)$ to $\rho'(\gamma)_*(X')$ as well. But then, since $\mathfrak{g}_\Lambda(\mathbb{B}) = \mathfrak{g}(\mathbb{B})$ for some neighborhood $\mathbb{B}$ of s it is possible within the infinite family of fields $\rho(\gamma)_*X$ to find n of them, $X_1, \ldots, X_n$ which are transverse at s and satisfy $[X_i, X_j] = 0$. Integrating the local flows of these fields near s will provide us with a new local chart $\psi : (\mathbb{R}^n, 0) \to (M, s)$. Then, the flows of the images $X'_1, \ldots, X'_n$ of these fields by c will give us a second local chart $\psi' : (\mathbb{R}^n, 0) \to (M, s')$ with the properties that both charts are analytic and $\psi' = c \circ \psi$. This means of course that c is analytic in a neighborhood of s.

We would now like to conclude this paper with a few natural questions.

Question 8.2. *Is the proximity assumption on ρ and ρ' necessary for theorem C to hold?*

Question 8.3. *In Section 6 we use the dynamical properties of transformations with a fixed point and a jacobian that lies in the Poincaré domain. Can something still be done when the fixed points of f and g lie within the Siegel domain?*

A complete answer to this question would be a prerequisite for the next one:

Question 8.4. *What can be said under the restrictive assumption that Γ preserves a volume form (or a symplectic structure, or whatever)?*

Finally, a small generic diffeomorphism of the 2-sphere has at least one source and one sink, so maybe our methods could be improved to yield:

Conjecture 8.5. *In the set of couples of $\mathrm{Diff}^\omega(\mathbb{S}^2)$ there is an open neighborhood of (Id, Id) and a dense and open subset U of it such that if (f, g) is in U then it spans a rigid group of transformations which acts minimally on every $J^k(\mathbb{S}^2)$.*

As for the answer to these questions, we admit that we haven't got a clue.

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