Comment. Math. Helv. 78 (2003) 394–417
0010-2571/03/020394-24

Commentarii Mathematici Helvetici

Restriction map in a regular reduction of $\mathbf{SU}(n)^{2g}$

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Abstract. The quasi-Hamiltonian reduction of $\mathbf{SU}(n)^{2g}$ at a regular value, in the centre of $\mathbf{SU}(n)$, of the moment map is isomorphic to a moduli-space of semi-stable vector bundles over a Riemann surface. We describe the restriction map from the equivariant cohomology of $\mathbf{SU}(n)^{2g}$ to the cohomology of the moduli space in terms of natural multiplicative generators of these cohomologies.

Mathematics Subject Classification (2000). 53D20, 14H60.

Keywords. Moduli spaces, symplectic reduction, quasi-Hamiltonian spaces, $\mathbf{SU}(n)$.

Motivations

All cohomologies will be taken with coefficients in the field $\mathbf{Q}$ of rational numbers. For a compact connected Lie group G, we denote $EG \longrightarrow BG$ the universal principal G-bundle. If G acts on a manifold M, we denote $(M)_G$ the space $M \times_G EG$. The equivariant cohomology $H^*_G(M)$ of M with respect to the action of G is by definition the Čech cohomology of $(M)_G$. For an account of equivariant cohomology see [6] and [14].

Let g be an integer bigger than 1. Let π be the group

$$\pi = \langle a_1, b_1, \dots, a_g, b_g, c; \prod_{k=1}^{g} [a_k, b_k] = c, c^n = 1 \rangle.$$

Let n and d be integers with n bigger than 1 and let ζ be the n-th root of unity $\zeta = e^{-2\pi i \frac{d}{n}}$. Put $\beta = \zeta \mathbf{I}$, where $\mathbf{I}$ is the identity matrix in the special unitary group $\mathbf{SU}(n)$. We define

$$S_\beta = \{\rho \in \mathrm{Hom}(\pi, \mathbf{SU}(n)) \mid \rho(c) = \beta\}$$

the space of $\mathbf{SU}(n)$-representations of π such that β is the image of c. Because β is in the centre of $\mathbf{SU}(n)$, the group $\mathbf{SU}(n)$ acts on S_β and its quotient

$$\mathfrak{m}_\beta = S_\beta / \mathbf{SU}(n)$$

is the moduli space of $\mathbf{SU}(n)$-representations of π that send c to β.

Narasimhan and Seshadri [17] have shown that $\mathfrak{m}_\beta$ is isomorphic to the moduli space of holomorphic semi-stable vector bundles of rank n, degree d, and fixed determinant over a compact Riemann surface X of genus g. For d and n co prime, $\mathfrak{m}_\beta$ is compact and smooth. In this case, Atiyah and Bott [1] showed that this space is symplectic, proposed a family of multiplicative generators of its cohomology and gave an inductive formula (on the rank n) for the Betti numbers of $\mathfrak{m}_\beta$. Their method consists in studying an infinite dimensional Hamiltonian space. In 1993, Huebschmann [8] and Jeffrey [10] independently gave a group cohomology construction of the symplectic form on $\mathfrak{m}_\beta$ (their results are summarised in a joint paper [9]). In 1998, Alekseev, Malkin and Meinrenken [2] showed that Huebschmann and Jeffrey's construction fits in a more general setting: one can get the moduli space $\mathfrak{m}_\beta$ (and many others, moduli spaces of flat connections on a principal bundle) as the Marsden–Weinstein reduction of a quasi-Hamiltonian space. This space is $\mathbf{SU}(n)^{2g}$, it is relatively simple in contrast with the usual descriptions of $\mathfrak{m}_\beta$ as a Hamiltonian reduction. A quasi-Hamiltonian (or q-Hamiltonian for short) space is a Hamiltonian space with a group valued moment map. Its 2-form is not symplectic in general but the Marsden–Weinstein reduction is well defined and the reduced space is symplectic.

An important result about Hamiltonian spaces is the

Theorem 0.1 (Kirwan)**.** *Let M be a symplectic manifold. Assume G is a compact Lie group acting symplectically on M and assume there exists a moment map ϕ for this action. Let 0 be the null vector of the dual of the Lie algebra of G. The restriction map*

$$H^*_G(M) \longrightarrow H^*_G(\phi^{-1}(0))$$

is surjective.

It is a natural question to ask if this theorem is still true for q-Hamiltonian spaces. It is quite easy to see that the answer is no. For example, to get $\mathfrak{m}_\beta$, one considers $\mathbf{SU}(n)^{2g}$ with moment map μ

$$\begin{array}{ccc} \mathbf{SU}(n)^{2g} & \longrightarrow & \mathbf{SU}(n) \\ (A_1, B_1, \ldots, A_g, B_g) & \longmapsto & \prod_{k=1}^{g}[A_k, B_k] \end{array}$$

the product of the commutators and a certain 2-form (see [2] for more details). Then the reduced space at β being symplectic and compact, its degree two cohomology (which is isomorphic to $H^2_{\mathbf{SU}(n)}(\mu^{-1}(\beta))$) contains a non trivial class whereas $H^2_{\mathbf{SU}(n)}(\mathbf{SU}(n)^{2g}) = \{0\}$. Thus the map

$$r : H^*_{\mathbf{SU}(n)}(\mathbf{SU}(n)^{2g}) \longrightarrow H^*_{\mathbf{SU}(n)}(\mu^{-1}(\beta))$$

is not surjective.

Our aim is to give a description of this last map r (Theorem 5.1) when d and n are co prime. Note that in [3], a theorem of localisation in the context of quasi-

Hamiltonian spaces is given. It may be interesting to see how our theorem could be used to apply this localisation theorem to the reduction of $\mathbf{SU}(n)^{2g}$ at β.

This paper is organised in the following way. Section 2 gives a (very short) review of the prerequisites on q-Hamiltonian spaces and semi-stable bundles. In particular, Narasimhan and Seshadri's theorem (see Theorems 2.9 and 2.13) is used throughout this article to identify $\mathfrak{m}_\beta$ with $\mu^{-1}(\beta)/\mathbf{SU}(n)$ and $H^*(\mathfrak{m}_\beta)$ with $H^*_{\mathbf{SU}(n)}(\mu^{-1}(\beta))$. In Section 3, we give a construction of a universal bundle on $\mathfrak{m}_\beta \times X$, we then recall how Biswas and Raghavendra [4] use this bundle to define a set $\{a_k, b_{k,j}, d_k, 2 \leq k \leq n, 1 \leq j \leq 2g\}$ of canonical multiplicative generators of the cohomology of $\mathfrak{m}_\beta$ (Theorem 3.4). In the next section we define a bundle on $\mathbf{SU}(n)^{2g} \times X - \{point\}$ and use it to get a set $\{c_k, \sigma_{k,j}, 2 \leq k \leq n, 1 \leq j \leq g\}$ of multiplicative generators for the equivariant cohomology of $\mathbf{SU}(n)^{2g}$ (Theorems 4.4 and 4.6). Finally in Section 5 we prove the

Theorem 5.1. *The restriction map*

$$r : H^*_{\mathbf{SU}(n)}(\mathbf{SU}(n)^{2g}) \longrightarrow H^*_{\mathbf{SU}(n)}(\mu^{-1}(\zeta\mathbf{I}))$$

is given by

$$\begin{aligned} r(c_k) &= a_k \;\; \textit{for } k = 2, \ldots, n \\ r(\sigma_{k,j}) &= b_{k,j} \;\; \textit{for } k = 2, \ldots, n,\ j = 1, \ldots, 2g. \end{aligned}$$

1. Acknowledgement

I would like to thank M. Audin and F. Kirwan for their help in writing this article. I would also like to thank J. Huebschmann and L. Jeffrey for sending their articles to me, they have been very useful. Finally, I would like to thank the referee for his reading and his comments.

2. Prerequisites

In paragraph 2.1, we recall the definition of semi-stability for holomorphic vector bundles and state Narasimhan and Seshadri's theorem (Theorem 2.9). Then in §2.2, we give the definition of a q-Hamiltonian space and restate Narasimhan and Seshadri's result in the language of q-Hamiltonian spaces (Theorem 2.13).

2.1. Semi-stable bundle

The following constructions are due to Narasimhan and Seshadri [17]. Apart from the proof of Proposition 2.1, everything in this paragraph is from their article.

Let X be a Riemann surface of genus g, $g \geq 2$. Fix a point x_0 of X. We will first give a construction of a ramified covering $Y \longrightarrow X$ used in [17].

Proposition 2.1. *There exists a simply connected covering*

$$p : Y \longrightarrow X$$

with only one point of ramification x_0 of order n. Outside of this point, the map

$$Y - \{p^{-1}(x_0)\} \longrightarrow X - \{x_0\}$$

is a covering with group

$$\pi = \langle a_1, b_1, \ldots, a_g, b_g, c; c = \prod_{k=1}^{g} [a_k, b_k], c^n = 1 \rangle.$$

Proof. We start by constructing Y, then we show that it is simply connected. Let D be an open neighbourhood of x_0 biholomorphic to an open disc in $\mathbf{C}$ centred at zero. Let $D' = D - \{x_0\}$. The fundamental group of $X' = X - \{x_0\}$ has a presentation

$$\pi_1(X') = \langle a_1, b_1, \ldots, a_g, b_g \rangle,$$

such that the element $\prod_{k=1}^{g} [a_k, b_k]$ is the class of a small circle γ included in D' and going counter clockwise around x_0. Let π be the group

$$\pi = \langle a_1, b_1, \ldots, a_g, b_g, c; c = \prod_{k=1}^{g} [a_k, b_k], c^n = 1 \rangle.$$

The natural surjective map

$$\pi_1(X') \longrightarrow \pi$$

defines a galoisian covering

$$p : Y' \longrightarrow X'$$

with group π. Fix a point x_1 of D'. We take it as the base point for the fundamental groups of X, X' and D'. Let us decompose $p^{-1}(D')$ in its different connected components

$$p^{-1}(D') = \bigcup_{\alpha \in \Lambda} V_\alpha.$$

Each

$$V_\alpha \xrightarrow{p} D'$$

is a connected covering. As D' is a disc minus its centre, this last covering group is generated by the element in π corresponding to the loop γ. The connected covering of a disc with its centre removed is either the upper half complex plane with the exponential as a projective map or a disc with its centre removed and a projection map of the type $z \longmapsto z^m$, where m is a positive integer. Here, the class of γ acts as c, thus V_α is a disc with its centre removed and, for each α,

$$V_\alpha \xrightarrow{p} D'$$

is the map $z \longmapsto z^n$. Let

$$Y = (Y' \bigcup_{\alpha \in \Lambda} (D, \alpha))/\sim$$

where for $y \in Y'$ and $(x, \alpha) \in (D, \alpha)$

$$y \sim (x, \alpha) \text{ if and only if } p(x) = y.$$

The natural projection

$$Y \longrightarrow X$$

is a covering with a unique ramification point at x_0 with order n. We now have to check that Y is simply connected. We get Y' from Y by removing a discrete set of points. Hence the map

$$\pi_1(Y') \longrightarrow \pi_1(Y)$$

is surjective. The sequence

$$\{1\} \longrightarrow \pi_1(Y') \longrightarrow \pi_1(X') \longrightarrow \pi \longrightarrow \{1\}$$

is exact. The kernel of $\pi_1(X') \longrightarrow \pi$ is the normal subgroup generated by c^n. Let a in $\pi_1(X')$ be the class of a loop $\eta : [0, 1] \longrightarrow X'$. The class of γ^n is c^n. Let us lift $\eta \cdot \gamma^n \cdot \eta^{-1}$ in Y'. To do so we have to take a lift $\widetilde{\eta}$ of η in Y' and then take a lift $\widetilde{\gamma}^n$ of γ^n satisfying $\widetilde{\gamma}^n(0) = \widetilde{\eta}(1)$. The loop we wanted is $\widetilde{\eta} \cdot \widetilde{\gamma}^n \cdot \widetilde{\eta}^{-1}$. There exists α such that

$$\widetilde{\gamma}^n \subset V_\alpha,$$

thus $\widetilde{\gamma}^n$ is homotopic to the constant loop in Y. The image of $a \cdot c^n \cdot a^{-1}$ by $\pi_1(Y') \longrightarrow \pi_1(Y)$ is 1, hence the image of $\pi_1(Y') \longrightarrow \pi_1(Y)$ is $\{1\}$ and

$$\pi_1(Y) = \{1\}.$$

□

Choose a y_0 in $p^{-1}(x_0)$. In the presentation

$$\pi = \langle a_1, b_1, \ldots, a_g, b_g, c; \prod_{k=1}^{g} [a_k, b_k] = c, c^n = 1\rangle$$

of π, we can assume that c is a generator of the isotropy group π_{y_0} of y_0.

For a representation $\rho : \pi \to \mathbf{GL}(n)$ of π, we denote $E_\pi(\rho)$ the vector bundle

$$Y \times \mathbf{C}^n \longrightarrow Y$$

with the action:

$$\begin{array}{ccc} \pi \times (Y \times \mathbf{C}^n) & \longrightarrow & Y \\ (\gamma, (y, v)) & \longmapsto & (\gamma \cdot y, \rho(\gamma) v). \end{array}$$

Let $\mathbf{E}$ be the sheaf of germs of holomorphic sections of $E_\pi(\rho)$. The group π acts on the image sheaf $p_*(\mathbf{E})$. Let $p_*^\pi(\mathbf{E})$ be the subsheaf of π-invariant elements of $p_*(\mathbf{E})$. It is a rank n locally free sheaf of $\mathbf{O_X}$-modules. It defines a holomorphic vector bundle, $p_*^\pi(E_\rho)$, of rank n on X. A set of transition functions is obtained in the following way. Let $\{U_i\}_{i=0}^m$ be a finite open covering of X satisfying:

(1) all non-empty intersections of sets of the type U_i is contractible,
(2) $x_0 \in U_0$ and $\bigcup_{i=1}^m U_i = X - \{x_0\}$,
(3) there exist discs $\{D_i\}_{i=0}^m$ in Y such that $y_0 \in D_0$ and U_0 is the quotient of D_0 by π_{y_0}, the restriction $p|_{D_i}$ is an homeomorphism of D_i with U_i, for all non zero i.

For each triplet i, j, k, choose a connected component $W_{ij,k}$ of $p^{-1}(U_i \cap U_j) \cap D_k$. If $U_i \cap U_j$ is not empty, we denote $\gamma_{i,j}$ the element of π satisfying $\gamma_{i,j} W_{ij,j} = W_{ji,i}$. According to [17, p. 550] :

Proposition 2.2. *On each $\{U_i\}_{i=0}^m$, the bundle $p_*^\pi(E_\rho)$ is trivial and a set of transition functions is given by:*

$$\begin{cases} g_{i,j} = \rho(\gamma_{i,j}) & \text{in } U_i \cap U_j, \text{ for } i, j \neq 0 \\ g_{0,i} = f_{0,i}\rho(\gamma_{0,i}) & \text{in } U_0 \cap U_i, \text{ for } i \neq 0 \end{cases}$$

where $f_{0,i} : U_0 \cap U_i \to \mathbf{C}^*$ depends only on τ.

Definition 2.3. Let W be a degree $d(W)$ and rank $r(W)$ holomorphic vector bundle on X. It is said to be stable, resp. semi-stable, if for each proper subbundle V, we have

$$\frac{d(V)}{r(V)} < \frac{d(W)}{r(W)}, \text{ resp. } \frac{d(V)}{r(V)} \leq \frac{d(W)}{r(W)}.$$

Remark 2.4. If $d(W)$ and $r(W)$ are co prime then the notions of stability and semi-stability are equivalent.

Recall that d is an integer, $0 \leq d \leq n-1$, and $\zeta = e^{-2\pi i \frac{d}{n}}$ is an n-th root of unity. Let z be a coordinate in a neighbourhood of y_0 such that π_{y_0} is the group of multiplications by ζ^k. Up to a change of generator c of π_{y_0}, we can assume that c acts by multiplication by $e^{\frac{2i\pi}{n}}$. Let τ be the character of π_{y_0} defined by $\tau(c) = \zeta$. A representation $\rho : \pi \to \mathbf{U}(n)$ is said to be of type τ if for all $\gamma \in \pi_{y_0}$, we have $\rho(\gamma) = \tau(\gamma)\mathbf{I}$. For any representation ρ of type τ we have:

$$d(p_*^\pi(E_\rho)) = d - n \qquad \text{(see [5, p. 13])}.$$

Again, according to [17]:

Theorem 2.5. *A holomorphic vector bundle of rank n and degree $d - n$ on X is semi-stable if and only if it is isomorphic to a $p_*^\pi(E_\rho)$, where $\rho : \pi \to \mathbf{U}(n)$ is a unitary representation of type τ. This bundle is stable if and only if the* *representation ρ is irreducible. Moreover, two such bundles are isomorphic if and only if their corresponding unitary representations are isomorphic.*

Remark 2.6. For d and n co prime, any representation $\rho : \pi \to \mathbf{U}(n)$ of type τ is irreducible [17, Prop. 9.3].

Let $\mathfrak{n}$ be the moduli space of rank n, degree d, stable holomorphic vector bundles over X.

Remark 2.7. Let M be a holomorphic line bundle of degree 1 over X (it always exists). The moduli space of rank n stable holomorphic vector bundles with fixed determinant of degree $d-n$ over X is isomorphic to the moduli space of rank n stable holomorphic vector bundles with fixed determinant of degree d over X. The isomorphism is induced by the map which to a bundle $E \to X$ associates $E \otimes M$.

We fix such a bundle M and use it to identify the two moduli spaces of Remark 2.7. Thus we have

Theorem 2.8. *The moduli space $\mathfrak{n}$ is isomorphic to the quotient of the space of unitary representations of type τ of π by the action of $\mathbf{U}(n)$.*

The map which to a class of bundles in $\mathfrak{n}$ associates its determinant is a fibration over the moduli space of line bundles of degree d. Its fibre is called the moduli space of rank n stable holomorphic line bundles over X with fixed determinant (of degree d). We get all such bundles by taking only representations $\rho : \pi \to \mathbf{SU}(n)$ of type τ. Let S be the set of such representations. We identify it with $\{(A_1, B_1, \ldots, A_g, B_g) \in \mathbf{SU}(n)^{2g} \mid \prod_{k=1}^{g}[A_k, B_k] = \zeta\mathbf{I}\}$ by:

$$\begin{aligned} S &\longrightarrow \{(A_1, B_1, \ldots, A_g, B_g) \in \mathbf{SU}(n)^{2g} \mid \prod_{k=1}^{g}[A_k, B_k] = \zeta\mathbf{I}\} \\ \rho &\longmapsto (\rho(a_1), \rho(b_1), \ldots, \rho(a_g), \rho(b_g)). \end{aligned}$$

The action of $\mathbf{SU}(n)$ on the representations becomes, under this identification, the diagonal action by conjugation of $\mathbf{SU}(n)$ on $\mathbf{SU}(n)^{2g}$.In this article we work with $\mathfrak{m}$ rather than $\mathfrak{n}$. We have:

Theorem 2.9. *Let d be an integer, $1 \leq d \leq n-1$, co prime with n. Let $\mathfrak{m}$ be the moduli space of rank n holomorphic stable vector bundles over X with fixed determinant (of degree d). The map*

$$\begin{aligned} S &\longrightarrow \mathfrak{m} \\ \rho &\longmapsto p_*^{\pi}(E_\rho) \end{aligned}$$

is a $\mathbf{PSU}(n)$-principal bundle.

Proof. The only thing that is left to check is that for a representation $\rho = (A_1, B_1, \ldots, A_g, B_g) \in S$ (recall we have identified S with a set of matrices), its stabiliser $\mathrm{Stab}(\rho)$ is the centre of $\mathbf{SU}(n)$. Let C be in the stabiliser of ρ. Let λ be an eigenvalue of C and let E_λ be its eigenspace. As C commutes with each of the A_i, B_i, the subspace E_λ is stable by the unitary representation ρ. As ρ is irreducible, $E_\lambda = \mathbf{C}^n$ and C is in the centre of $\mathbf{SU}(n)$. On the other hand, any matrix in the centre of $\mathbf{SU}(n)$ does leave ρ invariant. We have indeed a free action of $\mathbf{PSU}(n)$ on S. □

2.2. Quasi-Hamiltonian spaces

The definition of a q-Hamiltonian space is due to Alekseev, Malkin and Meinrenken. Roughly speaking this is a Hamiltonian space with a group valued moment map. When the group is a torus, the definition reduces to the usual one of a Hamiltonian torus action whose moment map takes its values in the torus itself (see McDuff [15] and Weitsmann [19]).

Let G be a compact Lie group. Let θ and $\overline{\theta}$ be respectively the left invariant and right invariant Maurer-Cartan forms on G. Choose a G-invariant scalar product $\langle\,,\rangle$ on the Lie algebra $\mathfrak{g}$ of G. Define a 3-form χ on G by

$$\chi = \frac{1}{12}\langle[\theta,\theta],\theta\rangle.$$

Definition 2.10 ([2]). Let (M, G, ω, μ) be a 4-tuple where M is a manifold acted on by a compact Lie group G, ω is a G-invariant 2-form on M and μ is an equivariant map from M to G (for the action by conjugation of G on itself). This 4-tuple (or simply M if there is no risk of confusion) is a q-Hamiltonian space if

(B1) $d\omega = -\mu^*\chi$

(B2) $\iota(v_\xi)\omega = \frac{1}{2}\mu^*\langle\theta + \bar{\theta}, \xi\rangle$

(B3) $\ker\omega_x = \{v_\xi(x) \mid \xi \in \ker \mathrm{Ad}_{\mu(x)} + 1\}$.

The map μ is called the moment map.

This definition is a generalisation of the definition of a Hamiltonian space in the sense that any compact Hamiltonian space can be endowed with a q-Hamiltonian structure (this is an easy corollary of [2, Prop. 3.4.]).

A first example of a q-Hamiltonian space is a conjugacy class in a Lie group with moment map the inclusion of the conjugacy class in the group (see [2, §3]). The example that will be of interest to us is

Theorem 2.11 ([2]). *Let G be a compact Lie group and $g \geq 1$ an integer. There exists a 2-form ω on G^{2g} such that the map*

$$\begin{array}{cccc} \mu: & G^{2g} & \longrightarrow & G \\ & (a_1, b_1, \ldots, a_g, b_g) & \longmapsto & \prod_{k=1}^{g}[a_k, b_k] \end{array}$$

and the diagonal action of G on G^{2g} by conjugation makes (G^{2g}, G, ω, μ) into a q-Hamiltonian space.

In particular we will apply this theorem with $G = \mathbf{SU}(n)$. An important fact about q-Hamiltonian spaces is that one can take their Marsden–Weinstein reduction. More precisely:

Theorem 2.12 ([2])**.** *Let (M, G, ω, μ) be a q-Hamiltonian space. Let h be in the centre of G. The moment map μ is a submersion at $x \in M$ if and only if the stabiliser of x in G is finite. If this is the case for any point of $\mu^{-1}(h)$, the reduced space $\mu^{-1}(h)/G$ is an orbifold (a manifold if the action of G on $\mu^{-1}(h)$ is principal) on which the restriction of ω to $\mu^{-1}(h)$ descends to define a symplectic form. We call this space the reduction of M at h.*

As a corollary of Theorems 2.2, 2.12 and 2.9 we have:

Theorem 2.13. *Let n, d be co prime integers, $n \geq 2$ and $0 \leq d \leq n-1$. Let $\zeta = e^{-2\pi i \frac{d}{n}}$ be an n-th root of unity and $\beta = \zeta \mathbf{I}$ in the centre of $\mathbf{SU}(n)$. The moduli space $\mathfrak{m}_\beta$ of rank n stable holomorphic vector bundles with fixed determinant (and degree d) over a Riemann surface X of genus g is isomorphic to the reduction of the q-Hamiltonian space $\mathbf{SU}(n)^{2g}$ at β. It is a compact smooth symplectic manifold.*

2.3. Characteristic classes of principal bundles

Following Biswas and Raghavendra [4], we define in this section some characteristic classes of a projective bundle. We will see that when the projective bundle comes from a vector bundle of degree 0, these characteristic classes are the same as the Chern classes of the vector bundle.

Let $\mathbf{Q}[X_1, \ldots, X_n]$ be a polynomial ring in n variables. The cohomology of $B\mathbf{U}(n)$ is isomorphic to the subalgebra of invariant polynomials in the algebra $\mathbf{Q}[X_1, \ldots, X_n]$, under the action of the symetric group S_n on the variables. For k an integer in $[1, n]$, the Chern class c_k in $H^*(B\mathbf{U}(n))$ corresponds to the Schur polynomial

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} X_{i_1} \ldots X_{i_k}.$$

The projection from $\mathbf{U}(n)$ to $\mathbf{PU}(n)$ defines a fibration $B\mathbf{U}(n) \longrightarrow B\mathbf{PU}(n)$ with fiber $B\mathbf{U}(1)$. This fibration is cohomologically trivial and $H^*(B\mathbf{PU}(n))$ injects into $H^*(B\mathbf{U}(n))$. Let us define

$$Y_k = X_k - \frac{1}{n} \sum_{k=1}^{n} X_k.$$

The image of $H^*(B\mathbf{PU}(n))$ in $H^*(B\mathbf{U}(n))$ is the ideal generated by the polynomials

$$p_k = \sum_{1 \le i_1 < \cdots < i_k \le n} Y_{i_1} \dots Y_{i_k}, \text{ for } 2 \le k \le n.$$

The k-th characteristic class of a projective bundle over a manifold M is the pull-back of p_k under the classifying map $M \longrightarrow B\mathbf{PU}(n)$.

For a vector bundle F of degree 0, that is when the first Chern class vanishes, we have $p_k(F) = c_k(F)$ for k in $[2, n]$. It will be the case in particular if the structure group of the vector bundle is $\mathbf{SU}(n)$. This corresponds to the fact that the projection $B\mathbf{SU}(n) \longrightarrow B\mathbf{PU}(n)$ defines an isomorphism in cohomology.

3. Construction of a universal bundle

In this section, we fix d, n, ζ and β as in Theorem 2.13. We use the notations of that theorem and of Theorem 2.2 with $G = \mathbf{SU}(n)$. We construct a universal bundle on $\mathfrak{m}_\beta$, that is a vector bundle U over $\mathfrak{m}_\beta \times X$, holomorphic in the X direction, such that for any class $[E]$ in $\mathfrak{m}_\beta$, the restriction of U to $\{[E]\} \times X$ is in the class $[E]$. We then use this bundle to define natural multiplicative generators of the cohomology of $\mathfrak{m}_\beta$.

Recall that we defined page 399 an open covering of X by subsets $\{U_i\}_{i=0}^m$. Define a complex vector bundle T over $S \times X$ (where we have identified S to $\mu^{-1}(\zeta \mathbf{I})$) as being trivial over the $S \times U_i$ and with transition functions:

$$\begin{array}{ccc} (S \times U_i) \cap (S \times U_j) & \longrightarrow & \mathbf{U}(n) \\ (\rho, x) & \longmapsto & \left\{ \begin{array}{cc} \rho(\gamma_{i,j}) & x \in U_i \cap U_j,\ i,j \neq 0 \\ f_{0,i}(x)\rho(\gamma_{0,i}) & x \in U_0 \cap U_i,\ i \neq 0 \end{array} \right. \end{array}.$$

According to Proposition 2.2:

Proposition 3.1. *The bundle T satisfies: for all ρ in S*

$$T|_{\{\rho\} \times X} \cong p_*^\pi(E_\rho).$$

Define an action of $\mathbf{SU}(n)$ on T by defining it on each $T|_{S \times U_i}$ by

$$\begin{array}{ccc} \mathbf{SU}(n) \times (S \times U_i \times \mathbf{C}^n) & \longrightarrow & S \times U_i \times \mathbf{C}^n \\ (g, (\rho, x, u)) & \longmapsto & (g \cdot \rho, x, g(u)). \end{array}$$

This action is well defined because if $x \in U_j \cap U_i$ and $t = (\rho, x, u)$ is in $S \times U_i \times \mathbf{C}^n$, then in the trivialisation $S \times U_j \times \mathbf{C}^n$, t is written $t = (\rho, x, \upsilon(x)\rho(\gamma_{i,j})(u))$ where $\upsilon(x)$ is a scalar and

$$\begin{aligned} g \cdot (\rho, x, \upsilon(x)\rho(\gamma_{i,j})(u)) &= (g \cdot \rho, x, g(\upsilon(x)\rho(\gamma_{i,j})(u))) \\ &= (g \cdot \rho, x, \upsilon(x) g \rho(\gamma_{i,j}) g^{-1} g(u)). \end{aligned}$$

This last term is $(g \cdot \rho, x, g(u))$ written in $S \times U_j \times \mathbf{C}^n$. This action is a lift for the action of $\mathbf{SU}(n)$ on $S \times X$. Unfortunately it does not come from an action of

$\mathbf{PSU}(n)$ and the bundle T does not descend to a bundle on $\mathfrak{m} \times X$. Indeed the centre $\mathbf{Z}/n\mathbf{Z}$ of $\mathbf{SU}(n)$ acts trivially on S but the generator $\zeta\mathbf{I}$ of $\mathbf{Z}/n\mathbf{Z}$ acts by multiplication by ζ in the fibres. To overcome this problem, we can construct a line bundle L on S with an action of $\mathbf{SU}(n)$ lifting the action on S and such that $\zeta\mathbf{I}$ also acts by multiplication by ζ in the fibres. We will also denote L the induced bundle on $S \times X$. The bundle $T \otimes L^*$ has the property of Proposition 3.1 but the action of $\mathbf{SU}(n)$ reduces to an action of $\mathbf{PSU}(n)$. By taking the quotient we get

Proposition 3.2. *Let M be the line bundle of Remark 2.7. The bundle*

$$U = M \otimes (T \otimes L^*)/\mathbf{PSU}(n) \longrightarrow \mathfrak{m} \times X$$

is a universal bundle for $\mathfrak{m}_\beta$. That is, if $[E] \in \mathfrak{m}_\beta$ is the class of a bundle $E \to X$ then $U|_{[E]\times X}$ is isomorphic to E.

We still have to prove the existence of the bundle L.

Lemma 3.3. *There exists a line bundle L over S with an action of $\mathbf{SU}(n)$ lifting the one of $\mathbf{SU}(n)$ on S. This action satisfies: $\zeta\mathbf{I}$ acts by multiplication by ζ in the fibres.*

Proof. The proof is inspired from [16].

The bundle $M \otimes T$ is a family (parameterised by S) of rank n, degree d stable holomorphic vector bundles. Let E be in this family and let k be an integer. By Serre duality,

$$H^1(E \otimes (\Omega^1_X)^k) = H^0(E^\vee \otimes (\Omega^1_X)^{1-k})^*$$

and this is the null vector space. Otherwise there would exist a non zero homomorphism $(\Omega^1_X)^{k-1} \to E^\vee$ and thus a subbundle of $E^\vee$ of degree bigger than or equal to $2(g-1)(k-1) \geq 0$. This is impossible because E is stable.

The $H^0(E \otimes (\Omega^1_X)^k)$ form a holomorphic bundle (see [13]) A_k over S of rank u_k the dimension of $H^0(E \otimes (\Omega^1_X)^k)$. By Riemann-Roch, we have

$$\begin{aligned} u_k &= d(E \otimes (\Omega^1_X)^k) + n(1-g) \\ &= d(E) + 2nk(g-1) + n(1-g) \\ &= d + n(g-1)(2k-1) \\ &= 2hk + d - h \qquad (\text{where } h = n(g-1)). \end{aligned}$$

We have

$$\begin{aligned} &(u_2, u_1) = 1 \Leftrightarrow (d+3h, d+h) = 1 \Leftrightarrow (2h, d+h) = 1 \\ &\Leftrightarrow d+h \text{ is odd and } (d,h) = 1. \end{aligned}$$

As d and n are co prime, d and h are co prime if and only if d and $g-1$ are co prime. If in addition we assume $g-1$ is odd then $d+n(g-1)$ is odd (d and n have different parities). In this case, there exist integers a and b such that $au_1 + bu_2 = 1$ and we can take

$$L = (\wedge^{u_1} A_1)^a \otimes (\wedge^{u_2} A_2)^b.$$

Otherwise, there exists $g' \geq g$ such that $g'-1$ is odd and $(d, g'-1) = 1$. The injection

$$\begin{array}{ccc} \mathbf{SU}(n)^{2g} & \longrightarrow & \mathbf{SU}(n)^{2g'} \\ (A_1, B_1, \ldots, A_g, B_g) & \longmapsto & (A_1, B_1, \ldots, A_g, B_g, 1, 1, \ldots, 1, 1) \end{array}$$

restricts to an equivariant injection

$$S \to S'$$

where S' is the set of $2g'$-tuple of matrices

$$S' = \{(A_1, B_1, \ldots, A_{g'}, B_{g'}), \prod_{k=1}^{g'} [A_k, B_k] = \zeta \mathbf{I}\}.$$

We have seen we can construct on S' a line bundle with the required properties. We take L to be the restriction of this bundle to S. □

Let us use the universal bundle to define classes in $H^*(\mathfrak{m}_\beta)$.

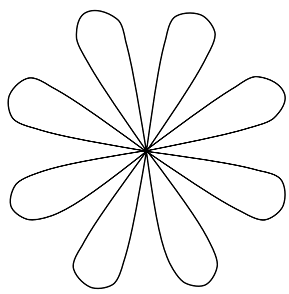

FIGURE 1. Bouquet of $2g$ circles (with $g = 4$)

Let B be a bouquet of $2g$ circles (Figure 1) embedded in X' in such a way that X' retracts on B. Each of the $2g$ circles defines a class in $H_1(X)$. Let $\alpha_1, \ldots, \alpha_{2g}$ be their Poincaré duals. They form a basis of $H^1(X)$. Let κ be the class of the volume form on X of volume 1. Let us decompose the characteristic classes of the projective bundle $P(U)$. For k in $[2, n]$:

$$p_k(P(U)) = a_k \otimes \mathbf{1} + \sum_{j=1}^{2g} b_{k,j} \otimes \alpha_j + d_k \otimes \kappa.$$

Then, according to Biswas and Raghavendra [4], we have

Theorem 3.4. *The family*

$$\{a_k,\ b_{k,j},\ d_k,\ 2 \le k \le n,\ 1 \le j \le 2g\}$$

is a multiplicative system of generators of $H^*(\mathfrak{m}_\beta) \simeq H^*_{\mathbf{SU}(n)}(\mu^{-1}(\beta))$.

4. A bundle over $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$ and its Chern classes

Let B be a bouquet of $2g$ circles (Figure 1) embedded in X' in such a way that X' retracts on B. The theory of vector bundles with their Chern classes is the same on B and X'. We want to construct a complex vector bundle on $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$. Denote B' the star with $2g$ branches (see Figure 2), that is $B' = (\cup_{i=1}^{2g}[0,1]_i)/\sim$,

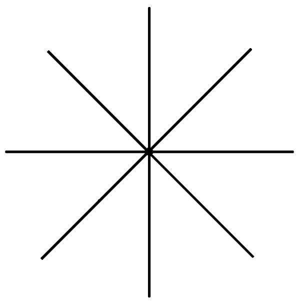

FIGURE 2. A star with $2g$ branches (with again $g = 4$)

where $\sim$ is the equivalence relation that identifies all the 0 to a point. There is a natural map

$$\eta : B' \longrightarrow B.$$

It is defined by means of the exponential $\exp : [0,1] \to S^1$. Denote

$$D_n = (\mathbf{SU}(n)^{2g} \times E\mathbf{U}(n) \times B' \times \mathbf{C}^n)/\sim$$

where $\sim$ is the relation:

$$((\rho_1, \dots, \rho_{2g}), e, 0, v) \sim ((\mathrm{Ad}_A \rho_1, \dots, \mathrm{Ad}_A \rho_{2g}), A \cdot e, 1_i, A \circ \rho_i(v)),$$
$$\forall i \in [1, 2g],\ \forall A \in \mathbf{SU}(n).$$

The projection

$$D_n \longrightarrow (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$$

makes D_n into a rank n complex vector bundle over $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$. We wish to compute the characteristic classes of the projectivised bundle $P(D_n)$ of D_n. Notice that as the structure group of D_n reduces to $\mathbf{SU}(n)$, the classes $p_k(P(D))$ are equal to the Chern classes $c_k(D)$.

Let us describe the cohomology of $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$. By the Künneth formula, we have

$$H^*((\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B) = H^*_{\mathbf{SU}(n)}(\mathbf{SU}(n)^{2g}) \otimes H^*(B).$$

Proposition 4.1. *Let G be a compact Lie group. Let k be an integer bigger than 0. Let G act on G^k diagonally by conjugation. The equivariant cohomology of G^k is isomorphic, as a graded algebra, to $H^*(G^k) \otimes H^*(BG)$.*

Proof. The fibration $(G^k)_G \longrightarrow BG$ is cohomologically trivial (see [3]) so that we have an isomorphism of graded vector spaces between $H^*_G(G^k)$ and $H^*(G^k) \otimes H^*(BG)$. The proposition then follows from the fact that for any compact Lie group, its cohomology is an exterior algebra on a finite number of elements and from the

Lemma 4.2. *Let $q : N \longrightarrow M$ be a cohomologically trivial fibration with fiber F. Assume that the cohomology of F is an exterior algebra on a family $\{\xi_1, \ldots, \xi_r\}$. Thenthe cohomology of N is isomorphic, as a graded algebra, to the tensor product of $H^*(F)$ and $H^*(M)$.*

Proof. Let $\mathfrak{I}$ be the set of strictly increasing sequences of integers $I = (i_1, \ldots, i_p)$ such that $i_1 \geq 1$ and $i_p \leq r$. For $I \in \mathfrak{I}$ with $I = (i_1, \ldots, i_p)$, let

$$\xi_I = \xi_{i_1} \wedge \cdots \wedge \xi_{i_p}.$$

The family $\{\xi_I\}_{I \in \mathfrak{I}}$ forms a basis of $H^*(F)$.

To say that the fibration $N \longrightarrow M$ is cohomologically trivial is equivalent (by the Leray-Hirsch Theorem) to saying that the inclusion of a fiber F into N induces a surjection $H^*(N) \longrightarrow H^*(F)$. For $i \in [1, r]$, let ζ_i, in $H^*(N)$, be a pre-image of ξ_i. For $I \in \mathfrak{I}$ with $I = (i_1, \ldots, i_p)$, let

$$\zeta_I = \zeta_{i_1} \wedge \cdots \wedge \zeta_{i_p}.$$

The map

$$\begin{array}{ccc} H^*(F) & \longrightarrow & H^*(N) \\ \sum \lambda_I \zeta_I & \longmapsto & \sum \lambda_I \xi_I \end{array}$$

is a morphism of algebra and the map

$$\begin{array}{ccc} H^*(F) \otimes H^*(M) & \longrightarrow & H^*(N) \\ (\sum \lambda_I \zeta_I) \otimes \chi & \longmapsto & (\sum \lambda_I \xi_I) \otimes q^*(\chi) \end{array}$$

is an isomorphism of graded algebra. □

□

According to the previous proposition, we have isomorphisms

$$\begin{aligned} H^*_{\mathbf{SU}(n)}(\mathbf{SU}(n)^{2g}) &\simeq H^*(\mathbf{SU}(n)^{2g}) \otimes H^*(B\mathbf{SU}(n)) \\ &\simeq \otimes_{j=1}^{2g} H^*(\mathbf{SU}(n)) \otimes H^*(B\mathbf{SU}(n)). \quad (3.1) \end{aligned}$$

For all $k \geq 2$, the fibration $\mathbf{SU}(k) \longrightarrow S^{2k-1}$ with fiber $\mathbf{SU}(k-1)$ is cohomologically trivial (see Hatcher [7]). Let γ_k be the volume form of volume 1 on S^{2k-1}. The cohomology of $\mathbf{SU}(n)$ is the exterior algebra freely generated by the family $\{\sigma_k,\, 2 \leq k \leq n\}$, where σ_k is a class of degree $2k-1$ which pulls-back under the restriction $\mathbf{SU}(k) \longrightarrow \mathbf{SU}(n)$ to the image of γ_k by $H^{2k-1}(S^{2k-1}) \longrightarrow H^{2k-1}(\mathbf{SU}(k))$. Denote $\sigma_{k,j}$ the image of $\sigma_k \in H^{2k-1}(\mathbf{SU}(n))$ by the homomorphism $H^*(\mathbf{SU}(n)) \to H^*(\mathbf{SU}(n)^{2g})$ induced by the projection on the j-th factor $\mathbf{SU}(n)^{2g} \to \mathbf{SU}(n)$. We have

Lemma 4.3. *The algebra $H^*(\mathbf{SU}(n)^{2g})$ is the exterior algebra freely generated by the family $\{\sigma_{k,j},\ 2 \leq k \leq n,\ 1 \leq j \leq 2g,\ \deg \sigma_{k,j} = 2k-1\}$.*

In addition, we know that $H^*(B\mathbf{SU}(n)) = \mathbf{Q}[c_2, \ldots, c_n]$. From the preceding lemma and Proposition 4.1, we deduce

Theorem 4.4. *Let Λ be the exterior algebra freely generated by the family $\{\sigma_{k,j},\ 2 \leq k \leq n,\ 1 \leq j \leq 2g,\ \deg \sigma_{k,j} = 2k-1\}$. The $\mathbf{SU}(n)$-equivariant cohomology of $\mathbf{SU}(n)^{2g}$ is isomorphic, as a graded algebra, to $\Lambda \otimes \mathbf{Q}[c_2, \ldots, c_n]$.*

When there is no risk of confusion, we will write c_k and $\sigma_{k,j}$ instead of respectively $1 \otimes c_k$ and $\sigma_{k,j} \otimes 1$.

Remark 4.5. The injection ι of $\mathbf{SU}(n)$ into $\mathbf{SU}(n+1)$ and the map $B\mathbf{SU}(n) \to B\mathbf{SU}(n+1)$ induce isomorphisms

$$H^k(\mathbf{SU}(n+1)) \xrightarrow{\sim} H^k(\mathbf{SU}(n)) \text{ for } k \leq 2n \text{ and } k = 2n+2 \tag{4.1}$$

and

$$H^k(B\mathbf{SU}(n+1)) \xrightarrow{\sim} H^k(B\mathbf{SU}(n)) \text{ for } k \leq 2n. \tag{4.2}$$

With the notations of Theorem 4.4, we have

Proposition 4.6. *The Chern classes of D_n are:*

$$\begin{aligned} c_0(D_n) &= 1, \\ c_1(D_n) &= 0, \\ c_k(D_n) &= (1 \otimes c_k) \otimes 1 + \textstyle\sum_{j=1}^{2g} (\sigma_{k,j} \otimes 1) \otimes \alpha_j \ \textit{for } k \geq 2. \end{aligned}$$

Proof. The classes $c_0(D_n)$ and $c_1(D_n)$ are trivially 1 and 0 (the structure group is $\mathbf{SU}(n)$). Assume from now on that $k \geq 2$. Let us write the Chern classes of D_n

in $H^*((\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B)$ as

$$c_k(D_n) = \gamma_k^{(n)} \otimes 1 + \sum_{j=1}^{2g} \beta_{k,j}^{(n)} \otimes \alpha_j.$$

We will prove the proposition by induction on n. For $n = 1$, $\mathbf{SU}(1)$ is just a point, the bundle D_1 is trivial and we are already done. Suppose the proposition to be true for a given n, $n \geq 1$ and let us prove it for $n+1$. We need to prove that

$$\gamma_k^{(n+1)} = 1 \otimes c_k \text{ and } \beta_{k,j}^{(n+1)} = \sigma_{k,j} \otimes 1.$$

Let

$$m : (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)},$$

the map induced by the inclusion $\mathbf{SU}(n) \longrightarrow \mathbf{SU}(n+1)$ and

$$\ell = m \times \mathrm{id}_B : (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)} \times B.$$

The bundle $\ell^* D_{n+1}$ is isomorphic to $D_n \oplus \mathbf{C}$. Hence, for all k, we have $c_k(\ell^* D_{n+1}) = c_k(D_n)$. Thus

$$(m^* \gamma_k^{(n+1)}) \otimes 1 + \sum_{j=1}^{2g} (m^* \beta_{k,j}^{(n+1)}) \otimes \alpha_j = \gamma_k^{(n)} \otimes 1 + \sum_{j=1}^{2g} \beta_{k,j}^{(n)} \otimes \alpha_j.$$

From this we deduce

$$m^* \gamma_k^{(n+1)} = \gamma_k^{(n)} \text{ and } m^* \beta_{k,j}^{(n+1)} = \beta_{k,j}^{(n)}.$$

Because of the isomorphisms (4.1), (4.2) and the induction hypothesis, we have:

$$\begin{aligned} \gamma_k^{(n+1)} &= 1 \otimes c_k \text{ for } k \leq n, \\ \beta_{k,j}^{(n+1)} &= \sigma_{k,j} \otimes 1 \text{ for } k \leq n. \end{aligned}$$

There only remains to compute $\gamma_{n+1}^{(n+1)}$ and the $\beta_{n+1,j}^{(n+1)}$. The class $\gamma_{n+1}^{(n+1)}$ belongs to

$$H^{2n+2}_{\mathbf{SU}(n+1)}(\mathbf{SU}(n+1)^{2g}) = \bigoplus_{p+q=2n+2} H^p(\mathbf{SU}(n+1)^{2g}) \otimes H^q(B\mathbf{SU}(n+1)).$$

Let us decompose it

$$\gamma_{n+1}^{(n+1)} = \sum_{k=0}^{n+1} \varepsilon_k^{(n+1)} \otimes c_k$$

where $\varepsilon_k^{(n+1)}$ is in $H^{2n+2-2k}(\mathbf{SU}(n+1)^{2g})$ and where we have put $c_0 = 1$ in $H^0(B\mathbf{SU}(n))$, $c_1 = 0$. The classes $\beta_{n+1,j}^{(n+1)}$ are in

$$H^{2n+1}_{\mathbf{SU}(n+1)}(\mathbf{SU}(n+1)^{2g}) = \bigoplus_{p+q=2n+1} H^p(\mathbf{SU}(n+1)^{2g}) \otimes H^q(B\mathbf{SU}(n+1)).$$

We decompose them in

$$\beta_{n+1,j}^{(n+1)} = \sum_{k=0}^{n} \delta_{k,j}^{(n+1)} \otimes c_k$$

where $\delta_{k,j}^{(n+1)}$ belongs to $H^{2n+1-2k}(\mathbf{SU}(n+1)^{2g})$. The bundle $\ell^* D_{n+1} = D_n \oplus \mathbf{C}$ has a nowhere vanishing section, hence its Euler class $\ell^* c_{n+1}(D_{n+1})$ vanishes. Because of the isomorphisms (4.1) and (4.2), we deduce that the $\{\varepsilon_k^{(n+1)},\ 1 \leq k \leq n\}$ and the $\{\delta_{k,j}^{(n+1)},\ 1 \leq k \leq n,\ 1 \leq j \leq 2g\}$ vanish. Remark that the $\{\delta_{n+1,j}^{(n+1)},\ 1 \leq j \leq 2g\}$ are linear combinations of the $\sigma_{2n+1,j},\ 1 \leq j \leq 2g$. Let us define a section

$$\begin{array}{rccc} s: & B\mathbf{SU}(n+1) & \longrightarrow & (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)} \times B \\ & [e] & \longmapsto & ([(\mathbf{I},\dots,\mathbf{I}),e],1) \end{array}$$

where, for e in $E\mathbf{U}(n+1)$, we denote by $[e]$ its class in $B\mathbf{SU}(n+1)$. The Euler class of the bundle $s^* D_{n+1}$ is εX_{n+1}. Since $s^* D_{n+1}$ is equal to $E\mathbf{U}(n+1) \times_{\mathbf{SU}(n+1)} \mathbf{C}^{n+1}$ we have $\varepsilon = 1$. As a conclusion we have

$$\gamma_{n+1}^{(n+1)} = 1 \otimes X_{n+1}.$$

Let

$$h : \mathbf{SU}(n+1)^{2g} \longrightarrow (\mathbf{SU}(n+1)^{2g})_{\mathbf{SU}(n+1)}$$

be the inclusion of a fiber (we will always write h this application, omitting the subscript n). The bundle

$$F_{n+1}^{2g} := (h \times \mathrm{id}_B)^* D_{n+1}$$

is isomorphic to

$$F_{n+1}^{2g} \cong (\mathbf{SU}(n+1)^{2g} \times B' \times \mathbf{C}^{n+1})/\sim,$$

where $\sim$ is the relation:

$$((\rho_1,\dots,\rho_{2g}),1_j,v) \sim ((\rho_1,\dots,\rho_{2g}),0,\rho_j^{-1}(v)), \text{ for all } j \text{ in } [1,2g].$$

The Euler class of F_{n+1}^{2g} is

$$c_{n+1}(F_{n+1}^{2g}) = \sum_{j=1}^{2g} \beta_{n+1,j}^{(n+1)} \otimes \alpha_j.$$

Let $f_j : S^1 \to B$ (resp. $g_j : \mathbf{SU}(n+1) \to \mathbf{SU}(n+1)^{2g}$) be the inclusion of the j-th circle (resp. $\mathbf{SU}(n+1)$) in B (resp. $\mathbf{SU}(n+1)^{2g}$). The $\beta_{n+1,j}^{(n+1)}$ are characterised by:

$$c_{n+1}((\mathrm{id}_{\mathbf{SU}(n+1)^{2g}} \times f_j)^* F_{2g}^{(n+1)}) = \beta_{n+1,j}^{(n+1)} \otimes f_j^* \alpha_j,$$

or

$$c_{n+1}((\mathrm{id}_{\mathbf{SU}(n+1)^{2g}} \times f_j)^* F_{2g}^{(n+1)}) = \beta_{n+1,j}^{(n+1)} \otimes \frac{\mathrm{d}\theta}{2\pi}. \tag{4.3}$$

Let us define a vector bundle E over $\mathbf{SU}(n+1)\times S^1$ by

$$E=(\mathbf{SU}(n+1)\times[0,1]\times\mathbf{C})/\sim,$$

where $\sim$ is the relation

$$(\rho,1,v)\sim(\rho,0,\rho^{-1}(v)).$$

The bundle $(\mathrm{id}_{\mathbf{SU}(n+1)^{2g}}\times f_j)^*F_{2g}^{(n+1)}$ is isomorphic to $(g_j\times\mathrm{id}_{S^1})^*E$. Hence there exists a real λ such that

$$c_{n+1}(E)=\lambda\sigma_{2n+1}\otimes\frac{\mathrm{d}\theta}{2\pi}.$$

If (ρ,t,v) belongs to $\mathbf{SU}(n+1)\times[0,1]\times\mathbf{C}^{n+1}$, let us write $[\rho,t,v]$ for its class in E. Let $(e_1,\ldots,e_{n+1})$ be the canonical basis, over the field $\mathbf{C}$, of $\mathbf{C}^{n+1}$. The family $(e_1,ie_1,\ldots,e_{n+1},ie_{n+1})$ is then a basis of $\mathbf{C}^{n+1}$ over $\mathbf{R}$. A section of E is given by:

$$\begin{array}{rcl} s:\mathbf{SU}(n+1)\times S^1 & \longrightarrow & E\\ (A,e^{2i\pi\theta}) & \longmapsto & [A,\theta,(\theta A+(1-\theta)\mathrm{id})e_1]. \end{array}$$

Let us determine its zeros. The vector $(\theta A+(1-\theta)\mathrm{id})e_1$ vanishes if $\theta=\frac{1}{2}$ and $A=\begin{bmatrix}-1&0\\0&\widetilde{A}\end{bmatrix}$, $\widetilde{A}\in\mathbf{U}(n)$, $\det\widetilde{A}=-1$. Fix ξ an n-th root of -1. The zero set Z of s is

$$Z=\left\{\left(\begin{bmatrix}-1&0\\0&\xi\widetilde{A}\end{bmatrix},\frac{1}{2}\right),\ \widetilde{A}\in\mathbf{SU}(n)\right\}.$$

Lemma 4.7. *The section s intersects the zero section s_0 transversally.*

Proof. We want to prove that for all x of Z

$$T_{s(x)}\mathrm{Im}s+T_{s(x)}\mathrm{Im}s_0=T_{(x,0)}E.$$

We have

$$T_{(x,0)}E\simeq T_x(\mathbf{SU}(n+1)\times S^1)\oplus\mathbf{C}^{n+1}\simeq\mathfrak{su}(n+1)\oplus\mathbf{R}\oplus\mathbf{C}^{n+1}$$

and

$$\begin{aligned} T_{s(x)}\mathrm{Im}s_0 &= \quad\mathfrak{su}(n+1)\oplus\mathbf{R}\oplus\{0\},\\ T_{s(x)}\mathrm{Im}s &= \mathrm{T}_x s(T_x(\mathbf{SU}(n+1)\times S^1)). \end{aligned}$$

Let x be the point $(A=\begin{bmatrix}-1&0\\0&\xi\widetilde{A}\end{bmatrix},\frac{1}{2})$,

$$\begin{aligned} \tfrac{d}{d\varepsilon}|_{\varepsilon=0}s(A,\tfrac{1}{2}+\varepsilon) &= [A,\tfrac{1}{2}+\varepsilon,((\tfrac{1}{2}+\varepsilon)A+(\tfrac{1}{2}-\varepsilon)\mathrm{id})e_1]\\ &= [A,\tfrac{1}{2}+\varepsilon,-2\varepsilon e_1]\\ &= (0,1,-2e_1). \end{aligned}$$

Let J be in $\mathfrak{su}(n+1)$,

$$\begin{aligned} \tfrac{d}{d\varepsilon}|_{\varepsilon=0} s(\exp(\varepsilon J)A, \tfrac{1}{2}) &= \tfrac{d}{d\varepsilon}|_{\varepsilon=0}[\exp(\varepsilon J)A, \tfrac{1}{2}, \tfrac{1}{2}(\exp(\varepsilon J)A + \mathrm{id})e_1] \\ &= \tfrac{d}{d\varepsilon}|_{\varepsilon=0}[\exp(\varepsilon J)A, \tfrac{1}{2}, \tfrac{1}{2}(\exp(\varepsilon J)(-e_1) + e_1)] \\ &= (J \cdot A, 0, \tfrac{1}{2}(-Je_1 + e_1)). \end{aligned}$$

We conclude the proof of Lemma 4.7 by noticing that, for any k, it is possible to find J in $\mathfrak{su}(n+1)$ such that Je_1 is equal to e_k or ie_k. □

Lemma 4.8. *The Euler class of the bundle E is*

$$c_{n+1}(E) = \sigma_{2n+1} \otimes \frac{\mathrm{d}\theta}{2\pi}.$$

Proof. According to the preceding lemma, the Euler class of E is Poincaré dual of Z, that is it is characterised by

$$\forall \nu \in H^{n^2-1}(\mathbf{SU}(n+1) \times S^1), \int_Z \nu = \int_{\mathbf{SU}(n+1)\times S^1} \nu \wedge c_{n+1}(E)$$

where $n^2 - 1 = \dim(\mathbf{SU}(n+1) \times S^1) - 2(n+1)$. This Euler class is of the type

$$c_{n+1}(E) = \lambda \sigma_{2n+1} \otimes \frac{\mathrm{d}\theta}{2\pi}$$

where λ is a real we are going to compute. The injection

$$\begin{array}{ccc} \mathbf{SU}(n) & \longrightarrow & \mathbf{SU}(n+1) \\ A & \longmapsto & \begin{bmatrix} -1 & 0 \\ 0 & \xi A \end{bmatrix} \end{array}$$

identifies $\mathbf{SU}(n)$ to the fibre above $(-1, 0, \ldots, 0)$ of the projection $\mathbf{SU}(n+1) \to S^{2n+1}$, that is Z. Let γ be the cohomology class of a volume form of volume 1 over $\mathbf{SU}(n)$. The decomposition $H^*(\mathbf{SU}(n+1)) = H^*(\mathbf{SU}(n)) \otimes H^*(S^{2n+1})$ defines a class

$$\nu = \gamma \otimes 1.$$

As the integral of ν on Z is 1, we have

$$\int_{\mathbf{SU}(n+1)\times S^1} \nu \wedge c_{n+1}(E) = 1,$$

that is

$$\lambda \int_{\mathbf{SU}(n+1)\times S^1} (\gamma \otimes 1) \wedge (\sigma_{2n+1} \otimes \frac{\mathrm{d}\theta}{2\pi}) = 1.$$

The conclusion follows since the integral in the left-hand side of the equality is equal to 1. □

Proposition 4.6 follows from this lemma. □

5. Description of the restriction map

Using results of the previous sections, we wish to prove:

Theorem 5.1. *The restriction map r is described by*

$$\begin{aligned} r(c_k) &= a_k \quad \textit{for } k = 2, \ldots, n \\ r(\sigma_{k,j}) &= b_{k,j} \quad \textit{for } k = 2, \ldots, n, j = 1, \ldots, 2g. \end{aligned}$$

In particular, $\mathrm{Im}(r)$ *is multiplicatively generated by*

$$\mathrm{Im}(r) = \langle a_k, b_{k,j}, k = 2, \ldots, n, j = 1, \ldots, 2g \rangle.$$

Notice that for n equals 2, we get that r is surjective modulo the symplectic form on $\mathfrak{m}_\beta$ (this result has been in [18]).

It is also very interesting to compare this theorem with [11, Theo. 7.1] where a group cohomological construction of multiplicative generators of $H^*(\mathfrak{m}_\beta)$ is given.

Proof. The key point of the proof is to compare the bundles U of Section 3 and D_n of Section 4.

From now on, if $g \in \mathbf{SU}(n)$, we denote $\bar{g}$ its class in $\mathbf{PSU}(n)$. Over each $S \times U_i$, $i = 0, \ldots, m$, the bundle $M \otimes T \otimes L^*$ is trivial. In each of these sets, the action of $\mathbf{PSU}(n)$ on $M \otimes T \otimes L^*$ is

$$\begin{array}{ccc} \mathbf{PSU}(n) \times M \otimes (S \times U_i \times \mathbf{C}^n) \otimes L^* & \longrightarrow & M \otimes (S \times U_i \times \mathbf{C}^n) \otimes L^* \\ m \otimes (\bar{g}, (\rho, x, u) \otimes l) & \longmapsto & m \otimes (g \cdot \rho, x, g(u)) \otimes (g \cdot l). \end{array}$$

Lemma 5.2. *We have*

$$\mathrm{P}(U) = \mathrm{P}(M \otimes (T \otimes L^*)/\mathbf{PSU}(n)) \cong \mathrm{P}(T)/\mathbf{PSU}(n).$$

Proof. This time, $\mathbf{PSU}(n)$ acts on $\mathrm{P}(T)$ by

$$\begin{array}{ccc} \mathbf{PSU}(n) \times (S \times U_i \times \mathbf{C}\mathrm{P}^n) & \longrightarrow & (S \times U_i \times \mathbf{C}\mathrm{P}^n) \\ (\bar{g}, (\rho, x, \overline{u})) & \longmapsto & (g \cdot \rho, x, \overline{g(u)}) \end{array}$$

and the announced isomorphism is

$$\begin{array}{ccc} \mathrm{P}(U) & \xrightarrow{\cong} & \mathrm{P}(T)/\mathbf{PSU}(n) \\ \text{class of } m \otimes (\rho, x, u) \otimes l & \longmapsto & \text{class of } (\rho, x, u). \end{array}$$

□

Lemma 5.3. *There exists an action of* $\pi \times \mathbf{PSU}(n)$ *on* $S \times Y' \times \mathbf{C}\mathrm{P}^{n-1}$ *such that the quotient*

$$(S \times Y' \times \mathbf{C}\mathrm{P}^{n-1})/(\pi \times \mathbf{PSU}(n))$$

is isomorphic to

$$\mathrm{P}(U)|_{\mathfrak{m}_\beta \times X'}.$$

Proof. The bundle T restricted to $S \times X'$ is trivial on each $S \times U_i$, $i \neq 0$ and transition functions are given by

$$\begin{array}{ccc} (S \times U_i) \cap (S \times U_j) & \longrightarrow & \mathbf{SU}(n) \\ (\rho, x) & \longmapsto & \rho(\gamma_{i,j}). \end{array}$$

The group π acts freely on Y' and $T|_{S\times X'}$ is $(S \times Y' \times \mathbf{C}^n)/\pi$, where the action of π is

$$\begin{array}{ccc} \pi \times (S \times Y' \times \mathbf{C}^n) & \longrightarrow & S \times Y' \times \mathbf{C}^n \\ (\gamma, (\rho, y, u)) & \longmapsto & (\rho, \gamma \cdot y, \rho(\gamma)u). \end{array}$$

Let us consider the projective bundle $\mathrm{P}(T)|_{S\times X'}$. It is isomorphic to $(S \times Y' \times \mathbf{C}\mathrm{P}^{n-1})/\pi$. The subspace $\mathrm{P}(T)|_{S\times X'}$ is stable by $\mathbf{PSU}(n)$ and the action comes from an action of $\mathbf{PSU}(n)$ on $S \times Y' \times \mathbf{C}\mathrm{P}^{n-1}$. That is

$$\begin{array}{ccc} \mathbf{PSU}(n) \times (S \times Y' \times \mathbf{C}\mathrm{P}^{n-1}) & \longrightarrow & S \times Y' \times \mathbf{C}\mathrm{P}^{n-1} \\ (\overline{g}, (\rho, y, \overline{u})) & \longmapsto & (g \cdot \rho, y, \overline{g(u)}). \end{array}$$

This action commutes indeed with the one of π, the result follows. □

The pull-back of the bundle $U \to (S/\mathbf{PSU}(n)) \times X'$ to $(S)_{\mathbf{SU}(n)} \times X'$ by the natural map

$$f : (S)_{\mathbf{SU}(n)} \times X' \longrightarrow (S/\mathbf{PSU}(n)) \times X'$$

is a vector bundle, we will denote it F. Its projectivised bundle is

$$\mathrm{P}(F) = (\mathrm{P}(T))_{\mathbf{SU}(n)} \longrightarrow (S)_{\mathbf{SU}(n)} \times X'.$$

We will now state a proposition which will be our main tool in the study of the map r:

Proposition 5.4. *There is a projective bundle* $\mathrm{P}(D)$ *over* $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$ *whose restriction to* $(S)_{\mathbf{SU}(n)} \times X'$ *is isomorphic to* $\mathrm{P}(F)$.

First proof. The projection $p : Y' \to X'$ is a covering. Its group is π. Let $q : \widetilde{Y}' \to Y'$ be the universal covering of Y'. The composed map $\widetilde{p} = p \circ q : \widetilde{Y}' \to X'$ is the universal covering of X'. Its group is

$$\pi_1(X') = \langle a_1, b_1, \ldots, a_g, b_g \rangle$$

and we have a projection $\pi_1(X') \xrightarrow{\theta} \pi$ whose kernel is the group of the covering $\widetilde{Y}' \to Y'$.

The open covering of X' by the $\{U_i\}_{i=1}^m$ is such that any intersection of open sets of the type U_i is contractible. In particular, for all i, there exists a disc $\widetilde{D}_i$ in $\widetilde{Y}'$ such that $\widetilde{p} : \widetilde{D}_i \to U_i$ is a diffeomorphism. Choose, for all i, j, k, a connected component $W_{ij,k}$ of $\widetilde{p}^{-1}(U_i \cap U_j) \cap \widetilde{D}_k$. If $U_i \cap U_j \neq \emptyset$, let $\widetilde{\gamma}_{i,j}$ be the element of $\pi_1(X')$ such that $\widetilde{\gamma}_{i,j}\widetilde{W}_{ij,j} = \widetilde{W}_{ji,i}$. In Proposition 2.2, we can take the $W_{ij,k}$ and $\gamma_{i,j}$ such that

$$W_{ij,k} = \widetilde{p}(\widetilde{W}_{ij,k}), \gamma_{i,j} = \theta(\widetilde{\gamma}_{i,j}).$$

Let us identify the set of representations $\rho : \pi_1(X') \to \mathbf{SU}(n)$ to $\mathbf{SU}(n)^{2g}$ by

$$\rho \mapsto (\rho(a_1), \rho(b_1), \dots, \rho(a_g), \rho(b_g)).$$

Let

$$T' \to \mathbf{SU}(n)^{2g} \times X'$$

be the rank n complex vector bundle defined by the following properties:

(1) $T'|_{\mathbf{SU}(n)^{2g} \times U_i}$ is trivial,

(2) the transition functions are

$$g_{i,j} = \rho(\widetilde{\gamma}_{i,j}) \text{ on } \mathbf{SU}(n)^{2g} \times (U_i \cap U_j).$$

The restriction of this bundle to $S \times X'$ is $T|_{S\times X'}$. The action of $\mathbf{SU}(n)$ on $T|_{S\times X'}$ is then the restriction of the $\mathbf{SU}(n)$ action on T' defined on each $T'|_{\mathbf{SU}(n)^{2g}\times U_i}$ by

$$\begin{array}{ccc} \mathbf{SU}(n) \times (\mathbf{SU}(n)^{2g} \times U_i \times \mathbf{C}^n) & \longrightarrow & \mathbf{SU}(n)^{2g} \times U_i \times \mathbf{C}^n \\ (g, (\rho, x, u)) & \longmapsto & (g \cdot \rho, x, g(u)). \end{array}$$

Notice that this action is a lift of the action of $\mathbf{SU}(n)$ on $\mathbf{SU}(n)^{2g} \times X'$. Thus the bundle

$$\mathrm{P}(F) = (\mathrm{P}(T))_{\mathbf{SU}(n)} \to (S)_{\mathbf{SU}(n)} \times X'$$

is the restriction of the bundle

$$(\mathrm{P}(T'))_{\mathbf{SU}(n)} \to (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'.$$

Second proof. We have seen that

$$\mathrm{P}(U)|_{\mathfrak{m}_\beta \times X'} \cong (S \times Y' \times \mathbf{C}\mathrm{P}^{n-1})/(\pi \times \mathbf{PSU}(n)),$$

hence

$$\mathrm{P}(F) \cong (S \times E\mathbf{U}(n) \times Y' \times \mathbf{C}\mathrm{P}^{n-1})/(\pi \times \mathbf{SU}(n)).$$

Let us define, in a similar way as before, an action of $\pi_1(X')$ on $\mathbf{SU}(n)^{2g} \times E\mathbf{U}(n) \times \widetilde{Y'} \times \mathbf{C}^n$ and denote D the bundle we obtain when quotienting by $\pi_1(X') \times \mathbf{SU}(n)$. The projection $S \times E\mathbf{U}(n) \times \widetilde{Y'} \times \mathbf{C}^n \to S \times E\mathbf{U}(n) \times Y' \times \mathbf{C}^n$ is equivariant for the respective actions of $\pi_1(X')$ and π. It induces an action on the quotient and defines an isomorphism between

$$(S \times E\mathbf{U}(n) \times \widetilde{Y'} \times \mathbf{C}^n)/(\pi_1(X') \times \mathbf{SU}(n))$$

and

$$(S \times E\mathbf{U}(n) \times Y' \times \mathbf{C}^n)/(\pi \times \mathbf{SU}(n)).$$

We deduce that $\mathrm{P}(F)$ is isomorphic to $\mathrm{P}(D)|_{(S)_{\mathbf{SU}(n)} \times X'}$. □

Remark 5.5. The bundle $D \to (\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$ is isomorphic to $(T' \times E\mathbf{U}(n))/\mathbf{SU}(n)$.

When restricted to $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times B$, the bundle D is isomorphic to D_n (restricted to $(\mu^{-1}(\zeta\mathbf{I}))_{\mathbf{SU}(n)} \times B$). Denote w the injection of $(S)_{\mathbf{SU}(n)} \times X'$ in $(\mathbf{SU}(n)^{2g})_{\mathbf{SU}(n)} \times X'$. The induced map w^* in cohomology is $r \times \mathrm{id}_{H^*(X')}$. The restriction $w^* D_n$ of D_n to $(S)_{\mathbf{SU}(n)} \times X'$ has the same projectivisation as F. Thus, because of Proposition 4.6, we have for every k

$$\begin{aligned} p_k(P(F)) &= a_k \otimes 1 + \sum_{j=1}^{2g} b_k^j \otimes \alpha_j && (5.1)\\ &= p_k(P(w^* D_n)) \\ &= w^* p_k(P(D_n)) \\ &= r(1 \otimes p_k) \otimes 1 + \sum_{j=1}^{2g} r(\sigma_{k,j} \otimes 1) \otimes \alpha_j . && (5.2) \end{aligned}$$

Theorem 5.1 follows from the comparison of Line (5.1) and Line (5.2). □

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(Received: May 28, 2002)