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On Waring's problem for several algebraic forms

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Abstract. We reconsider the classical problem of representing a finite number of forms of degree d in the polynomial ring over $n+1$ variables as scalar combinations of powers of linear forms. We define a geometric construct called a 'grove', which, in a number of cases, allows us to determine the dimension of the space of forms which can be so represented for a fixed number of summands. We also present two new examples, where this dimension turns out to be less than what a naïve parameter count would predict.

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1. Introduction

Waring's problem for algebraic forms is formulated in analogy with the number-theoretic version. Assume that $F_1, \dots, F_r$ are homogeneous forms of degree d in variables $x_0, \dots, x_n$. We would like to find linear forms $Q_1, \dots, Q_s$, such that each F_i is expressible as a linear combination of $Q_1^d, \dots, Q_s^d$. This problem, and especially the case $r = 1$, has received a great deal of attention classically. Indeed, since the representation

$$F = c_1 Q_1^d + \cdots + c_s Q_s^d \tag{1}$$

is computationally easy to work with, geometric results about the hypersurface $F = 0$ are sometimes more easily proved by reducing F to such an expression by a linear change of variables. For instance, the classical texts of Salmon [19, 18] frequently use this device.

Typically the forms F_i were assumed general, and the goal of the enquiry was to find the smallest s for which the problem is solvable. An elementary parameter count gives an expected value of s, which usually turns out to be correct. However, there are exceptional cases when the expected value does not suffice, and of course they are the ones of more interest. Here we consider a more general version of the problem, i.e., we fix s and ask for the dimension of the family of forms (F_i) which can be so expressed. See [10, 14] for an overview of the problem.

The formal set-up is as follows. Let V be a $\mathbf{C}$-vector space of dimension $n+1$, and consider the symmetric algebra $S = \bigoplus\limits_{d \geq 0} \operatorname{Sym}^d V$. Choosing a basis $\{x_0, \dots, x_n\}$ for V, an element in S_d may be written as a degree d form in the x_i.

Fix two positive integers $r \leq s$. Let $\underline{Q} = \{Q_1, \dots, Q_s\}$ denote a typical point of $\operatorname{Sym}^s(\mathbf{P}S_1)$, and consider the set

$$U_s = \{\underline{Q} : Q_1^d, \dots, Q_s^d \text{ are linearly independent over } \mathbf{C}\}.$$

This is an open set of $\operatorname{Sym}^s(\mathbf{P}S_1)$, and if $s \leq \dim S_d$, then it is nonempty. (Indeed, if the Q_i are chosen generally, then Q_i^d are linearly independent–see [14, p. 12 ff].) Henceforth we assume $s \leq \dim S_d$.

Let $G(r, S_d)$ denote the Grassmannian of r-dimensional subspaces of S_d and $\Lambda \in G(r, S_d)$ a typical point. Now consider the incidence correspondence $\Xi \subseteq G(r, S_d) \times U_s$, defined to be

$$\Xi = \{(\Lambda, \underline{Q}) : \Lambda \subseteq \operatorname{span}(Q_1^d, \dots, Q_s^d)\}. \tag{2}$$

Let Σ denote the image of the first projection $\pi_1 : \Xi \longrightarrow G(r, S_d)$. The chief preoccupation of this paper is calculating the dimension of Σ.

Remark 1.1. In general Σ may not be a quasiprojective variety. E.g., let $(n, d, r, s) = (1, 3, 1, 2)$. A binary cubic F lies in Σ, iff it is either a cube of a linear form, or has three distinct linear factors. Identify the set of cubes in $\mathbf{P}S_3$ with a twisted cubic curve C. Then its tangential developable T_C (i.e. the union of tangent lines to C) consists of forms which can be written as $Q_1^2 Q_2, (Q_i \in S_1)$. Hence

$$\Sigma = (\mathbf{P}S_3 \setminus T_C) \cup C.$$

In particular, the map $\pi_1|_\Xi$ may be dominant without being surjective. It is in general difficult to determine the smallest s such that it is surjective, and we do not address this problem here.

Definition 1.2. If $\underline{Q} \in U_s$ and $\Lambda \subseteq \operatorname{span}(Q_1^d, \dots, Q_s^d)$, then $\underline{Q}$ is called a polar s-hedron[1] of Λ.

Thus an element $\Lambda \in G(r, S_d)$ lies in Σ iff it admits a polar s-hedron. If $F_1, \dots, F_r$ span Λ, then we will speak of a polar s-hedron of the F_i.

The projection $\pi_2 : \Xi \longrightarrow U_s$ is a Grassmann bundle of relative dimension $r(s-r)$, hence $N_1 := \dim \Xi = sn + r(s-r)$. This is the number of parameters implicit in the right hand side of expression (1). Let $N_2 := \dim G(r, S_d) = r(\binom{n+d}{d} - r)$, then

$$\dim \Sigma \leq \min\{N_1, N_2\}. \tag{3}$$

We define the deficiency $\delta(\Sigma)$ as the difference $\min\{N_1, N_2\} - \dim \Sigma$. As we will see, positive deficiency is a rare phenomenon. A necessary condition for Σ to be

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[1] If $n = 2$, we will of course say polar triangle, quadrilateral etc.

dense in $G(r, S_d)$ is $N_1 \geq N_2$, i.e.,

$$s \geq \frac{r}{n+r}\binom{n+d}{d}. \tag{4}$$

If Σ is dense in G, then the general fibre of $\pi_1 : \Xi \longrightarrow \Sigma$ has dimension $N_1 - N_2$. An interesting case is $N_1 = N_2 = \dim \Sigma$, when a general Λ admits finitely many polar s-hedra. But in very few cases we know how many.

When $r = 1$, a complete answer to the problem of calculating $\dim \Sigma$ is known. Using apolarity (or equivalently Macaulay–Matlis duality), the question is reduced to a calculation of the Hilbert function of general fat points in $\mathbf{P}^n$. The final theorem is due to Alexander and Hirschowitz [1]. See [10, 14, 17] for further discussion and references.

Theorem 1.3 (Alexander–Hirschowitz)**.** *Assume $r = 1$ and $d \geq 3$. Then equality holds in* (3) *except when*

$$(n, d, s) = (2, 4, 5), (3, 4, 9), (4, 3, 7) \text{ or } (4, 4, 14).$$

For all exceptions, $\delta(\Sigma) = 1$.

The case $r = 1, d = 2$ is anomalous, in the sense that Σ is then almost always deficient. (See [12, Ch. 22] for the exact calculation.) Clebsch's discovery of the example $(2, 4, 5)$ (see [4]) was a surprise, as it showed that merely counting parameters was not sufficient to solve the problem. Thus a general planar quartic does not admit a polar pentagon, but a quartic which admits one (called a Clebsch quartic), admits at least ∞^1 of them. See [6] for some beautiful results on Clebsch quartics.

In this paper we consider the case $r > 1$, which remains open in general. Terracini's paper [21] addresses this problem, but it is not easy to follow. We know of only four examples when $r > 1$ and (3) is not an equality, viz.

$$(n, d, r, s) = (2, 3, 2, 5), (3, 2, 3, 5), (3, 2, 5, 6), (5, 2, 3, 8), \tag{5}$$

with $\delta(\Sigma) = 1$ in every case. The first two examples were classically known, see [16] for the first, and [5, p. 353], [22] for the second. The last two were found by the authors using a computer search.

The paper is organised as follows. In the next section we construct a morphism μ whose image is Σ. Then we differentiate the expression for μ to get a formula for the dimension of Σ (see Theorem 2.1). This motivates the definition of a geometric construct called a 'grove', which is, roughly speaking, a linear system of hypersurfaces with assigned singularities. In Theorem 2.6, we reinterpret the codimension of Σ as the dimension of a family of groves. In §3, we give several examples to show how geometic arguments can used to calculate $\dim \Sigma$. In the last section, we try to prove the deficiency of the four examples above using this method. For the last example, we do not succeed entirely.

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2. Groves and the dimension of Σ

2.1. An analytic representation of Σ

Let $\mathrm{Mat}^\circ(1,r;S_d)$ be the set of matrices of size $1\times r$ with entries in S_d, and columns independent over $\mathbf{C}$. (Similar definitions are understood below.) Then $G(r,S_d)$ is the quotient $\mathrm{Mat}^\circ(1,r;S_d)/GL_r(\mathbf{C})$. If $C\Sigma$ denotes the inverse image of Σ in $\mathrm{Mat}^\circ(1,r;S_d)$, then $\dim\Sigma=\dim C\Sigma - r^2$.

Consider the morphism of varieties

$$\begin{aligned} \mathrm{Mat}^\circ(1,s;S_1)\times\mathrm{Mat}^\circ(s,r;\mathbf{C}) &\xrightarrow{\mu} \mathrm{Mat}^\circ(1,r;S_d)\\ ([Q_1,\dots,Q_s],A) &\longrightarrow [Q_1^d,\dots,Q_s^d]\,A=([Z_1,\dots,Z_r]). \end{aligned} \tag{6}$$

The image of μ is $C\Sigma$, hence $\dim C\Sigma$ is the rank of the Jacobian matrix of μ at a general point in the domain of μ.

We can use this setup for a machine computation of $\dim\Sigma$. Write

$$Q_i=\sum_{j=0}^{n} q_{ij}x_j,\quad A=(\alpha_{ij}),\quad Z_k=\sum_{|I|=d} z_{k,I}\,\underline{x}^I,$$

where the q,α are indeterminates and $z_{k,I}$ polynomial functions in q,α. The Jacobian

$$\partial(z_{k,I})/\partial(q,\alpha)$$

is then easily written down, and in order to find its rank, we substitute random numbers for the q and α. We programmed this in Macaulay-2 to search for deficient examples. The search shows that in the intervals below, there are no examples of deficiencies other than those already mentioned.

- ∘ $n=2$, $2\le d\le 6$, all possible r,s (recall that $s<\dim S_d$),
- ∘ $n=3$, $2\le d\le 3$, all possible r,s,
- ∘ $n=4$, $d=2$, all possible r,s,
- ∘ $n=4$, $d=3$, $r\le 14, s\le 23$,
- ∘ $n=5$, $d=2$, all possible r,s,
- ∘ $n=5$, $d=3$, $r\le 9, s\le 34$.

A. Iarrobino pointed out that the deficient examples tend to occur for $s=n+2,n+3$, and when N_1,N_2 are close. However there are no further such examples in the following range:

$$2 \le n \le 10, 2 \le d \le 5, s = n+2, n+3, r = \left\lfloor \frac{ns}{\binom{n+d}{d} - s} \right\rfloor, \left\lceil \frac{ns}{\binom{n+d}{d} - s} \right\rceil .$$

The source code for the Macaulay–2 routine is available upon request, for which the reader should write to the second author.

2.2. A formula for $\dim \Sigma$

We will now use the morphism μ to describe a formula for $\dim \Sigma$. Let $R = \bigoplus_{d \ge 0} \operatorname{Sym}^d V^*$, so that $\mathbf{P}S_1 = \operatorname{Proj} R$. If $X \subseteq \mathbf{P}S_1 (= \mathbf{P}^n)$ is a closed subscheme, then I_X denotes its ideal and $I_X^{(2)}$ the second symbolic power of I_X.

Let $\underline{Q} = \{Q_1, \dots, Q_s\}$ be a set of s points in $\mathbf{P}S_1$. Given an $s \times r$ matrix $A = (\alpha_{ij})$ over $\mathbf{C}$, we have a morphism

$$\begin{aligned} \eta : \operatorname{Mat}(1, r; (I_{\underline{Q}})_d) &\longrightarrow \bigoplus_{i=1}^{s} R_d/(I_{Q_i}^{(2)})_d \\ [u_1, \dots, u_r] &\longrightarrow \left[\dots, \sum_{j=1}^{r} \alpha_{ij}.u_j + (I_{Q_i}^{(2)})_d, \dots \right]_{1 \le i \le s} \end{aligned} \tag{7}$$

Of course η depends on the choice of $A, \underline{Q}$, but we will write $\eta_{A,\underline{Q}}$ only if confusion is otherwise likely.

Theorem 2.1. *With notation as above, assume that the points $\underline{Q}$ and the matrix A are general. Then*

$$\operatorname{codim}(\Sigma, G(r, S_d)) = \dim \ker(\eta). \tag{8}$$

The proof uses the classical notion of apolarity. We introduce the essentials, see e.g. [8, 10, 11, 14] for details.

2.3. Apolarity

Recall that

$$R = \bigoplus_{d \ge 0} \operatorname{Sym}^d V^*, \quad S = \bigoplus_{d \ge 0} \operatorname{Sym}^d V.$$

Let $\{x_0, ..., x_n\}$ and $\{\partial_0, ..., \partial_n\}$ be the dual bases of V and V^* respectively. We interpret a polynomial $u(\partial_0, ..., \partial_n)$ in R as the differential operator $u(\frac{\partial}{\partial x_0}, ..., \frac{\partial}{\partial x_n})$.

Then we have maps $R_p \circ S_q \longrightarrow S_{q-p}$, and thus S acquires the structure of an R–module.

For a subspace $W \subseteq S_d$, let

$$W^{\perp} = \{u \in R_d : u \circ F = 0 \text{ for every } F \in W\},$$

which is a subspace of R_d, such that $\dim W^{\perp} + \dim W = \dim S_d$. In classical terminology, if $u \circ F = 0$ and $\deg u \leq \deg F$, then u, F are said to be apolar to each other. Thus $W^{\perp}$ is the set of differential operators in R_d, which are apolar to all forms in W.

In the following two instances $W^{\perp}$ can be concretely described (see [14, Lemma 2.2]). Let $Q \in S_1$ be a nonzero linear form, or equivalently a point in $\mathbf{P}S_1$.

i. If $W = \operatorname{span}(Q^d)$, then $W^{\perp} = (I_Q)_d$.

ii. If $W = \{Q^{d-1}Q' : Q' \in S_1\}$, then $W^{\perp} = (I_Q^{(2)})_d$.

Proof of Theorem 2.1. We will calculate the map on tangent spaces for the morphism μ in (6). Fix a general point $([Q_1, \dots, Q_s], A)$. Given arbitrary forms $Q'_1, \dots, Q'_s \in S_1$ and $B \in \operatorname{Mat}(s, r; \mathbf{C})$, we have

$$\begin{aligned} &\mu([Q_1 + \epsilon Q'_1, \dots, Q_s + \epsilon Q'_s], A + \epsilon B) - \mu([Q_1, \dots, Q_s], A) \\ &= \epsilon\{[Q_1^d, \dots, Q_s^d]B + d[Q_1^{d-1}Q'_1, \dots, Q_s^{d-1}Q'_s]A\} + O(\epsilon^2). \end{aligned}$$

Hence the tangent space to $C\Sigma$ at the point $\mu([Q_1, \dots, Q_s], A)$ is described as

$$\begin{aligned} \mathbb{T} = \{[Q_1^d, \dots, Q_s^d]B + [Q_1^{d-1}Q'_1, \dots, Q_s^{d-1}Q'_s]A :& \\ Q'_1, \dots, Q'_s \in S_1, B \in \operatorname{Mat}(s, r; \mathbf{C})\}&. \end{aligned}$$

Now $\dim \mathbb{T} = \dim C\Sigma = \dim \Sigma + r^2$. Define maps

$$\begin{aligned} \alpha : \operatorname{Mat}(1, s; S_1) &\longrightarrow \operatorname{Mat}(1, r; S_d) \\ [Q'_1, \dots, Q'_s] &\longrightarrow [Q_1^{d-1}Q'_1, \dots, Q_s^{d-1}Q'_s]A, \quad \text{and} \\ \beta : \operatorname{Mat}(s, r; \mathbf{C}) &\longrightarrow \operatorname{Mat}(1, r; S_d) \\ B &\longrightarrow [Q_1^d, \dots, Q_s^d]B, \end{aligned}$$

so that $\mathbb{T} = \operatorname{image} \alpha + \operatorname{image} \beta$. After dualising, we have a diagram

$$\begin{tikzcd} \operatorname{Mat}(1, r; R_d) \arrow[r, "\alpha^*"] \arrow[d, "\beta^*"'] & \operatorname{Mat}(1, s; R_1) \\ \operatorname{Mat}(s, r; \mathbf{C}) & \end{tikzcd}$$

Now $\underline{u} = [u_1, \dots, u_r] \in \ker \beta^* \iff$ for every $[F_1, \dots, F_r] \in \operatorname{image} \beta$, we have $u_i \circ F_i = 0$ for all i. For any pair of indices $1 \leq i_1, i_2 \leq r$, one can certainly arrange B such that $F_{i_1} = Q_{i_2}^d$. Thus $\underline{u} \in \ker \beta^*$ iff each u_i lies in $\cap_j \operatorname{span}(Q_j^d)^{\perp} = \cap_j (I_{Q_j})_d = (I_{\underline{Q}})_d$. Hence $\ker \beta^* = \operatorname{Mat}(1, r; (I_{\underline{Q}})_d)$.

By analogous reasoning, an element $\underline{u} \in \ker \beta^*$ will be in $\ker \alpha^*$ iff it annihilates all elements in $\operatorname{image} \alpha$, i.e., iff for every i, the operator $\sum_{j=1}^{r} \alpha_{ij}.u_j$ is apolar to $\{Q_i.Q' : Q' \in S_1\}$. Thus with the natural inclusion

$$\operatorname{Mat}(1, r; (I_{\underline{Q}})_d) \subseteq \operatorname{Mat}(1, r; R_d),$$

we have $\ker \eta = \ker \alpha^* \cap \ker \beta^*$. Finally

$$\begin{aligned}
\dim \ker \eta = {} & \dim \ker \alpha^* + \dim \ker \beta^* - \dim(\ker \alpha^* + \ker \beta^*) \\
& = (r \dim R_d - \dim \operatorname{image} \alpha) + (r \dim R_d - \dim \operatorname{image} \beta) \\
& -(r \dim R_d - \dim(\operatorname{image} \alpha \cap \operatorname{image} \beta)) \\
& = r \dim R_d - \dim(\operatorname{image} \alpha + \operatorname{image} \beta) \\
& = r \dim R_d - \dim \mathbb{T} = r \dim R_d - \dim \Sigma - r^2 = \dim G(r, S_d) - \dim \Sigma.
\end{aligned}$$

The theorem is proved. $\square$

If $r = 1$, then $\ker \eta = (I^{(2)}_{\underline{Q}})_d$. Hence we recover the formula (see [10, Theorem 6.1])

$$\dim \Sigma = \dim (R/I^{(2)}_{\underline{Q}})_d - 1. \tag{9}$$

Remark 2.2. Since $\dim \ker \eta$ is upper semicontinuous in the variables $A, \underline{Q}$ (see [13, p. 125, exer. 5.8])

$$\dim \ker \eta \geq \operatorname{codim} \Sigma \geq \max\{0, N_2 - N_1\}$$

for *any* choice of A and $\underline{Q}$. Hence if the first and the last terms coincide for some choice, then it follows that Σ is not deficient.

We will reformulate this theorem geometrically. In the sequel, assume that $\underline{Q} = \{Q_1, \dots, Q_s\}$ are points in a *fixed* copy of $\mathbf{P}^n$ ($\mathbf{P}S_1$ if you will) and similarly $\underline{p} = \{p_1, \dots, p_s\}$ are points in $\mathbf{P}^{r-1}$. Moreover, assume that the p_i span $\mathbf{P}^{r-1}$.

Definition 2.3. A *grove*[2] for the data $(\underline{p}, \underline{Q})$ consists of a triple (Γ, L, γ) such that

- $\Gamma \subseteq \mathbf{P}H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}}(d))$ is a linear system of dimension (say) $t \leq r - 1$,
- $L \subseteq \mathbf{P}^{r-1}$ is a linear space of dimension $r-(t+2)$ (thus defining a projection $\pi_L : \mathbf{P}^{r-1} - \to \mathbf{P}^t$), and
- $\gamma : \mathbf{P}^t \xrightarrow{\sim} \Gamma$ is an isomorphism,

satisfying the following conditions:

- all the Q_i belong to the base locus of Γ,
- for every i, either $p_i \in L$ or the hypersurface $\gamma \circ \pi_L(p_i)$ is singular at Q_i.

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[2] After some fitful experimentation, we decided to choose a name devoid of any mathematical associations.

We denote the collection of all groves by $\amalg(\underline{p}, \underline{Q})$. By the proposition below, it has a natural structure of a projective space.

Remark 2.4. To make the definition of π_L canonical, identify $\mathbf{P}^t$ with the set of linear subspaces of dimension $r-(t+1)$ containing L, and then let $\pi_L(p) = \overline{Lp}$. By our assumption, L misses at least one p_i.

If $t = r-1$, then L is empty and π_L the identity map. Then Γ is an $(r-1)$-dimensional system of degree d hypersurfaces passing through $\underline{Q}$, such that $\gamma(p_i)$ is singular at Q_i. (In the applications, almost always this will be the case.) If $r = 1$, then necessarily $t = 0, L = \emptyset$ and all p_i are the same point. Then a grove is a solitary hypersurface of degree d singular at all Q_i.

For the next proposition, we identify $\mathbf{P}^{r-1}$ with $\mathbf{P}\,\mathrm{Mat}(1, r; \mathbf{C})$. If $A \in \mathrm{Mat}(s, r; \mathbf{C})$ is a matrix with no zero rows, then we identify its i-th row as the point $p_i \in \mathbf{P}^{r-1}$.

Proposition 2.5. *Fix points $Q_1, \ldots, Q_s$ in $\mathbf{P}^n$, and assume that the p_i span $\mathbf{P}^{r-1}$ (hence* rank $A = r$*). Then with identifications as above, we have a bijection $\mathbf{P}(\ker \eta_{A,\underline{Q}}) \simeq \amalg(\underline{p}, \underline{Q})$.*

Proof. Let $\underline{u} = [u_1, \ldots, u_r]$ be a nonzero element of $\ker \eta$. Let Γ be the linear system generated by the u_i, and

$$L = \{X \in \mathrm{Mat}(1, r; \mathbf{C}) : [u_1, \ldots, u_r]\, X^t = 0\}.$$

Then π_L appears as the map

$$\mathbf{P}\,\mathrm{Mat}(1, r; \mathbf{C}) \dashrightarrow \mathbf{P}(\mathrm{Mat}(1, r; \mathbf{C})/L)(= \mathbf{P}^t).$$

Define

$$\gamma : \mathbf{P}^t \xrightarrow{\sim} \Gamma, \quad X + (L) \longrightarrow [u_1, \ldots, u_r]\, X^t.$$

By hypothesis, the form $\alpha_{i1}u_1 + \ldots \alpha_{ir}u_r$ lies in $(I^{(2)}_{Q_i})_d$. Hence, unless it is identically zero (i.e., $p_i = [\alpha_{i1}, \ldots, \alpha_{ir}] \in L$), the hypersurface it defines (which is $\gamma \circ \pi_L(p_i)$) is singular at Q_i.

Alternately, let (Γ, L, γ) be a grove for $\underline{p}, \underline{Q}$. Since Γ contains $\underline{Q}$ in its base locus, we can lift $\gamma : \mathbf{P}(\mathrm{Mat}(1, r)/L) \xrightarrow{\sim} \Gamma$ to an inclusion $\hat{\gamma} : \mathrm{Mat}(1, r)/L \longrightarrow (I_{\underline{Q}})_d$. Let $u_j = \hat{\gamma}([0, \ldots, 1, 0, \ldots])$ (the 1 in j-th place), for $1 \le j \le r$. By hypothesis, if $\pi_L(p_i)$ is defined, then $\gamma \circ \pi_L(p_i)$ is singular at Q_i. This is to say that $\sum\limits_{j=1}^{r} \alpha_{ij}u_j = \hat{\gamma}(p_i)$ lies in $I^{(2)}_{Q_i}$, i.e., $\underline{u} = [u_1, \ldots, u_r] \in \ker \eta$. Since the lift is unique up to a global scalar, $\underline{u}$ is well-defined as a point of $\mathbf{P}(\ker \eta)$. (Since rank $A = r$, $\underline{u} \neq 0$.) This defines the required bijection. □

The next result follows directly from Theorem 2.1. Nearly all subsequent results are based on this reformulation.

Theorem 2.6. *Let points $p_1, \dots, p_s \in \mathbf{P}^{r-1}$ and $Q_1, \dots, Q_s \in \mathbf{P}^n$ be chosen generally. Then Σ has codimension c in $G(r, S_d)$ if and only if, there are exactly ∞^{c-1} groves for $(\underline{p}, \underline{Q})$. In particular, Σ is dense in $G(r, S_d)$ if and only if, the points $(\underline{p}, \underline{Q})$ do not admit a grove.*

In the paper of Terracini cited above, he states something which resembles the last statement in the theorem. Unfortunately, neither his statement nor the argument leading to it are clear.

In the case $r = 1$, we recover the criterion of Ehrenborg and Rota [8, Theorem 4.2].

Corollary 2.7 (Ehrenborg, Rota)**.** *A general form in S_d cannot be written as a sum of d-th powers of s linear forms if and only if, given general points $Q_1, \dots, Q_s$ in $\mathbf{P}^n$, there exists a hypersurface of degree d singular at all of them.*

Now consider the collection

$$\amalg^\circ(\underline{p}, \underline{Q}) = \{(\Gamma, L, \gamma) : L \text{ contains none of the } p_i\}.$$

Lemma 2.8. *Assume that the points $(\underline{p}, \underline{Q})$ are general. Then $\amalg^\circ(\underline{p}, \underline{Q})$ is a nonempty Zariski open subset of $\amalg(\underline{p}, \underline{Q})$.*

Hence for purposes of calculating $\dim \Sigma$, we can assume that our groves lie in $\amalg^\circ$.

Proof. Let $\amalg_i \in \mathbf{P}(\ker \eta)$ be the open set of groves where $p_i \notin L$, then $\amalg^\circ = \cap_i \amalg_i$. Thus $\amalg^\circ$ fails to be dense only if some $\amalg_i$ is empty. But then by symmetry (here is where the generality is used) each $\amalg_i$ is empty, implying that every L contains all the p_i. Since the set $\underline{p}$ spans $\mathbf{P}^{r-1}$, this is impossible. $\square$

From Remark 2.2, we know that

$$\dim \amalg(\underline{p}, \underline{Q}) \geq \operatorname{codim} \Sigma - 1 \geq \max\{0, N_2 - N_1\} - 1,$$

for any choice of points $(\underline{p}, \underline{Q})$. If the end terms are equal for some configuration of points, then Σ is not deficient.

3. Examples

In this section we give a rather large number of examples illustrating the use of Theorem 2.6. All the results follow the same plan: we choose specific values of (n, d, r, s), then calculate the dimension of $\amalg$ and hence that of Σ. The choice of quadruples (n, d, r, s) does not follow any definite pattern, but we have given examples which we think are geometrically interesting. Some of the results proved here are known, and the novelty lies in the method used to obtain them.

We refer to [12] for the miscellaneous geometric facts needed. We mention two which will be used frequently. Recall that a set of points in $\mathbf{P}^N$ is said to be in linearly general position if no subset of m points ($m \leq N+1$) is contained in a $\mathbf{P}^{m-2}$.

- Given two sequences $\{A_1, \dots, A_{n+2}\}, \{B_1, \dots, B_{n+2}\} \subseteq \mathbf{P}^n$ in linearly general position, there is a unique automorphism γ of $\mathbf{P}^n$, such that $\gamma(A_i) = B_i$ for all i.
- Given $n+3$ points of $\mathbf{P}^n$ in linearly general position, there is a unique rational normal curve passing through all of them.

For every case treated in this section, $\dim \Sigma$ will coincide with the expected value $\min\{N_1, N_2\}$. The deficient examples are the subject of the next section.

The following result should be classically known, but we have been unable to trace a reference.

Theorem 3.1. *If $n = 1$, then Σ is not deficient for any d, r, s.*

Proof. Let $Q_1, \dots, Q_s$ and $A = (\alpha_{i,j})$ be as above. Consider the composite map of vector bundles on $\mathbf{P}^1$:

$$\rho_A : \left\{\mathcal{O}_{\mathbf{P}^1}\left(dH - \sum Q_i\right)\right\}^{\oplus r} \longrightarrow \{\mathcal{O}_{\mathbf{P}^1}(dH)\}^{\oplus r} \xrightarrow{\tilde{\eta}} \bigoplus_{i=1}^{s} \mathcal{O}_{2Q_i}(dH)$$

Here H denotes the hyperplane divisor on $\mathbf{P}^1$. The map on the left is the canonical inclusion, and the one on the right is induced by A. On local sections,

$$(\tilde{\eta}[u_1, \dots, u_r])_i = \sum_{j=1}^{r} \alpha_{ij} u_j, \text{ modulo functions vanishing to order at least 2 at } Q_i.$$

The map $H^0(\mathbf{P}^1, \rho_A)$ is identical to η in formula (7). Hence if $\mathcal{E} = \ker \rho_A$, then $h^0(\mathcal{E}) = \operatorname{codim} \Sigma$. The image of ρ_A is the skyscraper sheaf

$$\bigoplus_i \ker\left(\mathcal{O}_{2Q_i}(dH) \longrightarrow \mathcal{O}_{Q_i}(dH)\right) = \bigoplus_i \mathcal{O}_{Q_i}(dH - Q_i)$$

with degree s, hence $\mathcal{E}$ is a rank r-vector bundle of degree $\epsilon = r(d-s) - s$.

Now specialise A to the following matrix: write $s = r\alpha + \beta$, with $0 \leq \beta \leq r-1$ and let

$$A^t = [B_1 | \dots | B_{r-\beta} | C_{r+1-\beta} | \dots | C_r], \text{ where}$$

- the B_i (resp. C_i) are blocks of size $r \times \alpha$ (resp. $r \times (\alpha+1)$),
- each B_i or C_i is made of all 1's in the i-th row and zeros elsewhere.

Then $\mathcal{E}$ splits as a direct sum

$$\mathcal{O}_{\mathbf{P}^1}(d - \alpha - s)^{\oplus(r-\beta)} \oplus \mathcal{O}_{\mathbf{P}^1}(d - \alpha - s - 1)^{\oplus\beta}. \tag{10}$$

Now $N_1 = s + r(s-r)$ and $N_2 = r(d-r+1)$, so $N_2 - N_1 = \epsilon + r$. If $N_2 \leq N_1$, then all twists in (10) are negative, so $h^0(\mathcal{E}) = 0$. If $N_2 > N_1$, then all twists are at least -1, so $h^0(\mathcal{E}) = N_2 - N_1$. In either case $\operatorname{codim} \Sigma = \max\{0, N_2 - N_1\}$, hence by Remark 2.2 we are through. □

Remark 3.2. Fix points $\underline{Q}$, and think of $\mathcal{E}$ as moving in a family parametrised by A. By Grothendieck's theorem, $\mathcal{E}$ splits into a direct sum of line bundles. The point of the theorem is that if A is general, then its splitting type is balanced, i.e., it deviates from the sequence $(\deg \mathcal{E}/\operatorname{rank} \mathcal{E}, \dots, \deg \mathcal{E}/\operatorname{rank} \mathcal{E})$ as little as possible. Once the splitting type is known, $h^0(\mathcal{E})$ is known.

Example 3.3. This example might give some insight into the construction of A. Let $r = 3, s = 7$, so $\alpha = 2, \beta = 1$. Then

$$A^t = \begin{bmatrix} 1\,1\,0\,0\,0\,0\,0 \\ 0\,0\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,1\,1 \end{bmatrix}$$

and $\tilde{\eta}([u_1, u_2, u_3]) = [u_1, u_1, u_2, u_2, u_3, u_3, u_3]$. Thus a local section $[u_1, u_2, u_3]$ will lie in $\ker \rho_A$ iff u_1 (resp. u_2 and u_3) vanishes doubly at Q_1, Q_2 (resp. at Q_3, Q_4 and Q_5, Q_6, Q_7). Hence $\mathcal{E}$ is a direct sum

$$\mathcal{O}_{\mathbf{P}^1}\left(dH - Q_1 - Q_2 - \sum Q_i\right) \oplus \mathcal{O}_{\mathbf{P}^1}\left(dH - Q_3 - Q_4 - \sum Q_i\right) \oplus$$
$$\mathcal{O}_{\mathbf{P}^1}\left(dH - Q_5 - Q_6 - Q_7 - \sum Q_i\right).$$

Henceforth we use the same notation for a form $F \in S_d$ and the hypersurface in $\mathbf{P}R_1$ which it defines.

Proposition 3.4. *Two general plane conics have a unique polar triangle.* ($N_1 = N_2 = 8$.)

Firstly we will show that dim $\Sigma(2, 2, 2, 3) = 8$. Choose general points $p_1, p_2, p_3 \in \mathbf{P}^1, Q_1, Q_1, Q_3 \in \mathbf{P}^2$, and let $(\Gamma, L, \gamma) \in \Pi^\circ$ be a grove. Since there is no conic singular at all Q_i, $\dim \Gamma = 1$ and $L = \emptyset$. Now $\gamma(p_1)$ must be the line pair $Q_1Q_2 + Q_1Q_3$ and similarly for other p_i. Since any two elements $\gamma(p_i), \gamma(p_j)$ span Γ, all the three lines Q_iQ_j are in the base locus of Γ. This is absurd, hence there is no such grove.

Consequently, two general conics F_1, F_2 admit at least one polar triangle–say $\{Q_1, Q_2, Q_3\}$.[3] Now the pencil generated by the F_i contains a member belonging to $\operatorname{span}(Q_1^2, Q_2^2)$, and this member must be singular at the point $Q_1 \cap Q_2$. Hence the points $Q_i \cap Q_j$ must be the vertices of the three line pairs contained in the pencil.

---

[3] These Q_i are unrelated to those in the previous paragraph. By the nature of our arguments, the Q_i lead a double life: they are alternately linear forms and points.

This gives a geometric construction of the polar triangle and simultaneously shows that it is unique:

Let F_1, F_2 intersect in $\{Z_1, \ldots, Z_4\}$. Let A_1 be the point of intersection of the lines Z_1Z_2, Z_3Z_4, and similarly $A_2 = Z_1Z_3 \cap Z_2Z_4, A_3 = Z_1Z_4 \cap Z_2Z_3$. Define lines $Q_1 = A_2A_3, Q_2 = A_1A_3, Q_3 = A_1A_2$. Then $\{Q_1, Q_2, Q_3\}$ is the required triangle. □

Proposition 3.5. *Four general plane conics $F_1, \ldots, F_4$ have a unique polar quadrilateral.* $(N_1 = N_2 = 8.)$

Proof. Firstly let us show that $\Sigma\,(2,2,4,4)$ is dense in $G(4, S_2)$. Let $p_1, \ldots, p_4 \in \mathbf{P}^3, Q_1, \ldots, Q_4 \in \mathbf{P}^2$ be chosen generally, and $(\Gamma, L, \gamma) \in \amalg^\circ(\underline{p}, \underline{Q})$. Since there is no conic singular at all Q_i, we must have $\dim \Gamma = 1$. Then Γ is the pencil of conics through $\underline{Q}$, which has no members singular at any Q_i. This precludes any possibility of defining γ.

Thus four general conics $F_1, \ldots, F_4$ admit at least one polar quadrilateral, say $\{Q_1, \ldots, Q_4\}$. We may assume that Q_i are in linearly general position. Let $A = [\alpha_0, \alpha_1, \alpha_2]$ be the point of intersection of the lines Q_1, Q_2. (Thus as an element of $\mathbf{P}R_1$, $A = \alpha_0\partial_0 + \alpha_1\partial_1 + \alpha_2\partial_2$ up to a scalar). By hypothesis,

$$F_1 = c_1Q_1^2 + \cdots + c_4Q_4^2, \quad \text{for some constants } c_i.$$

Operate by A on the equality above, then

$$A \circ F_1 = \sum 2c_iQ_i(A)Q_i.$$

Now $Q_1(A) = Q_2(A) = 0$, hence $A \circ F_1$ (the polar line of F_1 with respect to A) belongs to the pencil generated by lines Q_3, Q_4. An identical argument applies to all F_i, hence we deduce that the four lines $A \circ F_1, \ldots, A \circ F_4$ are concurrent at the point $Q_3 \cap Q_4$. The line $A \circ F_i$ has equation

$$\frac{\partial F_i}{\partial x_0}(A)\, x_0 + \frac{\partial F_i}{\partial x_1}(A)\, x_1 + \frac{\partial F_i}{\partial x_2}(A)\, x_2 = 0,$$

hence the Jacobian matrix $J = \partial(F_1, \ldots, F_4)/\partial(x_0, x_1, x_2)$, has rank at most two at A.

Now consider the locus $X = \{\operatorname{rank} J \leq 2\} \subseteq \mathbf{P}^2$. It is easily seen that X must be a finite set. Hence we have a Hilbert–Burch (or Eagon–Northcott) resolution

$$0 \longrightarrow S(-4)^3 \longrightarrow S(-3)^4 \longrightarrow S \longrightarrow S/I_X \longrightarrow 0.$$

From the resolution (or the Porteous formula), we have $\deg X = 6$. By the argument above X contains the points $Q_i \cap Q_j$, so it can contain no others.

We claim that this forces the polar quadrilateral to be unique. Indeed let M_1 be a side of such a quadrilateral. The argument shows that M_1 must contain three of the points from X. This is impossible unless M_1 coincides with one of the Q_i. □

Proposition 3.6. *The variety* $\Sigma\,(2,2,3,3)$ *has dimension* 6. ($N_1 = 6$, $N_2 = 9$.)

Proof. Let $p_1, p_2, p_3, Q_1, Q_2, Q_3$ be general points in $\mathbf{P}^2$. We will show that $\underline{p}, \underline{Q}$ admit exactly ∞^2 groves. Let $(\Gamma, L, \gamma) \in \Pi^\circ$. Let G_1 be the line pair $Q_1Q_2 + Q_1Q_3$, and similarly for G_2, G_3. Evidently each G_i belongs to Γ, hence $\Gamma = \operatorname{span}(G_1, G_2, G_3)$ and $L = \emptyset$. Thus the only moving part of the grove is γ, and Π° is isomorphic to the variety

$$\{\gamma : \mathbf{P}^2 \xrightarrow{\sim} \Gamma \text{ such that } \gamma(p_i) = G_i \text{ for } i = 1, 2, 3\}.$$

Fix a point $Z \in \mathbf{P}^2$ such that p_1, p_2, p_3, Z are in linearly general position. Then γ is entirely determined by $\gamma(Z)$, so Π° is isomorphic to an open set of $\mathbf{P}^2$. □

We will frequently use Bézout's theorem in the following form: if a hypersurface of degree d intersects a curve of degree e in a scheme of length $> de$, then it must contain the curve. In such a circumstance we will loosely say that the hypersurface contains at least $de + 1$ points of the curve.

Theorem 3.7 (Sylvester's pentahedral theorem)**.** *A general cubic surface in* $\mathbf{P}^3$ *has a polar pentahedron.* ($N_1 = N_2 = 19$.)

The statement says that $\Sigma\,(3,3,1,5)$ is dense in $\mathbf{P}^{19}$, and it is covered by the Alexander–Hirschowitz theorem. We give a short geometric proof.

Proof. Choose general points $Q_1, \dots, Q_5$ in $\mathbf{P}^3$ and assume that a cubic F is singular at all of them. Choose a sixth general point Z and let C be the unique twisted cubic through $Q_1, \dots, Q_5, Z$. Since F contains at least 10 points of C (counting each Q_i as two points), it must contain C by Bézout's theorem. This implies the absurdity that F contains a general point of $\mathbf{P}^3$. Hence there is no such F and the claim is proved. □

In [20], Sylvester asserted that a general quaternary cubic has a unique polar pentahedron, and adduced some cryptic remarks in support. See [17] for a proof of the uniqueness.

The next result is a direct generalisation of Proposition 3.4.

Theorem 3.8. *The variety* $\Sigma(n, 2, 2, n+1)$ *is dense in* $G(2, S_2)$*, moreover two general quadrics in* $\mathbf{P}^n$ *admit a unique polar* $(n+1)$*-hedron.* ($N_1 = N_2 = n^2 + 3n - 2$.)

Proof. Choose general points $p_1, \dots, p_{n+1} \in \mathbf{P}^1$, and $Q_1, \dots, Q_{n+1} \in \mathbf{P}^n$ and let $(\Gamma, L, \gamma) \in \Pi^\circ$. There is no quadric singular at all Q_i (since the singular locus of a quadric is a linear space, and the $\underline{Q}$ are not contained in any proper linear subspace), hence $\dim \Gamma = 1$ and $L = \emptyset$. The quadric $\gamma(p_i)$ contains at least three points of the line Q_iQ_j (viz. Q_i twice and Q_j), so it must contain the line. Since any two quadrics $\gamma(p_i), \gamma(p_j)$ span Γ, it follows that all the lines Q_iQ_j lie in the base locus of Γ.

Let $F \in \Gamma$ and $F(-,-)$ its associated bilinear form. By what we have said, $F(Q_i + \lambda Q_j, Q_i + \lambda Q_j) = 0$ for all $\lambda \in \mathbf{C}$, hence $F(Q_i, Q_j) = 0$. Since the Q_i span $\mathbf{P}^n$, we have $F \equiv 0$. This is absurd, so $(\underline{p}, \underline{Q})$ do not admit a grove.

The proof of uniqueness is similar to Proposition 3.5. Let F_1, F_2 be general quadrics in $\mathbf{P}^n$ admitting a polar $(n+1)$-hedron $\{Q_1, \dots, Q_{n+1}\}$. Define points $A_i = \bigcap\limits_{j \neq i} Q_j \in \mathbf{P}^n$ for $1 \le i \le n+1$. For any i, the polar hyperplanes $A_i \circ F_1, A_i \circ F_2$ coincide, hence the Jacobian matrix $J = \partial(F_1, F_2)/\partial(x_0, \dots, x_n)$ must have rank one at each A_i. Now let $X = \{\operatorname{rank} J \le 1\}$, and use Hilbert–Burch together with Porteous to show that $X = \{A_1, \dots, A_{n+1}\}$. Then Q_i is uniquely determined as the linear span of the points A_j $(j \neq i)$. □

Remark 3.9. Before proceeding we record a small construction for later use. Let C be a twisted cubic in $\mathbf{P}^3$, and let $\Psi \subseteq \mathbf{P}H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}}(2))$ be the two-dimensional linear system of quadrics containing C. For every $x \in C$, there is a unique quadric (say ψ_x) in Ψ singular at x. Thus we have an imbedding

$$\tau : C \longrightarrow \Psi, \quad x \longrightarrow \psi_x.$$

Its image $\tau(C)$ is a smooth conic in Ψ.

This notation will come in force only when we explicitly refer to this remark. Otherwise C, Ψ etc may have unrelated meanings.

The following technical result will be useful later.

Lemma 3.10. *Let $f, v : \mathbf{P}^1 \longrightarrow \mathbf{P}^2$ be two morphisms. Assume that f is birational onto its image which is a curve of degree m, and v is an imbedding onto a smooth conic. Assume moreover, that there are $m+2$ points $\lambda_1, \dots, \lambda_{m+2}$ in $\mathbf{P}^1$, such that $f(\lambda_i) = v(\lambda_i)$ for all i. Then $v = f$.*

Proof. Choose a coordinate x on $\mathbf{P}^1$ such that $\lambda_{m+2} = \infty$. We may choose coordinates on $\mathbf{P}^2$ such that $v(x) = [1, x, x^2]$. Then $f(x) = [A_0, A_1, A_2]$, such that A_i are polynomials in x with no common factor and $\deg A_i \le m$. By hypothesis, $f(\infty) = [0, 0, 1]$, hence $\deg A_2 > \deg A_1, \deg A_0$. In particular, $\deg A_0 \le m-1$. Now the polynomial $A_1 - xA_0$ (which is of degree $\le m$), vanishes for $m+1$ values $\lambda_1, \dots, \lambda_{m+1}$, hence it vanishes identically. But then $\deg A_0 \le m-2$. By the same argument, $A_2 - x^2 A_0$ vanishes identically, hence $[A_0, A_1, A_2] = [1, x, x^2]$. □

Remark 3.11. If C is a curve isomorphic to $\mathbf{P}^1$ and $A_1, \dots, A_4$ distinct points on C, then $\langle A_1, A_2, A_3, A_4 \rangle_C$ will denote their cross-ratio as calculated on C. Of course, it depends on the choice C, for instance four points in $\mathbf{P}^2$ have different cross-ratios as calculated on different smooth conics passing through them.

In 1870, Darboux claimed that the case $\Sigma(3,2,4,6)$ is deficient (see [5, p. 357]). In [21], Terracini states (without proof) that Darboux's claim is wrong, and in fact there is no deficiency. Here we substantiate Terracini's statement.

Proposition 3.12. *The variety* $\Sigma(3,2,4,6)$ *is dense in* $G(4,S_2)$. ($N_1 = 26, N_2 = 24$.)

Proof. Choose general points $(\underline{p},\underline{Q})$ as usual, where $\underline{p}$ and $\underline{Q}$ lie in nominally distinct copies of $\mathbf{P}^3$. We can identify the copies in such a way that the following holds: $p_1,\dots,p_6,Q_1,\dots,Q_6$ are in the same $\mathbf{P}^3$ so that $p_i = Q_i$ for $1 \le i \le 5$ and p_6, Q_6 are distinct general points.

Let $(\Gamma, L, \gamma) \in \amalg^\circ(\underline{p},\underline{Q})$, and let C be the unique twisted cubic through the $\underline{Q}$. The quadric $\gamma \circ \pi_L(p_i)$ intersects C in at least seven points, so must contain C. Hence necessarily $\gamma \circ \pi_L(p_i) = \psi_{Q_i}$ in the notation of Remark 3.9. Thus $\Gamma = \Psi$ and L is a point in $\mathbf{P}^3$. Let $\mathbf{P}^2_{\langle L\rangle}$ be the set of lines through L (cf. Remark 2.4), so we have a map $\mathbf{P}^2_{\langle L\rangle} \xrightarrow{\gamma} \Psi$.

Now there are two maps $C \longrightarrow \mathbf{P}^2_{\langle L\rangle}$, namely π_L and $\gamma^{-1}\circ\tau$. The image of the latter (say D) is a smooth conic. Moreover, $\deg \operatorname{image}(\pi_L) \le 3$ and the two maps coincide on points $p_1,\dots,p_5\,(= Q_1,\dots,Q_5)$. Hence by Lemma 3.10, they must be the same. In particular, $\deg \pi_L(C) = 2$ which is only possible if L is a point *on* C. We claim that $\pi_L(p_6) = \pi_L(Q_6)$. Indeed, since π_L is an isomorphism on C,

$$\begin{aligned}
&\langle \pi_L(p_1), \pi_L(p_2), \pi_L(p_3), \pi_L(p_6)\rangle_D\\
&= \langle \psi_{Q_1}, \psi_{Q_2}, \psi_{Q_3}, \psi_{Q_6}\rangle_{\tau(C)},\\
&= \langle Q_1, Q_2, Q_3, Q_6\rangle_C\\
&= \langle \pi_L(Q_1), \pi_L(Q_2), \pi_L(Q_3), \pi_L(Q_6)\rangle_D
\end{aligned}$$

which shows the claim. This implies that the chord LQ_6 (in case $L \ne Q_6$) or the tangent to C at L (in case $L = Q_6$) passes through p_6. Now for a fixed Q_6, the chords $\{LQ_6\}_{L\in C}$ fill only a surface in $\mathbf{P}^3$. Hence if we choose p_6 off this surface, then no such configuration can exist. Thus general points $(\underline{p},\underline{Q})$ do not admit a grove, which proves the proposition. It follows that four general space quadrics have ∞^2 polar 6-hedra. ☐

Proposition 3.13. *The variety* $\Sigma\,(4,2,2,4)$ *has dimension* 20. ($N_1 = 20, N_2 = 26$.)

Proof. Choose general points $p_1,p_2,p_3,p_4 \in \mathbf{P}^1$ and $Q_1,\dots,Q_4 \in \mathbf{P}^4$. We will show that there are exactly ∞^5 groves for these data. Let Π denote the 3–space spanned by the Q_i, and choose $(\Gamma, L, \gamma) \in \amalg^\circ$. If $\dim\Gamma = 0$, then Γ is Π doubled, and L any point on $\mathbf{P}^1$. Since this is only a one-dimensional family, we may assume $\dim\Gamma = 1, L = \emptyset$.

Each of the quadrics $\gamma(p_i), \gamma(p_j)$ contains three points of the line Q_iQ_j, hence contains the line. Since these quadrics span Γ, all six lines Q_iQ_j are in the base locus of Γ. This forces Π to be in the base locus. Hence there exists a unique 2-plane $\Psi_\Gamma \subseteq \mathbf{P}^4$, such that

$$\Gamma = \Pi \,(\text{fixed component}) + \text{ pencil of 3-planes through } \Psi_\Gamma.$$

This leads to the following construction: let $\Psi \in G(3,5)$ be a 2–plane in $\mathbf{P}^4$ away from the Q_i and let $\psi_1, \dots, \psi_4$ be the 3–planes through Ψ containing the points $Q_1, \dots, Q_4$ respectively. Now we have a rational map

$$f : G(3,5) - \rightarrow \mathbf{P}^1, \quad \Psi \longrightarrow \langle \psi_1, \psi_2, \psi_3, \psi_4 \rangle.$$

It is easy to see that f is nonconstant, hence dominant. Now if Ψ belongs to the fibre $f^{-1}(\langle p_1, p_2, p_3, p_4 \rangle)$, then (and only then) we can define

$$\gamma : \mathbf{P}^1 \xrightarrow{\sim} \Gamma, \quad p_i \longrightarrow \Pi + \overline{\Psi Q_i} \text{ for } i = 1, \dots, 4.$$

Thus Π° is birational to the fibre $f^{-1}(\langle p_1, p_2, p_3, p_4 \rangle)$, which is five dimensional. □

Proposition 3.14 (London [16])**.** *The variety* $\Sigma(2,3,3,6)$ *is dense in* $G(3,S_3)$*, i.e., three general plane cubics admit a polar hexagon.* ($N_1 = N_2 = 21$.)

London's proof is laborious, and it may be doubted whether it meets modern standards of rigour.

Proof. It is enough to show that for *some* configuration $(\underline{p}, \underline{Q})$, there is no grove (cf. Remark 2.2).

Let $p_1, \dots, p_6$ be general points in $\mathbf{P}^2$. Fix a line M in $\mathbf{P}^2$, take Q_4, Q_5, Q_6 to be general points on M and Q_1, Q_2, Q_3 general points in $\mathbf{P}^2$ (away from M). Let (Δ, L, δ) be[4] in $\Pi^\circ(\underline{p}, \underline{Q})$. Since there is no cubic singular at all Q_i, $\dim \Delta \geq 1$. Now L is either a point or empty, in either case the cubics $\delta \circ \pi_L(p_i)$ $(i = 4,5,6)$ must span Δ. Now any of them intersects M in at least four points, so must contain it. Thus M lies in the base locus of Δ, and $\Delta = M$ (fixed component) $+\Gamma$, where Γ is a system of conics through Q_1, Q_2, Q_3. Since each of Q_1, Q_2, Q_3 is a singular point of some member of Γ, we have

$$\Gamma = \text{span}(G_1, G_2, G_3),$$

following the notation used in the proof of Proposition 3.6. In particular $L = \emptyset$. Composing the isomorphism $\Delta \longrightarrow \Gamma$ with δ, we have an isomorphism $\gamma : \mathbf{P}^2 \longrightarrow \Gamma$ such that $(\Gamma, \emptyset, \gamma)$ is a grove of conics for $(p_1, p_2, p_3, Q_1, Q_2, Q_3)$. Think of γ as belonging to the two–dimensional family in Proposition 3.6.

For $i = 4,5,6$, if $\lambda_i \subseteq \Gamma$ be the line consisting of conics passing through Q_i, then by hypothesis $\gamma(p_i) \in \lambda_i$. But the conditions $\gamma(p_4) \in \lambda_4, \gamma(p_5) \in \lambda_5$ determine γ uniquely. (To see this point, choose coordinates on $\mathbf{P}^2, \Gamma$ such that

$$p_1, G_1 = [1,0,0], \; p_2, G_2 = [0,1,0], \; p_3, G_3 = [0,0,1], \; p_4 = [1,1,1]$$

---

[4] The change in notation is of course deliberate.

and λ_4 has line coordinates $[1,1,1]$. Then the matrix of γ is diagonal, say equal to $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. Since $\gamma(p_4) \in \lambda_4$, we have $a+b+c=0$, and $\gamma(p_5) \in \lambda_5$ forces another independent condition. But then the matrix is uniquely determined up to a scalar.)

We conclude that the grove (Δ, L, δ) is entirely determined by the data $p_1, \dots, p_5$, $Q_1, \dots, Q_5$. This is absurd, since one can certainly choose p_6, Q_6 such that $\gamma(p_6) \notin \lambda_6$. Hence $(\underline{p}, \underline{Q})$ do not admit a grove. ☐

After a lengthy analysis, London concludes that three general cubics admit *two* polar hexagons. It would be worthwhile to re-examine his argument. We hope to take it up elsewhere.

4. Exceptional cases

In this section we will construct groves showing that Σ is deficient for the four quadruples mentioned in the introduction. Part I of our construction for the case $(3,2,3,5)$ is built on a hint in Terracini [21]. The rest we believe to be new. As we confessed earlier, we have only partial success in the last case.

Theorem 4.1. *The variety* $\Sigma(2,3,2,5)$ *has codimension* 1 *in* $G(2,S_3)$. ($N_1 = N_2 = 16$.)

Part I (construction of the grove). Choose general points $0, 1, \infty, \alpha, \beta$ in $\mathbf{P}^1$, and $Q_0, Q_1, Q_\infty, Q_\alpha, Q_\beta$ in $\mathbf{P}^2$. Let C be the unique (smooth) conic through the $\underline{Q}$. The proposed construction is as follows: let Z be a point in $\mathbf{P}^2$ and

$$\Gamma = C \,(\text{fixed component}) + \text{pencil of lines through } Z.$$

Then we define

$$\gamma : \mathbf{P}^1 \xrightarrow{\sim} \Gamma, \quad \star \longrightarrow C + \text{ line } ZQ_\star \text{ for } \star = 0, 1, \infty, \alpha, \beta.$$

Of course, for such a γ to exist, the cross–ratios must agree. Hence the position of Z is crucial.

Let D_α denote the unique smooth conic through $Q_0, Q_1, Q_\infty, Q_\alpha$ such that $\langle Q_0, Q_1, Q_\infty, Q_\alpha \rangle_{D_\alpha} = \alpha$. Similarly, let D_β be the unique smooth conic through $Q_0, Q_1, Q_\infty, Q_\beta$ such that $\langle Q_0, Q_1, Q_\infty, Q_\beta \rangle_{D_\beta} = \beta$.

Let $D_\alpha \cap D_\beta = \{Q_0, Q_1, Q_\infty, Z\}$. Since Z lies on D_α, we have

$$\langle ZQ_0, ZQ_1, ZQ_\infty, ZQ_\alpha \rangle = \alpha$$

and similarly for β. Hence the sequences

$$\{0, 1, \infty, \alpha, \beta\}, \quad \{ZQ_0, ZQ_1, ZQ_\infty, ZQ_\alpha, ZQ_\beta\},$$

are projectively equivalent. This ensures that γ is well-defined and we are through.

Part II (uniqueness of the grove). In part I, we have shown that $\dim \Pi \geq 0$ for general $(\underline{p}, \underline{Q})$, hence this is true of any $(\underline{p}, \underline{Q})$. If we show that the grove is unique for some configuration, it will follow that $\dim \Pi = 0$ for general $(\underline{p}, \underline{Q})$.

Let M, N be distinct lines in $\mathbf{P}^2$. Choose general points Q_0, Q_1, Q_∞ on M and Q_α, Q_β on N. Let $0, 1, \infty, \alpha, \beta$ be general points of $\mathbf{P}^1$, and assume that (Γ, L, γ) is a grove for these data. Since there is no cubic singular at all Q_i, $\dim \Gamma = 1, L = \emptyset$. By Bézout, the cubics $\gamma(0), \gamma(1)$ contain M, hence M is in the base locus of Γ. Now $\Gamma \setminus M$ is a pencil of conics, which, by the same argument on $\gamma(\alpha), \gamma(\beta)$, contains N in its base locus. Hence

$$\Gamma = M + N \text{ (fixed components)} + \text{ pencil of lines through a point } Z.$$

Now map $\mathbf{P}^1$ to M, by sending $0, 1, \infty$ to Q_0, Q_1, Q_∞ respectively, and via this map, think of α, β as points on M. Then Z is forced to be the point of intersection of the lines $\alpha.Q_\alpha, \beta.Q_\beta$. The grove is thus uniquely determined. The theorem is proved. $\square$

Remark 4.2. There is a simple explanation for the deficiency of Σ. Let F_1, F_2 be two plane cubics admitting a polar pentagon $\{Q_1, \ldots, Q_5\}$. Since $\operatorname{span}(F_1, F_2) \subseteq \operatorname{span}(Q_1^3, \ldots, Q_5^3)$, we deduce that the six partial derivatives $\partial F_i / \partial x_j$ $(i = 1, 2, j = 0, 1, 2)$ lie in $\operatorname{span}(Q_1^2, \ldots, Q_5^2)$. Hence they must be linearly dependent, which amounts to a nontrivial algebraic condition on the F_i. It is easy to write this condition as the vanishing of a 6×6 determinant whose entries are functions in the coefficients of F_i (see [16]).

For the next two theorems the notation of Remark 3.9 will remain in force.

Theorem 4.3. *The variety* $\Sigma(3, 2, 3, 5)$ *has codimension* 1 *in* $G(3, S_2)$. $(N_1 = N_2 = 21.)$

Proof. Choose general points $p_1, \ldots, p_5$ in $\mathbf{P}^2$ and $Q_1, \ldots, Q_5$ in $\mathbf{P}^3$. Let E be the smooth conic through the p_i, and consider the imbedding

$$E \longrightarrow \operatorname{Sym}^3 E, \quad p \longrightarrow 3p.$$

Abstractly $\operatorname{Sym}^3 E \simeq \mathbf{P}^3$, hence there is a unique isomorphism $\beta : \operatorname{Sym}^3 E \longrightarrow \mathbf{P}^3$, such that $\beta(3p_i) = Q_i$. Let C be the twisted cubic obtained as the image of the composite $E \longrightarrow \operatorname{Sym}^3 E \xrightarrow{\beta} \mathbf{P}^3$.

Part I (construction of the grove). Let $\Gamma = \Psi$ (in the notation of Remark 3.9) and define

$$\gamma : \mathbf{P}^2 \xrightarrow{\sim} \Gamma, \quad p_i \longrightarrow \psi_{Q_i} \text{ for } 1 \leq i \leq 4.$$

The sequences $\{p_1, \dots, p_5\} \subseteq E, \{\psi_{Q_1}, \dots, \psi_{Q_5}\} \subseteq \tau(C)$ are such that the cross-ratios of any two corresponding subsequences of four points are equal. Hence $\gamma(E) = \tau(C)$ and $\gamma(p_5) = \psi_{Q_5}$, implying that $(\Gamma, \emptyset, \gamma)$ is a grove.

Part II (uniqueness of the grove). We now show that $\text{II}^\circ = \text{II}^\circ(\underline{p}, \underline{Q})$ is a singleton set. The plan of the proof is to choose a general element $g \in (\Gamma, L, \gamma) \in \text{II}^\circ$, and then to show that the generality forces it to be the same as the grove constructed above. By construction, the functions

$$\text{II}^\circ \longrightarrow \dim L, \quad \text{II}^\circ \longrightarrow \operatorname{rank} \gamma \circ \pi_L(p_i) = \rho_i$$

are respectively upper and lower semicontinuous. (We mean the rank of $\gamma(-)$ as a quadric in $\mathbf{P}^3$.) Let $U_i \subseteq \text{II}^\circ$ be the open set where ρ_i is maximal, and let $g \in \cap U_i$. By symmetry, all ρ_i equal the same number ρ, which is either 2 or 3. (It cannot be 1 since no plane can contain all Q_i.)

Case $\rho = 3$. Each quadric $S_i = \gamma \circ \pi_L(p_i)$ is a cone with its vertex at Q_i. Then

$$S_i \cap S_j = (\text{line } Q_i Q_j) \cup C_{ij},$$

where C_{ij} is a twisted cubic through $Q_1, \dots, Q_5$. For any three indices i, j, k, the quadrics S_i, S_j, S_k span Γ. Hence the base locus of Γ equals $S_i \cap S_j \cap S_k$, which is set-theoretically just $C_{ij} \cap C_{ik} \cap C_{jk}$.

Assume that the base locus of Γ is zero dimensional, then it is supported only on $Q_1, \dots, Q_5$ (since two twisted cubics can have at most five points in common). Moreover the S_i intersect transversally at each Q_j, so each Q_j is a reduced point of the base locus. This is a contradiction, since by Bézout, the base locus is a scheme of length 8. Hence the base locus is positive dimensional, i.e., all C_{ij} are the same twisted cubic C.

It follows that $\Gamma = \Psi$ in the notation of Remark 3.9. Then $\gamma(p_i)$ must equal ψ_{Q_i} for each i, which determines γ uniquely. Hence $\text{II} = \text{II}^\circ$ is a singleton set whose "general" element is the one we have constructed in Part I.

Case $\rho = 2$. We will show that this case is impossible. Our claim is: *the base locus of* Γ *contains a line.*

Let us grant the claim for the moment. Let $U_{ij} \subseteq \text{II}^\circ$ be the open set of groves which do not contain the line $Q_i Q_j$ in their base locus. If (say) U_{12} is nonempty, then by symmetry each U_{ij} is nonempty. Then a general element $g \in \cap U_{ij}$ (which by hypothesis has $\rho = 2$) can contain none of the lines, which is a contradiction. Thus $U_{ij} = \emptyset$, implying that a general Γ must contain all ten lines $Q_i Q_j$ in the base locus. This is surely impossible, hence $\rho \neq 2$.

It remains to prove the claim. We will assume that no line is common to three of the S_i and deduce a contradiction. Each $S_i = \gamma \circ \pi_L(p_i)$ consists of two planes both of which pass through Q_i. Since S_1, S_2 contain the line $Q_1 Q_2$, none of the other S_i can contain it. Then S_3 is either the union of planes $Q_1 Q_3 Q_4 \cup Q_2 Q_3 Q_5$, or $Q_1 Q_3 Q_5 \cup Q_2 Q_3 Q_4$. In the former case, S_1, S_3, S_4 all contain $Q_1 Q_4$; and in the latter case, S_1, S_3, S_5 all contain $Q_1 Q_5$. The claim is proved, and so is the theorem. □

Example 4.4. Now let Π be a plane in $\mathbf{P}^3$, and $Q_1, \dots, Q_4$ general points in Π. Choose $Q_5 \in \mathbf{P}^3$ generally (away from Π) and $p_1, \dots, p_5$ general points in $\mathbf{P}^2$. We know that this configuration admits a grove, let (Γ, L, γ) be one. The quadric $\gamma \circ \pi_L(p_1)$ is singular at Q_1, moreover by Bézout, it contains the four lines Q_1Q_i. This would be impossible if the quadric were of rank 3, hence it must contain Π. The same argument applies to Q_2, Q_3, Q_4, hence $\Gamma = \Pi$ (as a fixed component) + a system of planes through Q_5. But then no member of Γ can be singular at Q_5, hence $\pi_L(p_5)$ is undefined, i.e., $L = p_5$. The base locus of the system of planes is a line, say N. This leads to the following construction: let $\mathbf{P}^2_{\langle Q_5 \rangle}$ denote the variety of lines through Q_5, and define

$$f : \mathbf{P}^2_{\langle Q_5 \rangle} - \to \mathbf{P}^1, \quad N \longrightarrow \langle NQ_1, NQ_2, NQ_3, NQ_4 \rangle.$$

Let λ denote the cross-ratio $\langle p_5p_1, p_5p_2, p_5p_3, p_5p_4 \rangle$. Now if $N \in f^{-1}(\lambda)$, then (and only then) we can define a grove as above. Thus $\Pi\,(\underline{p}, \underline{Q})$ is a one-dimensional family, which demonstrates the upper-semicontinuity of $\overline{\dim}\,\Pi$. Secondly, Lemma 2.8 fails for this set of points.

Remark 4.5. The following explanation of the deficiency is given by Salmon ([19, vol. I, Ch. IX, §235]). Let F_1, F_2, F_3 be quadratic forms in $x_0, \dots, x_3$. Introduce indeterminates a, b, c, and let $G = aF_1 + bF_2 + cF_3$. Then the discriminant Δ of G (as a quadratic form in the x_i) is a quartic in a, b, c. As F_i move through general quadrics, Δ assumes values over a dense set of planar quartics. However, if the F_i admit a polar pentahedron, then Δ is necessarily a Lüroth quartic (see [6]). Since Lüroth quartics form a hypersurface in $\mathbf{P}S_4$, this imposes an algebraic condition on F_i.

Theorem 4.6. *The variety* $\Sigma(3, 2, 5, 6)$ *has codimension* 3 *in* $G(5, S_2)$. ($N_1 = 23, N_2 = 25$.)

Proof. Choose general points $p_1, \dots, p_6 \in \mathbf{P}^4$ and $Q_1, \dots, Q_6 \in \mathbf{P}^3$. Let C be the unique twisted cubic through the Q_i. There is a unique imbedding

$$\alpha : C \longrightarrow \mathbf{P}^4, \quad \alpha(Q_i) = p_i \text{ for } 1 \le i \le 6.$$

Part I (construction of the groves). We will show that there are at least ∞^2 groves for these data. Let $\Gamma = \Psi$ in the notation of Remark 3.9. Let L be a chord or a tangent of the rational normal quartic $\alpha(C)$. Let $\mathbf{P}^2_{\langle L \rangle}$ denote the collection of 2–planes in $\mathbf{P}^4$ containing L, and

$$\pi_L : \mathbf{P}^4 - \to \mathbf{P}^2_{\langle L \rangle}, \quad p \longrightarrow \overline{Lp}$$

the natural projection. Now π_L is defined everywhere on $\alpha(C)$, and $\pi_L(\alpha(C)) = D_L$ is a smooth conic in $\mathbf{P}^2_{\langle L \rangle}$. The sequences

$$\{Q_1, \dots, Q_6\} \subseteq C, \{\pi_L(p_1), \dots, \pi_L(p_6)\} \subseteq D_L$$

are such that any corresponding subsequences of four points have the same cross-ratio. Define

$$\gamma_L : \mathbf{P}^2_{\langle L \rangle} \xrightarrow{\sim} \Gamma, \quad \pi_L(p_i) \longrightarrow \psi_{Q_i} \ \text{ for } 1 \leq i \leq 4.$$

By what we have said, $\gamma_L(D_L) = \tau(C)$ and $\gamma_L \circ \pi_L(p_i) = \psi_{Q_i}$ for $i = 5, 6$. Thus (Γ, L, γ_L) is a two-dimensional family of groves.

Part II (bounding the dimension of Π*).* We will show that we have already constructed a dense set of possible groves. Let $(\Gamma, L, \gamma) \in \Pi^\circ(\underline{p}, \underline{Q})$. Each $\gamma \circ \pi_L(p_i)$ contains at least seven points of C, hence contains C by Bézout. Thus C is in the base locus of Γ, i.e., $\Gamma \subseteq \Psi$. Since Ψ contains a unique element singular at p_i, $\Gamma = \Psi$ which in turn implies $\dim L = 1$. Let $\mathbf{P}^2_{\langle L \rangle}$ have the same meaning as above, so we have an isomorphism $\mathbf{P}^2_{\langle L \rangle} \xrightarrow{\gamma} \Psi$.

Now there are two maps $\alpha(C) \longrightarrow \mathbf{P}^2_{\langle L \rangle}$, namely π_L and $\gamma^{-1} \circ \tau \circ \alpha^{-1}$. The image of the latter is a smooth conic. Moreover, $\deg \operatorname{image}(\pi_L) \leq 4$ and the two maps coincide on points $p_1, \dots, p_6$. Hence by Lemma 3.10, they must be the same. In particular, $\deg \operatorname{image}(\pi_L) = 2$ which is only possible if L intersects $\alpha(C)$ twice. This implies that the grove belongs to the family constructed above. The theorem is proved. ☐

Remark 4.7. Here is an alternate explanation for the deficiency. Assume that $\Lambda \in G(5, S_2)$ admits a polar 6-hedron $\underline{Q} = \{Q_1, \dots, Q_6\} \subseteq \mathbf{P}S_1$, with the Q_i in linearly general position. (This will be true of a general Λ in Σ.) Let C be the twisted cubic through the Q_i, with ideal $I_C < R$. Then $\Lambda^\perp \supseteq (I_C)_2 \supseteq (I_{\underline{Q}})_2$.

Let $\mathcal{C}$ be the 12-dimensional space of twisted cubics in $\mathbf{P}^3$, and consider the correspondence $\Phi \subseteq \mathcal{C} \times G(5, S_2)$ defined as

$$\Phi = \{(C, \Lambda) : (I_C)_2 \subseteq \Lambda^\perp\}.$$

Now $\pi_1 : \Phi \longrightarrow \mathcal{C}$ is a $G(2,7)$-bundle, so $\dim \Phi = 22$. Since Φ dominates Σ, we have $\dim \Sigma \leq 22$.

The case $(5, 2, 3, 8)$ is perhaps more surprising than the rest of the exceptions. By counting parameters, we expect three general quadrics in $\mathbf{P}^5$ to have ∞^1 polar octahedra, but they do not have any.

4.1. The Segre–Gale transform

Consider the variety $(\mathbf{P}^1)^8$ with the group $\operatorname{Aut}(\mathbf{P}^1)$ acting componentwise. Let $U \subseteq (\mathbf{P}^1)^8$ be the open set of semistable points and $Y = U/\operatorname{Aut}(\mathbf{P}^1)$ the GIT quotient.

In the sequel, $\sigma : \mathbf{P}^1 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^5$ denotes the Segre imbedding. Let $\underline{A} = A_1, \ldots, A_8 \in \mathbf{P}^1, \underline{p} = p_1, \ldots, p_8 \in \mathbf{P}^2$ be general points, and C the unique rational normal quintic through the eight points $\sigma(A_i \times p_i)$. Choosing an isomorphism $\alpha : C \longrightarrow \mathbf{P}^1$, we get a point

$$\underline{B} = (\alpha \circ \sigma(A_1 \times p_1), \ldots, \alpha \circ \sigma(A_8 \times p_8)) \in Y,$$

which we call the Segre–Gale transform of $(\underline{A}, \underline{p})$. The passage via α between eight general points in $\mathbf{P}^5$ and eight points in $\mathbf{P}^1$ is an instance of the Gale transform–see [7, 9].

Lemma 4.8. *Fix eight general points* $\underline{p} \in \mathbf{P}^2$*. Then the rational map*

$$\omega(\underline{p}) : Y - \rightarrow Y, \quad \underline{A} \longrightarrow \underline{B}$$

is dominant. (The reader should check that it is well-defined.)

Proof. This is a direct computation using coordinates (and was done in Maple). Let

$$\underline{A} = (0, 1, \infty, a_1, \ldots, a_5), \quad p_i = [1, c_i, d_i].$$

Then $\underline{B} = (0, 1, \infty, b_1, \ldots, b_5)$, where the rational functions b_i are easy to calculate. The Jacobian determinant $|\partial(b_1, \ldots, b_5)/\partial(a_1, \ldots, a_5)|$ is not identically zero, hence it is not zero for general c_i, d_i. This implies that the image of $\omega(\underline{p})$ must be dense in Y. ☐

Theorem 4.9. *The variety* $\Sigma(5, 2, 3, 8)$ *has codimension at least one in* $G(3, S_2)$*.* $(N_1 = 55, N_2 = 54)$.

The machine computation shows that the codimension is exactly one, but we have not been able to prove this.

Proof. Let $z_0, \ldots, z_5$ be the coordinates on $\mathbf{P}^5$. Consider the matrix $\begin{bmatrix} z_0 & z_1 & z_2 \\ z_3 & z_4 & z_5 \end{bmatrix}$ and its minors

$$G_0 = z_1 z_5 - z_2 z_4, \; G_1 = z_2 z_3 - z_0 z_5, \; G_2 = z_0 z_4 - z_1 z_3.$$

The locus $G_0 = G_1 = G_2 = 0$ is the Segre threefold $\sigma(\mathbf{P}^1 \times \mathbf{P}^2)$.

For $[a, b, c] \in \mathbf{P}^2$, the quadric $a\,G_0 + b\,G_1 + c\,G_2$ is of rank 4, and singular exactly along the line joining the points $[a, b, c, 0, 0, 0], [0, 0, 0, a, b, c]$. Denote this line by $\mathbb{M}_{[a,b,c]}$.

Choose general points $p_1, \ldots, p_8 \in \mathbf{P}^2$ and $Q_1, \ldots, Q_8 \in \mathbf{P}^5$. By the lemma, there are points $A_1, \ldots, A_8 \in \mathbf{P}^1$ such that $\omega(\underline{p})(\underline{A})$ is the Gale transform of $\underline{Q}$. Hence we may as well assume that $Q_i = \sigma(A_i \times p_i)$, i.e., $Q_i \in \mathbb{M}_{p_i}$.

Let Γ be the net $\{[a, b, c] \in \mathbf{P}^2 : a\,G_0 + b\,G_1 + c\,G_2\}$, and define

$$\gamma : \mathbf{P}^2 \xrightarrow{\sim} \Gamma, \quad [a, b, c] \longrightarrow a\,G_0 + b\,G_1 + c\,G_2.$$

By construction, $\gamma(p_i)$ is singular at Q_i, hence $(\Gamma, \emptyset, \gamma)$ is a grove. $\square$

5. Questions

In this area, the open problems are certainly not in short supply. However, there are four specific themes which we find especially appealing.

1. One would like to have an analogue of the Alexander–Hirschowitz theorem, at least for a reasonably broad range of (n, d, r, s). In [21], Terracini claims the following result:

Assume $n = r = 2, d \geq 4$ and $s \geq (d^2 + 3d + 2)/4$ (this is the bound in (4)). Then Σ is dense in $G(2, S_d)$.

We do not understand his proof and a clarification would be welcome.

2. Since the imbedding $\Sigma \subseteq G(r, S_d)$ is GL_{n+1} equivariant, the equations defining the closure of Σ in G are in principle expressible in the language of classical invariant theory. For small values, there are results making these equations explicit. For instance, in the case $(2, 3, 1, 3)$ the hypersurface $\overline{\Sigma} \subseteq \mathbf{P}^9$ is defined by the Aronhold invariant of ternary cubics. Toeplitz [22] gives such a combinant for $(3, 2, 3, 5)$, which turns out to be a Pfaffian. One would like to have some general theoretical machinery for such problems.

3. Given $\Lambda \in G(r, S_d)$, the locus $\pi_2(\pi_1^{-1}(\Lambda))$ (as defined in the introduction) is called the variety of its polar s-hedra. It has a very rich geometry, see e.g. [6, 15, 17] for some old and new results. If $n = 1$, then it is an open subset of a projective space (see [2]), but much remains unknown for more than two variables.

4. We need interesting examples where the class of $\overline{\Sigma}$ in the cohomology ring $H^*(G, \mathbf{Z})$ can be calculated. For $n = 1$, such calculations can be done using the Porteous formula (see [3]) but in general it is not clear how to proceed.

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