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Four-manifold systoles and surjectivity of period map

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Abstract. P. Buser and P. Sarnak showed in 1994 that the maximum, over the moduli space of Riemann surfaces of genus s, of the least conformal length of a nonseparating loop, is logarithmic in s. We present an application of (polynomially) dense Euclidean packings, to estimates for an analogous 2-dimensional conformal systolic invariant of a 4-manifold X with indefinite intersection form. The estimate turns out to be polynomial, rather than logarithmic, in $\chi(X)$, if the conjectured surjectivity of the period map is correct. Such surjectivity is targeted by the current work in gauge theory. The surjectivity allows one to insert suitable lattices with metric properties prescribed in advance, into the second de Rham cohomology group of X, as its integer lattice. The idea is to adapt the well-known Lorentzian construction of the Leech lattice, by replacing the Leech lattice by the Conway–Thompson unimodular lattices which define asymptotically dense packings. The final step can be described, in terms of the successive minima λ_i of a lattice, as deforming a λ_2-bound into a λ_1-bound, illustrated by Figure 9.1.

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Contents

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1. Schottky problem, surjectivity conjecture, and main theorem

The work of P. Buser and P. Sarnak [BS94] on Riemann surfaces in connection with the Schottky problem shows that the maximum, over the moduli space, of the least conformal length of a nonseparating loop behaves *logarithmically* in the genus, *cf.* (6.1) below and also M. Gromov's result [Gro83, Theorem 5.5.C'].

We provide a lower bound which is *polynomial* in the second Betti number, for the analogous 2-dimensional conformal systolic invariant for a 4-manifold X with indefinite intersection form, modulo the conjectured surjectivity of the period map, targeted in the current work [ADK]. Our bound currently depends on such surjectivity, but see 1.3. In the case $b^+(X) = 1$ targeted in [ADK], such surjectivity is expressed in Hypothesis 1.1.

Let (X, g) be a Riemannian 4-manifold. Let $* : H^2_{dR}(X) \to H^2_{dR}(X)$ be the Hodge star operator in de Rham cohomology identified with the space $\mathcal{H}$ of harmonic 2-forms on X. Assume that $b^+(X) = 1$, so that the selfdual subspace (*i.e.* the $(+1)$-eigenspace of the Hodge star operator) is 1-dimensional. Recall that the cup-product form in $H^2(X)$ is dual to the intersection form in $H_2(X)$.

Hypothesis 1.1. For every line V in the positive cone in $H^2_{dR}(X)$ defined by the cup product form, there is a metric g on X whose selfdual subspace is exactly V.

Given a lattice L equipped with a norm $\| \, \|$, we denote by

$$\lambda_1(L) = \lambda_1(L, \| \, \|) \tag{1.1}$$

the least norm of a nonzero lattice vector. The λ_1 notation fits in with the successive minima λ_i of a lattice, studied in lattice theory, *cf.* [GruL87, p. 58], [BanK03, Section 4], and Definition 3.2.

Theorem 1.2. *Let $n \in \mathbb{N}$ and consider the complex projective plane blown up at n points, $P_n = \mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$, where bar denotes reversal of orientation, while $\#$ is connected sum. Assume that Hypothesis 1.1 is satisfied for such manifolds. Then*

$$C^{-1}\sqrt{n} < \sup_g \left\{ \lambda_1 \left(H^2(P_n, \mathbb{Z}), | \, |_{L^2} \right) \right\}^2 < Cn, \ \forall n > 0, \tag{1.2}$$

where $C > 0$ is a numerical constant, the supremum is over all smooth metrics g on $\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$, and $| \, |_{L^2}$ is the norm (3.1) *defined by g.*

Here the upper bound may be replaced by $\frac{2}{3}(n+1)$ by the estimate (4.3), while the lower bound, by $\sqrt{k(n)}$, where $k(n)$ is asymptotic to $\frac{n}{2\pi e}$ as $n \to \infty$, *cf.* Theorem 2.2. The theorem is proved in Section 11. The desired metric is specified in formula (11.1) in terms of inversion of the period map.

A number of systolic inequalities are now available in the literature. Nontrivial cup product relations lead to stable systolic inequalities [BanK03] (*cf.* inequality (4.3) below), some of them sharp [BanK04, NV03]. Meanwhile, nontrivial

Massey products also admit systolic repercussions, *cf.* [KKS]. For ordinary (rather than stable) systoles, systolic freedom prevails as soon as we go beyond loops. Here "systolic freedom" refers to the absence of systolic inequalities, *i.e.* the existence of sequences of metrics violating such potential inequalities. Such a phenomenon for the middle dimensional systole was first described in detail by the author in [Ka95B]. M. Gromov's original seminal example of 1993 is described in [Gro99, p. 268], as well as [CrK03, section 4.2]. Further generalisations of systolic freedom were obtained in [BabK98], [BKS98], [Fr99], [KS99, KS01], [Bab02], [Ka02]. See the recent survey [CrK03, Figure 4.1] for a 2-D map of systolic geometry, which places such results in mutual relation.

Question 1.3. Can one eliminate the dependence of our Theorem 1.2 on the surjectivity conjecture? Recent discussions with C. LeBrun and P. Biran suggest that one may be able to remove the dependence on the conjectured surjectivity of the period map, at least in the case of the blow-ups of the projective plane, by exploiting the action of the automorphism group of the intersection form, *cf.* Lemma 7.1 and Remark 7.2. This would work if one can show the existence of metrics adapted to symplectic forms which represent classes from a suitable fundamental domain for the action, *cf.* [Bi01, Theorem 3.2] and Remark 9.1.

Question 1.4. Can one improve the lower bound in (1.2) to *linear* dependence on n? Here one could envision an averaging argument, using Siegel's formula as in [MH73, Theorem 9.5], over integral vectors satisfying $q_{n,1}(v) = -p$. Here one seeks a vector $v \in \mathbb{R}^{n,1}$ such that the integer lattice $\mathbb{Z}^{n,1} \subset \mathbb{R}^{n,1}$ has the Conway–Thompson behavior (2.1) with respect to the positive definite form $SR(q_{n,1}, v)$.

Question 1.5. Is there an *asymptotically infinite* lower bound similar to Theorem 1.2 for the stable 2-systole in place of the conformal 2-systole? This is related to understanding the discrepancy between the comass norm and the L^2 norm in 2-dimensional cohomology. Note that Remark 4.1, concerning the 1-systoles of surfaces, suggests that *a priori* there may exist, instead, an *asymptotically vanishing* upper bound for the stable 2-systole, *cf.* (5.2) in the definite case.

The present work is organized as follows. Section 2 introduces the Conway–Thompson lattices and describes the idea of the proof. Section 3 defines the L^2-norm in cohomology, describes its relation to the intersection form, and discusses the successive minima of a lattice. Section 4 defines the conformal and stable systoles. Section 5 discusses the definite case. Note that our main Theorem 1.2 can be thought of as a higher-dimensional analogue of the Buser–Sarnak theorem, presented in Section 6. Section 7 explains a useful sign reversal relation between definite and indefinite forms. Section 8 describes a Lorentzian construction of lattices inspired by a result of J. Conway and N. Sloane, and presents a lower bound for the second successive minimum. Section 9 presents the necessary linear algebraic ingredient. Section 10 deforms a lower bound for the second successive minimum, into a lower bound for the first successive minimum. The proof is completed by a successive minimum calculation in Section 11.

2. Conway–Thompson lattices CT_n and idea of proof

The surjectivity of the period map (see Hypothesis 1.1) furnishes a lot of latitude in prescribing the position of the integer lattice in middle-dimensional de Rham cohomology, with respect to the L^2-norm. In particular, we show that the least norm, $\lambda_1\left(H^p(X,\mathbb{Z})_{\mathbb{R}}, |\,|_{L^2}\right)$, of a nonzero lattice element, can be made arbitrarily large as the Betti number grows. Here one relies on the existence of Euclidean unimodular lattices L with arbitrarily high $\lambda_1(L)$, as well as on the (elementary) classification of indefinite odd unimodular forms, *cf.* (8.1). We acknowledge the influence on our approach of the Lorentzian construction (*i.e.* using indefinite forms) of the Leech lattice of J. Conway and N. Sloane [CoS99, Chapter 26], namely the following result.

Theorem 2.1 (J. H. Conway, N. J. A. Sloane)**.** *If*

$$t = (3, 5, 7, \dots, 45, 47, 51|145)$$

is a vector with $q_{24,1}(t) = -1$ *in* $I_{24,1}$*, then* $t^{\perp} \cap I_{24,1}$ *is a copy of the Leech lattice.*

The *first* step of our approach can be described as adapting the Lorentzian construction by replacing the Leech lattice by the Conway–Thompson lattices. The latter are unimodular lattices which define packings of high asymptotic density. More precisely, we have the following result [MH73, Theorem 9.5].

Theorem 2.2 (Conway, Thompson)**.** *For any dimension* n*, there exists a positive definite inner product space, denoted* CT_n*, over* $\mathbb{Z}$ *of odd type and rank* n *with*

$$\min_{x \neq 0} x.x \geq k(n), \tag{2.1}$$

where $k(n)$ *is asymptotic to* $n/2\pi e$ *as* $n \to \infty$*.*

The *second* step of our approach is explained in Section 9.

3. Norms in cohomology and successive minima λ_i of lattices

Let (X, g) be a closed orientable Riemannian $(2p)$-dimensional manifold. Let $H^p(X,\mathbb{Z})_{\mathbb{R}} \subset \mathcal{H} = H^p_{dR}(X)$ be the lattice defined as the image of $H^p(X,\mathbb{Z})$ in $H^p(X,\mathbb{R})$ under the inclusion $\mathbb{Z} \subset \mathbb{R}$ of coefficients, *i.e.* quotient by its torsion subgroup. We will sometimes delete the subscript $_{\mathbb{R}}$, by abuse of notation, when the torsion subgroup is trivial. Consider the L^2-norm $|\ |_{L^2}$ in $\mathcal{H}$, defined by

$$|f|^2_{L^2} = \int_X f \wedge *f \tag{3.1}$$

for each harmonic p-form $f \in \mathcal{H}$, where $*$ is the Hodge operator for the metric g. The following lemma is obvious, *cf.* [FU84, Lemma 2.21].

Lemma 3.1. *Let p be even. Then the L^2-norm is related to the cup product form $\omega(f,g) = \int_X f \cup g$ by means of the "sign reversal" formula*

$$|f|^2_{L^2} = \langle f, f\rangle = \omega(f^+, f^+) - \omega(f^-, f^-) \tag{3.2}$$

where $f = f^+ + f^-$ is the decomposition given by the splitting $\mathcal{H} = V^+ + V^-$ into the (± 1)-eigenspaces of the involution $*$.

Similarly to the notation of formula (7.1) below, we can restate Lemma 3.1 as follows:

$$\langle\ ,\ \rangle = SR(\omega, V^-). \tag{3.3}$$

The lattice $H^p(X,\mathbb{Z})_\mathbb{R}$ is equipped with the L^2-norm defined by formula (3.1). The dual norm in the similarly defined lattice $H_p(X,\mathbb{Z})_\mathbb{R} \subset H_p(X,\mathbb{R})$ will also be denoted $|\ |_{L^2}$.

The successive minima are defined as follows. Note that the second successive minimum is exploited in Corollary 8.1 below.

Definition 3.2. Let i be an integer satisfying $1 \le i \le rk(L)$. The i-th successive minimum $\lambda_i(L, \|\ \|)$ is the least $\lambda > 0$ such that there exist i linearly independent vectors in L of norm at most λ:

$$\lambda_i(L, \|\ \|) = \inf_\lambda \left\{ \lambda \in \mathbb{R} \,\middle|\, \exists v_1, \ldots, v_i\ (l.i.)\ : \|v_1\| \le \lambda, \ldots, \|v_i\| \le \lambda \right\}.$$

4. Conformal length and systolic flavors

In this section, we define several flavors of systolic invariants of a $(2p)$-dimensional Riemannian manifold manifold (X, g). The (middle dimensional) conformal p-systole, denoted $\mathrm{confsys}_p(g)$, of the metric g, is the least norm of a nonzero element in the integer lattice in p-dimensional cohomology (or, equivalently, homology; see Remark 4.2), with respect to the L^2-norm (3.1) defined by g:

$$\begin{aligned}\mathrm{confsys}_p(g) &= \lambda_1\left(H^p(X^{2p},\mathbb{Z})_\mathbb{R}, |\ |_{L^2}\right)\\ &= \min\left\{ |v|_{L^2} \,\middle|\, v \in H^p(X,\mathbb{Z})_\mathbb{R} \setminus \{0\}\right\}.\end{aligned}$$

Meanwhile, the *stable p-systole* is the quantity

$$\mathrm{stsys}_p(g) = \lambda_1(H_p(X^{2p},\mathbb{Z})_\mathbb{R}, \|\ \|),$$

where $\| \ \|$ is the stable norm in homology, dual to the comass norm in cohomology, *cf.* [Fe74, 4.10], [BanK03]. The conformal systole is related to the stable systole as follows:

$$\operatorname{stsys}_p(g)\operatorname{vol}_{2p}(g)^{-\frac{1}{2}} \leq \binom{2p}{p}^{\frac{1}{2}} \operatorname{confsys}_p(g). \tag{4.1}$$

Here the binomial coefficient appears due to the discrepancy between the linear comass norm and the natural Euclidean norm on the space of p-forms, *cf.* [BanK03, section 7]. In the case $p = 1$, the binomial coefficient may be replaced by 1.

Remark 4.1 (1-systole asymptotics)**.** It should be kept in mind that the asymptotic behavior of the (stable) 1-systole as a function of the genus is completely different from the conformal systole. Thus, M. Gromov [Gro96, 2.C] reveals the existence of a universal constant C such that we have an asymptotically *vanishing* upper bound

$$\frac{\operatorname{sys}_1(\Sigma_s)^2}{\operatorname{area}(\Sigma_s)} \leq C\,\frac{(\log s)^2}{s},$$

for every orientable surface Σ_s of genus $s \geq 2$, with a Riemannian metric, see [CrK03, (2.9) and (2.10)] for related bounds. In contrast, P. Buser and P. Sarnak [BS94] provide an asymptotically *infinite* lower bound for the maximum of the conformal systole over the moduli space, *cf.* inequality (6.1).

Remark 4.2 (Conformal length)**.** The Poincaré duality map induces an isometry

$$PD : (H^p(X,\mathbb{Z})_{\mathbb{R}}, |\ |_{L^2}) \to (H_p(X,\mathbb{Z})_{\mathbb{R}}, |\ |_{L^2}), \tag{4.2}$$

proving that the integer lattice in middle dimension is *isodual* in the sense of [CoS94, BeM95]. Thus for $p = 1$, the invariant $\operatorname{confsys}_1$ is the conformal length of the surface.

We have the following upper bound on conformal systole:

$$\lambda_1(H^p(X^{2p},\mathbb{Z})_{\mathbb{R}}, |\ |_{L^2})^2 \leq \gamma_b < \frac{2}{3}\, b_p(X^{2p}) \quad \text{for } b_p(X) \geq 2, \tag{4.3}$$

see [BanK03] for stable systolic generalisations based on multiplicative relations in cohomology, and [CrK03] for an overview.

5. Systoles of definite intersection forms

Our main result is Theorem 1.2, which may be viewed as a higher dimensional generalisation of the Buser–Sarnak theorem (6.1). We briefly discuss the definite case. Consider the family of manifolds $n\mathbb{C}P^2$, defined as the connected sum of n copies of the complex projective plane with the standard orientation. Recall that these exhaust the smooth positive definite case by Donaldson's theorem, *cf.* [Ka95A].

In contrast to Theorem 1.2, the maximal conformal systole in the *definite* case is bounded as the second Betti number grows:

$$\lambda_1(H^2(n\mathbb{C}P^2,\mathbb{Z}),|\,|_{L^2}) = \lambda_1(H^2(n\mathbb{C}P^2,\mathbb{Z}),\sqrt{\omega}) = \lambda_1(\mathbb{Z}^n) = 1, \qquad (5.1)$$

for every Riemannian $n\mathbb{C}P^2$, $n=1,2,\ldots$. This is immediate from formula (3.2) which identifies the L^2-norm and the intersection form ω if the latter is positive definite. By inequality (4.1), we obtain the following result, pointed out by C. Lebrun: every Riemannian $n\mathbb{C}P^2$ satisfies the inequality

$$\mathrm{stsys}_2\left(n\mathbb{C}P^2\right)^2 \leq 6\ \mathrm{vol}_4\left(n\mathbb{C}P^2\right). \qquad (5.2)$$

6. Buser–Sarnak theorem

Our Theorem 1.2 may be viewed as a higher dimensional analogue of the theorem of P. Buser and P. Sarnak [BS94, formula (1.13)]. Let Σ_s be a closed orientable surface of genus s. Then the conformal 1-systole satisfies the bounds

$$C^{-1}\log s < \sup_g \left\{\lambda_1\left(H^1(\Sigma_s,\mathbb{Z}),|\,|_{L^2}\right)\right\}^2 < C\log s,\ \forall s\geq 2 \qquad (6.1)$$

where $C>0$ is a numerical constant, the supremum is over all metrics g on Σ_s, and $|\,|_{L^2}$ is the norm (3.1) associated with g. An explicit upper bound of $\frac{3}{\pi}\log(4s+3)$ is provided in [BS94, formula (1.13)].

Note that a (weaker) upper bound of $C\sqrt{s}$ (in place of $C\log s$) results from R. Lazarsfeld's work [La96, p. 441, Proposition, part (i)]. The systolic quantity $\lambda_1\left(H^1(\Sigma_s,\mathbb{Z}),|\,|_{L^2}\right)$ may be viewed as the conformal length of the surface, in view of the isomorphism of formula (4.2). By conformal invariance, the supremum in (6.1) may be restricted to the moduli space of hyperbolic metrics on the surface.

7. Sign reversal procedure SR and $Aut(I_{n,1})$-invariance

Let q be an indefinite quadratic form of index $+1$ (*i.e.* with a single negative direction) on a vector space E over $\mathbb{R}$, and let $v\in E$ be a vector satisfying $q(v)<0$. Denote by $v^\perp\subset E$ the q-orthogonal complement of $v\in E$, or, more precisely, the Q-orthogonal complement, where $Q(u,w)=\frac{1}{4}(q(u+w)-q(u-w))$ is the polarisation of q. Thus, we have a decomposition $E=v^\perp\oplus\mathbb{R}v$. The *sign reversal*, $SR(q,v)$, is the positive definite form on E obtained by reversing the sign of q in direction v, while keeping it fixed on $v^\perp\subset E$:

$$SR(q,v)(x) = q(x^+) - q(x^-), \qquad (7.1)$$

where $x=x^++x^-$ is the decomposition of $x\in E$ following the splitting $E=v^\perp\oplus\mathbb{R}v$, *cf.* formula (3.3). Let $\mathbb{R}^{p,q}$ denote the standard real vector space with quadratic form

$$q_{p,q}(x) = x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_{p+q}^2, \qquad (7.2)$$

and let $I_{p,q} \subset \mathbb{R}^{p,q}$ denote its integer lattice. For the purposes of the proof of Theorem 1.2, it is convenient to reverse the orientation and work instead on the manifold $n\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, while hoping that such a step may not prove baffling to an algebraic geometer.

Recall that the intersection form on $n\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ is $q_{n,1}$, and the integer lattice in two-dimensional homology becomes a copy of $I_{n,1}$.

Lemma 7.1. *The invariant* $\operatorname{confsys}_2\left(n\mathbb{C}P^2\#\overline{\mathbb{C}P}^2, g\right)$ *only depends on the orbit of the antiselfdual line of g in $H^2_{dR}\left(n\mathbb{C}P^2\#\overline{\mathbb{C}P}^2\right)$ under the action of the automorphism group of $I_{n,1}$.*

Proof. An endomorphism f of $H^2_{dR}(n\mathbb{C}P^2\#\overline{\mathbb{C}P}^2)$ which is an automorphism of the indefinite lattice $I_{n,1}$, induces an isometry of the definite form $SR(q_{n,1}, v)$, since f maps the subspace $v^{\perp_q}$ to the subspace $f(v)^{\perp_q}$, and hence

$$SR(q_{n,1}, v)(x) = SR(q_{n,1}, f(v))(f(x)).$$

Here if $x \in I_{n,1}$, then $f(x) \in I_{n,1}$ by the hypothesis that f preserves the integer lattice. Now the lemma follows from the formula

$$\operatorname{confsys}_2(g) = \lambda_1\left(H^2\left(n\mathbb{C}P^2\#\overline{\mathbb{C}P}^2, \mathbb{Z}\right), SR\left(q_{n,1}, V^-\right)^{\frac{1}{2}}\right),$$

where V^- is the antiselfdual direction of g. □

Remark 7.2. Note that not all automorphisms of the intersection form can be realized by a diffeomorphism of the manifold, *cf.* [Ko91].

8. Lorentz construction of Leech lattice and line $CT_n^\perp$

Let $CT_n \subset \mathbb{R}^{n,0}$ be a Conway–Thompson lattice as in Theorem 2.2, *i.e.* a unimodular lattice satisfying $\lambda_1(CT_n)^2 \geq k(n)$. Then the lattice $CT_n \oplus I_{0,1}$ is odd, indefinite, and unimodular, *cf.* (7.2) and notation there. The classification of odd indefinite unimodular forms [MH73, p. 22] implies that the lattice $I_{n,1}$ contains an isometric copy of CT_n such that the $q_{n,1}$-orthogonal complement of CT_n in $I_{n,1}$, is a copy of the line $I_{0,1}$. In formulas, there exists an isomorphism

$$\phi_n : CT_n \oplus I_{0,1} \to I_{n,1} \tag{8.1}$$

preserving the bilinear forms. We will use the following suggestive notation for the line identified by isomorphism (8.1): let

$$CT_n^\perp \subset I_{n,1} \tag{8.2}$$

be the $q_{n,1}$-orthogonal complement of $\phi_n(CT_n \oplus \{0\}) \subset I_{n,1}$.

Corollary 8.1. *Let $I_{n,1} \subset \mathbb{R}^{n,1}$ be the integer lattice. Let $v \in I_{n,1}$ be a generator of $CT_n^\perp \subset I_{n,1}$ as in* (8.2)*, i.e. $v = \phi_n(0,e)$, where $e \in I_{0,1}$ is a generator, as in isomorphism* (8.1)*. Consider the norm $\|x\|_v = \sqrt{SR(q_{n,1},v)(x)}$, in the notation of formula* (7.1)*. Then the integer lattice has successive minima $\lambda_1(I_{n,1}, \| \, \|_v) = +1$, and*

$$\lambda_2(I_{n,1}, \| \, \|_v)^2 \geq k(n),$$

cf. Definition 3.2, where $k(n)$ is as in Theorem 2.2. In other words, all vectors of square-norm smaller than $k(n)$ are proportional to each other.

Proof. For any lattice L with a positive definite form, we have the identity $SR(L \oplus I_{0,1}, \iota(e)) = L \oplus I_{1,0}$, where ι is the inclusion of the second factor. In particular,

$$SR(I_{n,1}, \phi_n(\iota(e))) = CT_n \oplus I_{1,0}, \tag{8.3}$$

proving the corollary. □

As an indication of how nontrivial the isomorphism ϕ as in formula (8.1) could be, consider Theorem 2.1, which exhibits an isomorphism $\Lambda_{24} \oplus I_{0,1} \to I_{24,1}$, where Λ_{24} is the Leech lattice.

With an eye on the lower bound of our main Theorem 1.2, we first prove Proposition 8.2 below. Recall that the intersection form on $n\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is the diagonal form $q_{n,1}$, *cf.* formula (7.2). Let ϕ_n be the isomorphism (8.1).

Proposition 8.2. *If g is a metric on $n\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ whose antiselfdual direction is the vector $\phi_n(0,e) \in I_{n,1}$, then all surfaces of "conformal area" smaller than $\sqrt{k(n)}$ with respect to g are homologous to multiples of one another.*

Proof. The integer lattice in the selfdual subspace V^+ is isometric to the Conway–Thompson lattice:

$$V^+ \cap H^2(n\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \mathbb{Z}) \simeq CT_n.$$

Moreover, this copy of the Conway–Thompson lattice is a direct summand, where the second summand is isometric to $I_{0,1}$. The sign reversal formula (3.2) shows that the integer lattice

$$\left(H^2(n\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \mathbb{Z}), SR\left(\omega, CT_n^\perp\right)^{\frac{1}{2}} \right),$$

is isometric to the positive definite lattice $CT_n \oplus I_{1,0}$, where $I_{0,1}$ has been replaced by $I_{1,0}$ as in formula (8.3). Thus the proposition is a restatement of Corollary 8.1. □

9. Three quadratic forms in the plane

The main result of this section is Lemma 9.3 below on the interplay of three quadratic forms in the plane, an indefinite one, q, and and a pair of definite ones, q_1 and q_s, where the parameter value s will be judiciously chosen in (10.7).

To go beyond Proposition 8.2 and prove our theorem, the lattice $CT_n \oplus I_{1,0}$ is not sufficient, as it contains vectors of unit norm in the second summand $I_{1,0}$, so that the quantity $\lambda_1(CT_n \oplus I_{1,0}) = 1$ is too small. In other words, we need to replace a lower bound for the successive minimum λ_2 of the integer lattice, by a lower bound for the successive minimum λ_1 for the same lattice, but with respect to a new norm. The idea is to deform appropriately the choice of the negative definite direction $v = \phi_n(0, e)$, responsible for the Conway–Thompson behavior of its complement.

Thus, to prove Theorem 1.2, we will apply the surjectivity of the period map, not to the line $CT_n^\perp \subset H^2_{dR}\left(n\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2\right)$, but rather to the image of $CT_n^\perp$ under a suitable "Lorentz deformation", *cf.* Figure 9.1 and formula (11.1).

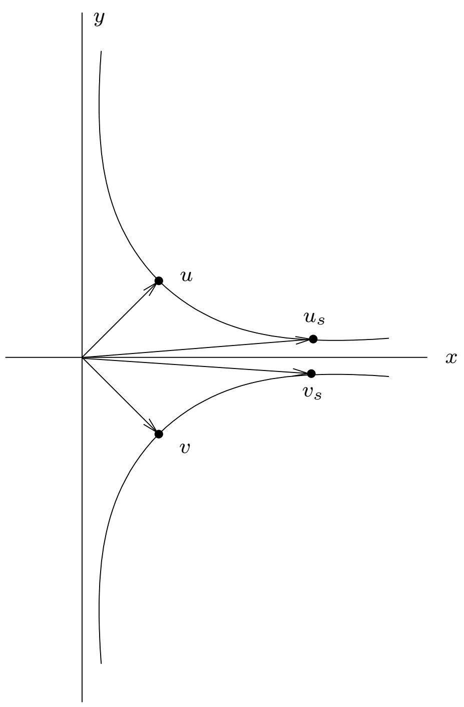

FIG. 9.1. Lorentz transformation A_s, *cf.* (10.4)

Remark 9.1. Since the quantity λ_1 (as in Definition 3.2) is continuous as a function on the space of positive definite lattices, while the form $SR(\omega, V)$ is continuous in both parameters, and V^- depends continuously on the metric, it follows that Hypothesis 1.1 can be relaxed to assume the density of the image in place of surjectivity.

The argument relies on a rather crude bound on the operator norm of the deformation. The deformation needs to be sufficient to eliminate short vectors, but with operator norm controlled so as not to negate entirely the Conway–Thompson effect.

Sign reversal on the line $CT_n^\perp \subset I_{n,1}$ produces a quadratic form with respect to which most vectors are suitably long, except for a single direction. To weed out the remaining short vector, we apply a suitable deformation, whose linear algebraic content is presented in Lemma 9.3 below.

Let π be the xy-plane. Let e_1, e_2 be the standard basis and x, y the standard coordinates. Consider the indefinite form $q = dxdy$, and let $s > 0$ be a real parameter.

Definition 9.2. Our "Lorentz transformation" $A_s : \pi \to \pi$ is defined by the matrix $A_s = \begin{pmatrix} s & 0 \\ 0 & \frac{1}{s} \end{pmatrix}$ with respect to the standard basis, and we set $u_s = A_s(e_1 + e_2)$ $= se_1 + \frac{1}{s}e_2$ and $v_s = A_s(e_1 - e_2) = se_1 - \frac{1}{s}e_2$, as illustrated in Figure 9.1.

Lemma 9.3. *Consider the positive definite quadratic form $q_s = SR(q, v_s)$ on π, obtained from q by reversing the sign in the direction v_s, as in formula* (7.1). *Then the map $A_s : (\pi, q_1) \to (\pi, q_s)$ is an isometry.*

Proof. Since the "Lorentz transformation" A_s preserves q and sends v_1 to v_s, it is clear that it also sends q_1 to q_s, but we will give a short explicit calculation. We have $q(u_s, v_s) = 0$. Let (x', y') be the coordinates with respect to the basis $\{u_s, v_s\}$ of the plane π. Then the two pairs of coordinates are related by $x = s(x' + y'), y = \frac{1}{s}(x' - y')$. Now $q = dxdy = s(dx' + dy')\,\frac{1}{s}(dx' - dy') = dx'^2 - dy'^2$. Therefore by definition, $q_s = SR(q, v_s) = dx'^2 + dy'^2$. Thus $q_s(u_s, v_s) = 0$ and $q_s(u_s) = q_s(v_s) = 1$. Similarly, the vectors $u = e_1 + e_2$ and $v = e_1 - e_2$ form an orthonormal basis for q_1, proving the Lemma. □

10. Replacing λ_1 by the geometric mean $(\lambda_1\lambda_2)^{1/2}$

Let $L = I_{n,1} \subset (\mathbb{R}^{n,1}, q_{n,1})$ be the integer lattice. Let $v \in L$ be a vector satisfying $q_{n,1}(v) = -1$ and

$$L = \mathbb{Z}v \oplus v^\perp \simeq I_{0,1} \oplus v^\perp, \tag{10.1}$$

where the sublattice $\left(v^\perp, (q_{n,1}|_{v^\perp})^{\frac{1}{2}}\right)$ is positive definite. Let $SR(q_{n,1}, v)$ be the positive definite form obtained by sign reversal. Let $\lambda_i = \lambda_i\left(L, SR(q_{n,1}, v)^{\frac{1}{2}}\right)$ be the successive minima with respect to the new form. We have $\lambda_1 = 1$ but we will ignore this in the statement of the proposition below, so as to emphasize the geometric mean inherent in the proof. Note that

$$\lambda_2 = \lambda_1\left(v^\perp, \left(q_{n,1}\left|_{v^\perp}\right.\right)^{\frac{1}{2}}\right). \tag{10.2}$$

Proposition 10.1. *There is a $q_{n,1}$-preserving transformation A of $\mathbb{R}^{n,1}$ such that $\lambda_1\left(L, SR\left(q_{n,1}, Av\right)^{\frac{1}{2}}\right) \geq \sqrt{\lambda_1\lambda_2}$.*

Proof. Let $\pi \subset \mathbb{R}^{n,1}$ be any 2-plane containing the vector v as in (10.1). We choose coordinates (x, y) in π with the following three properties:

(1) the union of the x-axis and the y-axis in π is the intersection of the isotropic cone of $q_{n,1}$ with π;
(2) the restriction of $q_{n,1}$ to π is the form q of Lemma 9.3;
(3) with respect to the standard basis e_1, e_2 in π, we have $v = e_1 - e_2$.

Now let $s \in \mathbb{R}$, and set $v_s = se_1 - \frac{1}{s}e_2$. Let q_s be the positive definite quadratic form obtained by sign reversal $q_s = SR(q_{n,1}, v_s)$. Thus, for $s = 1$, replacing q by q_1 has the effect of replacing $I_{0,1}$ by $I_{1,0}$ in the decomposition (10.1). Hence we have the following isometry of lattices:

$$(L, q_1) \simeq I_{1,0} \oplus v^\perp. \tag{10.3}$$

We wish to understand the position of the integer lattice L with respect to the definite form q_s "deforming" q_1. By Lemma 9.3, the map

$$A_s \oplus Id_{\pi^\perp}, \tag{10.4}$$

also denoted A_s, is an isometry from q_1 to q_s. Thus the pullback lattice $\left(A_s^{-1}(L), q_1\right)$ is isometric to (L, q_s). We have $A_s^{-1}(v) = \frac{1}{s}e_1 - se_2$, and hence

$$q_s(v) = q_1\left(A_s^{-1}v\right) = q_1\left(\tfrac{1}{s}e_1 - se_2\right) = \tfrac{1}{s^2} + s^2 \geq s^2. \tag{10.5}$$

Now consider an element $x \in L = \mathbb{Z}v \oplus v^\perp$ which is not proportional to the generator v of the first summand. By the Pythagorean theorem applied to formula (10.3), the element x satisfies $q_1(x)^{\frac{1}{2}} \geq \lambda_1\left(v^\perp\right) = \lambda_2\left(L, \sqrt{q_1}\right)$, by formula (10.2). Meanwhile, we have the following bound on the operator norm with respect to the form q_1: $\|A_s\| = \|A_s^{-1}\| \leq s$, and therefore

$$q_s(x) = q_1(A_s^{-1}x) \geq \frac{\lambda_2^2}{s^2}. \tag{10.6}$$

Combining (10.5) and (10.6), we obtain the lower bound $\lambda_1\left(L, \sqrt{q_s}\right) \geq \min\left\{s, \frac{\lambda_2}{s}\right\}$.

Choosing the parameter value

$$s = \sqrt{\frac{\lambda_2}{\lambda_1}} = \sqrt{\lambda_2}, \tag{10.7}$$

we complete the proof of the proposition. □

Corollary 10.2. *Let* $\mathbb{Z}v = CT_n^{\perp} \subset L = \mathbb{Z}^{n,1}$*, as in Theorem 2.2 and isomorphism* (8.1)*. Then there is a transformation* $A = A_{k(n)^{\frac{1}{4}}}$ *of* $\mathbb{R}^{n,1}$ *such that* $\lambda_1\left(L, SR(q_{n,1}, Av)^{\frac{1}{2}}\right) \geq k(n)^{\frac{1}{4}}$.

11. Period map and proof of main theorem

We are now in a position to prove Theorem 1.2. The inequality (4.3) proves the upper bound of estimate (1.2), insofar as $b_2(n\mathbb{C}P\#\overline{\mathbb{C}P}) = n+1$. Let us write down a formula, (11.1), for a metric g_n satisfying the lower bound. Let $X = n\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, so that $H^2(X,\mathbb{Z}) = I_{n,1}$, with cup-form $\omega = q_{n,1}$. Recall that the L^2-norm in $H^2(X,\mathbb{R})$ is related to the cup product form $\omega(f,g) = \int_X f \cup g$ by means of the "sign reversal" formula $|f|_{L^2}^2 = \langle f, f\rangle = \omega(f^+, f^+) - \omega(f^-, f^-)$, where $f = f^+ + f^-$ is the decomposition given by the splitting $H^2(X,\mathbb{R}) = V^+ + V^-$ into the (± 1)-eigenspaces of the Hodge involution $*$. It is convenient to introduce the notation SR, for the "sign reversal" procedure, whose effect is to replace an indefinite $(n,1)$ form by a positive definite form: $\langle\ ,\ \rangle = SR(\omega, V^-)$, *cf.* formula (3.3).

By the Conway–Thompson theorem [MH73, Theorem 9.5], there exist positive definite unimodular lattices CT_n of rank n satisfying $\lambda_1(CT_n)^2 \geq k(n)$, where $k(n)$ is asymptotic to $\frac{n}{2\pi e}$ as $n \to \infty$, while λ_1 is the least length of a nonzero lattice element, *cf.* (1.1). Furthermore, by the classification of odd indefinite unimodular forms [MH73, p. 22], there exists a vector $v \in I_{n,1}$ with $q_{n,1}(v) = -1$, whose orthogonal complement with respect to the polarisation of $q_{n,1}$ is the lattice CT_n. Denote by $CT_n^{\perp} \subset H^2(X,\mathbb{R})$ the negative definite line $\mathbb{R}v$. Proposition 10.1 yields a Lorentzian endomorphism A_s of $\mathbb{R}^{n,1}$ which replaces the first two successive minima, λ_1 and λ_2 (*cf.* Definition 3.2), of the lattice with respect to the definite quadratic form $SR(\omega, v)$, by their geometric mean, when one passes to the new definite form $SR(\omega, A_s v)$.

Let $\mathcal{M}(X)$ be the space of all Riemannian metrics on X, and let $\mathcal{G}$ be the projectivisation of the negative cone of the form ω. Let $\mathcal{P} : \mathcal{M} \to \mathcal{G}$ be the map assigning to each metric, its antiselfdual direction. Exploiting the surjectivity of $\mathcal{P}$, we set

$$g_n = \mathcal{P}^{-1} A_{k(n)^{\frac{1}{4}}} \left(CT_n^{\perp}\right), \tag{11.1}$$

where $\mathcal{P}^{-1}$ denotes a choice of an inverse image. Finally, the lower bound results from the following calculation:

$$\begin{aligned}
\mathrm{confsys}_2(g_n) &= \lambda_1(H^2(X), |\ |_{L^2}) \\
&= \lambda_1\left(H^2\left(n\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \mathbb{Z}\right),\ SR\left(q_{n,1}, A_{k(n)^{\frac{1}{4}}}\left(CT_n^{\perp}\right)\right)^{\frac{1}{2}}\right) \\
&\geq \sqrt{\lambda_2\left(H^2(n\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \mathbb{Z}),\ SR\left(q_{n,1}, CT_n^{\perp}\right)^{\frac{1}{2}}\right)} \\
&= \sqrt{\lambda_1(CT_n)} \\
&\geq k(n)^{\frac{1}{4}}.
\end{aligned}$$

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