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Pieri-type formulas for the non-symmetric Jack polynomials

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Abstract. In the theory of symmetric Jack polynomials the coefficients in the expansion of the pth elementary symmetric function $e_p(z)$ times a Jack polynomial expressed as a series in Jack polynomials are known explicitly. Here analogues of this result for the non-symmetric Jack polynomials $E_\eta(z)$ are explored. Necessary conditions for non-zero coefficients in the expansion of $e_p(z)E_\eta(z)$ as a series in non-symmetric Jack polynomials are given. A known expansion formula for $z_iE_\eta(z)$ is rederived by an induction procedure, and this expansion is used to deduce the corresponding result for the expansion of $\prod_{j=1,\, j\neq i}^{N} z_j\, E_\eta(z)$, and consequently the expansion of $e_{N-1}(z)E_\eta(z)$. In the general p case the coefficients for special terms in the expansion are presented.

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1. Introduction

Jack polynomials and Macdonald polynomials can be defined as homogeneous multivariable orthogonal polynomials, or as eigenfunctions of a family of commuting differential or difference operators respectively. From the latter viewpoint these polynomials occur in the study of certain quantum many body systems [3, 8]. In their most basic form the polynomials are non-symmetric, although eigenfunctions with a prescribed symmetry with respect to interchange of coordinates are often required in application [1]. The polynomials with a prescribed symmetry can be obtained from the non-symmetric polynomials by an appropriate symmetry operation. One consequence of this feature is that many properties of the symmetric Jack and Macdonald polynomials can be obtained from the corresponding properties of the non-symmetric polynomials [2, 10].

There are, however, a number of properties of the symmetric Jack and Macdonald polynomials which have no known relation to properties of the non-symmetric polynomials. One example is the so-called Pieri formula [14, 9, 4]. To present this formula requires some notation. Let κ and λ be partitions described by their diagrams and suppose $\kappa \subset \lambda$. A skew diagram λ/κ is said to be a vertical m-strip if it consists of m boxes, all of which are in distinct rows. For λ/κ a vertical m-strip define χ_m by $\lambda = \kappa + \chi_m$, and put

$$U^{(\alpha)}(\lambda/\kappa) := \frac{f_N^1(\alpha\kappa + \chi_m) f_N^{1/\alpha}(\kappa)}{f_N^1(\alpha\kappa) f_N^{1/\alpha}(\kappa + \chi_m)}$$

where

$$f_n^r(\kappa) = \prod_{1 \le i < j \le n} \frac{((j-i)r + \kappa_i - \kappa_j)_r}{((j-i)r)_r}, \qquad (u)_r := \frac{\Gamma(u+r)}{\Gamma(u)}.$$

With this notation the Pieri formula reads

$$e_p(z) P_\kappa(z) = \sum_{\substack{\lambda \\ \lambda/\kappa \text{ a vertical } m-\text{strip}}} U^{(\alpha)}(\lambda/\kappa) P_\lambda(z) \tag{1.1}$$

where

$$e_p(z) := \sum_{1 \le i_1 < \cdots < i_p \le N} z_{i_1} \cdots z_{i_p}$$

denotes the pth elementary symmetric function, and $P_\kappa(x) := P_\kappa(x;\alpha)$ denotes the symmetric Jack polynomial indexed by the partition κ and normalized so that when expanded in terms of monomial symmetric functions the coefficient of the monomial symmetric function m_κ is unity.

It is the objective of this paper to investigate non-symmetric analogues of the Pieri formula (1.1). Our original idea was to adapt the method used by Knop and Sahi [6] to derive (1.1), which involves the theory of the so-called shifted Jack polynomials. This was passed on to D. Marshall, who subsequently [11] obtained the explicit form of the coefficients in the expansions

$$z_i E_\eta(z) = \sum_{\nu : |\nu| = |\eta| + 1} c_{\eta\nu}^{(i)} E_\nu(z) \tag{1.2}$$

$$\Big(\sum_{i=1}^N z_i\Big) E_\eta(z) = \sum_{\nu : |\nu| = |\eta| + 1} C_{\eta\nu} E_\nu(z). \tag{1.3}$$

In this work we will give an inductive proof of the evaluation of the $c_{\eta\nu}^{(i)}$ which avoids all reference to the theory of the shifted Jack polynomials (the evaluation of the $C_{\eta\nu}$ follows as a simple corollary from knowledge of the $c_{\eta\nu}^{(i)}$).

In Section 3 of the paper we present necessary conditions on ν for the coefficients in the expansion

$$z_{i_1} \cdots z_{i_p} E_\eta(z) = \sum_{\nu : |\nu| = |\eta| + p} c_{\eta\nu}^{(i_1,\dots,i_p)} E_\nu(z) \tag{1.4}$$

to be non-zero. Here use is made of the theory of shifted Jack polynomials. In Section 4 the result of Marshall for the explicit value of $c_{\eta\nu}^{(i)}$ is revised, and in Section 5 we present our inductive proof of this result. The expansion (1.4) in the case $p = N - 1$, where N is the number of variables $z := (z_1, \dots, z_N)$, is given in Section 6. In the final section, Section 7, a coefficient in the expansion of $e_p(z)E_\eta(z)$ as a series in $\{E_\nu\}$ is evaluated for a special value of ν and the form of the evaluation further explored for a larger class of ν.

2. The non-symmetric Jack polynomials

The non-symmetric Jack polynomials $E_\eta(z)$ can be specified as the simultaneous polynomial eigenfunctions of the commuting operators

$$\xi_i := \alpha z_i \frac{\partial}{\partial z_i} + \sum_{p<i} \frac{z_i}{z_i - z_p}(1 - s_{ip}) + \sum_{p>i} \frac{z_p}{z_i - z_p}(1 - s_{ip}) + 1 - i,$$

where s_{ip} is the operator which permutes z_i and z_p, satisfying the eigenvalue equations

$$\xi_i E_\eta = \bar{\eta}_i E_\eta, \quad (i = 1, \dots, N) \tag{2.1}$$

and with coefficient of $z^\eta = z^{\eta_1} \cdots z^{\eta_N}$ unity. For a given composition $\eta := (\eta_1, \dots, \eta_N)$, the eigenvalue $\bar{\eta}_i$ in (2.1) is given by

$$\bar{\eta}_i := \alpha\eta_i - \#\{k < i | \eta_k \geq \eta_i\} - \#\{k > i | \eta_k > \eta_i\}. \tag{2.2}$$

An alternative characterization of the non-symmetric Jack polynomials is as multivariable orthogonal polynomials. With $z_j := e^{2\pi i x_j}$, introduce the inner product

$$\langle f|g\rangle := \int_0^1 dx_1 \cdots \int_0^1 dx_N \prod_{1\leq j<k\leq N} |z_k - z_j|^{2/\alpha} f^*(z_1, \dots, z_N) g(z_1, \dots, z_N), \tag{2.3}$$

where the $*$ denotes complex conjugation. Suppose $|\eta| = |\nu|$ for compositions $\eta \neq \nu$. Introduce the dominance partial ordering $<$ on compositions by the statement that $\nu < \eta$ if $\sum_{j=1}^p \nu_j < \sum_{j=1}^p \eta_j$ for each $p = 1, \dots, N$. Let η^+ denote the partition corresponding to the composition η. Introduce a further partial ordering $\lhd$ by the statement that $\nu \lhd \eta$ if $\nu^+ < \eta^+$, or in the case $\nu^+ = \eta^+$, if $\nu < \eta$. Then for a given value of $|\eta|$, the E_η can be constructed via a Gram-Schmidt procedure from the requirements that

$$\langle E_\eta | E_\nu \rangle = 0, \tag{2.4}$$

for $\eta \neq \nu$, and that

$$E_\eta(z) = z^\eta + \sum_{\nu \lhd \eta} c_{\eta\nu} z^\nu. \tag{2.5}$$

We will have future use for the explicit value of

$$\mathcal{N}_\eta := \langle E_\eta | E_\eta \rangle.$$

This requires the introduction of further quantities for its presentation. Following [13], define the arm and leg lengths at the node (i,j) of the diagram of a composition η by

$$a(i,j) = \eta_i - j, \qquad l(i,j) = \#\{k < i | j \le \eta_k + 1 \le \eta_i\} + \#\{k > i | j \le \eta_k \le \eta_i\} \tag{2.6}$$

and put

$$d'_\eta := \prod_{(i,j)\in\eta} \Big(\alpha(a(i,j)+1) + l(i,j)\Big), \qquad d_\eta := \prod_{(i,j)\in\eta} \Big(\alpha(a(i,j)+1) + l(i,j) + 1\Big). \tag{2.7}$$

Also, define the generalized factorial by

$$[u]^{(\alpha)}_{\eta^+} = \prod_{j=1}^{N} \frac{\Gamma(u-(j-1)/\alpha + \eta^+_j)}{\Gamma(u-(j-1)/\alpha)}$$

and put

$$e_\eta = \alpha^{|\eta|}[1+N/\alpha]^{(\alpha)}_{\eta^+}, \qquad e'_\eta = \alpha^{|\eta|}[1+(N-1)/\alpha]^{(\alpha)}_{\eta^+}. \tag{2.8}$$

In terms of the quantities (2.7) and (2.8) we have [12, 2]

$$\frac{\mathcal{N}_\eta}{\mathcal{N}_{(0^N)}} = \frac{d'_\eta e_\eta}{d_\eta e'_\eta}. \tag{2.9}$$

Starting with $E_{(0^N)}(z) = 1$, the non-symmetric Jack polynomials can be recursively generated from the action of just two fundamental operators. The first of these operators is the elementary permutation operator $s_i := s_{i\,i+1}$, which permutes z_i and z_{i+1}. It has the action [12]

$$s_i E_\eta(z) = \begin{cases} \frac{1}{\bar\delta_{i,\eta}} E_\eta(z) + \Big(1 - \frac{1}{\bar\delta^2_{i,\eta}}\Big) E_{s_i\eta}(z), & \eta_i > \eta_{i+1} \\ E_\eta(z), & \eta_i = \eta_{i+1} \\ \frac{1}{\bar\delta_{i,\eta}} E_\eta(z) + E_{s_i\eta}(z), & \eta_i < \eta_{i+1} \end{cases} \tag{2.10}$$

where

$$\bar\delta_{i,\eta} := \bar\eta_i - \bar\eta_{i+1}. \tag{2.11}$$

The second required operator is the raising type operator, defined when acting on functions according to

$$\Phi f(z_1,\dots,z_N) = z_N f(z_N, z_1, \dots, z_{N-1}),$$

which has the property [7]

$$\Phi E_\eta(z) = E_{\Phi\eta}(z), \qquad \Phi\eta := (\eta_2,\dots,\eta_N,\eta_1+1). \tag{2.12}$$

Starting from $\eta = (0^N)$, all compositions can be generated by the action of $\Phi\eta$ and $s_i\eta$, so (2.10) and (2.12) provide the recursive generation of all the E_η.

Future use will be made of the quantities (2.7) and (2.8) with η replaced by $s_i\eta$ and $\Phi\eta$. In particular, we require the formulas [13]

$$e_{s_i\eta} = e_\eta, \quad e'_{s_i\eta} = e'_\eta, \quad \frac{d_{s_i\eta}}{d_\eta} = \begin{cases} \frac{\bar\delta_{i,\eta}+1}{\bar\delta_{i,\eta}}, & \eta_i > \eta_{i+1} \\ \frac{\bar\delta_{i,\eta}}{\bar\delta_{i,\eta}-1}, & \eta_i < \eta_{i+1} \end{cases} \quad \frac{d'_{s_i\eta}}{d'_\eta} = \begin{cases} \frac{\bar\delta_{i,\eta}}{\bar\delta_{i,\eta}-1}, & \eta_i > \eta_{i+1} \\ \frac{\bar\delta_{i,\eta}+1}{\bar\delta_{i,\eta}}, & \eta_i < \eta_{i+1} \end{cases}$$

$$\frac{d_{\Phi\eta}}{d_\eta} = \frac{e_{\Phi\eta}}{e_\eta} = \bar\eta_1 + \alpha + N, \qquad \frac{d'_{\Phi\eta}}{d'_\eta} = \frac{e'_{\Phi\eta}}{e'_\eta} = \bar\eta_1 + \alpha + N - 1. \tag{2.13}$$

Let us now revise some aspects of the theory of non-symmetric shifted Jack polynomials E^*_η [5]. The polynomial $E^*_\eta(z)$ is the unique polynomial of degree $\le |\eta|$ with the property

$$E^*_\eta(\bar\rho/\alpha) = 0, \qquad |\rho| \le |\eta|, \ \ \rho \ne \eta$$

and $E^*_\eta(\bar\eta/\alpha) \ne 0$ with coefficient of z^η in its monomial expansion unity ($\bar\eta := (\bar\eta_1, \dots, \bar\eta_N)$ where the $\bar\eta_j$ are specified by (2.2)). The non-symmetric Jack polynomial E_η is the leading homogeneous term of E^*_η so that

$$E^*_\eta(z) = E_\eta(z) + \text{lower degree terms}. \tag{2.14}$$

A fundamental property of the E^*_η is the extra vanishing condition. Introduce the partial ordering $\preceq$ on compositions by writing $\nu \preceq \eta$ if there exisits a permutation π such that $\nu_i < \eta_{\pi(i)}$ for $i < \pi(i)$ and $\nu_i \le \eta_{\pi(i)}$ for $i \ge \pi(i)$. Note that for ν and η partitions the statement $\nu \preceq \eta$ is equivalent to $\nu \subseteq \eta$ (inclusion of diagrams) but for compositions, although $\nu \subseteq \eta$ implies $\nu \preceq \eta$ (take π to be the identity), the converse is not true in general. The extra vanishing condition states [5]

$$E^*_\eta(\bar\nu/\alpha) = 0 \quad \text{for} \quad \eta \not\preceq \nu. \tag{2.15}$$

3. Structure of the Pieri type expansions for the non-symmetric Jack polynomials

Our interest is in the coefficients $c_{\eta\nu}^{(i_1,\dots,i_p)}$ in the expansion (1.4). In this section we will use the theory of the non-symmetric shifted Jack polynomials to present necessary conditions for the coefficients to be non-zero.

Now the extra vanishing condition (2.15) implies that any analytic function vanishing on $\{\bar\rho/\alpha : \eta \not\preceq \rho\}$ can be written in the form

$$f(z) = \sum_{\nu:\eta\preceq\nu} c_{\eta\nu} E^*_\nu(z). \tag{3.1}$$

It follows from this that

$$z_{i_1} \cdots z_{i_p} E^*_\eta(z) = \sum_{\substack{\nu:\eta\preceq\nu \\ |\nu|\le|\eta|+p}} c_{\eta\nu}^{(i_1,\dots,i_p)} E^*_\nu(z) \tag{3.2}$$

for some coefficients $c_{\eta\nu}^{(i_1,\dots,i_p)}$. Taking the leading homogeneous term on both sides using (2.14) gives

$$z_{i_1}\cdots z_{i_p}E_\eta(z)=\sum_{\substack{\nu:\eta\preceq\nu\\ |\nu|=|\eta|+p}} c_{\eta\nu}^{(i_1,\dots,i_p)}E_\nu(z) \tag{3.3}$$

which is a refinement of (1.4).

The statement (3.3) can be further refined by making use of the orthogonality (2.4). Applying this orthogonality in (3.3) shows that

$$c_{\eta\nu}^{(i_1,\dots,i_p)}=\frac{\langle E_\nu|z_{i_1}\cdots z_{i_p}E_\eta\rangle}{\langle E_\nu|E_\nu\rangle}. \tag{3.4}$$

Using the facts that with $z^1 := z_1\cdots z_N$ we have

$$E_{\eta+(1^N)}(z)=z^1E_\eta(z),\quad \langle z^1f|z^1g\rangle=\langle f|g\rangle \text{ and } \langle f|g\rangle=\langle g|f\rangle$$

(the latter provided f and g have real coefficients) it follows from (3.4) that

$$\begin{aligned} c_{\eta\nu}^{(i_1,\dots,i_p)} &= \frac{\langle z_{j_1}\cdots z_{j_{N-p}}E_\nu|E_{\eta+(1^N)}\rangle}{\langle E_\nu|E_\nu\rangle}\\ &= \frac{\langle E_{\eta+(1^N)}|z_{j_1}\cdots z_{j_{N-p}}E_\nu\rangle}{\langle E_\nu|E_\nu\rangle}=c_{\nu\eta+(1^N)}^{(j_1,\dots,j_{N-p})}\frac{\langle E_\eta|E_\eta\rangle}{\langle E_\nu|E_\nu\rangle} \end{aligned} \tag{3.5}$$

where $j_1,\dots,j_{N-p}$ are such that $\{1,\dots,N\}=\{i_1,\dots,i_p\}\cup\{j_1,\dots,j_{N-p}\}$. But according to (3.3) $c_{\nu\,\eta+(1^N)}^{(j_1,\dots,j_{N-p})}=0$ for $\nu\not\preceq\eta+(1^N)$ and thus (3.5) implies

$$c_{\eta\nu}^{(i_1,\dots,i_p)}=0\quad\text{for}\quad \nu\not\preceq\eta+(1^N). \tag{3.6}$$

Hence in (3.3) we can make the additional restriction $\nu\preceq\eta+(1^N)$, and so obtain

$$z_{i_1}\cdots z_{i_p}E_\eta(z)=\sum_{\nu\in\mathbb{J}_{N,p}} c_{\eta\nu}^{(i_1,\dots,i_p)}E_\nu(z) \tag{3.7}$$

where

$$\mathbb{J}_{N,p}:=\{\nu:\eta\preceq\nu\preceq\eta+(1^N),\ |\nu|=|\eta|+p\}. \tag{3.8}$$

Note that by performing the sum $1\le i_1<\cdots<i_p\le N$ in (3.7) we obtain

$$e_p(z)E_\eta(z)=\sum_{\nu\in\mathbb{J}_{N,p}} A_{\eta\nu}^{(p)}E_\nu(z) \tag{3.9}$$

for some constants $A_{\eta\nu}^{(p)}$.

Next we seek a more explicit description of the set $\mathbb{J}_{N,p}$. Let w_η be the shortest element of S_N (the permutations of $\{1,\dots,N\}$) such that $w_\eta^{-1}(\eta)$ is a partition and similarly define w_ν. It is straightforward to show [5] that if $\nu\preceq\eta$ then the permutation π in the definition of the partial order can be represented $\pi=w_\nu\circ w_\eta^{-1}=:\pi_{\nu,\eta}$. Now, members ν of the set $\mathbb{J}_{N,p}$ require both $\eta\preceq\nu$ and $\nu\preceq\eta+(1^N)$ with $|\nu|=|\eta|+p$. For the former ordering constraint the relevant permutation is $\pi_{\eta,\nu} = \pi^{-1}_{\nu,\eta}$. Replacing π by π^{-1} in the definition of $\preceq$ shows we require

$$\eta_i < \nu_{\pi_{\nu,\eta}(i)} \quad \text{for} \quad i < \pi_{\nu,\eta}(i), \qquad \eta_i \le \nu_{\pi_{\nu,\eta}(i)} \quad \text{for} \quad i \ge \pi_{\nu,\eta}(i). \tag{3.10}$$

For the latter ordering constraint the relevant permutation is $\pi_{\nu,\eta}$. Replacing π by π^{-1} and i by $\pi(i)$ in the definition of $\preceq$ shows we require

$$\nu_{\pi_{\nu,\eta}(i)} < \eta_i + 1 \quad \text{for} \quad \pi_{\nu,\eta}(i) < i, \qquad \nu_{\pi_{\nu,\eta}(i)} \le \eta_i + 1 \quad \text{for} \quad \pi_{\nu,\eta}(i) \ge i. \tag{3.11}$$

Combining (3.10) and (3.11) gives

$$\eta_i < \nu_{\pi_{\nu,\eta}(i)} \le \eta_i + 1 \quad \text{for} \quad i < \pi_{\nu,\eta}(i), \qquad \eta_i \le \nu_{\pi_{\nu,\eta}(i)} < \eta_i + 1 \quad \text{for} \quad i > \pi_{\nu,\eta}(i),$$

and so

$$\nu_{\pi_{\nu,\eta}(i)} = \eta_i + 1 \quad \text{for} \quad i < \pi_{\nu,\eta}(i), \qquad \nu_{\pi_{\nu,\eta}(i)} = \eta_i \quad \text{for} \quad i > \pi_{\nu,\eta}(i). \tag{3.12}$$

In the case $i = \pi_{\nu,\eta}(i)$ (3.10) and (3.11) give $\eta_i \le \nu_{\pi_{\nu,\eta}(i)} \le \eta_i + 1$ and so

$$\nu_{\pi_{\nu,\eta}(i)} = \eta_i \quad \text{or} \quad \nu_{\pi_{\nu,\eta}(i)} = \eta_i + 1. \tag{3.13}$$

It remains to implement the requirement $|\nu| = |\eta| + p$. We see from (3.12) and (3.13) that we must have

$$\nu_{\pi_{\nu,\eta}(i_r)} = \eta_{i_r} + 1 \qquad (r = 1, \dots, p) \tag{3.14}$$

for some $1 \le i_1 < \cdots < i_p \le N$ and

$$\nu_{\pi_{\nu,\eta}(j_r)} = \eta_{j_r} \qquad (r = 1, \dots, N-p) \tag{3.15}$$

where $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_{N-p}\} = \{1, 2, \dots, N\}$. Combining (3.14) and (3.15) with (3.12) and (3.13) shows compositions $\nu \in \mathbb{J}_{n,p}$ are characterized by the properties

$$\begin{aligned} \nu_{\pi(i_r)} &= \eta_{i_r} + 1 \quad \text{for} \quad i_r \le \pi(i_r) \qquad r = 1, \dots, p \\ \nu_{\pi(j_r)} &= \eta_{j_r} \quad \text{for} \quad j_r \ge \pi(j_r) \qquad r = 1, \dots, N-p \end{aligned} \tag{3.16}$$

for some permutation π ($\pi = \pi_{\nu,\eta}$ suffices). The characterization (3.16) can be interpreted in terms of the diagram of η. We begin by adding one box to the rows $i_1, \dots, i_p$. Then we consider all rearrangements of the rows such the rows with a box added move downwards or stay stationary, while the rows with no box added move upwards or stay stationary. An example is given in Figure 1.

In the case $p = 1$ the compositions ν defined by (3.16) and thus belonging to the set $\mathbb{J}_{N,1}$ have the property of being the minimal elements lying above η [5]. Note that this set can be indexed by subsets $I = \{t_1, \dots, t_s\}$ of $\{1, \dots, N\}$ with $t_1 < \cdots < t_s$ which correspond to the element

$$\nu =: c_I(\eta) \in \mathbb{J}_{N,1} \tag{3.17}$$

where

$$\begin{aligned} \nu_{t_j} &= \eta_{t_{j+1}} \qquad j = 1, \dots, s-1 \\ \nu_{t_s} &= \eta_{t_1} + 1 \\ \nu_i &= \eta_i \qquad i \notin I \end{aligned} \tag{3.18}$$

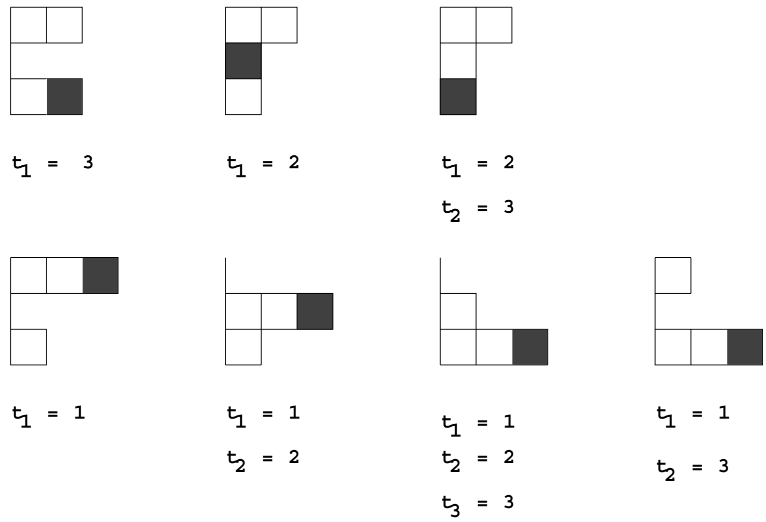

FIG. 1. Construction of the composition $\nu : \eta \preceq \nu \preceq \eta + (1)^3$ with $\eta = (2, 0, 1)$ and $|\nu| = |\eta| + 1$. The unshaded boxes originate from the diagram of η. With reference to the original diagram of η the row with the additional box (shaded) must move downwards or stay stationary, while the rows with no box added move upwards or stay stationary. The labels in the description (3.18) are also noted.

Furthermore, the subset I is called maximal with respect to η if $I \neq \emptyset$ and

$$\begin{aligned} &\eta_j \neq \eta_{t_u} \qquad j = t_{u-1} + 1, \dots, t_u - 1 \ (u = 1, \dots, s;\ t_0 := 0) \\ &\eta_j \neq \eta_{t_1} + 1 \qquad j = t_s + 1, \dots, N \end{aligned} \tag{3.19}$$

It follows from (3.18) that an equivalent way to characterize the maximal subsets is via the conditions

$$\begin{aligned} &\nu_j \neq \nu_{t_s} - 1, \qquad j = 1, \dots, t_1 - 1 \\ &\nu_j \neq \nu_{t_u}, \qquad j = t_u + 1, \dots, t_{u+1} - 1 \ (u = 1, \dots, s;\ t_{s+1} := N + 1). \end{aligned} \tag{3.20}$$

It is shown in [5] that it is only these maximal subsets which give distinct compositions ν (we illustrate this point in Figure 2). Thus we can write

$$\mathbb{J}_{N,1} := \mathbb{J}_{N,1}[\eta] = \{\nu : \ \nu = c_I(\eta),\ I \text{ maximal}\}. \tag{3.21}$$

It is also convenient to introduce the set $\mathbb{J}_\eta$ of maximal subsets

$$\mathbb{J}_\eta = \{I : \ I \text{ is maximal w.r.t. } \eta\}, \tag{3.22}$$

so that $\mathbb{J}_{N,1} = \{c_I(\eta) : \ I \in \mathbb{J}_\eta\}$.

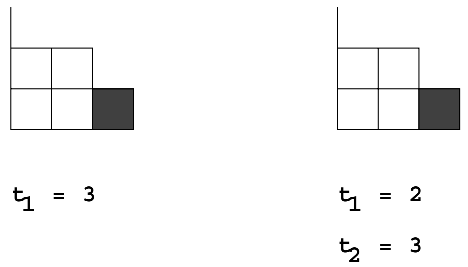

FIG. 2. In this example, starting with $\eta = (0, 2, 2)$, two different choices of subsets $I = \{t_1, \ldots, t_s\}$ give the same composition, but only the second subset is maximal (note that in the first diagram $\eta_2 = \eta_{t_1}$).

4. A Pieri type formula for the non-symmetric Jack polynomials in the case $p = 1$

So far the theory of shifted Jack polynomials [5] has been used to deduce the structural formula (3.7), and also notions from that theory are used to label the set $\mathbb{J}_{N,p}$ appearing in (3.7) in terms of certain maximal subsets I. To now evaluate the coefficients in (3.7), the most natural way to proceed is to make further use of theory from [5]. In the case $p = 1$ this part of the program has recently been successfully undertaken by Marshall [11]. The presentation of the result requires some notation.

First write

$$a(x, y) := \frac{1}{\alpha(x - y)}, \qquad b(x, y) := \frac{x - y - 1/\alpha}{x - y}. \tag{4.1}$$

For $I = \{t_1, \ldots, t_s\} \subseteq \{1, \ldots, N\}$, $I \neq \emptyset$, $t_1 < \cdots < t_s$ put

$$A_I(x) := \Big(\prod_{u=1}^{s-1} a(x_{t_u}, x_{t_{u+1}}) \Big) a(x_{t_s} - 1, x_{t_1}) \tag{4.2}$$

$$\begin{aligned} B_I(x) := & \Big(\prod_{u=1}^{s} \prod_{j=t_u+1}^{t_{u+1}-1} b(x_{t_u}, x_j) \Big) \\ & \times (x_{t_s} + (N-1)/\alpha) \prod_{j=1}^{t_1 - 1} b(x_{t_s} - 1, x_j), \quad t_{s+1} := N + 1 \end{aligned} \tag{4.3}$$

$$\begin{aligned} \tilde{B}_I(x) := & \Big(\prod_{u=1}^{s} \prod_{j=t_{u-1}+1}^{t_u - 1} b(x_{t_u}, x_j) \Big) \\ & \times \Big(\prod_{j=t_s+1}^{N} b(x_{t_1} + 1, x_j) \Big) (x_{t_1} + 1 + (N-1)/\alpha), \quad t_0 := 0 \end{aligned} \tag{4.4}$$

and for $i \in I$ write

$$\chi_I^{(i)}(x) = \begin{cases} \alpha(x_{t_{k-1}} - x_i), & i = t_k \ \ (k = 2, \dots, s) \\ \alpha(x_{t_s} - x_i - 1), & i = t_1 \end{cases}$$

$$\tilde{\chi}_I^{(i)}(x) = \begin{cases} \alpha(x_i - x_{t_{k+1}}), & i = t_k \ \ (k = 1, \dots, s-1) \\ \alpha(x_i - x_{t_1} - 1), & i = t_s. \end{cases} \tag{4.5}$$

In terms of these quantities, and the quantity d'_η of (2.7), the result of Marshall [11] reads

$$z_i E_\eta(z) = \alpha d'_\eta \sum_{\substack{I \in \mathbb{J}_\eta \\ \text{given } i \in I}} \frac{\chi_I^{(i)}(\overline{c_I(\eta)}/\alpha) A_I(\overline{c_I(\eta)}/\alpha) B_I(\overline{c_I(\eta)}/\alpha)}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z). \tag{4.6}$$

Also, noting from (4.5) that

$$\sum_{i \in I} \chi_I^{(i)}(x) = -\alpha \tag{4.7}$$

it follows from (4.6) that [11]

$$\Big(\sum_{i=1}^N z_i\Big) E_\eta(z) = -\alpha^2 d'_\eta \sum_{I \in \mathbb{J}_\eta} \frac{A_I((\overline{c_I(\eta)}/\alpha) B_I((\overline{c_I(\eta)}/\alpha)}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z). \tag{4.8}$$

We remark that it follows from the definition (4.3) of $B_I(x)$ that for I not maximal (i.e. cases for which the relations (3.20) are not obeyed), $B_I(\overline{c_I(\eta)}/\alpha) = 0$. Thus the restriction to maximal subsets in the summation of (4.6) and (4.8) is in fact a feature of the analytic form of the coefficients.

The dependence on $\overline{c_I(\eta)}$ in (4.6) and (4.8) can be replaced by a dependence on $\bar{\eta}$. Thus we note from the definitions (4.2), (4.4) and (4.5) that

$$A_I(\overline{c_I(\eta)}/\alpha) = A_I(\bar{\eta}/\alpha), \quad B_I(\overline{c_I(\eta)}/\alpha) = \tilde{B}_I(\bar{\eta}/\alpha),$$
$$\chi_I^{(i)}(\overline{c_I(\eta)}/\alpha) = \tilde{\chi}_I^{(i)}(\bar{\eta}/\alpha) \quad (i \in I),$$

which when substituted in (4.6) and (4.8) give

$$z_i E_\eta(z) = \alpha d'_\eta \sum_{\substack{I \in \mathbb{J}_\eta \\ \text{given } i \in I}} \frac{\tilde{\chi}_I^{(i)}(\bar{\eta}/\alpha) A_I(\bar{\eta}/\alpha) \tilde{B}_I(\bar{\eta}/\alpha)}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z) \tag{4.9}$$

$$\Big(\sum_{i=1}^N z_i\Big) E_\eta(z) = -\alpha^2 d'_\eta \sum_{I \in \mathbb{J}_\eta} \frac{A_I(\bar{\eta}/\alpha) \tilde{B}_I(\bar{\eta}/\alpha)}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z). \tag{4.10}$$

A still more useful form of (4.9) results by introducing

$$\begin{aligned} \hat{B}_I(\bar{\eta}/\alpha) &:= \alpha \frac{e'_\eta}{e'_{c_I(\eta)}} \tilde{B}_I(\bar{\eta}/\alpha) \\ &= \Big(\prod_{u=1}^s \prod_{j=t_{u-1}+1}^{t_u - 1} b(x_{t_u}, x_j)\Big) \Big(\prod_{j=t_s+1}^N b(x_{t_s} + 1, x_j)\Big), \quad t_0 := 0 \end{aligned} \tag{4.11}$$

where the equality follows from (2.8) and (4.4). In terms of this quantity (4.9) reads

$$z_i E_\eta(z) = \frac{d'_\eta}{e'_\eta} \sum_{\substack{I \in \mathbb{J}_\eta \\ \text{given } i \in I}} \frac{e'_{c_I(\eta)} \tilde{\chi}_I^{(i)}(\bar{\eta}/\alpha) A_I(\bar{\eta}/\alpha) \hat{B}_I(\bar{\eta}/\alpha)}{d'_{c_I(\eta)}} E_{c_I(\eta)}(z). \tag{4.12}$$

5. Inductive proof

In this section we will provide an inductive proof of (4.12). This has the advantage of being independent of the theory of the shifted Jack polynomials, relying only on the recurrence properties (2.10) and (2.12) of the non-symmetric Jack polynomials themselves.

Strategy

It has already been remarked that starting with $E_{(0^N)}(z) = 1$, the non-symmetric Jack polynomials can be generated recursively from the recurrence properties (2.10) and (2.12). To make use of these properties, suppose for a given η we know the coefficients $c^{(j)}_{\eta,\nu}$ in the expansion

$$z_j E_\eta(z) = \sum_{\nu \in \mathbb{J}_{N,1}[\eta]} c^{(j)}_{\eta,\nu} E_\nu(z) \tag{5.1}$$

for each $j = 1, \ldots, N$. Then, with $z_{N+1} := z_1$ and $c^{(N+1)}_{\eta,\nu} := c^{(1)}_{\eta,\nu}$, (2.12) gives

$$z_j E_{\Phi\eta}(z) = z_j \Phi E_\eta(z) = \Phi(z_{j+1} E_\eta(z)) = \sum_{\nu \in \mathbb{J}_{N,1}[\eta]} c^{(j+1)}_{\eta,\nu} E_{\Phi\nu}(z). \tag{5.2}$$

This shows $z_j E_{\Phi\eta}(z)$ can be computed from knowledge of the expansion (5.1) for the given η. Moreover, we can can give an explicit relationship between coefficients. To demonstrate this, for $I \subseteq \{1, \ldots, N\}$, $I \neq \emptyset$ put

$$\Phi(I) := \{j - 1 | j \in I \cap \{2, \ldots, N\}\} \cup \{N | 1 \in I\}. \tag{5.3}$$

Then we can check that $\Phi(I)$ is maximal with respect to $\Phi\eta$ if and only if I is maximal with respect to η. This means that in (5.2) we can replace the summation $\nu \in \mathbb{J}_{N,1}[\eta]$ by $\Phi\nu \in \mathbb{J}_{N,1}[\Phi\eta]$, which allows us to change variables $\Phi\nu \mapsto \nu$ to obtain

$$z_j E_{\Phi\eta}(z) = \sum_{\nu \in \mathbb{J}_{N,1}[\Phi\eta]} c^{(j+1)}_{\eta,\Phi^{-1}\nu} E_{\Phi\nu}(z). \tag{5.4}$$

On the other hand (5.1) gives

$$z_j E_{\Phi\eta}(z) = \sum_{\nu \in \mathbb{J}_{N,1}[\Phi\eta]} c^{(j)}_{\Phi\eta,\nu} E_{\Phi\nu}(z). \tag{5.5}$$

Comparing (5.4) and (5.5) shows we require

$$c^{(j)}_{\Phi\eta,\Phi\nu} = c^{(j+1)}_{\eta,\nu}, \quad (j = 1,\dots,N-1) \qquad c^{(N)}_{\Phi\eta,\Phi\nu} = c^{(1)}_{\eta,\nu}. \tag{5.6}$$

Let us now consider the computation of $z_j E_{s_i\eta}(z)$ for $\eta_i < \eta_{i+1}$ from knowledge of the expansion (5.1) for the given η. For this purpose we rewrite (5.1) as

$$z_j E_\eta(z) = \sum_\nu \alpha^{(j)}_{\eta,\nu} E_\nu(z) \tag{5.7}$$

where $\alpha^{(j)}_{\eta,\nu} = c^{(j)}_{\eta,\nu}$, $\nu \in \mathbb{J}_{N,1}[\eta]$ and $\alpha^{(j)}_{\eta,\nu} = 0$ otherwise. By doing this the sum over ν in (5.7) is unrestricted. Since from (2.10), with $\eta_i < \eta_{i+1}$,

$$\begin{aligned} z_j E_{s_i\eta}(z) &= z_j\Big(s_i E_\eta(z) - \frac{1}{\bar\delta_{i,\eta}} E_\eta(z)\Big) \\ &= \begin{cases} s_i(z_j E_\eta(z)), & j \ne i, i+1 \\ s_i(z_i E_\eta(z)), & j = i+1 \\ s_i(z_{i+1} E_\eta(z)), & j = i \end{cases} - \frac{1}{\bar\delta_{i,\eta}} z_j E_\eta(z) \end{aligned} \tag{5.8}$$

we see that knowledge of $z_j E_\eta(z)$ for each $j = 1,\dots,N$ implies the value of $z_j E_{s_i\eta}(z)$. We want to exhibit this feature as a recurrence for the coefficients $\alpha^{(j)}_{\eta,\nu}$. Now, from (5.7) and (2.10)

$$\begin{aligned} s_i(z_j E_\eta(z)) = &\sum_{\nu_i<\nu_{i+1}} \{\alpha^{(j)}_{\eta,\nu}\bar\delta^{-1}_{i,\nu} + \alpha^{(j)}_{\eta,s_i\nu}(1-\bar\delta^{-2}_{i,\nu})\}E_\nu(z) + \sum_{\nu_i=\nu_{i+1}} \alpha^{(j)}_{\eta,\nu}E_\nu(z) \\ &+ \sum_{\nu_i>\nu_{i+1}} \{\alpha^{(j)}_{\eta,\nu}\bar\delta^{-1}_{i,\nu} + \alpha^{(j)}_{\eta,s_i\nu}\}E_\nu(z) \end{aligned} \tag{5.9}$$

while (5.7) itself gives

$$z_j E_{s_i\eta}(z) = \sum_\nu \alpha^{(j)}_{s_i\eta,\nu} E_\nu(z). \tag{5.10}$$

Substituting (5.10), (5.9) and (5.7) in (5.8) and equating coefficients of $E_\nu(z)$ gives a recurrence allowing $\alpha^{(j)}_{\eta,s_i\nu}$ to be computed. In the recurrence it is necessary to distinguish the cases $\nu_i < \nu_{i+1}$ from $\nu_i > \nu_{i+1}$. However this can be avoided if we write the recurrence in terms of the quantity

$$\tilde\alpha^{(j)}_{\eta,\nu} := \frac{d'_\nu e'_\eta}{d'_\eta e'_\nu} \alpha^{(j)}_{\eta,\nu}$$

and make use of (2.13). We then find for $\nu_i \ne \nu_{i+1}$

$$\begin{aligned} (1+\bar\delta^{-1}_{i,\eta})\tilde\alpha^{(j)}_{s_i\eta,\nu} &= (1-\bar\delta^{-1}_{i,\nu})\tilde\alpha^{(j)}_{\eta,s_i\nu} + (\bar\delta^{-1}_{i,\nu} - \bar\delta^{-1}_{i,\eta})\tilde\alpha^{(j)}_{\eta,\nu} \quad (j \ne i, i+1) \\ (1+\bar\delta^{-1}_{i,\eta})\tilde\alpha^{(i+1)}_{s_i\eta,\nu} &= (1-\bar\delta^{-1}_{i,\nu})\tilde\alpha^{(i)}_{\eta,s_i\nu} + \bar\delta^{-1}_{i,\nu}\tilde\alpha^{(i)}_{\eta,\nu} - \bar\delta^{-1}_{i,\eta}\tilde\alpha^{(i+1)}_{\eta,\nu} \\ (1+\bar\delta^{-1}_{i,\eta})\tilde\alpha^{(i)}_{s_i\eta,\nu} &= (1-\bar\delta^{-1}_{i,\nu})\tilde\alpha^{(i+1)}_{\eta,s_i\nu} + \bar\delta^{-1}_{i,\nu}\tilde\alpha^{(i+1)}_{\eta,\nu} - \bar\delta^{-1}_{i,\eta}\tilde\alpha^{(i)}_{\eta,\nu} \end{aligned} \tag{5.11}$$

while for $\nu_i = \nu_{i+1}$

$$\begin{aligned}(1+\bar\delta_{i,\eta}^{-1})\tilde\alpha^{(j)}_{s_i\eta,\nu} &= (1-\bar\delta_{i,\eta}^{-1})\tilde\alpha^{(j)}_{\eta,\nu} \quad (j \neq i, i+1)\\ (1+\bar\delta_{i,\eta}^{-1})\tilde\alpha^{(i+1)}_{s_i\eta,\nu} &= \tilde\alpha^{(i)}_{\eta,\nu} - \bar\delta_{i,\eta}^{-1}\tilde\alpha^{(i+1)}_{\eta,\nu}\\ (1+\bar\delta_{i,\eta}^{-1})\tilde\alpha^{(i)}_{s_i\eta,\nu} &= \tilde\alpha^{(i+1)}_{\eta,\nu} - \bar\delta_{i,\eta}^{-1}\tilde\alpha^{(i)}_{\eta,\nu}. \end{aligned} \tag{5.12}$$

Noting that for $\nu_i = \nu_{i+1}$ we have $s_i\nu = \nu$, we see that the equations (5.11) remain valid in that they reduce to the equations (5.12), so it suffices to consider (5.11) for all ν.

Starting from knowledge of $c^{(j)}_{(0^N),\nu}$ and $\tilde\alpha^{(j)}_{(0)^N,\nu}$ for a particular j the recurrences (5.6) and (5.11) can be used to compute all the $c^{(j)}_{\eta,\nu}$ and $\tilde\alpha^{(j)}_{\eta,\nu}$. Thus, after independently establishing their validity in the case $\eta = (0^N)$ and a particular j, we want to show the functional forms

$$\tilde\alpha^{(j)}_{\eta,\nu} = \begin{cases} \tilde\chi^{(j)}_I(\bar\eta/\alpha)A_I(\bar\eta/\alpha)\hat B_I(\bar\eta/\alpha), & I \in \mathbb{J}_\eta \text{ and } j \in I \text{ with } \nu = c_I(\eta)\\ 0, & \text{otherwise}\end{cases} \tag{5.13}$$

$$c^{(j)}_{\eta,\nu} = \frac{d'_\eta e'_\nu}{d'_\nu e'_\eta}\tilde\chi^{(j)}_I(\bar\eta/\alpha)A_I(\bar\eta/\alpha)\hat B_I(\bar\eta/\alpha), \qquad \nu = c_I(\eta), \tag{5.14}$$

with $\tilde\chi^{(j)}$, A_I and $\hat B_I$ as specified by (4.5), (4.2) and (4.11) respectively, satisfy the recurrences (5.6) and (5.11) as appropriate.

Verification of the initial conditions

We require the expansion of z_i in terms of $\{E_\nu\}$. Any particular value of $i = 1, \dots, N$ is sufficient, although we will proceed with i arbitrary in this range. From the recurrences (5.6) and (5.11) we can readily show

$$E_{(0^k 1 0^{N-k-1})}(z) = z_{k+1} + \frac{1}{\alpha+k+1}(z_{k+2} + \cdots + z_N).$$

Thus the expansion of $\{E_{(0^k10^{N-k-1})}\}$, $k = 0, \dots, N-1$, in terms of $\{z_i\}$ has a triangular structure. This makes the task of inverting the formulas straightforward, provided we start with z_N and then compute the expansion of z_{N-1} etc.. We find

$$\begin{aligned} z_i = E_{(0^{i-1}10^{N-i})}(z) &- \frac{1}{\alpha+i}E_{(0^i10^{N-i-1})} - \frac{1}{\alpha+i+1}E_{(0^{i+1}10^{N-i-2})}(z) - \cdots\\ &- \frac{1}{\alpha+N-1}E_{(0^{N-1}1)}(z). \end{aligned} \tag{5.15}$$

On the other hand for $\nu = c_I(0^N)$, $I \in \mathbb{J}_{(0^N)}$ and $i \in I$ we see from (3.18) that the only possibilities are $\nu = (0^{j-1}10^{N-j})$ with $j \geq i$ so that (5.13) gives

$$z_i = \sum_{j=i}^{N} c^{(i)}_j E_{(0^{j-1}10^{N-j})}(z) \tag{5.16}$$

for some constants $c_j^{(i)}$. The general structure of (5.16) is in agreement with (5.15). To check that the coefficients agree we note from (3.18) that for $I \in \mathbb{J}_{(0^N)}$, we must have $I = \{1, 2, \ldots, j\}$ and $t_l = l$ $(l = 1, \ldots, j)$. Noting also that with $\eta = (0^N)$, $\bar{\eta}_i = -(i-1)$, we see that (4.5), (4.2) and (4.12) give

$$\begin{aligned} \tilde{\chi}_I^{(i)}(\overline{(0^N)}/\alpha) &= \begin{cases} 1, & i < j \\ -(N-1+\alpha), & i = j \end{cases} \\ A_I(\overline{(0^N)}/\alpha) &= -\frac{1}{j-1+\alpha}, \\ \hat{B}_I(\overline{(0^N)}/\alpha) &= \frac{j-1+\alpha}{N-1+\alpha}. \end{aligned} \tag{5.17}$$

Substituting (5.17), together with the evaluations

$$d'_{(0^N)} = e'_{(0^N)} = 1, \quad e'_{(0^{j-1}10^{N-j})} = \alpha + N - 1, \quad d'_{(0^{j-1}10^{N-j})} = \alpha + j - 1$$

from (2.8) and (2.7), in (5.13) we see that the coefficients in (5.16) are as required by (5.15).

Verification of the recurrences

Consider (5.6). From the definitions (2.2) and (2.12) we can check

$$(\overline{\Phi\eta})_i = (\bar{\eta})_{i+1}, \quad i \neq N \qquad (\overline{\Phi\eta})_N = (\bar{\eta})_1 + \alpha. \tag{5.18}$$

With $\Phi(I)$ defined by (5.3) and $I = \{t_1, \ldots, t_s\}$, making use of (5.18) it follows from the definition (4.2) that

$$\begin{aligned} &A_{\Phi(I)}(\overline{\Phi\eta}/\alpha) \\ &= \begin{cases} \Big(\prod_{u=1}^{s-1} a((\overline{\Phi\eta})_{t_u-1}/\alpha, (\overline{\Phi\eta})_{t_{u+1}-1}/\alpha)\Big) a((\overline{\Phi\eta})_{t_s-1}/\alpha - 1, (\overline{\Phi\eta})_{t_1-1}/\alpha), & t_1 \neq 1 \\ (a((\overline{\Phi\eta})_{t_{s-1}-1})/\alpha, (\overline{\Phi\eta})_N/\alpha) a((\overline{\Phi\eta})_N/\alpha - 1, (\overline{\Phi\eta})_{t_2-1}/\alpha) \\ \qquad \times \prod_{u=2}^{s-1} a((\overline{\Phi\eta})_{t_u-1}/\alpha, (\overline{\Phi\eta})_{t_{u+1}-1}/\alpha), & t_1 = 1 \end{cases} \\ &= \prod_{u=1}^{s-1} a(\bar{\eta}_{t_u}/\alpha, \bar{\eta}_{t_{u+1}}/\alpha)\, a(\bar{\eta}_{t_s}/\alpha - 1, \bar{\eta}_{t_1}/\alpha) = A_I(\bar{\eta}/\alpha). \end{aligned} \tag{5.19}$$

Similar calculations show

$$\hat{B}_{\Phi(I)}(\overline{\Phi\eta}/\alpha) = \hat{B}_I(\bar{\eta}/\alpha), \quad \tilde{\chi}_{\Phi(I)}^{(i)}(\overline{\Phi\eta}/\alpha) = \tilde{\chi}_I^{(i+1)}(\bar{\eta}/\alpha), \ (i = 1, \ldots, N-1),$$

$$\tilde{\chi}_{\Phi(I)}^{(N)}(\overline{\Phi\eta}/\alpha) = \tilde{\chi}_I^{(1)}(\bar{\eta}/\alpha), \quad c_{\Phi(I)}(\Phi\eta) = \Phi(c_I(\eta)).$$

These formulas together with the appropriate formula from (2.13) immediately imply (5.6) is satisfied by (5.13).

The recurrences (5.11) are not so straightforward. One complication is that the cases

$$\text{(i) } i,i+1\notin I \quad \text{(ii) } i\in I,\ i+1\notin I \quad \text{(iii) } i\notin I,\ i+1\in I \quad \text{(iv) } i,i+1\in I \tag{5.20}$$

must be treated separately, in addition to the division of cases depending on the value of (j). Independent of the division of cases (5.20), the fact that $\tilde{\alpha}^{(j)}_{\eta,\nu}=0$ for $\nu\notin\mathbb{J}_{N,1}[\eta]$ used in (5.11) gives

$$\tilde{\alpha}^{(j)}_{s_i\eta,\nu}=0, \qquad \nu\neq c_I(\eta),\ I\in\mathbb{J}_\eta \text{ and } \nu\neq s_ic_I(\eta),\ I\in\mathbb{J}_\eta. \tag{5.21}$$

For (5.21) to be consistent with (5.13) we must show

$$c_{I'}(s_i\eta)=c_I(\eta) \quad \text{or} \quad c_{I'}(s_i\eta)=s_ic_I(\eta) \text{ for some } I\in\mathbb{J}_\eta. \tag{5.22}$$

The validity of this statement will be verified for each of the cases separately.

First suppose $I'\in\mathbb{J}_{s_i\eta}\Big|_{i,i+1\notin I'}$. The definitions (3.18) and (3.19) give that this is equivalent to the statement that $I'=I$, $I\in\mathbb{J}_\eta\Big|_{i,i+1\in I}$, and

$$c_I(s_i\eta)=s_ic_I(\eta). \tag{5.23}$$

Suppose next $I'\in\mathbb{J}_{s_i\eta}\Big|_{i+1\in I',i\notin I'}$. Then there are two possibilities. The first is $I'=(I\cup\{i+1\})\backslash\{i\}$, $I\in\mathbb{J}_\eta\Big|_{i\in I,i+1\notin I}$, with

$$c_{(I\cup\{i+1\})\backslash\{i\}}(s_i\eta)=s_ic_I(\eta). \tag{5.24}$$

The second is $I'=I\backslash\{i\}$, $I\in\mathbb{J}_\eta\Big|_{i,i+1\in I}$ with

$$c_{I\backslash\{i\}}(s_i\eta)=c_I(\eta). \tag{5.25}$$

In the case $I'\in\mathbb{J}_{s_i\eta}\Big|_{i\in I,i+1\notin I'}$ the only possibility is $I'=(I\cup\{i\})\backslash\{i+1\}$, $I\in\mathbb{J}_\eta\Big|_{i\notin I,i+1\in I}$, with

$$c_{(I\cup\{i\})\backslash\{i+1\}}(s_i\eta)=s_ic_I(\eta). \tag{5.26}$$

The remaining case is $I'\in\mathbb{J}_{s_i\eta}\Big|_{i,i+1\in I'}$. Then we can have $I'=I\cup\{i\}$, $I\in\mathbb{J}_\eta\Big|_{i+1\in I,i\notin I}$ with

$$c_{I\cup\{i\}}(s_i\eta)=c_I(\eta). \tag{5.27}$$

These results together verify (5.22). Thus we can restrict attention to the cases

$$\nu=c_I(\eta), \qquad \nu=s_ic_I(\eta), \quad (I\in\mathbb{J}_\eta). \tag{5.28}$$

The case $i,i+1\notin I$

Because $\tilde{\alpha}^{(j)}_{\eta,\nu}$ requires $j\in I$ to be non-zero, while we are considering the case $i,i+1\notin I$, the second and third equations in (5.11) give $\tilde{\alpha}^{(i)}_{s_i\eta,\nu}=0$ and $\tilde{\alpha}^{(i+1)}_{s_i\eta,\nu}=0$, which is consistent with (5.13). Thus we can restrict attention to the first equation in (5.11). Also, if $\nu_i = \nu_{i+1}$ with $i, i+1 \notin I$ then we must have $\eta_i = \eta_{i+1}$. Since in the induction procedure it suffices to consider only the cases $\eta_{i+1} > \eta_i$ we can suppose $\nu_i \neq \nu_{i+1}$.

Suppose $\nu = c_I(\eta)$. The assumptions that $i, i+1 \notin I$ and $\eta_i \neq \eta_{i+1}$ together with (5.23) imply there is no $I' \in \mathbb{J}_\eta$ such that $c_{I'}(\eta) = s_i c_I(\eta)$ and thus $\tilde{\alpha}^{(j)}_{\eta, s_i\nu} = 0$. Also in this case it follows from the definition (2.11) that $\bar{\delta}_{i, c_I(\eta)} = \bar{\delta}_{i,\eta}$. Substituting these formulas in the first equation of (5.11) gives

$$\tilde{\alpha}^{(j)}_{s_i\eta, c_I(\eta)} = 0, \qquad j \neq i, i+1, \quad I \in \mathbb{J}_\eta\Big|_{i,i+1\notin I}. \tag{5.29}$$

This is consistent with (5.13) because (5.23) and the surrounding sentence implies there is no $I' \in \mathbb{J}_{s_i\eta}$ such that $c_{I'}(s_i\eta) = c_I(\eta)$.

According to (5.28) the remaining possibility for a non-zero value is $\nu = s_i c_I(\eta)$. From the above reasoning we know from this choice of ν, $\tilde{\alpha}^{(j)}_{\eta,\nu} = 0$, while (2.11) gives $\bar{\delta}_{i, s_i c_I(\eta)} = -\bar{\delta}_{i,\eta}$. Thus the first equation in (5.11) reduces to

$$\tilde{\alpha}^{(j)}_{s_i\eta, s_i c_I(\eta)} = \tilde{\alpha}^{(j)}_{\eta, c_I(\eta)} \qquad j \neq i, i+1, \quad I \in \mathbb{J}_\eta\Big|_{i,i+1\notin I}. \tag{5.30}$$

From (5.23) and the surrounding text we know that $s_i c_I(\eta) = c_I(s_i\eta)$ with $I \in \mathbb{J}_{s_i\eta}$. We note in general from (2.2) that

$$(\overline{s_i\eta})_i = \begin{cases} \bar{\eta}_{i+1}, & \eta_i \neq \eta_{i+1} \\ \bar{\eta}_i, & \eta_i = \eta_{i+1} \end{cases}, \ (\overline{s_i\eta})_{i+1} = \begin{cases} \bar{\eta}_i, & \eta_i \neq \eta_{i+1} \\ \bar{\eta}_{i+1}, & \eta_i = \eta_{i+1} \end{cases}, \ (\overline{s_i\eta})_j = \bar{\eta}_j \ (j \neq i, i+1). \tag{5.31}$$

Using (5.31), we see from the definitions (4.2), (4.11) that for $I \in \mathbb{J}_\eta\Big|_{i,i+1\notin I} = \mathbb{J}_{s_i\eta}\Big|_{i,i+1\notin I}$,

$$A_I(\bar{\eta}/\alpha) = A_I(\overline{s_i\eta}/\alpha), \qquad \hat{B}_I(\bar{\eta}/\alpha) = \hat{B}_I(\overline{s_i\eta}/\alpha)$$

while from (4.5) we see that

$$\tilde{\chi}^{(j)}_I(\bar{\eta}/\alpha) = \tilde{\chi}^{(j)}_I(\overline{s_i\eta}/\alpha).$$

Hence (5.13) satisfies (5.30).

The case $i \in I$, $i+1 \notin I$

We note that in this case $s_i c_I(\eta) \neq c_I(\eta)$. Consider the first equation in (5.11) and suppose $\nu = c_I(\eta)$. We can check from the definitions (3.18) and (3.19) that for $I \in \mathbb{J}_\eta\Big|_{i\in I, i+1\notin I}$,

$$s_i c_I(\eta) = c_{I\cup\{i+1\}}(\eta) \tag{5.32}$$

so the value of $\tilde{\alpha}^{(i)}_{\eta, s_i\nu}$ on the right hand side of the first equation in (5.11) is non-zero. Noting from (2.11) and the definition (3.18) of $c_I(\eta)$ that

$$\bar{\delta}_{i, c_I(\eta)} = (\overline{c_I(\eta)})_i - \bar{\eta}_{i+1},$$

this equation reads

$$\begin{aligned}(1+\bar{\delta}_{i,\eta}^{-1})\tilde{\alpha}^{(j)}_{s_i\eta,c_I(\eta)} &= \frac{\overline{(c_I(\eta))}_i-\bar{\eta}_{i+1}-1}{\overline{(c_I(\eta))}_i-\bar{\eta}_{i+1}}\tilde{\alpha}^{(j)}_{\eta,s_ic_I(\eta)} \\ &\quad -\Big(\frac{1}{\bar{\eta}_i-\bar{\eta}_{i+1}}-\frac{1}{\overline{(c_I(\eta))}_i-\bar{\eta}_{i+1}}\Big)\tilde{\alpha}^{(j)}_{\eta,c_I(\eta)}.\end{aligned} \tag{5.33}$$

The equation (5.32) allows the quantities $A_{I'}$, $\hat{B}_{I'}$ and $\tilde{\chi}^{(j)}_{I'}$, $I'=I\cup\{i+1\}$, making up $\tilde{\alpha}^{(j)}_{\eta,s_ic_I(\eta)}$ to be related to the corresponding quantities in $\tilde{\alpha}^{(j)}_{\eta,c_I(\eta)}$. Thus we can check from the definitions (4.2) and (4.11) that

$$\begin{aligned}A_{I\cup\{i+1\}}(\bar{\eta}/\alpha) &= \frac{(\bar{\eta}_i-\overline{(c_I(\eta))}_i)}{(\bar{\eta}_i-\bar{\eta}_{i+1})(\bar{\eta}_{i+1}-\overline{(c_I(\eta))}_i)}A_I(\bar{\eta}/\alpha) \\ \hat{B}_{I\cup\{i+1\}}(\bar{\eta}/\alpha) &= \frac{(\bar{\eta}_{i+1}-\overline{(c_I(\eta))}_i)}{\bar{\eta}_{i+1}-\overline{(c_I(\eta))}_i+1}\hat{B}_I(\bar{\eta}/\alpha) \\ \tilde{\chi}^{(j)}_{I\cup\{i+1\}}(\bar{\eta}/\alpha) &= \tilde{\chi}^{(j)}_I(\bar{\eta}/\alpha), \qquad j\neq i,i+1.\end{aligned} \tag{5.34}$$

When multiplied together according to (5.13) to form $\tilde{\alpha}^{(j)}_{\eta,s_ic_I(\eta)}$ and substituted in (5.33) we find all terms on the right hand side cancel giving the result

$$\tilde{\alpha}^{(j)}_{s_i\eta,c_I(\eta)}=0, \qquad j\neq i,i+1. \tag{5.35}$$

Consider now the second and third equation equations in (5.11) in the case $\nu=c_I(\eta)$. The requirement in (5.13) that $\tilde{\alpha}^{(j)}_{\eta,c_I(\eta)}\neq 0$ only if $j\in I$, while we are assuming $i\in I$, $i+1\notin I$, means the equations read

$$\begin{aligned}(1+\bar{\delta}_{i,\eta}^{-1})\tilde{\alpha}^{(i+1)}_{s_i\eta,c_I(\eta)} &= (1-\bar{\delta}_{i,\nu}^{-1})\tilde{\alpha}^{(i)}_{\eta,c_{I\cup\{i+1\}}(\eta)}+\bar{\delta}_{i,\nu}^{-1}\tilde{\alpha}^{(i)}_{\eta,c_I(\eta)} \\ (1+\bar{\delta}_{i,\eta}^{-1})\tilde{\alpha}^{(i)}_{s_i\eta,c_I(\eta)} &= (1-\bar{\delta}_{i,\nu}^{-1})\tilde{\alpha}^{(i+1)}_{\eta,c_{I\cup\{i+1\}}(\eta)}-\bar{\delta}_{i,\eta}^{-1}\tilde{\alpha}^{(i)}_{\eta,c_I(\eta)}\end{aligned} \tag{5.36}$$

where use has also been made of (5.32). To simplify the right hand sides of these equations we note from (4.5) that

$$\begin{aligned}\tilde{\chi}^{(i)}_{I\cup\{i+1\}}(\bar{\eta}/\alpha) &= \frac{\bar{\eta}_i-\bar{\eta}_{i+1}}{\bar{\eta}_i-\overline{(c_I(\eta))}_i}\tilde{\chi}^{(i)}_I(\bar{\eta}/\alpha), \\ \tilde{\chi}^{(i+1)}_{I\cup\{i+1\}}(\bar{\eta}/\alpha) &= \frac{\bar{\eta}_{i+1}-\overline{(c_I(\eta))}_i}{\bar{\eta}_i-\overline{(c_I(\eta))}_i}\tilde{\chi}^{(i)}_I(\bar{\eta}/\alpha).\end{aligned}$$

Use of these equation, together with the first two equations of (5.34) allows us to express $\tilde{\alpha}^{(i)}_{\eta,c_{I\cup\{i+1\}}(\eta)}$ and $\tilde{\alpha}^{(i+1)}_{\eta,c_{I\cup\{i+1\}}(\eta)}$ in terms of $\tilde{\alpha}^{(i)}_{\eta,c_I(\eta)}$. Doing this shows the right hand side is equal to zero in both cases and so for all $j=1,\dots,N$

$$\tilde{\alpha}^{(j)}_{s_i\eta,c_I(\eta)}=0. \tag{5.37}$$

For $s_i c_I(\eta) \neq c_I(\eta)$ the result (5.37) is consistent with (5.13) because the result (5.24) implies that for $I \in \mathbb{J}_\eta\Big|_{i+1 \notin I, i \in I}$ there is no $I' \in \mathbb{J}_{s_i\eta}$ such that $c_{I'}(s_i\eta) = c_I(\eta)$.

We now proceed to consider the equations (5.11) in the case $\nu = s_i c_I(\eta)$, $I \in \mathbb{J}_\eta\Big|_{i \in I, i+1 \notin I}$. Proceeding as in the derivation of (5.35) we find that in this case the first equation of (5.11) reads

$$\begin{aligned}\tilde{\alpha}^{(j)}_{s_i\eta, s_i c_I(\eta)} =& \frac{(\bar{\eta}_i - \bar{\eta}_{i+1} - 1)(\bar{\eta}_{i+1} - (\overline{c_I(\eta)})_i)}{(\bar{\eta}_i - \bar{\eta}_{i+1})(\bar{\eta}_{i+1} - (\overline{c_I(\eta)})_i + 1)} \tilde{\alpha}^{(j)}_{\eta, c_I(\eta)} \\ =& \tilde{\chi}^{(j)}_{(I \cup \{i+1\}) \setminus \{i\}}(\overline{s_i\eta}/\alpha) A_{(I \cup \{i+1\}) \setminus \{i\}}(\overline{s_i\eta}/\alpha) \hat{B}_{(I \cup \{i+1\}) \setminus \{i\}}(\overline{s_i\eta}/\alpha), \\ & j \neq i, i+1\end{aligned} \tag{5.38}$$

where the second equality follows after use of (5.13) to substitute for $\tilde{\alpha}^{(j)}_{\eta, c_I(\eta)}$ and use of the definitions (4.2), (4.11) and (4.5). An analogous calculation, involving the second and third equations of (5.11), gives

$$\tilde{\alpha}^{(i)}_{s_i\eta, s_i c_I(\eta)} = 0 \tag{5.39}$$

as well as the equation (5.38) in the case $j = i + 1$. Recalling (5.24) we see the equations (5.38) and (5.39) are consistent with (5.13).

The case $i \notin I$, $i + 1 \in I$

We distinguish the case

$$s_i c_I(\eta) = c_I(\eta) \tag{5.40}$$

from

$$s_i c_I(\eta) \neq c_I(\eta). \tag{5.41}$$

In the case (5.40) we can check that

$$(\overline{c_I(\eta)})_{i+1} = \bar{\eta}_i - 1, \tag{5.42}$$

while a feature of the case (5.41) is that there is no $I' \in \mathbb{J}_\eta$ such that $s_i c_I(\eta) = c_{I'}(\eta)$ and therefore

$$\tilde{\alpha}^{(j)}_{\eta, s_i c_I(\eta)} = 0. \tag{5.43}$$

Consider first the equations (5.11) in the case (5.40) (as already noted, the equations (5.11) are equivalent to the equations (5.12) for $\nu = s_i c_I(\eta) = c_I(\eta)$). The equations read

$$\begin{aligned}(1 + \bar{\delta}^{-1}_{i,\eta}) \tilde{\alpha}^{(j)}_{s_i\eta, c_I(\eta)} &= (1 - \bar{\delta}^{-1}_{i,\eta}) \tilde{\alpha}^{(j)}_{\eta, c_I(\eta)}, \quad j \neq i, i+1 \\ (1 + \bar{\delta}^{-1}_{i,\eta}) \tilde{\alpha}^{(i+1)}_{s_i\eta, c_I(\eta)} &= -\bar{\delta}^{-1}_{i,\eta} \tilde{\alpha}^{(i+1)}_{\eta, c_I(\eta)} \\ (1 + \bar{\delta}^{-1}_{i,\eta}) \tilde{\alpha}^{(i)}_{s_i\eta, c_I(\eta)} &= \tilde{\alpha}^{(i+1)}_{\eta, c_I(\eta)}.\end{aligned} \tag{5.44}$$

To verify that (5.13) satisfies these equations we note that in the case (5.40) the equation (5.27) is valid, so we should seek to express (5.44) in terms of $\tilde{\alpha}_{s_i\eta, c_{I\cup\{i\}}(s_i\eta)}$. Now (5.31), (4.2), (4.11) and (4.5) give

$$\begin{aligned} A_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) &= \frac{\bar{\eta}_i - \bar{\eta}_{i+1} - 1}{\bar{\eta}_i - \bar{\eta}_{i+1}} A_I(\bar{\eta}/\alpha) \\ \hat{B}_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) &= \frac{\bar{\eta}_i - \bar{\eta}_{i+1}}{\bar{\eta}_i - \bar{\eta}_{i+1} + 1} \hat{B}_I(\bar{\eta}/\alpha) \\ \tilde{\chi}^{(j)}_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) &= \tilde{\chi}^{(j)}_I(\bar{\eta}/\alpha), \quad j \neq i, i+1 \\ \tilde{\chi}^{(i+1)}_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) &= \frac{1}{\bar{\eta}_{i+1} - \bar{\eta}_i + 1} \tilde{\chi}^{(i+1)}_I(\bar{\eta}/\alpha) \\ \tilde{\chi}^{(i)}_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) &= \frac{\bar{\eta}_{i+1} - \bar{\eta}_i}{\bar{\eta}_{i+1} - \bar{\eta}_i + 1} \tilde{\chi}^{(i+1)}_I(\bar{\eta}/\alpha). \end{aligned} \tag{5.45}$$

Making use of these equations in the right hand side of (5.44) we find that for each $j = 1, 2, \dots, N$

$$\tilde{\alpha}^{(j)}_{s_i\eta, c_I(\eta)} = \tilde{\chi}^{(j)}_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) A_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) \hat{B}_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) \tag{5.46}$$

which by virtue of (5.27) is consistent with (5.13).

Consider now the equations (5.11) with $\nu = c_I(\eta)$ in the case (5.41). Then (3.12) holds, so (5.11) can be appropriately simplified. Furthermore, we can check that the second and third members of (5.45) remain valid, while the remaining equations are to be replaced by

$$\begin{aligned} A_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) &= -\frac{(\bar{\eta}_{i+1} - (\overline{c_I(\eta)})_{i+1})}{(\bar{\eta}_i - \bar{\eta}_{i+1})(\bar{\eta}_i - (\overline{c_I(\eta)})_{i+1})} A_I(\bar{\eta}/\alpha) \\ \tilde{\chi}^{(i+1)}_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) &= \frac{\bar{\eta}_i - (\overline{c_I(\eta)})_{i+1}}{\bar{\eta}_{i+1} - (\overline{c_I(\eta)})_{i+1}} \tilde{\chi}^{(i+1)}_I((\bar{\eta}/\alpha) \\ \tilde{\chi}^{(i)}_{I\cup\{i\}}(\overline{s_i\eta}/\alpha) &= \frac{\bar{\eta}_{i+1} - \bar{\eta}_i}{\bar{\eta}_{i+1} - (\overline{c_I(\eta)})_{i+1}} \tilde{\chi}^{(i+1)}_I((\bar{\eta}/\alpha). \end{aligned} \tag{5.47}$$

Using these equations to further simplify (5.11) again gives (5.46), which we know is consistent with (5.13). It remains to consider the case $\nu = s_i c_I(\eta)$, for which it suffices to restrict attention to the subcase (5.41) as the subcase (5.41) is included in the above working. We first simplify the equations (5.11) according to (5.43) and then obtain the analogues of (5.45) for the quantities $A_{(I\cup\{i\})\setminus\{i+1\}}(\overline{s_i\eta}/\alpha)$ etc.. We find, for $j \neq i+1$,

$$\tilde{\alpha}^{(j)}_{s_i\eta, c_I(\eta)} = \tilde{\chi}^{(j)}_{I\cup\{i\}\setminus\{i+1\}}(\overline{s_i\eta}/\alpha) A_{(I\cup\{i\})\setminus\{i+1\}}(\overline{s_i\eta}/\alpha) \hat{B}_{I\cup\{i\}\setminus\{i+1\}}(\overline{s_i\eta}/\alpha)$$

while

$$\tilde{\alpha}^{(i+1)}_{s_i\eta, c_I(\eta)} = 0.$$

We see from (5.26) that these equations are consistent with (5.13).

The case $i, i+1 \in I$

Analogous to the case $i \notin I$, $i+1 \in I$ we distinguish the case $s_i c_I(\eta) = c_I(s_i\eta)$ from $s_i c_I(\eta) \notin c_I(s_i\eta)$. In the latter case

$$s_i c_I(\eta) = c_{I\backslash\{i+1\}}(\eta). \tag{5.48}$$

This tells us that this case is the same as that with $I' \in \mathbb{J}_\eta$, $i \in I'$, $i+1 \notin I'$, which has already been dealt with. Thus we can restrict attention to the case $s_i c_I(\eta) = c_I(s_i\eta)$, when the equations (5.11) reduce to the equations (5.12). Obtaining the analogue of (5.45) but for $A_{I\backslash\{i\}}(\overline{s_i\eta}/\alpha)$ etc. we find, for $j \neq i$,

$$\tilde{\alpha}^{(j)}_{s_i\eta, c_I(\eta)} = \tilde{\chi}^{(j)}_{I\backslash\{i\}}(\overline{s_i\eta}/\alpha) A_{I\backslash\{i\}}(\overline{s_i\eta}/\alpha) \hat{B}_{I\backslash\{i\}}(\overline{s_i\eta}/\alpha)$$

while

$$\tilde{\alpha}^{(i)}_{s_i\eta, c_I(\eta)} = 0.$$

By virtue of (5.25) these equations are consistent with (5.13).

This completes consideration of the choices of ν (5.28) in all four cases (5.20). In each case it was found (5.13) satisfies the recurrences (5.11), thereby completing the demonstration that for general ν (5.13) satisfies (5.11). Since the other fundamental recurrence (5.6) has also been shown to be satisfied, as has the initial condition, our inductive proof is complete.

6. An equivalent expansion formula

The formula (4.10) is the non-symmetric analogue of the Pieri formula (1.1) in the case $p = 1$. Here we will use this result and the formula (3.5) to derive the analogue of (1.1) in the case $p = N - 1$.

First we note that in the case $p = N-1$, analogous to the case $p = 1$, the set $\mathbb{J}_{N,p}$ appearing in (3.7) and (3.9) can be indexed by subsets $I = \{t_1, \dots, t_s\}$ of $\{1, \dots, N\}$ with $t_1 < \cdots < t_s$. Each such subset corresponds to the element

$$\nu =: \hat{c}_I(\eta)$$

where

$$\begin{aligned} (\hat{c}_I(\eta))_{t_1} &= \eta_{t_s}, \\ (\hat{c}_I(\eta))_{t_u} &= \eta_{t_{u-1}} + 1, \quad u = 2, \dots, s \\ (\hat{c}_I(\eta))_k &= \eta_k + 1, \quad k \notin I \end{aligned} \tag{6.1}$$

(c.f. (3.18)). Furthermore, to avoid duplication within the set $\mathbb{J}_{N,N-1}$, as with the description (3.18) of $\mathbb{J}_{N,1}$, we must restrict I to maximal subsets with respect to η, in this case specified by the requirements

$$\begin{aligned} &\eta_j \neq \eta_{t_s} - 1, \qquad j = 1, \dots, t_1 - 1 \\ &\eta_j \neq \eta_{t_u}, \qquad j = t_u + 1, \dots, t_{u+1} - 1 \end{aligned} \tag{6.2}$$

for $u = 1, \ldots, s$ with $t_{s+1} := N + 1$. With this definition of maximal, analogous to (3.22) we define

$$\hat{\mathbb{J}}_\eta = \{I : I \text{ is maximal with respect to } \eta\}.$$

According to (3.5)

$$c_{\eta\nu}^{(i_1,\ldots,i_{N-1})} = c_{\nu\,\eta+(1^N)}^{(j_1)} \frac{\langle E_\eta | E_\eta \rangle}{\langle E_\nu | E_\nu \rangle}. \tag{6.3}$$

Now $c_{\nu\,\eta+(1^N)}^{(j_1)}$ is non-zero only if $\eta + (1^N) = c_I(\nu)$, $I \in \mathbb{J}_\nu$, $j_1 \in I$. With $I = \{t_1, \ldots, t_s\}$ we see from (3.18) that $\eta + (1^N) = c_I(\nu)$ gives

$$\begin{aligned} \eta_{t_j} &= \nu_{t_{j+1}} - 1, \qquad j = 1, \ldots, s-1 \\ \eta_{t_s} &= \nu_{t_1} \\ \eta_i &= \nu_i - 1, \qquad i \notin I \end{aligned} \tag{6.4}$$

while from (3.20) the condition $I \in \mathbb{J}_\nu$ gives

$$\eta_j \neq \eta_{t_s} - 1 \qquad j = 1, \ldots, t_1 - 1 \tag{6.5}$$

$$\eta_j \neq \eta_{t_u} \qquad j = t_u + 1, \ldots, t_{u+1} - 1 \tag{6.6}$$

for $u = 1, \ldots, s$ with $t_{N+1} := N + 1$. These are precisely the equations (6.1) with $\nu := \hat{c}_I(\eta)$ and the equations (6.2) for $I \in \hat{\mathbb{J}}_\eta$, so we conclude

$$\eta + (1^N) = c_I(\nu)\Big|_{I \in \mathbb{J}_\nu} \quad \text{iff} \quad \nu = \hat{c}_I(\eta)\Big|_{I \in \hat{\mathbb{J}}_\eta}.$$

It remains to substitute for the explicit values in the right hand side of (6.3). With $\eta + (1^N) = c_I(\nu)$, $I \in \mathbb{J}_\nu$ and thus $\nu = \hat{c}_I(\eta)$, $I \in \hat{\mathbb{J}}_\eta$ we read off from (4.12) and (2.9) that

$$c_{\eta\nu}^{(i_1,\ldots,i_{N-1})} = \frac{e_\eta d_\nu}{d_\eta e_\nu} \frac{d_\eta e_{\eta+(1^N)}}{d_{\eta+(1^N)} e_\eta} \tilde{\chi}_I^{(j_1)}(\overline{\eta}/\alpha) A_I(\overline{\eta}/\alpha) \hat{B}_I(\overline{\eta}/\alpha) \tag{6.7}$$

where use has been made of the facts that $\mathcal{N}_\eta = \mathcal{N}_{\eta+(1^N)}$ and $\tilde{\chi}_I^{(j_1)}((\overline{\eta}+c)/\alpha) = \tilde{\chi}_I^{(j_1)}(\overline{\eta}/\alpha)$ etc. for any constant c. This further simplifies by noting from (2.7), (2.6) and (2.2) that

$$\frac{d_\eta}{d_{\eta+(1^N)}} = \frac{1}{\prod_{j=1}^N (\bar{\eta}_j + \alpha + N)}$$

while (2.8) together with (5.31) and (2.2) implies

$$\frac{e_{\eta+(1^N)}}{e_\eta} = \prod_{j=1}^N (\bar{\eta}_j + \alpha + N).$$

Substituting these formulas in (6.7) gives

$$c_{\eta\nu}^{(i_1,\ldots,i_{N-1})} = \frac{e_\eta d_\nu}{d_\eta e_\nu} \tilde{\chi}_I^{(j_1)}(\overline{\eta}/\alpha) A_I(\overline{\eta}/\alpha) \hat{B}_I(\overline{\eta}/\alpha), \quad \nu = \hat{c}_I(\eta),\ I \in \hat{\mathbb{J}}_\eta \tag{6.8}$$

(c.f. (5.14)).

As in the derivation of (4.8) from (4.6), if follows from (6.8) that

$$e_{N-1}(z)E_\eta(z) = -\alpha\frac{e_\eta}{d_\eta}\sum_{I\in\hat{\mathbb{J}}_\eta}\frac{e_{\hat{c}_I(\eta)}A_I(\bar{\eta}/\alpha)\hat{B}_I(\bar{\eta}/\alpha)}{d_{\hat{c}_I(\eta)}}E_{\hat{c}_I(\eta)}(z). \tag{6.9}$$

7. The coefficient $A^{(p)}_{\eta,\nu}$ of the Pieri type formula for general p

In this final section we will consider features of the coefficient $A^{(p)}_{\eta,\nu}$ in the expansion (3.9) for general p. Our first result concerns the value of $A^{(p)}_{\eta,\nu}$ for a particular value of ν. Denote by $\mathcal{M}$ the set of all sets of the form $\{j_1,\dots,j_p|1\le j_1<\cdots<j_p\le N\}$. For a given $M\in\mathcal{M}$, let

$$\chi_M = ((\chi_M)_1,(\chi_M)_2,\dots,(\chi_M)_N) \qquad \text{where} \qquad (\chi_M)_i = \begin{cases} 1, & i\in M\\ 0, & \text{otherwise}\end{cases}$$

and note that the pth monomial symmetric function can be written

$$e_p(z) = \sum_{M\in\mathcal{M}} z^{\chi_M}.$$

Let M^* be the particular member of $\mathcal{M}$ such that

$$\eta+\chi_M \lhd \eta+\chi_{M^*}$$

for all $M\neq M^*$. Noting from (2.5) that the coefficient of z^η in $E_\eta(z)$ is unity and all other monomials are smaller with respect to the ordering $\lhd$, it follows that we must have

$$A^{(p)}_{\eta,\eta+\chi_{M^*}} = 1. \tag{7.1}$$

Moreover, with

$$l'_\eta(i) := \#\{k<i|\eta_k\ge\eta_i\}+\#\{k>i|\eta_k>\eta_i\}$$

it follows from the definition of $\lhd$ that

$$(\eta+\chi_{M^*})_i = \begin{cases}\eta_i+1, & l'_\eta(i)\le p-1\\ \eta_i, & l'_\eta(i)\ge p\end{cases} \tag{7.2}$$

The result (7.1) suggests an alternative way to write $A^{(1)}_{\eta,\nu}$ in (3.9) for general ν. To see this, first observe that associated with (7.2) are the sets

$$G_0 := \{i\in\{1,\dots,N\} : l'_\eta(i)\ge p\}, \qquad G_1 := \{i\in\{1,\dots,N\} : l'_\eta(i)\le p-1\}. \tag{7.3}$$

An alternative characterization follows by noting that since $\eta\subseteq\eta+\chi_{M^*}$ we have $\eta\preceq\eta+\chi_{M^*}$ and so from the definition of $\preceq$, for $\nu=\eta+\chi_{M^*}$,

$$G_0 := \{i\in\{1,\dots,N\} : \nu_{\pi(j)}=\eta_j\}, \quad G_1 := \{i\in\{1,\dots,N\} : \nu_{\pi(j)}=\eta_j+1\}. \tag{7.4}$$

Let us now put

$$B^{(p)}_{\eta,\nu} := \frac{d'_\nu e'_\eta}{e'_\nu d'_\eta} A^{(p)}_{\eta,\nu}. \tag{7.5}$$

Then it follows from the definitions (2.7) and (2.8), together with (7.1), that

$$B^{(p)}_{\eta,\eta+\chi_{M^*}} = \Big(\prod_{\substack{j\in G_0,\, k\in G_1 \\ j<k}} \frac{\bar\eta_j - \bar\eta_k + 1}{\bar\eta_j - \bar\eta_k} \Big) \Big(\prod_{\substack{j\in G_1,\, k\in G_0 \\ \pi(j)<\pi(k)}} \frac{\bar\eta_j - \bar\eta_k + \alpha - 1}{\bar\eta_j - \bar\eta_k + \alpha} \Big). \tag{7.6}$$

In the case $p = 1$, comparison with the expression

$$B^{(1)}_{\eta,\nu} = -\alpha A_I(\bar\eta/\alpha) \hat B_I(\bar\eta/\alpha), \quad I \in \mathbb{J}_\eta, \tag{7.7}$$

which follows from (4.12), (4.7) and (7.5), we see that as written (7.6) is in fact valid for all $\nu = c_I(\eta)$ with I consisting of a single element t_1. More explicitly, we then have

$$A_I(\bar\eta/\alpha) = a(\bar\eta_{t_1}/\alpha - 1, \bar\eta_{t_1}) = -\frac{1}{\alpha} \tag{7.8}$$

$$\hat B_1(\bar\eta/\alpha) = \prod_{j=1}^{t_1-1} b(\bar\eta_{t_1}/\alpha, \bar\eta_j/\alpha) \prod_{j=t_1+1}^{N} b(\bar\eta_{t_1}/\alpha + 1, \bar\eta_j/\alpha); \tag{7.9}$$

the factor (7.8) cancels with $-\alpha$ in (7.7) while the two products (7.9) correspond with the two products in (7.6) respectively. The structure exhibited by (7.6) suggests an extension with the property that for $p = 1$ there is agreement with (7.7). The extended form is

$$\begin{aligned} B^{(p)}_{\eta,\nu} = {} & \Big(\prod_{\substack{j\in G_0, k\in G_1 \\ j<k}} \frac{\bar\eta_j - \bar\eta_k + 1}{\bar\eta_j - \bar\eta_k} \Big) \Big(\prod_{\substack{j\in G_1,\, k\in G_0 \\ \pi(j)<\pi(k)}} \frac{\bar\eta_j - \bar\eta_k + \alpha - 1}{\bar\eta_j - \bar\eta_k + \alpha} \Big) \\ & \times \prod_{\pi^2(j)<\pi(j)<j} \frac{1}{\bar\eta_{\pi(j)} - \bar\eta_j} \prod_{j\le\pi^2(j)\le\pi(j)} \frac{1}{\bar\eta_{\pi(j)} - \bar\eta_j - \alpha}, \end{aligned} \tag{7.10}$$

valid for $p = 1$ and $\nu = c_I(\eta)$, $I \in \mathbb{J}_\eta$.

The significant feature of (7.10) is that in the general p case, with $\nu = c_I(\eta) \in \mathbb{J}_{N,p}$ and I such that at most one part of η in the formation of ν according to the prescription below (3.16) move downwards, explicit small N calculations indicate it remains valid. However we have no proof of this empirical observation.

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