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The representation theory of cyclotomic Temperley–Lieb algebras

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Abstract. A class of associative algebras called cyclotomic Temperley–Lieb algebras is introduced in terms of generators and relations. They are closely related to the group algebras of complex reflection groups on the one hand and generalizations of the usual Temperley–Lieb algebras on the other hand. It is shown that the cyclotomic Temperley–Lieb algebras can be defined by means of labelled Temperley–Lieb diagrams and are cellular in the sense of Graham and Lehrer. One thus obtains not only a description of the irreducible representations, but also a criterion for their quasi-heredity in the sense of Cline, Parshall and Scott. The branching rule for cell modules and the determinants of Gram matrices for certain cell modules are calculated, they can be expressed in terms of generalized Tchebychev polynomials, which therefore play an important role for semisimplicity.

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1. Introduction

The Temperley–Lieb algebras were first introduced in 1971 in the paper [15] where they were used to study the single bond transfer matrices for the Ising model. Later they were independently found by Jones when he characterized the algebras arising from the tower construction of semisimple algebras in the study of subfactors. Their relationship with knot theory comes from their role in the definition of the Jones polynomial. The theory of quantum invariants of links nowadays involves many of research fields. Thus, many important kinds of algebras related to the invariants of braids or links, such as Birman–Wenzl algebras [3], Hecke algebras and Brauer algebras, have been of great interest in mathematics and physics. They are all deformations of certain group algebras or other well-known algebras. Recently, several interesting type of such algebras have emerged: the cyclotomic Birman–Murakami–Wenzl algebras are introduced in [6] and cyclotomic Brauer algebras are investigated in [14] (see also [6]), while the cyclotomic Hecke algebras were already introduced by Broué and Malle in [4], and independently by Ariki and Koike for type $G(m,1,n)$ in [1]. They are deformations of the unitary reflection groups.

In the present paper, we focus our attention on the study of cyclotomic Temperley–Lieb algebras, which are generalizations of the classical Temperley–Lieb algebras. They are also subalgebras of cyclotomic Brauer algebras, which are closely related to complex reflection groups. We first present the ring theoretic definition of the cyclotomic Temperley–Lieb algebras in terms of generators and relations. Then we show that our definition can be reformulated geometrically by means of labelled Temperley–Lieb diagrams. Using this description we are able to prove that the cyclotomic Temperley–Lieb algebras are cellular, a notion introduced in [7]. As a consequence, we obtain both, the classification of the irreducible representations of the cyclotomic Temperley–Lieb algebras, and a criterion for a cyclotomic Temperley–Lieb algebra to be quasi-hereditary. For cell modules, the branching rule is discussed, and also the discriminants of certain bilinear forms are calculated. This leads us to introduce the n-th generalized Tchebychev polynomials. It turns out that a necessary condition for a cyclotomic Temperley–Lieb algebra to be semisimple is that certain generalized Tchebychev polynomials do not vanish on its defining parameters.

2. The ring theoretic definition of cyclotomic Temperley–Lieb algebras

Throughout the paper, let R be a commutative ring containing an identity 1 and elements $\delta_0, \delta_1, \dots, \delta_{m-1}$. Let $n, m \in \mathbb{N}$ be two positive integers. In this section, we introduce the cyclotomic Temperley–Lieb algebra $TL_{n,m}(\delta_0, \dots, \delta_{m-1})$ of type $G(m,1,n)$ over R. We shall prove that the R-rank of $TL_{n,m}(\delta_0, \dots, \delta_{m-1})$ is at most $\frac{m^n}{n+1}\binom{2n}{n}$.

Definition 2.1. The *cyclotomic Temperley–Lieb algebra* $TL_{n,m}(\delta_0, \dots, \delta_{m-1})$ (or $TL_{n,m}$ for simplicity) is the associative algebra over R with generators 1 (*the identity*), $e_1, \dots, e_{n-1}, t_1, \dots, t_n$ subject to the following conditions:

$$\begin{aligned}
&(1)\quad e_i e_j e_i = e_i \ \ \textit{if}\ \ |j-i| = 1,\\
&(2)\quad e_i e_j = e_j e_i \ \ \textit{if}\ \ |j-i| > 1,\\
&(3)\quad e_i^2 = \delta_0 e_i \ \ \textit{for}\ \ 1 \leqslant i \leqslant n-1,\\
&(4)\quad t_i^m = 1 \ \ \textit{for}\ \ 1 \leqslant i \leqslant n,\\
&(5)\quad t_i t_j = t_j t_i \ \ \textit{for}\ \ 1 \leqslant i, j \leqslant n,\\
&(6)\quad e_i t_i^k e_i = \delta_k e_i \ \ \textit{for}\ \ 1 \leqslant k \leqslant m-1, 1 \leqslant i \leqslant n-1,\\
&(7)\quad t_i t_{i+1} e_i = e_i,\ e_i t_i t_{i+1} = e_i \ \ \textit{for}\ \ 1 \leqslant i \leqslant n-1,\\
&(8)\quad e_i t_j = t_j e_i \ \ \textit{if}\ \ j \notin \{i, i+1\}.
\end{aligned}$$

If $m = 1$, then $TL_{n,m}$ is the usual Temperley–Lieb algebra, which is denoted by $TL_n(\delta_0)$ or TL_n for simplicity. This algebra was first introduced in [15] to describe the transfer matrices for the Ising model and for the Potts model in statistical mechanics (see also [12]). It is known that

$$\dim_R TL_n = \frac{1}{n+1}\binom{2n}{n} \text{ if } R \text{ is a field.}$$

The following lemma is due to Jones [8]. Recall that an expression of a monomial $w \in TL_n(\delta_0)$ (in the variables $e_1, e_2, \ldots, e_{n-1}$) is called reduced if the number of e_i in the expression is minimal.

Lemma 2.2. (1) *Any monomial $w \in TL_n(\delta_0)$ has a reduced expression*

$$(e_{j_1}e_{j_1-1}\cdots e_{k_1})(e_{j_2}e_{j_2-1}\cdots e_{k_2})\cdots(e_{j_p}e_{j_p-1}\cdots e_{k_p}),$$

where $j_{i+1} > j_i \geq k_i, k_{i+1} > k_i$ for any $1 \leqslant i \leqslant p-1$.

(2) *For any n, there is an isomorphism of TL_{n-1}-modules*

$$TL_n(\delta_0) \cong TL_{n-1}(\delta_0) \oplus TL_{n-1}(\delta_0)e_{n-1}TL_{n-1}(\delta_0),$$

where $TL_{n-1}(\delta_0)$ is the subalgebra of $TL_n(\delta_0)$ generated by $1, e_1, \ldots, e_{n-2}$.

To obtain an upper bound on the rank of a cyclotomic Temperley–Lieb algebra, we need the following lemma.

Lemma 2.3. *For any n, the cyclotomic Temperley–Lieb algebra*

$$TL_{n,m}(\delta_0, \ldots, \delta_{m-1})$$

is spanned over R by the set

$$M_n = \{t_1^{k_1}t_2^{k_2}\cdots t_n^{k_n}xt_1^{l_1}t_2^{l_2}\cdots t_n^{l_n} \mid 0 \leqslant k_i, l_i \leqslant m-1, 1 \leqslant i \leqslant n, x \in TL_n(\delta_0)\}.$$

Proof. We claim that the R-module $\tilde{T}L_{n,m}$ spanned by M_n is a left $TL_{n,m}$-module. This claim implies $\tilde{T}L_{n,m} = TL_{n,m}(\delta_0, \ldots, \delta_{m-1})$ since $1 \in M_n$.

By the definition of M_n, we see that $\tilde{T}L_{n,m}$ is stable under the left multiplication of t_i, $1 \leqslant i \leqslant n$. So we have to prove that for $1 \leqslant j \leqslant n-1$,

$$(*) \qquad e_jt_1^{k_1}t_2^{k_2}\cdots t_n^{k_n}xt_1^{l_1}t_2^{l_2}\cdots t_n^{l_n} \in \tilde{T}L_{n,m}.$$

Without loss of generality, we may assume that x is a monomial in $e_1, e_2, \ldots, e_{n-1}$. First, we consider the case $j = n-1$. By Lemma 2.2,

$$x = (e_{j_1}e_{j_1-1}\cdots e_{k_1})(e_{j_2}e_{j_2-1}\cdots e_{k_2})\cdots(e_{j_p}e_{j_p-1}\cdots e_{k_p}).$$

By 2.1(8), $xt_n = t_nx$ if $j_p \neq n-1$. It follows from 2.1(7) that

$$e_{n-1}\Big(\prod_{i=1}^{n} t_i^{k_i}\Big)x\Big(\prod_{i=1}^{n} t_i^{l_i}\Big) = \Big(\prod_{i=1}^{n-2} t_i^{k_i}\Big)e_{n-1}x\Big(\prod_{i=1}^{n-1} t_i^{l_i}\Big)t_n^{l_n+k_n-k_{n-1}} \in \tilde{T}L_{n,m}.$$

Suppose $j_p = n-1$. If e_{n-2} does not occur in the expression $e_{j_1}\ \cdots\ e_{k_1}\ \cdots\ e_{j_{p-1}}\ \cdots\ e_{k_{p-1}}$, then $(*)$ follows from the following equality

$$e_{n-1}t_{n-1}^{k_{n-1}-k_n}x = e_{n-1}t_{n-1}^{k_{n-1}-k_n}e_{n-1}(e_{j_1}\cdots e_{k_{p-1}})(e_{n-2}\cdots e_{k_p}) = \delta_k x,$$

where $k_{n-1}-k_n \equiv k \pmod m$. If e_{n-2} occurs in the expression of $e_{j_1}\ \cdots\ e_{j_{p-1}}\ \cdots\ e_{k_{p-1}}$, then $e_{j_{p-1}} = e_{n-2}$. In this case, we have

$$\begin{aligned}&e_{n-1}t_{n-1}^{k_{n-1}-k_n}x\\ &= (e_{j_1}\cdots e_{k_{p-2}})(e_{n-1}t_n^{k_n-k_{n-1}}e_{n-2}e_{n-1})(e_{n-3}\cdots e_{k_{p-1}})(e_{n-2}\cdots e_{k_p})\\ &= (e_{j_1}\cdots e_{k_{p-2}})t_{n-2}^{k_n-k_{n-1}}(e_{n-3}\cdots e_{k_{p-1}})(e_{n-1}e_{n-2}\cdots e_{k_p}).\end{aligned}$$

If e_{n-3} does not occur in $e_{j_1}\cdots e_{k_{p-2}}$, then

$$(e_{j_1}\cdots e_{k_{p-2}})t_{n-2}^{k_n-k_{n-1}} = t_{n-2}^{k_n-k_{n-1}}(e_{j_1}\cdots e_{k_{p-2}}),$$

and $(*)$ follows. If e_{n-3} occurs in the expression of $e_{j_1}\cdots e_{j_{p-2}}\cdots e_{k_{p-2}}$, then $e_{j_{p-2}} = e_{n-3}$. In this case, $(*)$ follows from the argument similar to the case $e_{j_{p-1}} = e_{n-2}$ together with an induction. Thus we have proved $(*)$ in the case $j = n-1$.

For $1 \leqslant j \leqslant n-2$, we use induction on n. In this case, $e_j t_n = t_n e_j$. If e_{n-1} does not occur in the expression of x, then $(*)$ follows from the induction assumption on $n-1$. Now suppose that $x = y(e_{n-1}e_{n-2}\cdots e_k)$ for some $y \in TL_{n-1}$ and $k \in \mathbb{N}$. Note that $e_{j+1}e_jt_{j+2}^l = (e_{j+1}t_{j+2}^l)e_j = e_{j+1}t_{j+1}^{-l}e_j = e_{j+1}(t_{j+1}^{-l}e_j) = e_{j+1}t_j^l e_j = t_j^l e_{j+1}e_j$ for all l and j. By a direct computation, we have

$$(**)\qquad e_{n-1}\cdots e_k t_1^{l_1}\cdots t_n^{l_n} = t_1^{l_1}\cdots t_{k-1}^{l_{k-1}}t_k^{l_{k+2}}\cdots t_{n-2}^{l_n}e_{n-1}\cdots e_k t_k^{l_k-l_{k+1}}.$$

Again by the induction hypothesis on $n-1$, we see that

$$e_j\Big(\prod_{i=1}^{n-1}t_i^{k_i}\Big)y t_1^{l_1}\cdots t_{k-1}^{l_{k-1}}t_k^{l_{k+2}}\cdots t_{n-2}^{l_n}$$

can be expressed as a linear combination of the elements in M_{n-1}. Now, $(**)$ together with the 2.1(7)-(8) yields the desired form $(*)$. This completes the proof of the result.

The following lemma gives more explicit information on the elements in M_n, which leads to an upper bound on the rank of $TL_{n,m}$.

Lemma 2.4. *For any $x \in TL_n$, the element $w = \big(\prod_{i=1}^n t_i^{k_i}\big)x\big(\prod_{j=1}^n t_j^{l_j}\big) \in M_n$ with $0 \leqslant k_i, l_j \leqslant m-1$ can be written as $\big(\prod_{i=1}^p t_{j_i}^{k_i'}\big)x\big(\prod_{i=p+1}^n t_{j_i}^{l_i'}\big)$ with $0 \leqslant k_i', l_j' \leqslant m-1$.*

Proof. Without loss of generality, we may assume that

$$x = (e_{j_1}e_{j_1-1}\cdots e_{k_1})(e_{j_2}e_{j_2-1}\cdots e_{k_2})\cdots(e_{j_p}e_{j_p-1}\cdots e_{k_p}).$$

Suppose $j_p \neq n-1$. Then $x \in TL_{n-1,m}$ and hence $t_n x = x t_n$. Therefore,

$$\Big(\prod_{i=1}^{n} t_i^{k_i}\Big) x \prod_{j=1}^{n} t_j^{l_j} = \Big(\prod_{i=1}^{n-1} t_i^{k_i}\Big) x \Big(\prod_{j=1}^{n-1} t_j^{l_j}\Big) t_n^{k_n+l_n}.$$

By induction on n, the element $\big(\prod_{i=1}^{n-1} t_i^{k_i}\big) x \prod_{j=1}^{n-1} t_j^{l_j}$ can be written as

$$\Big(\prod_{i=1}^{p} t_{j_i}^{k_i'}\Big) x \prod_{i=p+1}^{n-1} t_{j_i}^{l_i'} \quad \text{with} \quad 0 \leqslant k_i', l_j' \leqslant m-1.$$

This proves the result.

Suppose $j_p = n-1$. By $(**)$,

$$\begin{aligned} w &= \prod_{i=1}^{n} t_i^{k_i} \prod_{i=1}^{p-1} (e_{j_i} e_{j_i-1} \cdots e_{k_i}) \prod_{i=1}^{k-1} t_i^{l_i} \prod_{i=k}^{n-2} t_i^{l_{i+2}} \cdot (e_{n-1} \cdots e_k) t_k^{l_k - l_{k+1}} \\ &= \prod_{i=1}^{n-1} t_i^{k_i} \prod_{i=1}^{p-1} (e_{j_i} e_{j_i-1} \cdots e_{k_i}) \prod_{i=1}^{k-1} t_i^{l_i} \prod_{i=k}^{n-2} t_i^{l_{i+2}} \cdot t_{n-1}^{m-k_n} (e_{n-1} \cdots e_k) t_k^{l_k - l_{k+1}}. \end{aligned}$$

Now the result follows immediately from the induction assumption, 2.1(8) and $(**)$. This completes the proof of Lemma 2.4.

Let us remark that the proof of this lemma also shows that for a fixed $x \in TL_n$, when we write w as the form $\big(\prod_{i=1}^{p} t_{j_i}^{k_i'}\big) x \big(\prod_{i=p+1}^{n} t_{j_i}^{l_i'}\big)$ with $0 \leqslant k_i', l_j' \leqslant m-1$, the lower index sets $\{j_1, \ldots, j_p\}$ and $\{j_{p+1}, \ldots, j_n\}$ depend only on x.

Corollary 2.5. *If R is a field, then*

$$\dim_R TL_{n,m} \leqslant m^n \dim_R TL_n = \frac{m^n}{n+1}\binom{2n}{n}.$$

In the next section, we shall show that over a commutative ring R the rank of $TL_{n,m}$ is equal to $\frac{m^n}{n+1}\binom{2n}{n}$.

Finally, let us point out that the notion of B-type Temperley–Lieb algebras was introduced by tom Dieck [16], whose approach was based on the knot theoretic point of view and root systems. In fact, these algebras are completely different from our cyclotomic Temperley–Lieb algebras since the dimension of the B-type Temperley–Lieb algebra over a field is always of the form $\binom{2n}{n}$ (see [16]). However, the algebra $TL_{n,m}$ is closely related to the complex reflection groups $W_{n,m}$ of type $G(m,1,n)$. Recall that $W_{n,m}$ is generated by $s_0, s_1, \ldots, s_{n-1}$ satisfying the relations (1) $s_i^2 = 1$ for $i \geq 1$ and the braid relations for $s_1, \ldots, s_{n-1}$; (2) $s_0^m = 1$, and (3) $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$, $s_0 s_i = s_i s_0$ for $i \geq 2$. If we define $t_1 = s_0$, $t_i = s_{i-1} t_{i-1} s_{i-1}$, then $t_i^m = 1$. Thus, a deformation of the group algebra of $W_{n,m}$ is the cyclotomic Brauer algebra, which is clearly related to cyclotomic Birman–Wenzl algebra as mentioned in [6]. $TL_{n,m}$ is a subalgebra of the cyclotomic Brauer algebra. Thus it is related in this way to both, the complex reflection group $W_{n,m}$, and the cyclotomic Brauer algebra.

As we know, Ariki-Koike algebras are deformations of the unitary reflection groups. But these algebras can also be viewed as deformations of certain products of cyclic groups and Hecke algebras. In this same way, the cyclotomic Temperley–Lieb algebras are deformations of certain products of cyclic groups and Temperley–Lieb algebras. On the other hand, it is known that there are nice relationships between Temperley–Lieb algebra and the quantum group $U_q(sl_2)$ (see [12]).

3. The graphical definition of cyclotomic Temperley–Lieb algebras

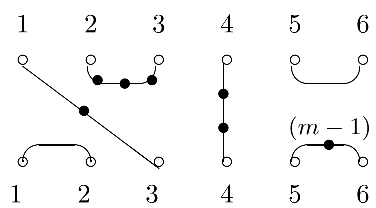
In this section, we shall redefine the cyclotomic Temperley–Lieb algebra in a geometrical way. This is motivated by knot theory. Let us denote by $\tilde{T}L_{n,m}$ the graphical cyclotomic Temperley–Lieb algebra. The main result in this section is that the ring theoretic and the graphical definitions of cyclotomic Temperley–Lieb algebras are equivalent, namely, $TL_{n,m} \cong \tilde{T}L_{n,m}$ for any n and m.

First, we introduce labelled Temperley–Lieb diagrams. These are special cases of dotted Brauer graphs introduced in [6](see also [14]).

Definition 3.1. A *labelled Temperley–Lieb diagram D* of type $G(m, 1, n)$ is a Temperley–Lieb diagram with $2n$ vertices in which the arcs are labelled by the elements of $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$.

In the following a labelled Temperley–Lieb diagram D will simply be called a labelled TL-diagram; if i and j are the endpoints of an arc in D, we shall simply write $\{i, j\} \in D$.

Graphically, we may represent a labelled TL-diagram D of type $G(m, 1, n)$ in a rectangle of the plane, where there are n numbers $\{1, 2, \ldots, n\}$ on the top row from left to right, and there are another n numbers $\{1, 2, \ldots, n\}$ on the bottom row again from left to right. To indicate the label $i \in \mathbb{Z}_m$ on an arc, we mark the arc with a dot and write the label i in parentheses above or below the dot. Sometimes we draw i dots directly on the arc. For example, the following is a labelled TL-diagram of type $G(m, 1, 6)$ with $m \geq 4$.



An arc in a labelled TL-diagram is said to be horizontal if its endpoints both lie in the top row or in the bottom row; and otherwise it is said to be vertical.

In order to have a graphical version of cyclotomic Temperley–Lieb algebras, we need to define the multiplication of two labelled TL-diagrams. Here we follow the definition in [6] (see also [14]).

From here onward, we make the following convention: Given a horizontal arc $\{i, j\}$ with $i < j$, we call i (resp. j) the left (resp. right) endpoint of the arc $\{i, j\}$, and always assume that all dots in a labelled TL-diagram are marked at the left endpoints of the arcs. A dot marked at the left (or right) endpoint of an arc will be called a left (or right) dot of the arc. For a vertical arc we do not define its left endpoint and its right endpoint.

The rule for movements of dots. We allow dots to move along an arc from left to right. They may also move to another arc.

(1) A left dot of a horizontal arc $\{i, j\}$ is equal to $m - 1$ right dots of the arc $\{i, j\}$, and conversely, a right dot of an horizontal arc is equal to $m - 1$ left dots.

(2) A dot on a vertical arc can move freely to the endpoints of the arc.

(3) Given two distinct arcs $\{i, j\}$ and $\{j, k\}$, we allow that a dot at the endpoint j of the arc $\{i, j\}$ can be replaced by a dot at the endpoint j of the arc $\{j, k\}$.

The rule for compositions. Given two labelled TL-diagrams D_1 and D_2 of type $G(m, 1, n)$, we define a new labelled TL- diagram $D_1 \circ D_2$, called the composition of D_1 and D_2, in the following way: First, we compose D_1 and D_2 in the same way as was done for Temperley–Lieb algebras. Thus we have a new Temperley–Lieb diagram P (which is possibly not a labelled TL-diagram). Second, we apply the rule for movements to relabel each arc in P, and thus obtain a labelled TL-diagram graph, denoted by $D_1 \circ D_2$.

The rule for counting closed cycles. For each closed cycle appearing in the above natural concatenation of D_1 and D_2 we apply the rule for movements of dots to relabel the cycle.

Note that the number of dots in each cycle lies in $\mathbb{Z}/(m)$. We denote by $n(\bar{i}, D_1, D_2)$ the number of relabelled closed cycles in which there are i dots.

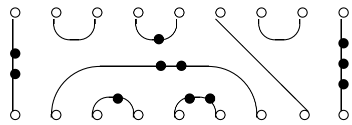
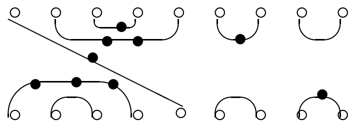
The following lemma can be proved easily.

Lemma 3.2. *Given two labelled TL-diagrams D_1 and D_2, we define $D_1 \cdot D_2 = \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, D_1, D_2)} D_1 \circ D_2$. Then $(D_1 \cdot D_2) \cdot D_3 = D_1 \cdot (D_2 \cdot D_3)$ for arbitrary labelled TL-diagrams D_1, D_2 and D_3.*

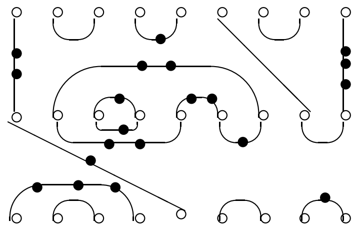
Definition 3.3. Let R be a commutative ring containing 1 and $\delta_0, \dots, \delta_{m-1}$. A *graphical cyclotomic Temperley–Lieb algebra* $(\tilde{T}L_{n,m}, \cdot)$ is an associative algebra over R with a basis consisting of all labelled TL-diagrams of type $G(m, 1, n)$. The multiplication is given by $D_1 \cdot D_2 = \prod_{i=0}^{m-1} \delta_i^{n(\bar{i}, D_1, D_2)} D_1 \circ D_2$.

It is easy to see that $\tilde{T}L_{n,m}$ is the usual Temperley–Lieb algebra if $m = 1$ and that $\tilde{T}L_{n,m}$ is a subalgebra of the cyclotomic Brauer algebra of type $G(m, 1, n)$ (see [6]).

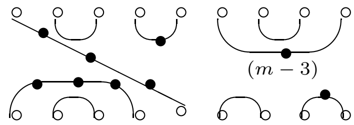
Now let us illustrate this definition by an example. If

$D_1 =$  , $D_2 =$  ,

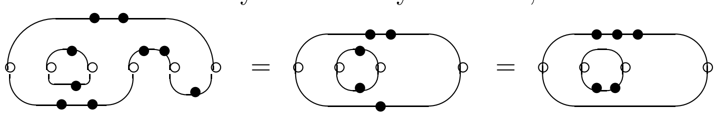
then we have a diagram

$D =$  .

Thus the composition $D_1 \circ D_2$ of D_1 and D_2 is as follows:

$D_1 \circ D_2 =$ 

Now we relabel the closed cycles in D. By definition,



In this case, $n(\bar{0}, D_1, D_2) = n(\bar{1}, D_1, D_2) = 0$ and $n(\bar{2}, D_1, D_2) = n(\bar{3}, D_1, D_2) = 1$ for $m \geq 4$. Thus $D_1 \cdot D_2 = \delta_1^1 \delta_3^1 D_1 \circ D_2$ for $m \geq 4$.

Now let us prove that the graphical definition and the ring theoretic definition of cyclotomic Temperley–Lieb algebras coincide.

Theorem 3.4. *Suppose that R is a commutative ring containing $1, \delta_0, \ldots, \delta_{m-1}$. Then $TL_{n,m} \cong \tilde{T}L_{n,m}$ for any m and n. Therefore, $TL_{n,m}$ is a free R-module of rank $\frac{m^n}{n+1}\binom{2n}{n}$. In particular, if R is a field, then*

$$\dim_R \, TL_{m,n} = \frac{m^n}{n+1}\binom{2n}{n}.$$

Proof. Let E_i be the labelled TL-diagram in which $\{i, i+1\}$ is an arc in the top row and also an arc in the bottom row, and the other vertex j ($\neq i, i+1$) in the top row connects with the vertex j in the bottom row. Let T_i be the labelled TL-diagram in which the j-th vertex in the top row connects with the j-th vertex in the bottom row for $j = 1, 2, \ldots, n$, and the i-th vertical arc carries one dot. If we replace e_i with E_i and t_i with T_i and apply the three rules above, then we know that all E_i and T_i satisfy the relations in Definition 2.1. This induces an algebra homomorphism $\phi : TL_{n,m} \to \tilde{T}L_{n,m}$ with $\phi(t_i) = T_i$ and $\phi(e_i) = E_i$. Since $\tilde{T}L_{n,m}$ is generated as an R-algebra by E_i and T_j with $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, the map ϕ is surjective.

We show that $TL_{n,m}$ is a free R-module. Put $r = \frac{m^n}{n+1}\binom{2n}{n}$. By Lemma 2.4, there is a surjective R-module homomorphism $f : R^r \longrightarrow TL_{n,m}$. Thus, we have a surjective R-module homomorphism ϕf from the free R-module R^r to the free R-module $\tilde{T}L_{n,m}$ of rank r. Let K be the kernel of ϕf. Then we have a split exact sequence of R-modules:

$$0 \longrightarrow K \longrightarrow R^r \longrightarrow R^r \longrightarrow 0.$$

Here we identify the R-module $\tilde{T}L_{n,m}$ with R^r. This sequence also shows that K is a finitely generated projective R-module. We claim $K = 0$.

Let $\mathfrak{p}$ be a maximal ideal in R. Since localization preserves (split) exact sequences, we have a split exact sequence

$$0 \longrightarrow K_{\mathfrak{p}} \longrightarrow (R_{\mathfrak{p}})^r \longrightarrow (R_{\mathfrak{p}})^r \longrightarrow 0,$$

where $M_{\mathfrak{p}}$ stands for the localization of an R-module M at $\mathfrak{p}$. Thus $(R_{\mathfrak{p}})^r \simeq (R_{\mathfrak{p}})^r \oplus K_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$-modules. Since $R_{\mathfrak{p}}$ is a local ring and every finitely generated projective module over a local ring is free, we see that the $R_{\mathfrak{p}}$-module $K_{\mathfrak{p}}$ is free. Note that any commutative ring with identity has the invariant dimension property. It follows from $(R_{\mathfrak{p}})^r \simeq (R_{\mathfrak{p}})^r \oplus K_{\mathfrak{p}}$ that $K_{\mathfrak{p}} = 0$, and therefore $K = 0$. (All facts on localization used in the above argument can be found in standard text books on commutative rings, for example [2].)

If $K = 0$, then ϕf is an isomorphism of R-modules and f must be injective. Thus $TL_{n,m}$ is a free R-module of rank r and ϕ is an isomorphism of R-modules. This also implies that ϕ is an isomorphism of R-algebras. The proof is complete.

Finally, let us remark that in [13] the so called blob algebras are considered, but those algebras have different defining relations and are therefore completely different from our cyclotomic Temperley–Lieb algebras.

4. Cellular algebras

Now let us recall the definition of cellular algebras due to Graham and Lehrer.

Definition 4.1. (Graham and Lehrer [7]) An associative R-algebra A is called a *cellular algebra* with cell datum (I, M, C, i) if the following conditions are satisfied:

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R-basis $C^{\lambda}_{S,T}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R-linear anti-automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$, the product $aC^{\lambda}_{S,T}$ can be written as

$$aC^{\lambda}_{S,T} = \sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T} + r',$$

where r' belongs to $A^{<\lambda}$ consisting of all R- linear combination of basis elements with upper index μ strictly smaller than λ, and the coefficients $r_a(U,S) \in R$ do not depend on T.

In this paper, we call an R-linear anti-automorphism i of A with $i^2 = id$ an *involution* of A. The following is a basis-free definition of cellular algebras in [9] which is equivalent to that given by Graham and Lehrer.

Definition 4.2. Let A be an R-algebra. Assume there is an anti-automorphism i on A with $i^2 = id$. A two-sided ideal J in A is called a *cell ideal* if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A-bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i-image of Δ) making the following diagram commutative:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \\ i \big\downarrow & & \big\downarrow x \otimes y \mapsto i(y) \otimes i(x) \\ J & \xrightarrow{\alpha} & \Delta \otimes_R i(\Delta) \end{array}$$

The algebra A (with the involution i) is called *cellular* if and only if there is an R-module decomposition $A = J_1' \oplus J_2' \oplus \cdots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}. (We call this chain a cell chain for the cellular algebra A.)

Cellular algebras include a large variety of important algebras related to links in knot theory such as cyclotomic Hecke algebras, Temperley–Lieb algebras [7] and cyclotomic Brauer algebras [14] as well as Birman–Wenzl algebras [18].

Given a cellular algebra A with the cell datum (I, M, C, i), for each $\lambda \in I$, one can define a cell module $\Delta(\lambda)$ and a symmetric, associative bilinear form

$\Phi_\lambda : \Delta(\lambda) \otimes_R \Delta(\lambda) \to R$ in the following way (see [7]): As an R-module, $\Delta(\lambda)$ has an R-basis $\{C_S^\lambda \mid S \in M(\lambda)\}$ and the module structure is given by

$$aC_S^\lambda = \sum_{U \in M(\lambda)} r_a(U,S) C_U^\lambda,$$

where the coefficients $r_a(U,S)$ are determined by $(C3)$ in Definition 4.1.

The bilinear form Φ_λ is defined by

$$\Phi_\lambda(C_S^\lambda, C_T^\lambda) C_{U,V}^\lambda \equiv C_{U,S}^\lambda C_{T,V}^\lambda \ (\text{mod } A^{<\lambda}),$$

where U and V are arbitrary elements in $M(\lambda)$.

Let $\operatorname{rad}\Delta(\lambda) = \{c \in \Delta(\lambda) \mid \Phi_\lambda(c,c') = 0$ for all $c' \in \Delta(\lambda)\}$. Then $\operatorname{rad}\Delta(\lambda)$ is a submodule of $\Delta(\lambda)$. Put $L(\lambda) = \Delta(\lambda)/\mathrm{rad}\Delta(\lambda)$. Then a complete set of irreducible representations of A can be described as follows:

Lemma 4.3. (Graham and Lehrer [7]) *Suppose R is a field. Then*

(1) $\{L(\lambda) \mid \Phi_\lambda \neq 0\}$ *is a complete set of non-isomorphic irreducible A-modules.*

(2) *The algebra A is semisimple if and only if all cell modules are simple and pairwise non-isomorphic.*

In the following, we shall see an easy example of cellular algebras, which will be used later on.

Let $G_{m,n}$ be the R-subalgebra of $TL_{n,m}$ generated by $t_1, t_2, \cdots, t_n$. Note that $G_{m,n}$ is isomorphic to the group algebra of the abelian group $\bigoplus_{i=1}^n \mathbb{Z}/(m)$.

Suppose that R is a splitting field of $x^m - 1$. Therefore the relation $t_i^m = 1$ implies that $t_i^m - 1 = \prod_{j=1}^m (t_i - u_j) = 0$ for some $u_1, \cdots, u_m \in R$. Let $\Lambda(m,n) = \{(i_1, i_2, \cdots, i_n) \mid 1 \leqslant i_j \leqslant m\}$. We assume that in case $n = 0$ the set $\Lambda(m,n)$ consists of only one element $\emptyset$. Now we define $(i_1, i_2, \cdots, i_n) \leqslant (j_1, j_2, \cdots, j_n)$ if and only if $i_k \leqslant j_k$ for all $1 \leqslant k \leqslant n$. For each $\mathbf{i} = (i_1, i_2, \cdots, i_n)$, define

$$C_{1,1}^{\mathbf{i}} = \prod_{j=1}^{n} \prod_{l=i_j+1}^{m} (t_j - u_l).$$

(Here the product over the empty set is 1.) Note that $\{C_{1,1}^{\mathbf{i}} \mid \mathbf{i} \in \Lambda(m,n)\}$ is a cellular basis for the algebra $G_{m,n}$ with respect to the identity involution. Let us remark that in this case each cell $G_{m,n}$-module $\Delta(\mathbf{i})$ is one-dimensional. In fact, this cell $G_{m,n}$-module corresponds to the subquotient $G_{m,n}^{\leqslant \mathbf{i}}/G_{m,n}^{<\mathbf{i}}$. The simple $G_{m,n}$-modules are parametrized by the following set.

Lemma 4.4. *Suppose R is a splitting field of $x^m - 1$ with characteristic p.*

(1) *If p divides m, say $m = p^t s$ with $(p,s) = 1$, then a complete set of non-isomorphic simple $G_{m,n}$-modules can be chosen as*

$$\{L(\mathbf{i}) \mid \mathbf{i} = (i_1, i_2, \ldots, i_n) \text{ with } p^t \text{ divides } i_j \text{ for all } j\}$$

its cardinality is s^n.

(2) *If p does not divide m (for example, $p = 0$), then a complete set of non-isomorphic simple $G_{m,n}$-modules is $\{L(\mathbf{i}) \mid \mathbf{i} \in \Lambda(m,n)\}$. In this case, the algebra $G_{m,n}$ is semisimple.*

Proof. It is easy to check that

$$(t_j - u_l)\prod_{k>i}^{m}(t_j - u_k) = (u_i - u_l)\prod_{k>i}^{m}(t_j - u_k) + \prod_{k>i-1}^{m}(t_j - u_k).$$

It follows from the above equality that

$$C^{\mathbf{i}}_{1,1}C^{\mathbf{i}}_{1,1} \equiv \Big(\prod_{j=1}^{n}\prod_{k>i_j}^{m}(u_{i_j} - u_k)\Big)C^{\mathbf{i}}_{1,1} \pmod{G^{<\mathbf{i}}_{m,n}}.$$

If p divides m, then we see that each root of the polynomial $x^s - 1$ is a root of $x^m - 1$ with multiplicity p^t. But all roots of $x^s - 1$ are simple roots. Hence we may assume that $(u_1, u_2, \dots, u_m) = (1, \dots, 1, \xi, \dots, \xi, \dots, \xi^{s-1}, \dots, \xi^{s-1})$, where ξ is a primitive s-th root of $x^s - 1$. Thus (1) follows.

If p does not divide m, then the algebra $G_{m,n}$ is semisimple, and therefore (2) follows.

5. Irreducible representations of $TL_{n,m}$

In this section, we assume that R is a splitting field of $x^m - 1$. We shall prove that $TL_{n,m}$ is a cellular algebra in the sense of [7]. Using the standard results on cellular algebras, we classify the irreducible representations of $TL_{n,m}$ over the field R. Let us first introduce some auxiliary notions.

An (n,k)-*labelled parenthesized graph* is a graph consisting of n vertices $\{1, 2, \dots, n\}$ and k horizontal arcs (hence $2k \leqslant n$ and there are $n - 2k$ "free" vertices which do not belong to any arc), and satisfying the following conditions:

(1) there are at most $m - 1$ dots on an arc,

(2) there are no arcs $\{i, j\}$ and $\{q, l\}$ satisfying $i < q < j < l$, and

(3) there is no arc $\{i, j\}$ and free vertex q such that $i < q < j$. (Given an (n,k)-labelled parenthesized graph, the vertices which do not belong to any arc are called free vertices.)

Let $P(n,k)$ be the set of all (n,k)-labelled parenthesized graphs and let $V(n,k)$ be the free R-module with $P(n,k)$ as its basis. Recall that $G_{m,n}$ is the R-subalgebra of $TL_{n,m}$ generated by $t_1, t_2, \cdots, t_n$.

Lemma 5.1. *There is an R-module isomorphism $V(n,k)\otimes_R V(n,k)\otimes_R G_{m,n-2k} \cong M_{n,k}$, where $M_{n,k}$ is the free R-module spanned by all labelled TL-diagrams with $2n$ vertices and $2k$ horizontal arcs.*

Proof. Given a labelled TL-diagram D, we can write it uniquely as $D_1 \otimes D_2 \otimes x$, where D_i is obtained from D in the following manner: After cutting all vertical arcs and forgetting all dots on the vertical arcs, the top row is defined to be the D_1 and the bottom is the D_2. Suppose that in D_1 the free vertices are $\{i_1, i_2, \dots, i_{n-2k}\}$ and that in D_2 the free vertices are $\{j_1, j_2, \dots, j_{n-2k}\}$. Then in D the vertical arcs are $\{i_1, j_1\}, \dots, \{i_{n-2k}, j_{n-2k}\}$. Suppose there are m_s dots on the arc $\{i_s, j_s\}$. Then we define $x = t_1^{m_1} t_2^{m_2} \dots t_{n-2k}^{m_{n-2k}} \in G_{m,n-2k}$. Conversely, given such an expression $D_1 \otimes D_2 \otimes x$, we have a unique labelled TL-diagram D in $M_{n,k}$. Hence the result follows.

Thus we have the following equivalent description of the graphical basis of $TL_{n,m}$. Usually, this basis is not a cellular basis.

Corollary 5.2. *The set* $\{v_1 \otimes v_2 \otimes x \mid 0 \leqslant k \leqslant [n/2], v_1, v_2 \in P(n,k), x \in G_{m,n-2k}\}$ *is a basis of* $TL_{n,m}$.

In the following, we shall construct a cellular basis for $TL_{n,m}$. Here we keep the notation introduced in the previous section.

Let $\Lambda_{m,n} = \{(k, \mathbf{i}) \mid 0 \leqslant k \leqslant [n/2], \mathbf{i} \in \Lambda(m, n-2k)\}$. We define a partial order on $\Lambda_{m,n}$ by saying that $(k, \mathbf{i}) \leqslant (l, \mathbf{j})$ if $k > l$; or if $k = l$ and $\mathbf{i} \leqslant \mathbf{j}$. Then $(\Lambda_{m,n}, \leqslant)$ is a finite poset. For each $(k, \mathbf{i}) \in \Lambda_{n,m}$, let $I(k, \mathbf{i}) = \{(v, \mathbf{i}) \mid v \in P(n,k)\}$. In the following, we shall show that this datum defines a cellular algebra.

Theorem 5.3. *Let R be a splitting field of $x^m - 1$. Then $TL_{n,m}$ is a cellular algebra with respect to the involution σ which sends $v_1 \otimes v_2 \otimes x$ to $v_2 \otimes v_1 \otimes x$ for all $v_1, v_2 \in P(n,k)$ and $x \in G_{m,n-2k}$, $0 \leqslant k \leqslant [n/2]$.*

Proof. For any $(k, \mathbf{i}) \in \Lambda_{m,n}$ and $v_1, v_2 \in P(n,k)$, we define $C_{v_1,v_2}^{(k,\mathbf{i})} = v_1 \otimes v_2 \otimes C_{1,1}^{\mathbf{i}}$. By 5.2, the set $\{C_{v_1,v_2}^{(k,\mathbf{i})} \mid (k, \mathbf{i}) \in \Lambda_{n,m}, v_1, v_2 \in P(n,k)\}$ is a basis of $TL_{n,m}$. We show that it is a cellular basis. Let us verify the conditions in Definition 4.1. By definition, 4.1(C1)-(C2) follow. It remains to check the condition 4.1(C3). Take a labelled TL-diagram $D_1 \otimes D_2 \otimes x$ with $D_1, D_2 \in P(n,k)$ and $x = t_1^{m_1} t_2^{m_2} \dots t_{n-2k}^{m_{n-2k}} \in G_{m,n-2k}$. Suppose that $i_1, i_2, \dots, i_{n-2k}$ are the free vertices in D_1 and that $j_1, j_2, \dots, j_{n-2k}$ are the free vertices of D_2, where $1 \leqslant i_s \leqslant n$ and $1 \leqslant j_s \leqslant n$ for all $s = 1, 2, \dots, n-2k$. Then $D_1 \otimes D_2 \otimes x = X \cdot (D_1 \otimes D_2 \otimes id_{n-2k}) = (D_1 \otimes D_2 \otimes id_{n-2k}) \cdot Y$, where $X = T_{i_1}^{m_1} T_{i_2}^{m_2} \cdots T_{i_{n-2k}}^{m_{n-2k}}$ and $Y = T_{j_1}^{m_1} T_{j_2}^{m_2} \cdots T_{j_{n-2k}}^{m_{n-2k}}$ (see 3.4 for the definition of T_i). Thus, for any labelled TL-diagram $D_1 \otimes D_2 \otimes x$,

$$(D_1 \otimes D_2 \otimes x) \cdot C_{v_1,v_2}^{(k,\mathbf{i})} \in TL_{n,m}^{\leqslant (k,\mathbf{i})}$$

where $TL_{n,m}^{\leqslant (k,\mathbf{i})}$ is the free R-submodule spanned by $C_{v_1,v_2}^{(k',\mathbf{i}')}$ with $(k', \mathbf{i}') \leqslant (k, \mathbf{i})$ and $v_1, v_2 \in P(n,k')$. Suppose that $(D_1 \otimes D_2 \otimes x) \cdot C_{v_1,v_2}^{(k,\mathbf{i})} \in TL_{n,m}^{(k,\mathbf{i})}$, where $TL_{n,m}^{(k,\mathbf{i})}$

is the free R-submodule spanned by $C^{(k,\mathbf{i})}_{v_1,v_2}$ with $v_1, v_2 \in P(n,k)$. Then

$$(D_1 \otimes D_2 \otimes x) \cdot C^{(k,\mathbf{i})}_{v_1,v_2} = D_1' \otimes v_2 \otimes x' C^{\mathbf{i}}_{1,1}$$

for some D_1' in $P(n,k)$ and some $x' \in G_{m,n-2k}$, here x' does not depend on v_2. Write $x' = \prod_{j=1}^{n-2k} t_j^{k_j}$ for some $0 \leqslant k_j \leqslant m-1, 1 \leqslant j \leqslant n-2k$. By an easy calculation, we know that

$$x' C^{\mathbf{i}}_{1,1} \equiv \prod_{j=1}^{n-2k} u_{i_j}^{k_j} C^{\mathbf{i}}_{1,1} \ (\mathrm{mod}\ G^{<\mathbf{i}}_{m,n-2k}),$$

where $G^{<\mathbf{i}}_{m,n-2k}$ is the free R-submodule spanned by $C^{\mathbf{j}}_{1,1}$ with $\mathbf{j} < \mathbf{i}$. Note also that the coefficient $\prod_{j=1}^{n-2k} u_{i_j}^{k_j}$ is independent of v_2. This implies that 4.1 (C3) is true.

As a corollary of Theorem 5.3, we classify the irreducible representations of cyclotomic Temperley–Lieb algebras.

Corollary 5.4. *Suppose R is a splitting field of $x^m - 1$. Let p be the characteristic of R. Then:*

(i) *suppose n is odd.*

If $m = p^t s$ with $(p,s) = 1$ and $t \geq 0$, then the set

$$\{L(k,\mathbf{i}) \mid 0 \leqslant k \leqslant [n/2], \mathbf{i} = (i_1, i_2, \ldots, i_{n-2k}) \in \Lambda(m, n-2k) \text{ with all } i_j \text{ divisible by } p^t\}$$

is a complete set of pairwise non-isomorphic simple $TL_{n,m}$-modules.

(ii) *Suppose n is even.*

1) *If not all δ_i are zero and if $m = p^t s$ with $(p,s) = 1$ and $t \geq 0$, then the set*

$$\{L(k,\mathbf{i}) \mid 0 \leqslant k \leqslant [n/2], \mathbf{i} = (i_1, i_2, \ldots, i_{n-2k}) \in \Lambda(m, n-2k) \text{ with all } i_j \text{ divisible by } p^t\}$$

is a complete set of pairwise non-isomorphic simple $TL_{n,m}$-modules.

2) *Suppose all δ_i are zero. If $m = p^t s$ with $(p,s) = 1$ and $t \geq 0$, then a complete set of pairwise non-isomorphic simple $TL_{n,m}$-modules can be parametrized by* $\{(k,\mathbf{i}) \mid 0 \leqslant k < [n/2], \mathbf{i} = (i_1, i_2, \ldots, i_{n-2k})$ with all i_j divisible by $p^t\}\}$.

Proof. For any $D_1, D_2 \in P(n,k)$ and $\mathbf{i} \in \Lambda(m, n-2k)$, we have

$$(D_1 \otimes D_2 \otimes C^{\mathbf{i}}_{1,1})(D_1 \otimes D_2 \otimes C^{\mathbf{i}}_{1,1}) = D_1 \otimes D_2 \otimes x C^{\mathbf{i}}_{1,1} C^{\mathbf{i}}_{1,1}, \quad x \in G_{m,n-2k}.$$

If this product is not equal to zero, then $C^{\mathbf{i}}_{1,1} C^{\mathbf{i}}_{1,1} \neq 0$. Now suppose that n is odd. If $D_1 = E_1 E_3 \cdots E_{2k-1}$ and $D_2 = E_2 E_4 \cdots E_{2k}$, then $x = id$. Hence statement (i) follows from 4.4.

Assume that n is even. First case: there is some $\delta_j \neq 0$ and p does not divide m. Then for $k = n/2$ and $\mathbf{i} = \emptyset$, the bilinear form $\Phi_{(k,\mathbf{i})} \neq 0$. For the other $(k, \mathbf{i})$, we take D_1 and D_2 as above. This implies $\Phi_{(k,\mathbf{i})} \neq 0$. Hence the index set of non-isomorphic simple modules is $\Lambda_{m,n}$. Second case: there is some $\delta_j \neq 0$ and p divides m. By arguments similar to the above, we have that a complete set of non-isomorphic simple modules is $\{(k, \mathbf{i}) \mid 0 \leqslant k \leqslant [n/2], \mathbf{i} = (i_1, i_2, \ldots, i_{n-2k})$ with all i_j divisible by $p^t\}$.

Assume $\delta_j = 0, 0 \leq j \leq m-1$. In this case, $\Phi_{(k,\mathbf{i})} = 0$ for $k = n/2$ and $\mathbf{i} = \emptyset$. For $k \neq n/2$, our discussion will be the same as above, namely, if $m = p^t s$, then index set of simple modules is $\{(k, \mathbf{i}) \mid 0 \leqslant k < [n/2], \mathbf{i} = (i_1, i_2, \ldots, i_{n-2k})$ with all i_j divisible by $p^t\}$; if p does not divide m, then the index set of simple modules is $\Lambda_{m,n} \setminus \{(n/2, \emptyset)\}$.

The following result follows from the proof of Theorem 5.3.

Corollary 5.5. *Let $\Delta(k, \mathbf{i})$ be the cell module corresponding to $(k, \mathbf{i}) \in \Lambda_{n,m}$. Then*

$$\dim_R \Delta(k, \mathbf{i}) = m^k [\tbinom{n}{k} - \tbinom{n}{k-1}].$$

6. Quasi-heredity of $TL_{n,m}$

In this section, we shall characterize the parameters for which the cyclotomic Temperley–Lieb algebras are quasi-hereditary in the sense of [5]. First, we recall the definition of quasi-hereditary algebras.

Definition 6.1. (Cline, Parshall and Scott [5]) Let R be a field and let A be an R-algebra. An ideal J in A is called a *heredity ideal* if J is idempotent, $J(\mathrm{rad}\,(A))J = 0$ and J is a projective left (or, right) A-module, where $\mathrm{rad}\,(A)$ is the Jacobson radical of A. The algebra A is called *quasi-hereditary* provided there is a finite chain $0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = A$ of ideals in A such that J_j/J_{j-1} is a heredity ideal in A/J_{j-1} for all j. Such a chain is then called a heredity chain of the quasi-hereditary algebra A.

From the ring theoretic definition of cellular algebras, we see immediately that there is a large intersection of the class of cellular algebras with that of quasi-hereditary algebras. Typical examples of quasi-hereditary cellular algebras include Temperley–Lieb algebras with non-zero parameters [17] and Birman–Wenzl algebras for most choices of parameters [18] as well as certain cyclotomic Brauer algebras [14].

The main result in this section is the following theorem.

Theorem 6.2. *Suppose R is a splitting field of the polynomial $x^m - 1$. Then the cyclotomic Temperley–Lieb algebra $TL_{n,m}$ is quasi-hereditary if and only if the characteristic of R does not divide m and one of the following is true:*

(1) *n is odd;*
(2) *n is even and $\delta_j \neq 0$ for some $0 \leq j \leq m-1$.*

Proof. In [7, Remark 3.10] it is shown that A is quasi-hereditary if the index set of the non-isomorphic simple modules over a cellular algebra A with cell datum (I, M, C, i) is I. Conversely, A is not quasi-hereditary if there is a cell datum (I, M, C, i) of A such that the index set of the non-isomorphic simple modules is not I [10, Theorem 3.1]. In other words, every chain of ideals in A is not a heredity chain. Thus Theorem 6.2 follows immediately from Corollary 5.4.

For the cases which are not included in Theorem 6.2, we can get a quasi-hereditary quotient of $TL_{n,m}$. In order to make $TL_{n,m}$ quasi-hereditary, we need first to ensure that the group algebra $G_{m,n}$ is semisimple. The following result follows from the above fact and the definition 4.1.

Proposition 6.3. *Suppose that R is a splitting field of $x^m - 1$ and $p \nmid m$, $2|n$ and $\delta_j = 0$ for all $0 \leq j \leq m-1$. Suppose J is the two-sided ideal of $TL_{n,m}$ generated by all $(n, n/2)$-labelled TL-diagrams. Then the quotient $TL_{n,m}/J$ is quasi-hereditary.*

7. Restriction and induction of the cell modules

In this section, we assume that R is a splitting field of $x^m - 1$. The main result of this section is the branching rule for the cell modules of $TL_{n,m}$.

Recall that $V(n,k)$ is the R-space spanned by all labelled parenthesized graphs with k arcs. Let $J_i := \bigoplus_{j=i}^{[n/2]} V(n,j) \otimes_R V(n,j) \otimes_R G_{m,n-2j}$. Then we have a chain

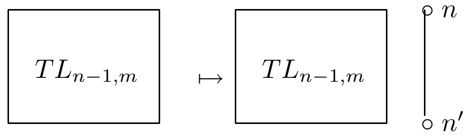
$$0 \subset J_{[n/2]} \subset \cdots \subset J_{i+1} \subset J_i \subset \cdots \subset J_\epsilon = TL_{n,m}$$

of ideals in $TL_{m,n}$, where ϵ is zero if n is even, and 1 if n is odd. For any $(k, \mathbf{i}) \in \Lambda_{n,m}$, the cell module

$$\Delta(k, \mathbf{i}) = V(n,k) \otimes_R v_0 \otimes_R \Delta(\mathbf{i}),$$

where $v_0 \in P(n,k)$ is a fixed diagram and $\Delta(\mathbf{i})$ is the cell module of the algebra $G_{m,n}$ with respect to $\mathbf{i}$. In the sequel, we choose v_0 to be the (n,k)-labelled parenthesized graph with arcs $\{1,2\}, \ldots, \{2k-1, 2k\}$ and free vertices $2k+1, 2k+2, \ldots, n$. Note that the subquotient $V(n,j) \otimes_R V(n,j) \otimes_R G_{m,n-2j}$ is a $TL_{n,m}$-module and the cell module structure on $V(n,k) \otimes_R v_0 \otimes_R \Delta(\mathbf{i})$ is induced from this subquotient. We make the following convention:

Throughout this section we fix an m and the parameters $\delta_0, \delta_1, \ldots, \delta_{m-1}$ and consider the algebra $TL_{n-1,m}$ canonically as a subalgebra of $TL_{n,m}$ by adding the vertical arc $\{n, n'\}$ to the right side of each labelled TL-diagram in $TL_{n-1,m}$. This embedding can be visualized as follows:



Note that the identity in $TL_{n-1,m}$ is sent to the identity of $TL_{n,m}$. Thus every $TL_{n,m}$-module is also a $TL_{n-1,m}$-module via this embedding. The cell modules $V(n,k) \otimes_R v_0 \otimes_R \Delta(\mathbf{i})$ over $TL_{n,m}$ will be denoted by $\Delta(n,k;i_1,i_2,\ldots,i_{n-2k})$. Then we have

Proposition 7.1. (a) *For all n and $0 \leqslant k \leqslant [n/2]$, there is an exact sequence*

$$\begin{aligned}0 \longrightarrow \Delta(\ n-1,k;i_1,i_2,\ldots,\ i_{n-2k-1}) &\xrightarrow{\alpha} \Delta(n,k;i_1,i_2,\ldots,i_{n-2k}) \downarrow \\ &\xrightarrow{\beta} \textstyle\bigoplus_{j=0}^{m-1} V(n-1,k-1) \otimes_R \underline{v}_0 \otimes_R \Delta(i_1,i_2,\ldots,i_{n-2k}) t_{n-2k+1}^j \longrightarrow 0,\end{aligned}$$

where $M\downarrow$ is the restriction of a $TL_{n,m}$-module M to a $TL_{n-1,m}$-module, and $\Delta(i_1,i_2,\ldots,i_{n-2k})t_{n-2k+1}^j$ stands for $\Delta(i_1,i_2,\ldots,i_{n-2k}) \otimes_R Rt_{n-2k+1}^j$.

(b) *If $I_0 = 0 \subset I_1 \subset \ldots \subset I_m = R\langle t_{n-2k+1}\rangle$ is a cell chain of the group algebra $R\langle t_{n-2k+1}\rangle = G_{m,1}$, that is, I_j is the free R-module generated by $\{\prod_{l>s}^m (t_{n-2k+1} - u_l) \mid 1 \le s \le j\}$, then there are $m-1$ short exact sequences*

$$\begin{aligned}0 \to V(n-1,k-1) &\otimes_R \underline{v}_0 \otimes_R \Delta(i_1,i_2\ldots,i_{n-2k}) \otimes I_{j-1} \\ &\xrightarrow{\gamma} V(n-1,k-1) \otimes_R \underline{v}_0 \otimes_R \Delta(i_1,i_2\ldots,i_{n-2k}) \otimes I_j \\ &\xrightarrow{\delta} \Delta(n-1,k-1;i_1,i_2,\ldots,i_{n-2k},j) \to 0.\end{aligned}$$

(c) *If $TL_{n-1,m}$ is semisimple, then*

$$\begin{aligned}\Delta(n,k;i_1,i_2,\ldots,i_{n-2k})\downarrow \cong \Delta(n-1,k;i_1,i_2,\ldots,i_{n-2k-1}) &\oplus \\ \bigoplus_{j=1}^{m} \Delta(n-1,k-1;i_1,i_2,\ldots,i_{n-2k},j)&.\end{aligned}$$

Proof. If $TL_{n-1,m}$ is semisimple, then every $TL_{n-1,m}$-module is projective. Therefore, each short exact sequence in (a) and (b) splits. Now the statement (c) follows immediately from (a) and (b). The map γ in (b) is the canonical injective map and the map δ in (b) comes from the canonical projection $I_i \to I_i/I_{i-1}$. One can easily prove that (b) is a short exact sequence of vector spaces. Obviously, both γ and δ in (b) are $TL_{n-1,m}$-module homomorphisms. Now let us prove the statement (a).

Since we may consider $\Delta(n-1,k;i_1,i_2,\ldots,i_{n-2k-1})$ as a subset of $TL_{n-1,m}$, the map α is just the restriction of the above embedding. It is obvious that α is an injective map. Note that $TL_{n,m}$ is generated as an algebra by $\{e_i, t_j \mid 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n\}$. To show that α is a $TL_{n-1,m}$-module homomorphism, it suffices to prove that for $D \in \{e_i, t_j \mid 1 \leqslant i \leqslant n-2, 1 \leqslant j \leqslant n-1\}$,

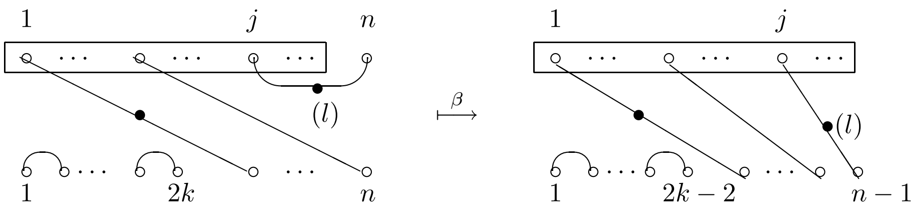
$$\alpha(Dv \otimes v_0 \otimes C^{(i_1,i_2,\ldots,i_{n-2k})}) = D\alpha(v \otimes v_0 \otimes C^{(i_1,i_2,\ldots,i_{n-2k})}).$$

However, a vertex ($\neq n$) in v is free if and only if it is free in v', and $\{i, j\}$ is an arc in v if and only if it is an arc in v', where v' is the $(n, k-1)$-labelled parenthesized obtained from v by adding the vertex n. By the multiplication of labelled TL-diagrams in 3.2, we can see immediately that the above equation holds. Hence α is a $TL_{n-1,m}$-module homomorphism.

Now let us define the map β. Given an (n, k)-labelled parenthesized $v \in P(n, k)$, we denote by $\bar{v}$ the labelled parenthesized obtained from v by deleting the vertex n and removing the arc connected with n if it exists.

Let v be in $P(n, k)$. If the vertex n in v is a free vertex, then β sends $v \otimes v_0 \otimes C_{1,1}^{(i_1,i_2,\dots,i_{n-2k})}$ to zero. If the vertex n in v is connected by an arc in which there are l dots, then β sends $v\otimes v_0\otimes C_{1,1}^{(i_1,i_2,\dots,i_{n-2k})}$ to $\bar{v}\otimes \underline{v}_0\otimes C_{1,1}^{(i_1,i_2,\dots,i_{n-2k})}t_{n-2k+1}^{l}$, where $\underline{v}_0$ is the $(n–1, k–1)$-labelled parenthesized with arcs $\{1, 2\}, \{3, 4\}, \dots, \{2k–3, 2k–2\}$ and $n - 2k + 1$ free vertices.

In fact, we can extend β to a map from $V(n, k)\otimes v_0\otimes G_{m,n-2k}$ to $V(n-1, k-1) \otimes\underline{v}_0 \otimes G_{m,n-2k+1}$. This map β can be illustrated as follows:



(The image of an (n, k)-labelled TL-diagram under the map β is obtained from the given (n, k)-labelled TL-diagram by deleting both the arc $\{2k - 1, 2k\}$ and its endpoints from the bottom row, and then shifting the vertex n from the top row to the bottom row, and finally renaming the vertices at the bottom from left to right.)

It is trivial that the sequence is an exact sequence of vector spaces. To finish the proof, it remains to show that β is also a $TL_{n-1,m}$-module homomorphism. Since β restricted to the image of α preserves the module structure, we need only to prove that β preserves the $TL_{n-1,m}$-module structure on the elements of the form $a(v \otimes v_0 \otimes C_{1,1}^{(i_1,i_2,\dots,i_{n-2k})})$, where $a \in \{e_i, t_j \mid 1 \leqslant i \leqslant n-2, 1 \leqslant j \leqslant n-1\}$ and the vertex n in v is not free. In the following, we show more generally that the extended map β is a $TL_{n-1,m}$-module homomorphism.

Let $v \otimes v_0 \otimes x$ with $v \in P(n, k)$ such that n is connected to j by an arc in v. Suppose $a = t_s$ or $a = e_r$ with $r \notin \{j-1, j\}$. In this case, by inspecting the picture, it is easy to see that β preserves the module structure on the element $a(v\otimes v_0\otimes x)$. Now suppose $a = e_s$ with $s = j - 1$ or $s = j$. In the latter case, since there are no free vertices between j and n in v, the labelled TL-diagram $\beta(e_j(v \otimes v_0 \otimes x))$ is just the graph $e_j\beta(v \otimes v_0 \otimes x)$. This is what we wanted to prove. In the former case, if $j - 1$ is a free vertex in v, then $e_{j-1}(v \otimes v_0 \otimes x)$ lies in the image of α, which is mapped to zero under β. Moreover, the element $e_{j-1}\beta(v\otimes v_0\otimes x)$ is also zero since it contains one more arc. Now assume that $j-1$ is adjacent to a vertex s in v. Then $s < j-1$. In this case, there are no free vertices between s and n in v. Again by inspecting picture we see that $\beta(e_j(v\otimes v_0\otimes x)) = e_j\beta(v\otimes v_0\otimes x)$. This completes the proof.

The following result follows from Proposition 7.1 and Frobenius reciprocity.

Proposition 7.2. *If $TL_{n,m}$ is semisimple, then*

$$\Delta(n-1,k;i_1,\ldots,i_{n-2k-1})\uparrow\simeq \Delta(n,k+1;i_1,i_2,\ldots,i_{n-2k-2})\oplus \bigoplus_{j=1}^{m}\Delta(n,k;i_1,\ldots,i_{n-2k-1},j),$$

where $\Delta(n-1,k;i_1,\ldots,i_{n-2k-1})\uparrow$ stands for the $TL_{n,m}$-module induced from the $TL_{n-1,m}$-module $\Delta(n-1,k;i_1,\ldots,i_{n-2k-1})$.

8. Gram matrices and their determinants

In this section, we assume that the field R contains a primitive m-th root of unity (for example, if R is algebraically closed and of characteristic p which does not divide m, then our assumptions are satisfied). The goal in this section is to calculate the discriminant of the bilinear form $\Phi_{(k,\mathbf{i})}$ for certain $(k,\mathbf{i})$, where $0\leqslant k\leqslant[\frac{n}{2}]$ and $\mathbf{i}=(i_1,i_2,\ldots,i_{n-2k})$.

Recall that $\Phi_{(k,\mathbf{i})}$ is defined on the cell module $\Delta(k,\mathbf{i})$ in the following way.

$$(v_1\otimes v_1\otimes C^{\mathbf{i}}_{1,1})(v_2\otimes v_2\otimes C^{\mathbf{i}}_{1,1})\equiv \Phi_{(k,\mathbf{i})}(v_1,v_2)v_1\otimes v_2\otimes C^{\mathbf{i}}_{1,1}\pmod{TL^{<(k,\mathbf{i})}_{n,m}},$$

where $TL^{<(k,\mathbf{i})}_{n,m}$ is a free R-submodule spanned by $C^{(k',\mathbf{i}')}_{v_1,v_2}$ with $(k',\mathbf{i}')<(k,\mathbf{i})$ and $v_1,v_2\in P(n,k')$.

According to a general construction in [11], there is a bilinear form $\phi^{(n,k)}$ from $V(n,k)\otimes_R V(n,k)$ to $G_{m,n-2k}$ such that the product can be written as

$$(v_1\otimes v_1\otimes C^{\mathbf{i}}_{1,1})(v_2\otimes v_2\otimes C^{\mathbf{i}}_{1,1})\equiv v_1\otimes v_2\otimes \phi^{(n,k)}_{v_1,v_2}(t_1,t_2,\ldots,t_{n-2k})(C^{\mathbf{i}}_{1,1})^2, \quad \pmod{TL^{<(k,\mathbf{i})}_{n,m}},$$

where $\phi^{(n,k)}_{v_1,v_2}(t_1,t_2,\ldots,t_{n-2k})$ is an element in $G_{m,n-2k}$.

Define $a(k,\mathbf{i})=\prod_{j=1}^{n-2k}\prod_{l>i_j}^{m}(u_{i_j}-u_l)$. It follows that

$$\Phi_{(k,\mathbf{i})}(v_1,v_2)=a(k,\mathbf{i})\phi^{(n,k)}_{v_1,v_2}(u_{i_1},u_{i_2},\ldots,u_{i_{n-2k}}),$$

where $u_1,u_2,\ldots,u_m$ are the roots of x^m-1.

Now let us compute the matrix $\Psi(n,k):=(\phi^{(n,k)}_{v_1,v_2})$ for the case $k=1$. Let v_i be the element in $P(n,1)$ whose unique arc is $\{i,i+1\}$ and let $v_i^{(j)}$ be the $(n,1)$-labelled parenthesis in which there are j dots on the arc $\{i,i+1\}$. The elements in $P(n,1)$ can be ordered as follows: $v_1^{(0)} := v_1, v_1^{(1)}, \dots, v_1^{(m-1)}, v_2^{(0)} := v_2, v_2^{(1)}, \dots, v_2^{(m-1)}, \dots, v_{n-1}^{(0)} := v_{n-1}, v_{n-1}^{(1)}, \dots, v_{n-1}^{(m-1)}$. Thus:

$$\Psi(n,1) = \begin{pmatrix} A & B_1 & & & & \\ B_1^T & A & B_2 & & & \\ & B_2^T & A & B_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & A & B_{n-2} \\ & & & & B_{n-2}^T & A \end{pmatrix},$$

where B_i is the matrix with the (s,t)-entry t_i^{s-t} for $1 \leqslant s,t \leqslant m$, the matrix B_i^T stands for the transpose of B_i, and

$$A = \begin{pmatrix} \delta_0 & \delta_1 & \cdots & \delta_{m-1} \\ \delta_1 & \delta_2 & \cdots & \delta_0 \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{m-1} & \delta_0 & \cdots & \delta_{m-2} \end{pmatrix}.$$

Let us look at a simple example. We consider the algebra $TL_{(4,3)}(\delta_0, \delta_1, \delta_2)$, that is, $n = 4$ and $m = 3$. In this case we have $t_i^3 = 1$ for $i = 1, 2$. Thus

$$\Psi(n,1) = \begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & 1 & t_1^2 & t_1 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & \delta_0 & t_1 & 1 & t_1^2 & 0 & 0 & 0 \\ \delta_2 & \delta_0 & \delta_1 & t_1^2 & t_1 & 1 & 0 & 0 & 0 \\ 1 & t_1 & t_1^2 & \delta_0 & \delta_1 & \delta_2 & 1 & t_2^2 & t_2 \\ t_1^2 & 1 & t_1 & \delta_1 & \delta_2 & \delta_0 & t_2 & 1 & t_2^2 \\ t_1 & t_1^2 & 1 & \delta_2 & \delta_0 & \delta_1 & t_2^2 & t_2 & 1 \\ 0 & 0 & 0 & 1 & t_2 & t_2^2 & \delta_0 & \delta_1 & \delta_2 \\ 0 & 0 & 0 & t_2^2 & 1 & t_2 & \delta_1 & \delta_2 & \delta_0 \\ 0 & 0 & 0 & t_2 & t_2^2 & 1 & \delta_2 & \delta_0 & \delta_1 \end{pmatrix}.$$

Suppose u_1 is a primitive m-th root of unity. Define $u_j = u_1^j$ for $j = 2, \dots, m$ and $u_m = u_0 = 1$. Then $u_k^{-1} = u_{m-k}$. Let $V = V_m(1, u_1, \dots, u_{m-1})$ be the Vandermonde matrix of order m:

$$V_m(1, u_1, \dots, u_{m-1}) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ u_1 & u_2 & \cdots & u_m \\ u_1^2 & u_2^2 & \cdots & u_m^2 \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{m-1} & u_2^{m-1} & \cdots & u_m^{m-1} \end{pmatrix}.$$

Since we shall evaluate each t_j as some u_{i_j}, when we calculate the value of $\Phi_{(k,\mathbf{i})}(v_1, v_2)$, we may suppose that $t_j = u_{i_j}$ for all $1 \leqslant j \leqslant n-2$. Thus the matrix B_j is of the form

$$B_j = \begin{pmatrix} 1 \\ u_{i_j} \\ \vdots \\ u_{i_j}^{m-1} \end{pmatrix} (1, u_{i_j}^{-1}, \dots, u_{i_j}^{-(m-1)}) = \begin{pmatrix} 1 \\ u_{i_j} \\ \vdots \\ u_{i_j}^{m-1} \end{pmatrix} (1, u_{m-i_j}, \dots, u_{m-i_j}^{m-1}).$$

Now we define Y_j to be the matrix of order m with 1 at the $(i_j, m - i_j)$-position and 0 otherwise. For $i_j = m$, we define Y_j to be the matrix with (m, m)-entry 1 and all other entries 0. Then $B_j = VY_jV^T$.

Let $p(x) = \delta_0 x^{m-1} + \delta_1 x^{m-2} + \cdots + \delta_{m-1} \in R[x]$. We write

$$\frac{p(x)}{x^m - 1} = \frac{\bar{\delta}_1}{x - u_1} + \frac{\bar{\delta}_2}{x - u_2} + \cdots + \frac{\bar{\delta}_m}{x - u_m}.$$

Since u_1 is a primitive m-th root of unity and $u_i \neq u_j$ for $i \neq j$, we have $\bar{\delta}_j = p(u_j)/\prod_{i\neq j}(u_j - u_i)$ for all $j = 1, 2, \dots, m$. Now we can rewrite $\delta_k = \sum_{j=1}^m \bar{\delta}_j u_j^k$. Note that the index k can be an arbitrary natural numbers and that $\delta_l = \delta_k$ if $l \equiv k \pmod{m}$. Thus the matrix A can be written as $(\delta_{k+l})_{0\leqslant k,l\leqslant m-1}$. Furthermore, we have $A = V\bar{A}V^T$, where $\bar{A} = \operatorname{diag}(\bar{\delta}_1, \bar{\delta}_2, \dots, \bar{\delta}_m)$ is the diagonal matrix.

Since $x^m - 1 = (x - u_1)(x - u_2)\cdots(x - u_m)$, we know that the k-th elementary symmetric polynomial in $u_1, u_2, \dots, u_m$ is zero for $1 \leqslant k \leqslant m - 1$. Hence Newton's identities imply that

$$\sigma_k(u_1, u_2, \dots, u_m) := \sum_{j=1}^m u_j^k = \begin{cases} m, & \text{if } k = m, \\ 0, & \text{if } 1 \leqslant k \leqslant m - 1. \end{cases}$$

Thus we have

$$VV^T = \begin{pmatrix} m & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & m \\ 0 & 0 & \cdots & m & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & m & \cdots & 0 & 0 \end{pmatrix}.$$

This implies that $(\det(V))^2 = (-1)^{(m-1)(m-2)/2} m^m$. Thus $\det(\Psi(n, 1)) = (-1)^{\frac{1}{2}(m-1)(m-2)(n-1)} m^{m(n-1)} \det(\overline{\Psi}(n, 1))$, where

$$\overline{\Psi}(n, 1) = \begin{pmatrix} \bar{A} & Y_1 & & & & \\ Y_1^T & \bar{A} & Y_2 & & & \\ & Y_2^T & \bar{A} & Y_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \bar{A} & Y_{n-2} \\ & & & & Y_{n-2}^T & \bar{A} \end{pmatrix}.$$

Now let us calculate the determinant of $\overline{\Psi}(n, 1)$. Since each matrix Y_j is of special form, we partition $(i_1, i_2, \dots, i_{n-2})$ into $(i_{1,1}, i_{1,2}, ..., i_{1,j_1}, i_{2,1}, i_{2,2}, ..., i_{2,j_2}, ..., i_{r,j_r})$

with $j_1 + j_2 + \cdots + j_r = n-2$ such that m divides $i_{p,q} + i_{p,q+1}$ for all p with $1 \leqslant q < j_p$ and that m does not divide $i_{p,j_p} + i_{p+1,1}$ for all $1 \leqslant p < r$. Thus

$$\det(\overline{\Psi}(n,1)) = \frac{(\bar\delta_1\bar\delta_2\dots\bar\delta_m)^{n-1}}{\prod_{p=1}^r(\bar\delta_{m-i_{p,j_p}}\prod_{q=1}^{j_p}\bar\delta_{i_{p,q}})}\prod_{p=1}^r \det\begin{pmatrix} \bar\delta_{i_{p,1}} & 1 & & & & \\ 1 & \bar\delta_{i_{p,2}} & 1 & & & \\ & 1 & \bar\delta_{i_{p,3}} & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \bar\delta_{i_{p,j_p-1}} & 1 \\ & & & & 1 & \bar\delta_{i_{p,j_p}} \end{pmatrix}.$$

Let

$$P(x_1,\dots,x_n) = \det\begin{pmatrix} x_1 & 1 & & & & \\ 1 & x_2 & 1 & & & \\ & 1 & x_3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & x_{n-1} & 1 \\ & & & & 1 & x_n \end{pmatrix}.$$

Then

$$\det(\Psi(n,1)) = (-1)^{\frac{1}{2}m(m-1)(n-1)}m^{m(n-1)}\frac{(\bar\delta_1\bar\delta_2\dots\bar\delta_m)^{n-1}}{\prod_{p=1}^r(\bar\delta_{m-i_{p,j_p}}\prod_{q=1}^{j_p}\bar\delta_{i_{p,q}})}\times \\ \times\prod_{p=1}^r P(\bar\delta_{i_{p,1}},\bar\delta_{i_{p,2}},\dots,\bar\delta_{i_{p,j_p}}).$$

We have proved the following proposition.

Proposition 8.1. *Let R be a field containing a primitive m-th root of unity. Then the determinant of the Gram matrix of the bilinear form $\Phi_{(1,\mathbf{i})}$ is*

$$\det\Phi_{(1,\mathbf{i})} = (-1)^{\frac{1}{2}m(m-1)(n-1)}a(1,\mathbf{i})^{m(n-1)}\frac{m^{m(n-1)}(\bar\delta_1\bar\delta_2\dots\bar\delta_m)^{n-1}}{\prod_{p=1}^r(\bar\delta_{m-i_{p,j_p}}\prod_{q=1}^{j_p}\bar\delta_{i_{p,q}})}\times \\ \times\prod_{p=1}^r P(\bar\delta_{i_{p,1}},\bar\delta_{i_{p,2}},\dots,\bar\delta_{i_{p,j_p}}).$$

As a consequence of Proposition 8.1, we know that under the above assumption, a necessary condition for $TL_{n,m}$ to be semisimple is that all the polynomials $P(\bar\delta_{i_{p,1}},\bar\delta_{i_{p,2}},\dots,\bar\delta_{i_{p,j_p}})$ are non zero.

The following is a description of the polynomial $P(x_1,\dots,x_n)$.

Let $I(n) := \{n, n-2, n-4, \dots\} \subset \{0\} \cup \mathbb{N}$ and define $\Gamma(n,r) := \{(i_1, i_2, \dots, i_r) \mid 1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n, i_r \equiv n \pmod 2; i_{j+1} \equiv i_j + 1 \pmod 2 \text{ for all } 1 \leqslant j \leqslant r-1\}$ for all $r \in I(n)$. If $\alpha = (i_1, i_2, \dots, i_r) \in \Gamma(n,r)$ we write x_α for $x_{i_1} x_{i_2} \dots x_{i_r}$. Then

$$P(x_1, x_2, \dots, x_n) = \sum_{r \in I(n)} \sum_{\alpha \in \Gamma(n,r)} (-1)^{(n-r)/2} x_\alpha.$$

This can be proved by induction on n and the recursive formula $P(x_1, x_2, \dots, x_n) = x_n P(x_1, x_2, \dots, x_{n-1}) - P(x_1, x_2, \dots, x_{n-2})$. In fact, the set $\Gamma(n,r)$ is a disjoint union of $\{(i_1, i_2, \dots, i_{r-1}, n) \mid 1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n-1, i_{r-1} \equiv n-1 \pmod 2; i_{j+1} \equiv i_j + 1 \pmod 2 \text{ for all } 1 \leqslant j \leqslant r-2\}$ and $\Gamma(n-2, r)$. Thus this decomposition of $\Gamma(n,r)$ corresponds just to the two summands in the recursive formula of $P(x_1, x_2, \dots, x_n)$.

Note that if $m = 1$ or if $x_1 = x_2 = \cdots = x_n$, then both det $\Phi_{(1,\mathbf{i})}$ and $P(x, x, \dots, x)$ are Tchebychev-type polynomials which play an important role in the study of Temperley–Lieb algebras (see [7] and [17]). Hence we call $P(x_1, x_2, \dots, x_n)$ the n-th generalized Tchebychev polynomial. It follows from the recursive formula that $P(x_1, x_2, \dots, x_n)$ is irreducible in the polynomial ring $R[x_1, x_2, \dots, x_n]$ with n variables $x_1, x_2, \dots, x_n$.

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