

Automorphisms of partially commutative groups I: Linear subgroups*

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Abstract. We construct and describe several arithmetic subgroups of the automorphism group of a partially commutative group. More precisely, given an arbitrary finite graph Γ we construct arithmetic subgroups $\text{St}(L_Y)$ and $\text{St}(L^{\max})$, represented as subgroups of $\text{GL}(n, \mathbb{Z})$, where n is the number of vertices of the graph Γ . Here L_Y and L^{\max} are certain lattices of subsets of $X = V(\Gamma)$ and $\text{St}(K)$ is the stabiliser of the subgroup generated by K . In addition we give a description of the decomposition of the group $\text{St}^{\text{conj}}(L_X)$, which stabilises L_X up to conjugacy, as a semidirect product of the group of conjugating automorphisms and $\text{St}(L_X)$.

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Introduction

Recently a wave of interest in groups of automorphisms of partially commutative groups has risen; see [3], [2], [4], [10]. This emergence of interest may be attributable to the fact that numerous geometric and arithmetic groups are subgroups of these groups.

The goal of this paper is to construct and describe certain arithmetic subgroups of the automorphism group of a partially commutative group. More precisely, given an arbitrary finite graph Γ we construct an arithmetic subgroup $\text{St}(L(\Gamma))$ (see Section 2.1 for definitions), represented as a subgroup of $\text{GL}(n, \mathbb{Z})$, where n is the number of vertices of the graph Γ ; see Theorems 2.4 and 2.12. Note that our proof is independent of the results of Laurence and Servatius, [11], [12], which give a description of the generating set of the automorphism group $\text{Aut}(G(\Gamma))$. One of the advantages of this proof is that it is largely combinatorial, rather than group-theoretic, so could be adjusted to obtain analogous results for partially commutative algebras determined by the graph Γ (in various varieties of algebras).

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In the last section of the paper we give a description of the decomposition of the group $\text{St}^{\text{conj}}(L(\Gamma))$ (see Section 2.1 for definitions) as a semidirect product of the group of conjugating automorphisms $\text{Conj}(G)$ and $\text{St}(L(\Gamma))$. This result is closely related to Theorem 1.4 of [10], but the situations considered in [10] and in this paper are somewhat different.

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1. Preliminaries

1.1. Graphs and lattices of closed subsets. In this section we give definitions and a summary of the facts we need concerning graphs, orthogonal systems and closed subsets of a graph. For further details the reader is referred to [7]. Graph will mean undirected, finite, simple graph throughout this paper. If x and y are vertices of a graph then we define the *distance* $d(x, y)$ from x to y to be the minimum of the lengths of all paths from x to y in Γ . A subgraph S of a graph Γ is called a *full* subgraph if vertices a and b of S are joined by an edge of S whenever they are joined by an edge of Γ .

Let Γ be a graph with $V(\Gamma) = X$. A subset Y of X is called a *simplex* if the full subgraph of Γ with vertices Y is isomorphic to a complete graph. Given a subset Y of X the *orthogonal complement* of Y is defined to be

$$Y^\perp = \{u \in X \mid d(u, y) \leq 1 \text{ for all } y \in Y\}.$$

By convention we set $\emptyset^\perp = X$. It is not hard to see that $Y \subseteq Y^{\perp\perp}$ and $Y^\perp = Y^{\perp\perp\perp}$ [7], Lemma 2.1. We define the *closure* of Y to be $\text{cl}(Y) = Y^{\perp\perp}$. The closure operator in Γ satisfies, among others, the properties that $Y \subseteq \text{cl}(Y)$, $\text{cl}(Y^\perp) = Y^\perp$ and $\text{cl}(\text{cl}(Y)) = \text{cl}(Y)$ [7], Lemma 2.4. Moreover if $Y_1 \subseteq Y_2 \subseteq X$ then $\text{cl}(Y_1) \subseteq \text{cl}(Y_2)$.

Definition 1.1. A subset Y of X is called *closed* (with respect to Γ) if $Y = \text{cl}(Y)$. Denote by $L = L(\Gamma)$ the set of all closed subsets of X .

Then $\text{cl}(Y) \in L$, for all $Y \subseteq X$ and $U \in L$ if and only if $U = V^\perp$ for some $V \subseteq X$ [7], Lemma 2.7. The relation $Y_1 \subseteq Y_2$ defines a partial order on the set L . As the closure operator cl is inclusion preserving and maps arbitrary subsets of X into closed sets, L is a lattice where the infimum $Y_1 \wedge Y_2$ of Y_1 and Y_2 is $Y_1 \wedge Y_2 = \text{cl}(Y_1 \cap Y_2) = Y_1 \cap Y_2$ and the supremum is $Y_1 \vee Y_2 = \text{cl}(Y_1 \cup Y_2)$.

1.2. Partially commutative groups. Let Γ be a finite, undirected, simple graph. Let $X = V(\Gamma)$ be the set of vertices of Γ and let $F(X)$ be the free group on X . For

elements g, h of a group we denote the commutator $g^{-1}h^{-1}gh$ of g and h by $[g, h]$. Let

$$R = \{[x_i, x_j] \in F(X) \mid x_i, x_j \in X \text{ and there is an edge from } x_i \text{ to } x_j \text{ in } \Gamma\}.$$

We define the *partially commutative group with (commutation) graph* Γ to be the group $G(\Gamma)$ with presentation $\langle X \mid R \rangle$. (Note that these are the groups which are called finitely generated free partially commutative groups in [5].)

The subgroup generated by a subset $Y \subseteq X$ is called a *canonical parabolic subgroup* of G and denoted $G(Y)$. This subgroup is equal to the partially commutative group with commutation graph the full subgraph of Γ generated by Y [1]. The connection between closed sets and the group $G(\Gamma)$ is established by Proposition 3.9 of [8]: a subgroup $G(Y)^g$ of G is a centraliser if and only if $Y \in L(\Gamma)$. If Y is a closed subset of Γ and $g \in G$ then the subgroup $G(Y)^g = C_G(Y^\perp)^g$ is called a *parabolic centraliser*.

Let Γ be a simple graph, $G = G(\Gamma)$ and let $w \in G$. Denote by $\text{lg}(w)$ the minimum of the lengths of words that represents the element w . We say that $w \in G$ is *cyclically minimal* if and only if

$$\text{lg}(g^{-1}wg) \geq \text{lg}(w) \quad \text{for all } g \in G.$$

We write $u \circ v$ to express the fact that $\text{lg}(uv) = l(u) + l(v)$. We say that u is a *left divisor (right divisor)* of w if there exists v such that $w = u \circ v$ ($w = v \circ u$). If $g \in G$ and w is a word of minimal length representing g then we write $\alpha(g)$ for the set of elements $x \in X$ such that $x^{\pm 1}$ occurs in w . It is shown in [1] that $\alpha(g)$ depends only on g and not on the choice of w .

The *non-commutation graph* of the partially commutative group $G(\Gamma)$ is the graph Δ , dual to Γ , with vertex set $V(\Delta) = X$ and an edge connecting x_i and x_j if and only if $[x_i, x_j] \neq 1$. The graph Δ is a union of its connected components $\Delta_1, \dots, \Delta_k$ and words that depend on letters from distinct components commute. For any graph Γ , if S is a subset of $V(\Gamma)$ we shall write $\Gamma(S)$ for the full subgraph of Γ with vertices S . Now, if the vertex set of Δ_k is I_k and $\Gamma_k = \Gamma(I_k)$ then $G = G(\Gamma_1) \times \dots \times G(\Gamma_k)$. For $g \in G$ let $\alpha(g)$ be the set of elements x of X such that $x^{\pm 1}$ occurs in a minimal word w representing g . Now suppose that the full subgraph $\Delta(\alpha(w))$ of Δ with vertices $\alpha(w)$ has connected components $\Delta_1, \dots, \Delta_l$ and let the vertex set of Δ_j be I_j . Then, since $[I_j, I_k] = 1$, we can split w into the product of commuting words, $w = w_1 \circ \dots \circ w_l$, where $w_j \in G(\Gamma(I_j))$, so $[w_j, w_k] = 1$ for all j, k . If w is cyclically minimal then we call this expression for w a *block decomposition* of w and say w_j is a *block* of w for $j = 1, \dots, l$. Thus w itself is a block if and only if $\Delta(\alpha(w))$ is connected.

As in [8] we make the following definition. An element $g \in G$ is called a *root element* if it has the property that $g = h^n$, for some $h \in G$, implies that $n = \pm 1$.

Definition 1.2. Let w be a cyclically minimal root element of G with block decomposition $w = w_1 \dots w_k$ and let Z be a subset of X such that $Z \subseteq \alpha(w)^\perp$. Then

the subgroup $Q = Q(w, Z) = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Z)$ is called a *canonical quasiparabolic subgroup* of G .

A subgroup is called *quasiparabolic* if it is conjugate to a canonical quasiparabolic subgroup. In [8] centralisers of arbitrary subsets of a partially commutative group are characterised in terms of quasiparabolic subgroups and we shall use this result in Section 2.3.

1.3. Conjugating automorphisms. Automorphisms which act locally by conjugation play an important role in the structure of $\text{Aut}(G)$.

For $S \subseteq X$ define Γ_S to be $\Gamma \setminus S$, the graph obtained from Γ by removing all vertices of S and all their incident edges.

Definition 1.3. Let $x \in X$ and let C be a connected component of Γ_{x^\perp} . The automorphism $\alpha_C(x)$ given by

$$y \mapsto \begin{cases} y^x & \text{if } y \in C, \\ y & \text{otherwise} \end{cases}$$

is called an *elementary conjugating automorphism* of Γ . The subgroup of $\text{Aut}(G)$ generated by all elementary conjugating automorphisms (over all connected components of Γ_{x^\perp} and all $x \in X$) is called the group of *conjugating automorphisms* and is denoted $\text{Conj}(G)$.

Theorem 1.4 (M. Laurence [11]). *An element $\psi \in \text{Aut}(G)$ is called a conjugating automorphism if, for all $x \in X$, there exists $g_x \in G$ such that $x^\psi = x^{g_x}$.*

From Theorem 1.4 it follows that the group of inner automorphisms $\text{Inn}(G)$ is a subgroup of $\text{Conj}(G)$; and is therefore a normal subgroup.

2. Stabilisers of parabolic centralisers

2.1. Stabiliser subgroups. Throughout the remainder of this paper let Γ be a finite graph with vertices X , let $G = G(\Gamma)$ and let $L = L(\Gamma)$ be the lattice of closed sets of Γ . We denote the automorphism group of G by $\text{Aut}(G)$.

Definition 2.1. The *stabiliser* of L is defined to be

$$\text{St}(L) = \{\phi \in \text{Aut}(G) \mid G(Y)^\phi = G(Y) \text{ for all } Y \in L\}$$

and the *conjugate-stabiliser* of L is defined to be

$$\text{St}^{\text{conj}}(L) = \{\phi \in \text{Aut}(G) \mid \text{for all } Y \in L \text{ there exists } g_Y \text{ such that } G(Y)^\phi = G(Y)^{g_Y}\}.$$

If $\phi \in \text{St}^{\text{conj}}(L)$, $Y \in L$ and g_Y is such that $G(Y)^\phi = G(Y)^{g_Y}$ then we say that ϕ acts as g_Y on Y .

Proposition 2.2. *Both the stabiliser $\text{St}(L)$ and the conjugate-stabiliser $\text{St}^{\text{conj}}(L)$ of L are subgroups of $\text{Aut}(G)$ and $\text{St}(L) < \text{St}^{\text{conj}}(L)$.*

Proof. It is clear that the stabiliser of L is a group and that it is contained in the conjugate-stabiliser. If $\phi \in \text{St}^{\text{conj}}(L)$ acts as g_Y on $Y \in L$ then $G(Y) = (G(Y)^h)^\phi$, where $h = (g_Y^{-1})^{\phi^{-1}}$. Thus $G(Y)^{\phi^{-1}} = G(Y)^h$. If $\phi' \in \text{St}^{\text{conj}}(L)$ acts as g'_Y on Y then $G(Y)^{\phi'\phi} = G(Y)^k$, where $k = g_Y(g'_Y)^\phi$. \square

2.2. Generators for the stabiliser of L . We introduce three sets of maps J , V_\perp and Tr_\perp : which will turn out to be automorphisms and to generate $\text{St}(L)$ (cf. [12], [11]). First we recall some notation from [7] and establish some background information.

As in [7] we define an equivalence relation \sim_\perp on X by $x \sim_\perp y$ if and only if $x^\perp = y^\perp$. Denote the equivalence class of x under \sim_\perp by $[x]_\perp$. Then $[x]_\perp$ is a simplex for all $x \in X$. The set N_2 consists of those $x \in X$ such that $|[x]_\perp| \geq 2$. Define $N_1 = X \setminus N_2$, $X' = X / \sim_\perp$ and $N'_2 = \{[x]_\perp \in X' \mid x \in N_2\}$. If $x \in N_1$ then $[x]_\perp = \{x\}$, so X is the disjoint union

$$X = \bigsqcup_{[x]_\perp \in N'_2} [x]_\perp \sqcup N_1.$$

For $x \in X$ we write $G[x]_\perp$ for $G([x]_\perp) = \langle [x]_\perp \rangle$, so $G[x]_\perp \subseteq G$. For all $x, y, z \in X$ such that $y \in [x]_\perp$ we have $[y, z] = 1$ if and only if $[u, z] = 1$ for all $u \in [x]_\perp$. It follows that we may extend an automorphism ϕ of $G[x]_\perp$ to an automorphism ϕ^{ε_x} of G by setting $g^{\phi^{\varepsilon_x}} = g^\phi$ for all $g \in G[x]_\perp$, and $y^{\phi^{\varepsilon_x}} = y$ for all $y \in X \setminus [x]_\perp$. The map ε_x such that $\phi \mapsto \phi^{\varepsilon_x}$ is then an monomorphism from $\text{Aut}(G[x]_\perp)$ into $\text{Aut}(G)$. Moreover, if $[x]_\perp \neq [y]_\perp$ then $G[x]_\perp \cap G[y]_\perp = 1$. Therefore $\text{Aut}(G[x]_\perp)^{\varepsilon_x} \cap \text{Aut}(G[y]_\perp)^{\varepsilon_y} = 1$ and, for all $\phi \in \text{Aut}(G[x]_\perp)$ and $\psi \in \text{Aut}(G[y]_\perp)$, we have $\phi^{\varepsilon_x} \psi^{\varepsilon_y} = \psi^{\varepsilon_y} \phi^{\varepsilon_x}$. Hence $\text{Aut}(G)$ contains a subgroup

$$V = \prod_{[x]_\perp \in N'_2} \text{Aut}(G[x]_\perp)^{\varepsilon_x} \times \prod_{x \in N_1} \text{Aut}(G[x]_\perp)^{\varepsilon_x}.$$

- (1) The set J consists of the extension to G of all maps $X \rightarrow G$ such that $x^\phi = x$ or x^{-1} for all $x \in X$. Then $J \leq \text{Aut}(G)$ and $|J| = 2^{|X|}$.
- (2) We define

$$V_\perp = \prod_{[x]_\perp \in N'_2} \text{Aut}(G[x]_\perp)^{\varepsilon_x} \leq V.$$

- (3) For distinct $x, y \in X$ define a map $\text{tr}_{x,y}: X \rightarrow G$ by $\text{tr}_{x,y}(x) = xy$, and $\text{tr}_{x,y}(z) = z$ for $z \neq x$. If $x^\perp \setminus \{x\} \subseteq y^\perp$ then $\text{tr}_{x,y}$ extends to an automorphism

of G which we also call $\text{tr}_{x,y}$. Define Tr to be the set consisting of the extension to G of all the maps $\text{tr}_{x,y}$, where $x^\perp \setminus \{x\} \subseteq y^\perp$. Then $\text{Tr} \subseteq \text{Aut}(G)$. We define

$$\text{Tr}_\perp = \{\text{tr}_{x,y} \in \text{Tr} \mid \text{cl}(y) < \text{cl}(x)\} \subseteq \text{Tr}.$$

Note that we have excluded those $\text{tr}_{x,y}$ where $\text{cl}(y) = \text{cl}(x)$ and that $\text{cl}(y) < \text{cl}(x)$ implies that $y \in \text{cl}(x)$ so that $x^\perp \subset y^\perp$ and $[x, y] = 1$. Therefore the map $\text{tr}_{x,y} \in \text{Tr}_\perp$ if and only if $x^\perp \subset y^\perp$ (the inclusion being strict).

We remark that the subgroup generated by J and V_\perp is V . Moreover Laurence [11], building on results of Servatius [12], showed that $\text{Aut}(G)$ is generated by J , Tr and $\text{Conj}(G)$, together with automorphisms which permute the vertices of Γ (see for example [7]).

Proposition 2.3. *The sets of maps J , V and V_\perp are subgroups of $\text{St}(L)$ and the set Tr_\perp is contained in $\text{St}(L)$.*

Proof. As J fixes every parabolic subgroup it is clear that $J \leq \text{St}(L)$. To see that V_\perp is a subgroup of $\text{St}(L)$ consider $Y \in L$. If $x \in Y$ and $z \in [x]_\perp$ then $Y^\perp \subseteq x^\perp = z^\perp$ so $z \in Y$. Hence $x \in Y$ implies that $[x]_\perp \subseteq Y$. Now suppose that $\phi \in V_\perp$. If $y \in Y \cap N_2$ then $y^\phi \in G[y]_\perp \leq G(Y)$, by the above. If $y \in Y \setminus N_2$ then $y^\phi = y \in G(Y)$. Hence $G(Y)^\phi \subseteq G(Y)$ and since V_\perp is a subgroup of $\text{Aut}(G)$ it follows that $G(Y)^\phi = G(Y)$. Therefore $V_\perp \leq \text{St}(L)$. Finally, let $\tau = \text{tr}_{x,y} \in \text{Tr}_\perp$ and let $Y \in L$. If $x \notin Y$ then τ fixes Y pointwise, so we assume that $x \in Y$. In this case $\text{cl}(x) \subseteq Y$ and $y \in \text{cl}(x)$, from the remark following the definition of Tr_\perp . Hence $x^\tau = xy \in G(Y)$ and $G(Y)^\tau \leq G(Y)$. As $x = (xy^{-1})^\tau$ it follows that $G(Y)^\tau = G(Y)$. As V is generated by J and V_\perp it is also a subgroup of $\text{St}(L)$. \square

Before stating the next theorem we shall briefly explain what is meant by an arithmetic group. Two subgroups A and B of a group G are said to be commensurable if $A \cap B$ is of finite index in both A and B . A linear algebraic group is a group which is also an affine algebraic variety, such that multiplication and inversion are morphisms of affine algebraic varieties. A linear algebraic group is said to be \mathbb{Q} -defined if it is a subgroup of $\text{GL}(n, \mathbb{C})$ which can be defined by polynomials over \mathbb{Q} and is such that the group operations are morphisms defined over \mathbb{Q} . Let G be a \mathbb{Q} -defined algebraic group. A subgroup $A \subseteq G \cap \text{GL}(n, \mathbb{Q})$ is called an *arithmetic subgroup* of G if it is commensurable with $G \cap \text{GL}(n, \mathbb{Z})$. A group is called *arithmetic* if it is isomorphic to an arithmetic subgroup of a \mathbb{Q} -defined linear algebraic group.

Theorem 2.4. *The stabiliser $\text{St}(L)$ is an arithmetic group generated by the elements of J , V_\perp and Tr_\perp .*

We defer the proof of this theorem, which is part of the more technical Theorem 2.12 below.

2.3. Ordering L and the stabiliser of L -maximal elements. We shall now define a partial order on elements of X which reflects the lattice structure of L . We shall then describe a subgroup of the automorphism group of G which stabilises subgroups generated by closures of the maximal elements in this order. First note that if $Y \in L$ then $Y = \bigcup_{y \in Y} \text{cl}(y)$. Therefore if $\phi \in \text{Aut}(G)$ and $G(\text{cl}(y))^\phi = G(\text{cl}(y))$ for all $y \in Y$ then $G(Y)^\phi = G(Y)$. This implies that if $G(\text{cl}(x))^\phi = G(\text{cl}(x))$ for all $x \in X$ then $\phi \in \text{St}(L)$. Setting $L_X = \{Y \in L \mid Y = \text{cl}(x) \text{ for some } x \in X\}$ and $\text{St}(L_X) = \{\phi \in \text{Aut}(G) \mid G(Y)^\phi = G(Y) \text{ for all } Y \in L_X\}$ we have

$$\text{St}(L) = \text{St}(L_X). \tag{2.1}$$

Definition 2.5. Let $<_L$ be the partial order on X given by $x <_L y$ if and only if $\text{cl}(x) \subset \text{cl}(y)$ and $\text{cl}(x) \neq \text{cl}(y)$. By $x =_L y$ we mean $\text{cl}(x) = \text{cl}(y)$. We say x is L -minimal (L -maximal) if $y \leq_L x$ ($x \leq_L y$) implies $\text{cl}(y) = \text{cl}(x)$.

Note that $y <_L x$ if and only if $\text{cl}(y) \subset \text{cl}(x)$ and $x \notin \text{cl}(y)$.

Write $L^{\max} = \{\text{cl}(x) \in L_X \mid x \text{ is } <_L\text{-maximal}\}$ and denote $\text{St}(L^{\max}) = \{\phi \in \text{Aut}(G) \mid G(Y)^\phi = G(Y) \text{ for all } Y \in L^{\max}\}$.

Proposition 2.6. $\text{St}(L)$ and $\text{St}(L^{\max})$ are commensurable.

Proof. Since $\text{St}(L) \subseteq \text{St}(L^{\max})$, it suffices to prove that $\text{St}(L)$ has finite index in $\text{St}(L^{\max})$. Let $\phi \in \text{St}(L^{\max})$, let $x \in X$, let $Z \in L^{\max}$, say $Z = \text{cl}(z)$ for some L -maximal element $z \in X$, and let $Y \in L$ such that $Y \subseteq Z$. By definition of $\text{St}(L^{\max})$ we have $G(Z)^\phi = G(Z)$ and so $G(Y)^\phi \subseteq G(Z)^\phi = G(Z)$. As $Y \in L$ we have $Y = U^\perp$ for some $U \in L$, so $G(Y) = C_G(U)$, a canonical parabolic centraliser. Hence $G(Y)^\phi$ is a centraliser and from [8], Theorem 3.12, is conjugate to a quasiparabolic subgroup. As $G(Z)$ is Abelian it follows (loc. cit.) that $G(Y)^\phi$ is a canonical parabolic centraliser: that is $G(Y)^\phi = G(V)$ for some $V \in L$ with $V \subseteq G(Z)$. As ϕ is an automorphism it therefore permutes the subgroups $G(Y)$ where Y is an element of the set $L(Z) = \{V \in L \mid V \subseteq Z\}$ and this induces a permutation on the set $L(Z)$. This holds for all elements $Z \in L^{\max}$ and setting $M = \bigcup_{Z \in L^{\max}} L(Z)$ we obtain a permutation $\sigma(\phi)$ of M . It is clear that if ψ is another element of $\text{St}(L^{\max})$ then $\sigma(\phi\psi) = \sigma(\phi)\sigma(\psi)$. Thus, writing S_M for the permutation group of the finite set M , we may view σ as a homomorphism $\sigma: \text{St}(L^{\max}) \rightarrow S_M$. If $\phi \in \ker(\sigma)$ then $\sigma(\phi)$ fixes every element of M and in particular every element of L_X . Hence $\ker(\sigma) \subseteq \text{St}(L_X)$ and since from (2.1) we have $\text{St}(L_X) = \text{St}(L)$, this completes the proof. \square

2.4. Ordering X and closures of elements. Using the order induced from L we shall define a total order on X . This order will be used to index a basis of \mathbb{Z}^n with respect to which elements of $\text{St}(L)$ will be described as matrices. The ordering depends on the following stratification of the closures of single elements of X .

Proposition 2.7. $[x]_{\perp} = \text{cl}(x) \setminus \{u \in \text{cl}(y) \mid y <_L x\}$ for all $x \in X$. In particular, if x is L -minimal then $[x]_{\perp} = \text{cl}(x)$.

Proof. First recall $\text{cl}(z) = \text{cl}(x)$ if and only if $z^{\perp} = x^{\perp}$ (as $Y^{\perp} = Y^{\perp\perp\perp}$), so $z \in [x]_{\perp}$ if and only if $\text{cl}(z) = \text{cl}(x)$. If $u \in \text{cl}(y)$, where $y <_L x$ then $\text{cl}(u) \subseteq \text{cl}(y) < \text{cl}(x)$ so $u^{\perp} \neq x^{\perp}$. Hence $[x]_{\perp} \subseteq \text{cl}(x) \setminus \{u \in \text{cl}(y) \mid y <_L x\}$. On the other hand if $z \in \text{cl}(x)$ then $\text{cl}(z) \subseteq \text{cl}(x)$. If also $z \notin \text{cl}(y)$ for all $y <_L x$ then $z \not<_L x$, so $\text{cl}(z) = \text{cl}(x)$, as required. \square

We now define a total order $<$ on X , which will have the properties that

- (1) if $x <_L y$ then $y < x$, and
- (2) if $z < y < x$ and $z \in [x]_{\perp}$ then $y \in [x]_{\perp}$.

To begin with let

$$B_0 = \{Y \in L_X \mid Y = \text{cl}(x) \text{ where } x \text{ is } L\text{-minimal}\}.$$

Suppose that B_0 has k elements and choose an ordering $Y_1 < \dots < Y_k$ of these elements. If $i \neq j$ then $Y_i \cap Y_j \in L$ and from the remark at the beginning of this section and the fact that the Y_i 's are L -minimal it follows that $Y_i \cap Y_j = \emptyset$. Therefore we may define the ordering $<$ on $\bigcup_{i=1}^k Y_i$ in such a way that if $x_i \in Y_i$ and $x_j \in Y_j$ and $Y_i < Y_j$ then $x_j < x_i$: merely by choosing an ordering for elements of each Y_i .

We recursively define sets B_i of elements of L_X for $i \geq 0$ as follows. Assume that we have defined sets B_0, \dots, B_i , set $U_i = \bigcup_{j=0}^i B_j$ and define $X_i = \{u \in X \mid u \in Y \text{ for some } Y \in U_i\}$. If $U_i \neq L_X$ define B_{i+1} by

$$B_{i+1} = \{Y = \text{cl}(x) \in L_X \mid Y \notin U_i \text{ and } y <_L x \text{ implies that } \text{cl}(y) \in U_i\}.$$

If $U_i \neq L_X$ then $X_i \neq X$ and $B_{i+1} \neq \emptyset$. We assume inductively that we have ordered the set X_i in such a way that if $0 \leq a < b \leq i$ then $x_a \in Y_a$ where $Y_a \in B_a$ and $x_b \in Y_b$ where $Y_b \in B_b$ implies that $x_b < x_a$. From Proposition 2.7, if $Y = \text{cl}(x) \in B_{i+1}$ then

$$[x]_{\perp} = Y \setminus \{u \in \text{cl}(y) : y <_L x\} = Y \setminus \{u \in X_i\}.$$

Therefore we have defined $<$ on the set $Y \setminus [x]_{\perp}$. Moreover, if $Y_1 \neq Y_2$ and $Y_1, Y_2 \in B_{i+1}$ then $Y_1 \cap Y_2 \in L$, so $z \in Y_1 \cap Y_2$ implies $\text{cl}(z) \subseteq Y_1 \cap Y_2$. As $Y_1 \neq Y_2$ this implies that $\text{cl}(z)$ is strictly contained in $Y_i, i = 1, 2$. If $Y_i = \text{cl}(x_i)$ then $z <_L x_i$ and so $z \notin [x_i]_{\perp}, i = 1, 2$. That is, $[x_1]_{\perp} \cap [x_2]_{\perp} = \emptyset$. Now choose an ordering on the set of elements of B_{i+1} : $Z_1 < \dots < Z_k$ say, where $Z_j = \text{cl}(x_j)$. Then $Z_j \setminus [x_j]_{\perp} \subseteq X_i, j = 1, \dots, k$. We can extend the total order $<$ on X_i to

$$X_{i+1} = X_i \cup \bigcup_{j=1}^k Z_j = X_i \cup \bigcup_{j=1}^k [x_j]_{\perp}$$

as follows. Assume the order has already been extended to $X_i \cup_{j=1}^{s-1} [x_j]_{\perp}$. Extend the order further by choosing the ordering $<$ on the elements of $[x_s]_{\perp}$ and then setting its greatest element less than the least element of $X_i \cup_{j=1}^{s-1} [x_j]_{\perp}$. At the final stage $s = k$ and the order on X_i is extended to X_{i+1} . We continue until $U_i = L_X$, at which point $X = X_i$ and we have the required total order on X . Note that, by construction, if $x, y \in X$ and $x <_L y$ then $y < x$. Also, if $x < y < z$ and $[z]_{\perp} = [x]_{\perp}$ then $[y]_{\perp} = [x]_{\perp}$. Thus (1) and (2) above hold. If $\text{cl}(x)$ belongs to B_i we shall say that $x, \text{cl}(x)$ and $[x]_{\perp}$ have *height* i and write $h(x) = h(\text{cl}(x)) = h([x]_{\perp}) = i$.

2.5. A matrix representation of $\text{St}(L)$. Suppose that $X = \{x_1, \dots, x_k\}$ with $x_1 < \dots < x_k$. If $x \in X$ and $\phi \in \text{St}(L)$ then we have $x^{\phi} \in G(\text{cl}(x))$. If $\text{cl}(x) = \{y_1, \dots, y_r\}$, where $x = y_1$ say, then as $G(\text{cl}(x))$ is a free Abelian group we may write

$$x^{\phi} = y_1^{b_1} \dots y_r^{b_r}. \tag{2.2}$$

Setting $a_j = 0$ if $x_j \notin \text{cl}(x)$, and $a_j = b_i$ if $x_j = y_i$ for some i , we can write

$$x^{\phi} = x_1^{a_1} \dots x_k^{a_k}.$$

As this holds for all $x \in X$ we have

$$x_1^{\phi} = x_1^{a_{1,1}} \dots x_k^{a_{1,k}}, \dots, x_k^{\phi} = x_1^{a_{k,1}} \dots x_k^{a_{k,k}}.$$

Assume now that $Y \in L$. Then $Y = \bigcup_{y \in Y} \text{cl}(y)$ and for $1 \leq i \leq k$, either $[x_i]_{\perp} \subseteq Y$ or $[x_i]_{\perp} \cap Y = \emptyset$. Let $I = \{i \mid 1 \leq i \leq k \text{ and } [x_i]_{\perp} \subseteq Y\}$. Then, from Proposition 2.7, it follows that $Y = \bigcup_{i \in I} [x_i]_{\perp}$. Moreover for all i such that $x_i \in Y$ we have

$$x_i^{\phi} = \prod_{j \in I} x_j^{a_{i,j}}.$$

We denote the restriction $\phi|_{G(Y)}$ of ϕ to $G(Y)$ by ϕ_Y for any subset Y of X and $\phi \in \text{St}(L)$. We shall also write ϕ_x instead of $\phi_{\text{cl}(x)}$ for $x \in X$.

Definition 2.8. In the above notation, given $Y \in L$ we define the matrix corresponding to the restriction ϕ_Y of $\phi \in \text{St}(L)$ to $G(Y)$ to be $[\phi_Y] = (a_{i,j})_{i,j \in I}$. If $Y = X$ we write $[\phi]$ for $[\phi_X]$.

Definition 2.9. Let $Y = \{y_1, \dots, y_r\} \in L$ with $y_1 < \dots < y_r$ and let Z be a subset of Y . Let $I = \{i \mid 1 \leq i \leq r, y_i \in Z\}$. Given $A = (a_{i,j}) \in \text{GL}(r, \mathbb{Z})$ we define the Z -minor of A to be the matrix $M(A, Y, Z) = (a_{i,j})_{i,j \in I}$. If $Y = X$ we write $M(A, Z)$ for $M(A, X, Z)$.

The Z -minor of a matrix A is therefore the matrix obtained from A by deleting the i th row and column for all i such that $y_i \in Y \setminus Z$. From these definitions we have the following lemma.

Lemma 2.10. *Let $\phi \in \text{St}(L)$ and $Y \in L$ then $[\phi_Y] = M([\phi], Y)$. If $Z \subseteq Y \subseteq W$ are elements of L then $M(A, W, Z) = M(M(A, W, Y), Y, Z)$.*

For $x \in X$ we say that $y \in [x]_{\perp}$ is the *minimal* element of $[x]_{\perp}$ if $y < z$ for all $z \in [x]_{\perp}$. Let $X^{\min} = \{x \in X \mid x \text{ is the minimal element of } [x]_{\perp}\}$. Then $X = \bigcup_{x \in X} [x]_{\perp} = \bigsqcup_{x \in X^{\min}} [x]_{\perp}$. We extend this notation to arbitrary $Y \in L$ by defining $Y^{\min} = X^{\min} \cap Y$; so $Y = \bigsqcup_{y \in Y^{\min}} [y]_{\perp}$.

2.6. Sets of matrices corresponding to closed sets. We define a set of integer valued upper block-triangular matrices corresponding to a closed set. Let $Y \in L$ and write $Y^{\min} = \{v_1, \dots, v_m\}$, where $v_1 < \dots < v_m$. Assume further that $Y = \{u_1, \dots, u_r\}$, where $u_1 < \dots < u_r$. The set S_Y is defined to be the set of $r \times r$ integer valued matrices $A = (a_{i,j})$ such that the following conditions hold.

- (1) A has m diagonal blocks A_1, \dots, A_m , such that $A_i \in \text{GL}([v_i]_{\perp}, \mathbb{Z})$.
- (2) If $i > j$ and $a_{i,j}$ is not part of a diagonal block then $a_{i,j} = 0$.
- (3) If $i < j$ and $a_{i,j}$ is not part of a diagonal block then $a_{i,j} = 0$ unless $u_j <_L u_i$, in which case $a_{i,j}$ may be any element of \mathbb{Z} .

The first two of these conditions imply that A is an upper block-triangular matrix. Suppose that $A \in S_Y$ and has diagonal blocks A_1, \dots, A_m . Define the matrix B to be the block-diagonal matrix with diagonal blocks $A_1^{-1}, \dots, A_m^{-1}$. Then AB is a unipotent matrix and it follows that $A \in \text{GL}(r, \mathbb{Z})$. Therefore S_Y is a subset of $\text{GL}(r, \mathbb{Z})$.

Lemma 2.11. (1) *Elements of S_Y are upper block-triangular elements of $\text{GL}(r, \mathbb{Z})$.*

(2) *If ϕ is an element of J , V_{\perp} or Tr_{\perp} then $[\phi] \in S_X$.*

(3) *The set S_Y (with matrix multiplication) is a monoid for all $Y \in L$.*

Proof. If $\phi \in J$ then $[\phi]$ is a diagonal matrix with diagonal entries ± 1 , so belongs to S_X . If $\phi \in V_{\perp}$ then $\phi = \prod_{[x]_{\perp} \in N'_2} \phi_x^{\varepsilon_x}$ for some automorphisms $\phi_x \in \text{Aut}(G[x]_{\perp})$. Hence $[\phi]$ is block diagonal, with a blocks of dimension $|[x]_{\perp}|$ for each $[x]_{\perp} \in N'_2$ and of dimension 1 for all $x \in N_1$. It follows that $[\phi] \in S_X$. Finally let $\phi = \text{tr}_{x,y} \in \text{Tr}_{\perp}$. Then $\text{cl}(y) < \text{cl}(x)$ so $x < y$ and if $x = x_i$ and $y = x_j$ then $[\phi]$ is the matrix with 1's on the leading diagonal, $a_{i,j} = 1$ and 0's elsewhere. As $x_i < x_j$ we have $i < j$ and, as $x_j <_L x_i$, $a_{i,j}$ is not in a diagonal block so this matrix belongs to S_X . Thus statement (2) holds.

To prove statement (3) assume that $Y = \{u_1, \dots, u_r\}$ where $u_1 < \dots < u_r$. Let $A, B \in S_Y$, $A = (a_{i,j})$, $B = (b_{i,j})$ and let $AB = C = (c_{i,j}) \in \text{GL}(r, \mathbb{Z})$. Then $c_{i,j} = \sum_{k=1}^r a_{i,k} b_{k,j}$. Suppose that A has diagonal blocks A_1, \dots, A_m . Let A_D be the block-diagonal matrix with diagonal blocks A_1, \dots, A_m and 0's elsewhere. Let $A_N = A - A_D$. Define B_D and B_N similarly, so B_D is block diagonal and $B = B_D + B_N$. Then $C = A_D B_D + A_D B_N + A_N (B_D + B_N)$. Therefore C is upper block-triangular with diagonal blocks $A_1 B_1, \dots, A_m B_m$ and $A_i B_i \in \text{GL}([A_i], \mathbb{Z})$.

Suppose that $i < j$ and $c_{i,j}$ does not belong to a diagonal block. If i and k are such that $u_k \not\leq_L u_i$ then $a_{i,k} = 0$ and if k, j are such that $u_j \not\leq_L u_k$ then $b_{k,j} = 0$. Hence $a_{i,k}b_{k,j} \neq 0$ implies that $u_k \leq_L u_i$ and $u_j \leq_L u_k$. If $u_i =_L u_j$ then $c_{i,j}$ belongs to a diagonal block, a contradiction. Hence $u_j <_L u_i$. Thus $c_{i,j} \neq 0$ implies $u_j <_L u_i$ and so $C \in S_Y$. As the identity matrix is in S_Y it follows that S_Y is a monoid. \square

We are now ready to prove Theorem 2.4, which is the second statement of the following result.

Theorem 2.12. *The map $\pi : \text{St}(L) \rightarrow \text{GL}(|X|, \mathbb{Z})$ given by $\phi \mapsto [\phi]$ is an injective homomorphism with image S_X . In particular S_X is a group. Moreover the group $\text{St}(L)$ is generated by the elements of J, V_\perp and Tr_\perp , and is an arithmetic group.*

Proof. We shall first show that π is an injective group homomorphism, from $\text{St}(L)$ to $\text{GL}(|X|, \mathbb{Z})$. Assume that $|X| = k$, let $\phi, \psi \in \text{St}(L)$ and let $[\phi] = (a_{i,j})$ and $[\psi] = (b_{i,j})$. In the notation of Section 2.4 for $x_i \in X$ we have $x_i^{\phi\psi} = (x_1^{a_{i,1}})^\psi \dots (x_k^{a_{i,k}})^\psi = \prod_{r=1}^k (x_1^{b_{r,1}} \dots x_k^{b_{r,k}})^{a_{i,r}} = \prod_{j=1}^k x_j^{c_{i,j}}$, where $c_{i,j} = \sum_{r=1}^k a_{i,r}b_{r,j}$. Hence $[\phi\psi] = (c_{i,j}) = [\phi][\psi]$. Therefore π is a homomorphism and it is immediate from the definition that π is injective.

Let T denote the subgroup of $\text{St}(L)$ generated by J, V_\perp and Tr_\perp and recall that V is the subgroup of $\text{St}(L)$ generated by J and V_\perp . From Lemma 2.11 it follows that $[\phi] \in S_X$ for all $\phi \in T$. Therefore once we have proved the final statement of the Lemma it will also follow that the image of π is contained in S_X . The proof of the final statement will be broken into three cases and in each case we shall also verify that π maps surjectively onto S_X .

Let $\phi \in \text{St}(L)$ and $x \in X$ and suppose that $\text{cl}(x) = \{y_1, \dots, y_r\}$. Then we can express x^ϕ as in (2.2). Assume further that $[x]_\perp = \{y_1, \dots, y_s\}$, where $s \leq r$. In the notation of (2.2), if $b_j = 0$ for all $j > s$, then $x^\phi \in G[x]_\perp$. Suppose this holds for all $x \in [x]_\perp$; so $\phi_{[x]_\perp} \in \text{Aut}(G[x]_\perp)$. In this case we call ϕ a *block-diagonal automorphism*.

Case 1. Let $\phi \in \text{St}(L)$ be a block-diagonal automorphism and let $x \in X$. If $x \in N_2$ then $[x]_\perp \in N'_2$ and $\phi_{[x]_\perp}^{\varepsilon_x} \in \text{Aut}(G[x]_\perp)^{\varepsilon_x} \subseteq V_\perp$. If $x \notin N_2$ then $[x]_\perp = \{x\}$ and $\phi_{[x]_\perp}^{\varepsilon_x} \in J$. In either case $\phi_{[x]_\perp}^{\varepsilon_x} \in V$ and, as the same is true of all $x \in X$,

$$\phi = \prod_{[x]_\perp \in N'_2} \phi_{[x]_\perp}^{\varepsilon_x} \cdot \prod_{x \in N_1} \phi_{[x]_\perp}^{\varepsilon_x} \subseteq V \subseteq T.$$

Now let $X = \{x_1, \dots, x_k\}$, where $x_1 < \dots < x_k$ and write $[\phi] = (a_{i,j})$. Let $X^{\min} = \{x_{i_1}, \dots, x_{i_m}\}$ for some $m \geq 1$, with $x_{i_1} < \dots < x_{i_m}$. In this terminology what we have shown is the following.

If $a_{i,j} = 0$ for all i, j such that $x_i \in [x_{i_n}]_\perp, x_j \notin [x_{i_n}]_\perp$, then $\phi_{[x_{i_n}]_\perp}^{\varepsilon_{i_n}} \in V$; and if this holds for $n = 1, \dots, m$, then $\phi = \prod_{n=1}^m \phi_{[x_{i_n}]_\perp}^{\varepsilon_{i_n}} \in V \subseteq T$.

That is, if the above holds then $\phi \in T$, and so $[\phi] \in S_X$.

On the other hand, let $A \in S_X$ be a block-diagonal matrix. Then A has diagonal blocks $A_n \in \text{GL}([x_{i_n}]_{\perp}, \mathbb{Z})$ for $n = 1, \dots, m$. Here A_n determines an automorphism, ϕ_n say, of $G[x_{i_n}]_{\perp}$, and $\phi = \prod_{n=1}^m \phi_n^{\varepsilon_{x_{i_n}}} \in V$. Moreover $[\phi] = A$; so all block-diagonal matrices in S_X are in the image of π .

Case 2. Let $\phi \in \text{St}(L)$ and $A = [\phi]$. Write $A = A_D + A_N$ as in the proof of Lemma 2.11. In this case we assume that A_D is the identity matrix. This means that $x_i^{\phi} = x_i w_i$, where $w_i \in G(Y)$ for some $Y \subseteq \text{cl}(x) \setminus [x_i]_{\perp}$ for $i = 1, \dots, k$. Define $r = r(\phi)$ to be the maximal integer such that $x_r \in [x_{i_n}]_{\perp}$ for some n , and there exists x_j such that $x_j \notin [x_{i_n}]_{\perp}$ but $a_{r,j} \neq 0$. Let $j = c = c(\phi)$ be maximal with this property. (The argument of case 1 covers the case $r = 0$.) As $a_{r,c} \neq 0$ we have $w_r = u x_c^{a_{r,c}}$ for some $u \in G(Z)$, where $x_r \notin Z$ and, since A_D is the identity $w_s \in G(Z_s)$, where $x_r \notin Z_s$ for all $s > r$.

As $\phi \in \text{St}(L)$ we have $x_r^{\phi} \in \text{cl}(x_r)$, thus $x_c \in \text{cl}(x_r)$, which implies that $\text{cl}(x_c) \subseteq \text{cl}(x_r)$. Since $x_c \notin [x_r]_{\perp}$ it follows that $\text{cl}(x_c) \neq \text{cl}(x_r)$. Hence $\text{tr}_{x_r, x_c} \in \text{Tr}_{\perp} \subseteq \text{St}(L)$. Let $\phi_1 = (\text{tr}_{x_r, x_c})^{-a_{r,c}} \in T$. Then $x_r^{\phi_1} = x_r x_c^{-a_{r,c}}$ and $x_l^{\phi_1} = x_l$ for all $l \neq r$. Let $\phi_0 = \phi \phi_1$; so $\phi_0 \in \text{St}(L)$. We have

$$x_r^{\phi_0} = (x_r u x_c^{a_{r,c}})^{\phi_1} = x_r u \quad \text{and} \quad x_s^{\phi_0} = (x_s w_s)^{\phi_1} = x_s w_s = x_s^{\phi}$$

for $s > r$. If $s < r$ then

$$x_s^{\phi_0} = x_s w_s^{\phi_1} = \begin{cases} x_s w_s x_c^{a_{r,c}} = x_s^{\phi} x_c^{a_{r,c}} & \text{if } x_r \text{ occurs in } w_s, \\ x_s w_s = x_s^{\phi} & \text{otherwise.} \end{cases}$$

Therefore all diagonal blocks of $[\phi_0]$ are the identity matrix and either $r([\phi_0]) < r$ or $c([\phi_0]) < c$. We may then assume inductively that $[\phi_0] \in S_X$ and $\phi_0 \in T$: so $\phi \in T$. Now define E to be the matrix which has zero in every position except row r column c , which is equal to $a_{r,c}$. Then $[\phi_1^{\pm 1}] = I \mp E \in S_X$ and from Lemma 2.11 it follows that $[\phi] = [\phi_0][\phi_1^{-1}] \in S_X$. By induction it follows that for all ϕ such that $[\phi] = A = A_D + A_N$, with A_D the identity, we have $\phi \in T$ and $\phi^{\pi} \in S_X$. The same argument shows that if $A \in S_X$ and $A = A_D + A_N$, with A_D the identity, then $A = \phi^{\pi}$ for some $\phi \in T$.

Case 3. In the general case let $\phi \in \text{St}(L)$ and write $[\phi] = A = A_D + A_N$ as before. Let $B = A_D^{-1}$. Then from case 1, $B^{\pm 1} \in S_X$ and $B = [\sigma_B]$ for some $\sigma_B \in T$. Let $\zeta = \phi \sigma_B$. Then all diagonal blocks of $[\zeta]$ are the identity, so $\zeta \in T$ and $[\zeta] \in S_X$, from case 2. Therefore $\phi \in T$ and $[\phi] \in S_X$. If we begin this argument with an arbitrary element A of S_X instead of an element of $\text{St}(L)$ it shows again that $A \in T^{\pi}$ and $A \in S_X$.

We now show that the group $\text{St}(L)$ is an arithmetic group. Let K be the subgroup of $\text{GL}(|X|, \mathbb{C})$ satisfying conditions (1), (2) and (3) in the definition of S_X above. Then K is a \mathbb{Q} -defined linear algebraic group. As $S_X = K \cap \text{GL}(|X|, \mathbb{Z})$ it now follows that $\text{St}(L)$ is arithmetic. □

Combining the final statement of the theorem with Proposition 2.6 we obtain the following corollary.

Corollary 2.13. *St(L^{max}) is an arithmetic group.*

Proof. From the proof of Proposition 2.6 it follows that St(L) is a finite index subgroup of St(L^{max}). Also the proof of Theorem 2.12 goes through to show that St(L^{max}) is isomorphic to a subgroup S^{max} of GL(|X|, Z). To see this for each x ∈ X, let M_x = {z ∈ X | z is L-maximal and x <_L z}. Then G(M_x) is Abelian and contains x^ϕ for all ϕ ∈ St(L^{max}). Let ϕ ∈ St(L^{max}) and, as at the beginning of Section 2.5, write x^ϕ = x₁^{a₁} . . . x_k^{a_k}, where a_j ≠ 0 only if x_j ∈ M_x. As before this allows us to associate an integer valued matrix [ϕ] to ϕ. The proof that the map ϕ ↦ [ϕ] is a monomorphism from St(L^{max}) into GL(|X|, Z) is exactly the same as the first paragraph of the proof of Theorem 2.12. Thus St(L^{max}) is isomorphic to its image S^{max} ⊆ GL(|X|, Z). Moreover this monomorphism restricts to St(L) to give the map π and so S_X is a finite index subgroup of S^{max}.

Keeping the notation of the proof of the previous theorem we have S_X = K ∩ GL(|X|, C). Now choose a transversal a₁, . . . , a_s for cosets of S_X in S^{max}. Then g ∈ S^{max} if and only if ga_r⁻¹ ∈ S_X ⊆ K for some r. As S^{max} ∈ GL(|X|, Z) so a_r⁻¹ ∈ GL(|X|, Z) for all r. Hence the condition that an element h ∈ GL(|X|, C) satisfies h = ga_r⁻¹ for some g, can be expressed using |X|² polynomials with integer coefficients (namely the entries of the matrix a_r⁻¹). Set p = |X| and let these polynomials be m_{r,1,1}, . . . , m_{r,p,p}. (Thus if h = ga_r⁻¹ = (h_{ij}) then substitution of entries of g for variables of the m_{r,i,j} gives h_{ij} = m_{r,i,j}(g) for all i, j.) Suppose that the algebraic variety K is defined by polynomials f₁, . . . , f_l. Then g ∈ Ka_r if and only if g satisfies the polynomial equations f_i(m_{r,1,1}, . . . , m_{r,p,p}), i = 1, . . . , l. As f_i and m_{r,i,j} are polynomials with integer coefficients, this implies that Ka_r is a Q-defined affine algebraic variety. Thus ∪_{r=1}^s Ka_r is a variety and

$$\left(\bigcup_{r=1}^s Ka_r \right) \cap \text{GL}(|X|, \mathbb{Z}) = \bigcup_{r=1}^s (K \cap \text{GL}(|X|, \mathbb{Z}))a_r = \bigcup_{r=1}^s S_X a_r = S^{\max},$$

so St(L^{max}) is an arithmetic group. □

In the previous theorem we restricted attention to the entire group St(L) and its isomorphic image S_X. However, we shall now show that the set S_Y is a group for all closed sets Y in L, and in fact all these groups are arithmetic. By defining appropriate maps corresponding to inclusion, as follows, it can be seen that the lattice L maps contravariantly to a sublattice of the lattice of subgroups of Aut(G). If Y, Z ∈ L with Y ⊆ Z then M(A, Y, Z) ∈ GL(|Z|, Z) and so ρ(Y, Z): A ↦ M(A, Y, Z) is a map from S_Y to GL(|Z|, Z).

Lemma 2.14. *Let Z, Y ∈ L with Z ⊆ Y. The set S_Y is an arithmetic group and the map ρ(Y, Z) is a surjective homomorphism from S_Y to S_Z. There is an injective*

homomorphism $\varepsilon(Z, Y)$ from S_Z to S_Y such that $\varepsilon(Z, Y)\rho(Y, Z)$ is the identity on S_Z .

Proof. We show that $\rho(Y, Z)$ is an surjective monoid homomorphism for all $Z \subseteq Y \in L$. Since S_X is a group it will then follow that S_Y is a group for all $Y \in L$. The proof that S_Y is arithmetic is then the same as for S_X , replacing X by Y throughout. Let $Y^{\min} = \{v_1, \dots, v_m\}$, where $v_1 < \dots < v_m$. Also let $Y = \{u_1, \dots, u_r\}$, where $u_1 < \dots < u_r$ and let $I = \{i \mid 1 \leq i \leq r \text{ and } u_i \in Z\}$. Let $A = (a_{i,j}) \in S_Y$ and suppose that A has diagonal blocks A_1, \dots, A_m . As A is upper block-triangular it follows from the definition that $M(A, Y, Z)$ is upper block-triangular. If $v_i \in Z$ then $[v_i]_{\perp} \subseteq Z$ and the diagonal block containing A_i is unaffected in the transformation of A to $M(A, Y, Z)$. On the other hand if $v_i \notin Z$ then the diagonal block A_i is deleted in forming $M(A, Y, Z)$. As $Z^{\min} = Y^{\min} \cap X$, the diagonal blocks of $M(A, Y, Z)$ satisfy condition (1) of the definition of S_Z .

It remains to verify condition (3). Suppose that $i, j \in I$ and that $a_{i,j} \neq 0$ and $a_{i,j}$ does not belong to a diagonal block of $M(A, Y, Z)$. From the above, $a_{i,j}$ does not belong to a diagonal block of A , and since A is upper block-triangular and satisfies condition (3), $i < j$ and $u_j <_L u_i$. Then the same holds for $M(A, Y, Z)$, as required. Therefore $\rho(Y, Z)$ maps S_Y into S_Z .

To see that $\rho(Y, Z)$ is a homomorphism let $A = (a_{i,j})$ and $B = (b_{i,j})$ be elements of S_Y and let $C = (c_{i,j}) = AB$. From Lemma 2.11, we have $C \in S_Y$. Suppose that $i, j \in I$ and that $a_{i,k}b_{k,j} \neq 0$ for some k . Then $u_k \leq_L u_i$ and $u_i \in Z$. Hence $u_k \in \text{cl}(u_k) \subseteq \text{cl}(u_i) \subseteq Z$. Therefore $i, j, k \in I$ and $c_{i,j} = \sum_{k \in I} a_{i,k}b_{k,j}$. It follows that $M(AB, Y, Z) = M(A, Y, Z)M(B, Y, Z)$, so $\rho(Y, Z)$ is a homomorphism.

To construct $\varepsilon(Z, Y)$ note that if $P \in S_Z$ then we may write $P = (p_{i,j})_{i,j \in I}$, by expressing Z as a subset of $\{u_1, \dots, u_r\}$. Then let the diagonal blocks of P be P_i , where $i \in I$. With this notation define an $r \times r$ integer matrix A by first of all setting $a_{i,j} = p_{i,j}$ for $i, j \in I$, then setting $a_{i,i} = 1$ for $i \notin I$, and finally setting $a_{i,j} = 0$ for all other i, j . Then A is upper block-triangular and has blocks A_1, \dots, A_m , where $A_i = P_i$ if $i \in I$, and A_i is the identity matrix in $\text{GL}([v_i]_{\perp}, \mathbb{Z})$ otherwise. As P satisfies condition (3) then so does A . Hence A belongs to S_Y . Define $M(P, Z, Y) = A$ (where $Z \subseteq Y$) and $P^{\varepsilon(Y,Z)} = M(P, Z, Y)$ for all $P \in S_Z$. By definition, $P^{\varepsilon(Y,Z)}\rho(Y,Z) = P$ for all $P \in S_Z$, so $\varepsilon(Y, Z)$ is injective and $\rho(Y, Z)$ is surjective. Moreover, from the definition, $\varepsilon(Y, Z)$ is a homomorphism. \square

2.7. Restriction to closed sets. Here we consider the restriction of automorphisms in $\text{St}(L)$ to subgroups $G(Y)$, where Y is in L . Given $Y \in L$ let us define $L(Y) = \{Z \in L \mid Z \subseteq Y\}$. Note that $L(Y)$ is not in general the same as $L(\Gamma(Y))$, the set of closed sets of the full subgraph $\Gamma(Y)$ of Γ generated by Y ; although $L\Gamma(Y) \subseteq L(Y)$. We define $\text{St}_Y(L) = \{\phi_Y \mid \phi \in L\}$. Then $\text{St}_Y(L)$ is a subgroup of $\text{Aut}(G(Y))$ and is contained in the subgroup of stabilisers, in $\text{Aut}(G(Y))$, of $L(Y)$.

Lemma 2.15. *The map $\rho_Y : \text{St}(L) \rightarrow \text{St}_Y(L)$ given by $\phi \mapsto \phi_Y$ is a surjective homomorphism. The map $\pi_Y : \text{St}_Y(L) \rightarrow S_Y$ given by $\phi_Y \mapsto [\phi_Y]$ is an isomorphism, so $\text{St}_Y(L)$ is arithmetic for all $Y \in L$. Moreover $\rho_Y \pi_Y = \pi_X \rho(X, Y)$.*

Proof. Let $\phi, \psi \in \text{St}(L)$. Then $x^\phi \in G(Y)$ for all $x \in Y$, so $x^{\phi\psi} = (x^\phi)^\psi = (x^{\phi_Y})^{\psi_Y}$ for all $x \in Y$. Hence $(\phi\psi)_Y = \phi_Y \psi_Y$ and ρ_Y is a homomorphism; surjective onto its image which is, by definition, $\text{St}_Y(L)$.

From Theorem 2.12 the map $\pi_X = \pi$ is an isomorphism from $\text{St}(L)$ to S_X . From Lemma 2.14 the map $\rho(X, Y)$ is a surjective homomorphism from S_X to S_Y . Let $\theta = \pi_X \rho(X, Y)$. An element $\phi \in \text{St}(L)$ belongs to $\ker(\rho_Y)$ if $x^\phi = x$ for all $x \in Y$: in which case $[\phi_Y]$ is the identity matrix of dimension $|Y|$. Hence the diagonal blocks of ϕ^{π_X} corresponding to $[x]_\perp \subseteq Y$ are identity matrices; and $\phi^\theta = I$, the $|Y|$ -dimensional identity matrix. This shows that $\ker(\rho_Y) \subseteq \ker(\theta)$, so θ induces a homomorphism from $\text{St}_Y(L)$ to S_Y . The image of ϕ_Y under this homomorphism is $\phi^\theta = [\phi]^\rho(X, Y) = M([\phi], Y)$ and from the definitions we have $\phi_Y^{\pi_Y} = M([\phi], Y)$. Therefore π_Y is a homomorphism and $\rho_Y \pi_Y = \pi_X \rho(X, Y)$. As θ is surjective, so is π_Y . If $\phi_Y^{\pi_Y} = I$ then $x^{\phi_Y} = x$ for all $x \in Y$, so ϕ_Y is the identity on $G(Y)$ and π_Y is injective. □

If $Z \subseteq Y \in L$ and $\phi \in \text{St}(L)$ we define $\rho_{Y,Z}$ to be the map sending $\phi \in \text{St}_Y(L)$ to $\phi|_{G(Z)} \in \text{St}_Z(L)$.

Corollary 2.16. *Let $Y, Z \in L$ with $Z \subseteq Y$. Then $\rho(X, Z) = \rho(X, Y)\rho(Y, Z)$ and $\rho_Z = \rho_Y \rho_{Y,Z}$. Moreover $\rho_{Y,Z}$ is surjective and $\pi_Y \rho(Y, Z) = \rho_{Y,Z} \pi_Z$.*

Proof. This follows from Lemmas 2.10, 2.14 and 2.15. □

The various maps we have defined are illustrated in the commutative diagram of Figure 1.

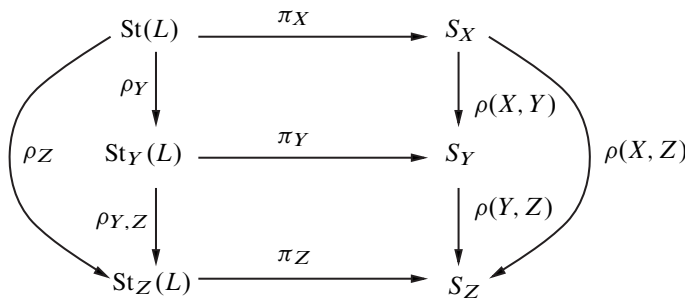


Figure 1. Maps defined on subgroups of $\text{St}(L)$.

2.8. The structure of $\text{St}^{\text{conj}}(L)$. First we examine the structure of $\text{St}_Y(L)$ and S_Y for an arbitrary closed set Y .

Definition 2.17. Let $Y \in L$ and $Y = \bigcup_{i=1}^m [v_i]_{\perp}$, where $Y^{\min} = \{v_1, \dots, v_m\}$ with $v_1 < \dots < v_m$. Denote by D_Y the set of block diagonal elements of S_Y with diagonal blocks A_1, \dots, A_m such that $A_i \in \text{GL}([v_i]_{\perp}, \mathbb{Z})$. Denote by U_Y the subset of S_Y consisting of matrices for which A_i is the identity matrix of $\text{GL}([v_i]_{\perp}, \mathbb{Z})$ for $i = 1, \dots, m$.

Lemma 2.18. Let $Y \in L$ and $Y = \bigcup_{i=1}^m [v_i]_{\perp}$, where $Y^{\min} = \{v_1, \dots, v_m\}$ with $v_1 < \dots < v_m$. Both U_Y and D_Y are subgroups of S_Y and $S_Y = U_Y \rtimes D_Y$. Moreover

$$D_Y = \prod_{i=1}^m G[v_i]_{\perp}.$$

Proof. Let $A \in S_Y$ with diagonal blocks A_1, \dots, A_m and define A_D to be the block-diagonal matrix with diagonal blocks A_1, \dots, A_m , and let d be the map sending A to A_D . Then $(A^d)^{\pi_Y^{-1}}$ is clearly an element of $\text{St}_Y(L)$ and so $A^d \in S_Y$. Hence $A^d \in D_Y$ and d is a surjective map from S_Y to D_Y . If B is also in S_Y and has diagonal blocks B_1, \dots, B_m then, as in the proof of Lemma 2.11, AB has blocks A_1B_1, \dots, A_mB_m , so d is a homomorphism and D_Y is a group. If $A \in S_Y$ then $A \in \ker(d)$ if and only if every diagonal block of A is an identity matrix. Hence $\ker(d) = U_Y$. If i is the inclusion of D_Y in S_Y then id is the identity map on D_Y and so $S_Y = U_Y \rtimes D_Y$, as claimed. That $D_Y = \prod_{i=1}^m G[v_i]_{\perp}$ is immediate from the definitions. □

In the case where Y is the closure of a single element of X we have the following corollary of the above results.

Corollary 2.19. (1) $[\phi_x] \in S_x$ for all $\phi \in \text{St}(L)$.

(2) There are automorphisms $\phi_{x,s}$ and $\phi_{x,u}$ of $G(\text{cl}(x))$ such that $\phi_x = \phi_{x,s}\phi_{x,u}$ and $[\phi_{x,s}]$ is the block-diagonal matrix with diagonal blocks A_1, \dots, A_m and $[\phi_{x,u}]$ is an upper unitriangular matrix (the unipotent part of $[\phi_x]$).

(3) Given $A \in S_x$ there exists $\psi \in \text{St}(L)$ such that $[\psi_x] = A$.

(4) If $y <_L x$ then $[\phi_y] = M([\phi_x], \text{cl}(y))$.

(5) The set S_x is a group.

Proof. Assertions (1), (3) and (4) follow from Lemma 2.15. Statement (5) follows from Lemma 2.10, and claim (2) follows from Lemma 2.18. □

Theorem 2.20. $\text{St}^{\text{conj}}(L) = \text{Conj}(G) \rtimes \text{St}(L)$.

Proof. First we show that $\text{Conj}(G) \subseteq \text{St}^{\text{conj}}(L)$. As $\text{Conj}(G)$ is generated by automorphisms of the form given in Definition 1.3 it suffices to show that $\phi = \alpha_C(y)$ belongs to $\text{St}^{\text{conj}}(L)$, where $y \in X$ and C is a connected component of the full subgraph on $X \setminus y^\perp$. Let $V \in L$, so $V = T^\perp$ for some $T \in L$. If $y \in V$ then $G(Y)^\phi = G(V)$, so assume $y \notin V$. Let $v_1, v_2 \in V$. If $v_1 \in y^\perp$ then $v_1^\phi = v_2^\phi$, so assume $v_i \notin y^\perp$ for $i = 1, 2$. Now $y \notin V$ implies there exists some $t \in T$ such that $[y, t] \neq 1$. As $v_i \in T^\perp$ and $v_i \notin y^\perp$ it follows that v_1, v_2 and t lie in a connected component of $\Gamma(X \setminus y^\perp)$. In particular $v_1^\phi = v_2^\phi$. Therefore either $G(V)^\phi = G(V)$ or $G(V)^\phi = G(V)^y$ and so $\phi \in \text{St}^{\text{conj}}(L)$, as required.

Next we show that $\text{Conj}(G) \triangleleft \text{St}^{\text{conj}}(L)$. It suffices to show that $\theta^{-1}\phi\theta \in \text{Conj}(G)$, where ϕ is defined as above and $\theta \in \text{St}^{\text{conj}}(L)$. Let $x \in X$ and let $g \in G$ such that $G(\text{cl}(x))^\theta = G(\text{cl}(x))^g$. Then $G(\text{cl}(x))^{\theta^{-1}} = G(\text{cl}(x))^h$, where $h = (g^{-1})^{\theta^{-1}}$. Thus $x^{\theta^{-1}} = w^h$ for some $w \in G(\text{cl}(x))$, so $x = (w^\theta)^{h\theta}$ and $x^g = w^\theta$. Now $x^{\theta^{-1}\phi\theta} = (w^h)^{\phi\theta} = ((w^\phi)^{h\phi})^\theta$, so

$$x^{\theta^{-1}\phi\theta} = \begin{cases} ((w^y)^{h\phi})^\theta = (w^\theta)^{y^\theta h^{\phi\theta}} = x^{g y^\theta h^{\phi\theta}} & \text{if } \text{cl}(x) \subseteq C \cup y^\perp, \\ (w^{h\phi})^\theta = (w^\theta)^{h^{\phi\theta}} = x^{g h^{\phi\theta}} & \text{otherwise.} \end{cases}$$

Therefore $\theta^{-1}\phi\theta$ is a conjugating automorphism.

Now we shall show that any element $\theta \in \text{St}^{\text{conj}}(L)$ can be expressed as $\theta = \tau\phi$ for some $\tau \in \text{Conj}(G)$ and $\phi \in \text{St}(L)$. Fix $\theta \in \text{St}^{\text{conj}}(L)$ and for all $Y \in L$ fix $g_Y \in G$ such that $G(Y)^\theta = G(Y)^{g_Y}$. Without loss of generality we may choose g_Y so that none of its left divisors belong to $G(Y)$ or to $C_G(Y) = G(Y^\perp)$. Given two non-empty closed sets $Y, Z \in L$ with $Y \subseteq Z$, we claim that $g_Y g_Z^{-1} = ab$ where $a \in G(Z)$ and $b \in C_G(Y)$. To see this suppose that $u \in G(Y)$ and let $r \in G(Y)$, $s \in G(Z)$ such that $u^\theta = r^{g_Y} = s^{g_Z}$. From [6], Corollary 2.4, and the choice of g_Y and g_Z there exist $c, c', d_1, d'_1, d_2, d'_2, v \in G$ such that $g_Y = c \circ d_2$, $g_Z = c' \circ d'_2$, $r = d_1^{-1} \circ v \circ d_1$, $s = d'_1{}^{-1} \circ v \circ d'_1$, and with $d = d_1 \circ d_2$ and $d' = d'_1 \circ d'_2$, $r^{g_Y} = d^{-1} \circ v \circ d$ and $s^{g_Z} = d'^{-1} \circ v \circ d'$.

By definition of θ for $x \in Y$ there exists $u \in G(Y)$ such that $u^\theta = x^{g_Y}$. We may then take $r = x$ and $s \in G(Z)$ such that $u^\theta = s^{g_Z} = x^{g_Y}$. In this case we have $r = x = v$ and so $d_1 = 1$ and, again by [6], Corollary 2.4, $c, c' \in C_G(x)$. Allowing x to range over Y we see that $c, c' \in C_G(Y)$, and by choice of g_Y it follows that $c = 1$ and $g_Y = d_2$. Now, with $r = x \in Y$, again we have $d_2^{-1} \circ x \circ d_2 = r^{g_Y} = s^{g_Z} = d'^{-1} \circ x \circ d'$, so $d' = d_2$. As $\alpha(d'_1) \subseteq \alpha(s) \subseteq Z$ we have $d'_1 \in G(Z)$. If d'_1 has a left divisor in $G(Y)$ then so does g_Y , and $c' \notin C_G(Z)$ since it is a left divisor of g_Z . This completes the proof of the claim as $g_Y g_Z^{-1} = d'_1 c'^{-1}$.

Next we use θ to construct a homomorphism from G to itself and subsequently show that this homomorphism is an element of $\text{St}(L)$. Let $x \in X$, so $G(\text{cl}(x))^\theta = G(\text{cl}(x))^{g_x}$. Then there exists $u_x \in G(\text{cl}(x))$ such that $x^\theta = u_x^{g_x}$. Define a map $\phi: X \rightarrow G$ by $x^\phi = u_x$ for all $x \in X$. Suppose that $x, y \in X$ with $[x, y] = 1$. Then $x, y \in x^\perp \cap y^\perp$, hence $\text{cl}(x) \cup \text{cl}(y) \subseteq x^\perp \cap y^\perp$. Let $Z = x^\perp \cap y^\perp$ and

so $G(Z)^\theta = G(Z)^{g_Z}$. As $\text{cl}(x) \subseteq Z$ we have, from the above, $g_x = abg_Z$, with $a \in G(Z)$ and $b \in C_G(\text{cl}(x)) = G(x^\perp)$. Thus $ab \in G(x^\perp)$ and $u_x^{ab} = u_x$. Hence $u_x^{g_x} = u_x^{g_Z}$ and similarly $u_y^{g_y} = u_y^{g_Z}$. Because θ is an automorphism, we have $1 = [u_x^{g_x}, u_y^{g_y}] = [u_x, u_y]^{g_Z}$, so $[u_x, u_y] = 1$. Therefore ϕ extends to an endomorphism of G .

The next step is to show that ϕ is surjective. To this end suppose that $y, z \in X$ and $\text{cl}(y) \subseteq \text{cl}(z)$. If $u \in \text{cl}(y)$ then $[u, v] = 1$ for all $v \in \text{cl}(z)$ since $\text{cl}(z) \subseteq z^\perp \subseteq y^\perp$. We have $g_y = abg_z$, where $a \in G(\text{cl}(z))$ and $b \in G(y^\perp)$. Hence $u_y^{ab} = u_y$ and so $u_y^{g_y} = u_y^{g_z}$. Now let $x \in X$. As $G(\text{cl}(x))^\theta = G(\text{cl}(x))^{g_x}$ there exists $w \in G(\text{cl}(x))$ such that $w^\theta = x^{g_x}$. Assume that $w = y_1^{\varepsilon_1} \dots y_n^{\varepsilon_n}$ for some $y_i \in \text{cl}(x)$ and $\varepsilon_i = \pm 1$. Let $u_i = u_{y_i}$. Then $\text{cl}(y_i) \subseteq \text{cl}(x)$, hence $y_i^\theta = u_i^{g_x}$ from the preceding argument, and $w^\theta = (u_1^{\varepsilon_1} \dots u_n^{\varepsilon_n})^{g_x} = x^{g_x}$. Hence $w^\phi = u_1^{\varepsilon_1} \dots u_n^{\varepsilon_n} = x$ and ϕ is surjective.

To show that ϕ is injective consider the automorphism θ^{-1} and let $h_x = (g_x^{-1})^{\theta^{-1}}$ for all $x \in X$. Choose, for all $x \in X$, an element $k_x \in G$ and $v_x \in G(\text{cl}(x))$ such that $G(\text{cl}(x))^{\theta^{-1}} = G(\text{cl}(x))^{k_x}$, $x^{\theta^{-1}} = v_x^{k_x}$ and k_x has no left divisor in $G(\text{cl}(x))$. Then, as in the case of θ and ϕ above, the map $\bar{\phi}: x \rightarrow v_x$ extends to an endomorphism of G . Moreover $h_x = j_x k_x$ for some $j_x \in G(x^\perp)$. Suppose that $u_x = y_1^{\varepsilon_1} \dots y_n^{\varepsilon_n}$, where $y_i \in \text{cl}(x)$. Write $v_i = v_{y_i} = y_i^{\bar{\phi}}$ for $i = 1, \dots, n$. Then, from the above, $y_i^{\theta^{-1}} = v_i^{k_x} = v_i^{h_x}$ since $\text{cl}(y_i) \subseteq \text{cl}(x)$, $v_i \in \text{cl}(y_i)$ and $j_x \in G(x^\perp)$. Now $x = x^{\theta\theta^{-1}} = (u_x^{g_x})^{\theta^{-1}} = (u_x^{\theta^{-1}})^{h_x^{-1}} = (v_1^{\varepsilon_1 h_x} \dots v_n^{\varepsilon_n h_x})^{h_x^{-1}} = v_1^{\varepsilon_1} \dots v_n^{\varepsilon_n}$, so $x^{\phi\bar{\phi}} = u_x^{\bar{\phi}} = v_1^{\varepsilon_1} \dots v_n^{\varepsilon_n} = x$. It follows that ϕ is a bijection and hence is an automorphism. By definition ϕ maps $G(\text{cl}(x))$ to itself for all $x \in X$, and so belongs to $\text{St}(L)$.

Now define $l_x = g_x^{\phi^{-1}}$ for all $x \in X$. Then $x^\theta = u_x^{g_x} = (x^\phi)^{l_x} = (x^{l_x})^\phi$, so $\tau = \theta\phi^{-1}$ is a conjugating automorphism. Note that if $\rho \in \text{Conj}(G) \cap \text{St}(L)$ then $x^\rho = x^{w_x}$ for some $w_x \in G$, and $G(\text{cl}(x))^\rho = G(\text{cl}(x))$. Therefore, $x^{w_x} \in G(\text{cl}(x))$. It follows that $w_x \in G(x^\perp)$, so $x^\rho = x$ and ρ is the identity map. Hence $\text{Conj}(G) \cap \text{St}(L) = \{1\}$. Now suppose that $\tau, \tau' \in \text{Conj}(G)$ and $\phi, \phi' \in \text{St}(L)$. Then $\tau\phi = \tau'\phi'$ implies $\tau'^{-1}\tau = \phi'\phi \in \text{Conj}(G) \cap \text{St}(L)$, so $\tau = \tau'$ and $\phi = \phi'$. What we have shown is that every element $\theta \in \text{St}^{\text{conj}}(L)$ can be uniquely expressed as $\theta = \tau\phi$ with $\tau \in \text{Conj}(G)$ and $\phi \in \text{St}(L)$. The theorem now follows. \square

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