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Fixity and free group actions on products of spheres

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Abstract. A representation $G \subset U(n)$ of degree n has *fixity* equal to the smallest integer f such that the induced action of G on $U(n)/U(n-f-1)$ is free. Using bundle theory we show that if G admits a representation of fixity one, then it acts freely and smoothly on $\mathbb{S}^{2n-1} \times \mathbb{S}^{4n-5}$. We use this to prove that a finite p-group (for $p > 3$) acts freely and smoothly on a product of two spheres if and only if it does not contain $(\mathbb{Z}/p)^3$ as a subgroup.

We use propagation methods from surgery theory to show that a representation of fixity $f < n - 1$ gives rise to a free action of G on a product of $f + 1$ spheres provided the order of G is relatively prime to $(n - 1)!$. We give an infinite collection of new examples of finite p-groups of rank r which act freely on a product of r spheres, hence verifying a strong form of a well-known conjecture for these groups. In addition we show that groups of fixity two act freely on a finite complex with the homotopy type of a product of three spheres. A number of examples are explicitly described.

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1. Introduction

A well known result in topology is the characterization of those finite groups that can act freely on a sphere, namely groups with either cyclic or generalized quaternion Sylow subgroups and such that every involution is central (see [23]). Free linear actions on spheres are the most basic examples. These can be constructed as follows: a subgroup G of $U(n)$ acts on the homogeneous space $\mathbb{S}^{2n-1} \cong U(n)/U(n-1)$; if no conjugate of G intersects $U(n-1)$ non-trivially, then this gives rise to a free linear action on the sphere.

In this paper we consider the situation for products of spheres. Here the problem is much more complex, and in particular we still do not have a characterization of those finite groups that can act freely on a product of *two* spheres. A key new ingredient in our approach is the use of the G actions on the homogeneous spaces $U(n)/U(k)$; our view is that these are also important building blocks for actions

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of groups of larger rank. On the algebraic side, group theorists have studied representations such that the action of G on $U(n)/U(n-f-1)$ is free; the smallest such f is called the *fixity* of the representation. In particular there is an explicit classification of those finite p-groups having a representation of fixity $f < p$.

Our main results are in the direction of propagating the natural free actions on $U(n)/U(k)$ to free actions on an actual product of spheres. For low fixity we can use explicit arguments involving equivariant vector bundles; in the case of fixity equal to one we have

Theorem 1.1. *A subgroup $G \subset U(n)$ of fixity one acts freely and smoothly on $X = \mathbb{S}^{2n-1} \times \mathbb{S}^{4n-5}$. In particular if G is any finite subgroup of $SU(3)$, then it will act freely and smoothly on $\mathbb{S}^5 \times \mathbb{S}^7$.*

From this we conclude that A_5, $SL_3(\mathbb{F}_2)$ and the triple cover $3A_6$ all act freely and smoothly on $\mathbb{S}^5 \times \mathbb{S}^7$.

Using the explicit classification of rank two p-groups (see [3]) as well as the description of p-groups of fixity equal to one in [24] we see that our result can be used to construct free actions for the exceptional (i.e. non-metacyclic) p-groups on the list. Combining this with the well-known fact that metacyclic groups act freely on a product of two linear spheres, we obtain

Theorem 1.2. *Let $p > 3$ be a prime. Then a p-group P acts freely and smoothly on $\mathbb{S}^m \times \mathbb{S}^n$ for some $m, n > 0$ if and only if P does not contain $\mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$ as a subgroup.*

This geometric result improves on the homotopy theoretic version recently established in [2]. Indeed this paper arose out of efforts to promote the results there to actions on manifolds. Moreover we obtain all of these actions explicitly, thus providing rank 2 models for group actions which may play a special role analogous to that of linear spheres. In contrast, the situation at the primes $p = 2, 3$ remains unresolved, reflecting the complications in the corresponding group theory at these primes. Recently Ünlü [31] has shown that among the 396 groups of order dividing 256 and which have rank equal to two, there is exactly **one** group which is not known to act freely and smoothly on a product of two spheres!

Using the methods developed in [2] we also obtain interesting results for the case of fixity equal to two, namely

Theorem 1.3. *If $G \subset U(n)$ is of fixity equal to two, then G acts freely on a finite complex $X \simeq \mathbb{S}^{2n-1} \times \mathbb{S}^{4n-5} \times \mathbb{S}^M$ for some $M > 0$. In particular if $G \subset SU(4)$, then G acts freely on a finite complex $X \simeq \mathbb{S}^7 \times \mathbb{S}^{11} \times \mathbb{S}^M$ for some $M > 0$.*

For example, this can be used to show that $Sp_4(\mathbb{F}_3)$ acts freely on a finite complex $X \simeq \mathbb{S}^7 \times \mathbb{S}^{11} \times \mathbb{S}^M$ for some $M > 0$.

For arbitrary fixity we must make use of methods from propagation theory,

involving homotopy theory and surgery. Observe that the Stiefel manifolds $U(n)/U(n - f - 1)$ have the cohomology of a product of $f + 1$ spheres (see 4.1); our goal is to propagate this natural model to an action on a product of spheres. The main result in this paper is the following

Theorem 1.4. *Let G denote either a finite subgroup of $SU(n)$ and let $k = 1$, or let G denote a finite subgroup of $U(n)$ which acts freely on $U(n)/U(k)$ with $k \geq 1$. If the order of G is prime to $(n - 1)!$, then G acts freely, smoothly and homologically trivially on $\mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1}$.*

If $G \subset U(n)$ and $(|G|, (n - 1)!) = 1$, our methods yield a free G-action on a finite complex $X \simeq \mathbb{S}^{2n-1} \times \cdots \times \mathbb{S}^3 \times \mathbb{S}^M$ for some $M > 1$; we conjecture that an analogue of our main theorem should also hold in this case, but there are surgery-theoretic difficulties to overcome which we hope to address in a subsequent paper.

Applying the available characterization of low fixity p-groups, we obtain interesting examples of group actions:

Theorem 1.5. *Let P denote a finite non-abelian p-group with cyclic center and having an abelian maximal subgroup. If the rank of P is $r < p$, then there exists a free and homologically trivial action of P on $M = \mathbb{S}^{2p-1} \times \mathbb{S}^{2p-3} \times \cdots \times \mathbb{S}^{2(p-r)+1}$, a product of r spheres.*

We should point out that this produces an infinite number of new examples of free actions by rank r groups on a product of r spheres. More generally it is conjectured that a rank r finite group will act freely on a finite complex X having the homotopy type of a product of r spheres. A related conjecture is that every finite group acts freely and homologically trivially on a product of spheres. Although the condition $(|G|, (n - 1)!) = 1$ is somewhat restrictive, our approach does yield a new method for approaching the conjectures mentioned above; most importantly we have constructed many interesting *geometric* actions.

We are grateful to R. L. Griess for providing information concerning the classification of finite linear groups.

2. Basic definitions and properties

In this section we will recall the notion of fixity for a complex representation of a finite group G (see [24]) and relate it to properties of associated actions on complex Stiefel manifolds $U(n)/U(k)$.

First we introduce some notation. Let V denote a finite dimensional $\mathbb{C}G$-module. For a subgroup $H \subset G$, we denote by V^H the subspace of vectors in V fixed by all $h \in H$. We denote by $\langle g \rangle$ the subgroup generated by an element g of G.

Definition 2.1. The *fixity* of a finite dimensional $\mathbb{C}G$-module V is

$$\mathrm{Fix}_G(V) = \max_{1 \neq g \in G} \{\dim_{\mathbb{C}} V^{\langle g \rangle}\}.$$

Given a faithful complex representation V of G, we can always obtain an equivalent unitary representation. Hence in what follows we will restrict our attention to faithful unitary representations and the associated embeddings $G \hookrightarrow U(n)$, where $U(n)$ denotes the group of $n \times n$ unitary matrices and $n = \dim V$. Given such an embedding and a closed subgroup $H \subset U(n)$, we have a natural G-action on the homogeneous space $U(n)/H$. In particular we will be interested in the subgroups $U(k) \subset U(n)$, defined by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

where I_{n-k} is the $(n-k) \times (n-k)$ identity matrix.

Note that $\mathrm{Fix}_G(V)$ can be expressed as the maximum (for $g \in G$) of the dimensions of the eigenspaces $\ker(g - I)$ for $g \neq 1$.

Lemma 2.2. $\mathrm{Fix}_G(V) \leq f$ *if and only if the induced action of G on* $U(n)/U(n-f-1)$ *is a free action.*

Proof. If $\mathrm{Fix}_G(V) \leq f$, then given any $1 \neq g \in G$, $\ker(g - I)$ can be at most f-dimensional, hence g cannot be conjugated into the subgroup $U(n-f-1)$ and so the action of G on the homogeneous space $U(n)/U(n-f-1)$ must be free. Conversely if the action is free, no element can be conjugated into $U(n-f-1)$, hence $\mathrm{Fix}_G(V) \leq f$. $\square$

Corollary 2.3. *A faithful $\mathbb{C}G$-module V has fixity f if and only if f is the smallest integer such that the induced G-action on $U(n)/U(n-f-1)$ is free.*

We can now reformulate the notion of fixity.

Definition 2.4. A faithful unitary representation $G \subset U(n)$ has *fixity* f if f is the smallest integer so that the induced action of G on $U(n)/U(n-f-1)$ is free.

From this we derive an invariant associated to G.

Definition 2.5. For a finite group G, we define its *fixity*, $\mathrm{Fix}(G)$, as the minimum value of $\mathrm{Fix}_G(V)$, as V ranges over all faithful, finite dimensional complex representations of G.

The case of fixity zero coincides with that of linear space forms. Indeed if an n-dimensional representation V has fixity zero then the induced action on $U(n)/U(n-1)$ is free; this can be identified with the action on the sphere $S(V)$

of unit vectors in V. Examples of higher fixity are less well-known from the point of view of transformation groups, and they will provide building blocks for new examples of group actions.

Proposition 2.6. *Let $G \subset SU(n)$. Then G acts freely on $U(n)/U(1)$. Hence the fixity of G is at most $n - 2$.*

Proof. Note $\det : U(1) \xrightarrow{\cong} S^1$. If $g \in G$ has a fixed point $hU(1) \in U(n)/U(1)$, then $h^{-1}gh \in U(1)$. Since $h^{-1}gh$ has determinant one, $h^{-1}gh = 1$ so $g = 1$. $\qquad\square$

We now relate fixity to another group-theoretic invariant.

Definition 2.7. For a finite group G and a prime p, we define its *p-rank* as $r_p(G) = \max \{r \mid (\mathbb{Z}/p)^r \subset G\}$ and its *rank* as $r(G) = \max \{r_p(G) \mid p$ divides $|\mathrm{G}|\}$.

The following is a basic result relating rank to fixity ([28], Lemma 3.1):

Proposition 2.8. *For any finite group G, $r(G) \leq \mathrm{Fix}(G) + 1$.*

Note that this inequality may be strict. For example, if G is a non-abelian split extension of $\mathbb{Z}/p$ by $\mathbb{Z}/q$ where p and q are prime, then this is a rank one group which does not have a fixed point free representation (i.e. G is not a Frobenius complement, as it does not satisfy the pq condition); hence its fixity is greater than zero. For us the most interesting case occurs when $r(G) = \mathrm{Fix}(G) + 1$.

Assume that $G \subset U(n)$ has fixity equal to f. Then the G action on $U(n)/U(n-f)$ is not free. In fact we have

Proposition 2.9. *If $G \subset U(n)$ has fixity f, then all of the isotropy subgroups for the G-action on $U(n)/U(n-f)$ have rank equal to one.*

Proof. Let H denote an isotropy subgroup for the G-action on $U(n)/U(n-f)$; this means that there exists a $g \in U(n)$ such that $g^{-1}Hg \subset U(n-f)$. This subgroup then acts freely on the quotient $U(n-f)/U(n-f-1) \cong \mathbb{S}^{2(n-f)-1}$ as otherwise a conjugate of G would intersect $U(n-f-1)$ non-trivially. Hence H must be a group of rank equal to one. $\qquad\square$

Remark 2.10. More generally if we let $M_k = U(n)/U(k)$, this defines a sequence of G-manifolds $\{M_0, M_1, \ldots, M_n\}$ and equivariant maps $M_k \to M_{k+1}$, for $k = 0, \ldots, n$ where M_0 is a free G-space, $M_{n-1} = \mathbb{S}^{2n-1}$ and $M_n = \{x_0\}$. As we go up this tower, the isotropy must increase from rank zero to rank $r(G)$. Using an argument similar to the one above, we see that the rank can only increase by one at each stage. There are n steps and a total increase by $r(G)$ must happen. Hence we have a partition of $[0, n] \cap \mathbb{Z}$, given by integers $0 \leq s_0 < s_1 < \cdots < s_{r(G)} \leq n$ such that the isotropy of M_k has rank t if $s_t \leq k < s_{t+1}$. The patterns which arise

in this process seem to be an interesting invariant of the representation; we shall make use of this in our applications. Note that when $n = r(G)$, the rank must increase exactly by one at every step; this is the case of maximal fixity $n - 1$.

We now describe a characterization of p-groups of fixity $f < p$, which appears in [24].

Theorem 2.11. *Let f be a non-negative integer and let p be a prime number greater than f. A non-abelian p-group P has fixity f if and only if the following hold:*

- *P has p-rank equal to $f + 1$;*
- *P has cyclic center and an abelian maximal subgroup.*

Moreover, for any such group there exists a faithful irreducible P-module of fixity f and dimension p.

The p-groups of fixity $f < p$ have been explicitly described by Conlon in [9], and enumerated in [24]. Using this one can verify for example that for $p \geq 5$ and $n \geq 4$, there are exactly *four* non-abelian p-groups of order p^n and fixity one. They can be listed as follows in terms of generators and relations (see [24], page 228 and [18], page 343):

- $< a, b \mid a^{p^{n-1}} = b^p = 1, a^b = a^{1+p^{n-2}} >$
- $< a, x, y \mid a^{p^{n-3}} = [x, y], a^{p^{n-2}} = [a, x] = [a, y] = x^p = y^p = 1 >$
- $< a, x \mid a^{-\lambda p^{n-3}} = [x, a, x], a^{p^{n-2}} = x^p = [x, a]^p = [x, a, a] = 1 >$, where λ is equal to 1 or to a non-quadratic residue modulo p.

Later we will construct actions of these p-groups on a product of two spheres. Note that the first group on this list is a metacyclic group; the other three are said to be of *exceptional type* and they appear in the classification of rank two p-groups which we will make use of in a subsequent section. The situation for $p = 2$ is rather different: there are $3n - 8$ nonabelian groups of order 2^n having fixity equal to one.

In [28] a detailed analysis of group theoretic properties of groups of low fixity is carried out. The main result is that there exists a function $\theta : \mathbb{N} \to \mathbb{N}$ such that if V is a $\mathbb{C}G$-module of fixity f, then there exists a normal subgroup $N \lhd G$ such that $[G : N] \leq \theta(f)$ and N is solvable of derived length at most 3. Another result is that there exists a function $\Phi : \mathbb{N} \to \mathbb{N}$ such that if V is a $\mathbb{C}G$-module of fixity f and $\dim V > \Phi(f)$, then either G is solvable or $G/O(G)$ has a subgroup of index at most 2 which is isomorphic to $\mathrm{SL}(2, 5)$. Here $O(G)$ denotes the largest normal subgroup of G whose order is odd, hence it is solvable.

There are substantial restrictions for groups of low fixity. For example, if G is a finite group acting on V with fixity equal to one, then we have

- $\dim V \leq |G|$. This is in contrast to the situation for fixity zero, where we can take modules of arbitrarily large dimension. In fact this is the only fixity where the modules do not have bounded dimension.

- The p-rank of G is one for all primes $p < \dim V - 1$. For example this means that as soon as $\dim V > 3$, the 2-Sylow subgroup of G must be either cyclic or generalized quaternion. This excludes all non-abelian simple groups.
- If $p = 2, 3$ do not divide $|G|$, then there exists at most one prime p with the property that G has a non-abelian Sylow p-subgroup.

As we shall see later, groups of fixity one constitute a class of non-periodic groups that act freely and smoothly on a product of two spheres, and their structure may shed some light on the general conjecture that any finite group of rank equal to two acts freely on a product of two spheres. In this paper we prove this conjecture for p-groups with $p > 3$.

3. Actions of groups of fixity one and two

In this section we begin by discussing the special case of groups having fixity equal to one. As we have seen they will act freely on some $U(n)/U(n-2)$, which happens to be a rather nice manifold which we can approach using very direct methods.

Theorem 3.1. *A subgroup $G \subset U(n)$ of fixity one acts freely and smoothly on* $X = \mathbb{S}^{2n-1} \times \mathbb{S}^{4n-5}$.

Proof. We have a fiber bundle

$$\mathbb{S}^{2n-3} = U(n-1)/U(n-2) \to U(n)/U(n-2) \to U(n)/U(n-1) = \mathbb{S}^{2n-1},$$

with structure group $U(n-1)$. It can be identified with the sphere bundle of an associated $(n-1)$-dimensional complex vector bundle ξ. In the next section we will see that the vector bundle ξ is always non-trivial, but here we will note that $\xi \oplus \xi$ is trivial since it is classified by an element of $\pi_{2n-1}(BU(2n-2)) \cong \pi_{2n-2}(U(2n-2)) \cong \pi_{2n-2}(U)$, which is zero by Bott periodicity.

Note that ξ is a G-vector bundle, and G acts freely away from the zero section. By considering the pullback diagram

$$
\begin{array}{ccc}
E(\xi \oplus \xi) & \longrightarrow & E(\xi) \times E(\xi) \\
\downarrow & & \downarrow \\
\mathbb{S}^{2n-1} & \xrightarrow{\ \Delta\ } & \mathbb{S}^{2n-1} \times \mathbb{S}^{2n-1},
\end{array}
$$

we see that $\xi \oplus \xi$ is also a G-vector bundle with action free away from the zero section. We can take $X = S(\xi \oplus \xi)$ (the associated sphere bundle) to complete the proof. $\qquad\square$

Corollary 3.2. *Any finite subgroup of $SU(3)$ acts freely and smoothly on $\mathbb{S}^5 \times \mathbb{S}^7$.*

Example 3.3. The finite linear groups (to be precise we are referring to primitive unimodular irreducible groups) in low dimensions have been completely classified

and listed by Blichtfeldt and others (see [4], [15]). For a fixed prime p let H_p denote the semidirect product of the form $E_p \times_T SL_2(\mathbb{F}_p)$, where

$$E_p = \langle a, b, c \mid a^p = b^p = c^p = 1, [a, b] = c \rangle$$

is the extraspecial p-group of order p^3 and exponent p. Note that $|H_p| = p^4(p-1)(p+1)$. We find four interesting subgroups of $SU(3)$: the alternating group A_5, the simple linear group $SL_3(\mathbb{F}_2) \cong PSL_2(\mathbb{F}_7)$, the triple cover $3A_6$, and H_3. We obtain

Corollary 3.4. *The groups A_5, $SL_3(\mathbb{F}_2)$, $3A_6$, and H_3 all act freely and smoothly on $\mathbb{S}^5 \times \mathbb{S}^7$.*

Example 3.5. In [28] page 290, a group $G_p \subset U(p)$ of fixity equal to one and having order $2p^3(p-1)$ is explicitly described as follows. Let $v_0, v_1, \ldots, v_{p-1}$ denote a basis for the underlying complex vector space and let ω denote a primitive p-th root of unity. Take the two linear transformations $A, B \in U(p)$ defined by $Av_i = \omega v_i$ and $Bv_i = \omega^i v_i$ where $0 \leq i \leq p-1$. Now identify the set $\{0, 1, \ldots, p-1\}$ with $\mathbb{F}_p$, and let x be a generator of $\mathbb{F}_p^*$. Let $\mu \in \mathbb{C}$ be a primitive $2(p-1)$-th root of unity. We define $C, D \in U(p)$ via $Cv_i = v_{i+1}$, $Dv_i = \mu v_{xi}$, where $i \in \mathbb{F}_p$. Let G_p denote the subgroup of $U(p)$ generated by A, B, C, D. The subgroup $H = \langle A, B \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p$ is normal in G_p, and G_p/H is isomorphic to a double cover of the semi-direct product $\mathbb{Z}/p \times_T \mathbb{Z}/(p-1)$. The group G_p acts freely and smoothly on $\mathbb{S}^{2p-1} \times \mathbb{S}^{4p-5}$.

These techniques are particularly effective for constructing actions of small p-groups.

Proposition 3.6. *Let P denote a p-group of rank equal to two and with order $|P| \leq p^4$. Then P acts freely and smoothly on $\mathbb{S}^m \times \mathbb{S}^n$ for some $m, n > 0$.*

Proof. If the group P has center $Z(P)$ of rank equal to two, then every element of order p in P is central. In this case the group acts freely on a product of two representations (see [2], page 422). Hence we can assume that $Z(P)$ is cyclic. For any p-group of order p^4, there exists an abelian subgroup of order p^3 (see [30], page 85), and so we can assume that P has an abelian maximal subgroup. By Theorem 2.11 we infer that P has fixity equal to one, whence by Theorem 3.1 it will act freely and smoothly on a product of two spheres, completing the proof. $\square$

As we have seen, if $G \subset U(n)$ has fixity one, then it will act on $U(n)/U(n-1) = \mathbb{S}^{2n-1}$ with rank one isotropy subgroups. In [2], it was shown that given a linear G-sphere $S(V)$ satisfying this condition, there exists a finite G-CW complex X with a free G-action, and such that $X \simeq S(V) \times \mathbb{S}^M$ for some integer M (see [31] for a more direct proof). From the above we see that for a group G of fixity one, this construction can be realized explicitly via a free and smooth G-action

on $\mathbb{S}^{4n-5} \times \mathbb{S}^{2n-1}$. However there are groups which act on a linear sphere with periodic isotropy which do not have fixity equal to one. Indeed in [2] it was shown that *every* rank two p-group admits a representation in $U(|P : Z(P)|)$ such that the action on $U(|P : Z(P)|)/U(|P : Z(P)| - 1)$ has periodic isotropy; on the other hand we have seen that p-groups of fixity equal to one are rather restricted. However in the case of rank two p-groups we have complete information:

Proposition 3.7. *If $p > 3$ and P is a finite p-group of rank equal to two, then P is either a metacyclic group or a group of exceptional type, with fixity equal to one.*

This proposition is proved in [24], page 228 by using the classification of rank two p-groups appearing in [3], as well as the characterization of p-groups of fixity equal to one described in Lemma 2.11. Next we recall the proof of an elementary

Lemma 3.8. *If P is a metacyclic p-group then it acts freely on a product of two linear spheres.*

Proof. The group P is an extension of the form $1 \to \mathbb{Z}/p^t \to P \to \mathbb{Z}/p^s \to 1$. Let χ denote a one dimensional character of the subgroup $B \cong \mathbb{Z}/p^t$ which maps a generator of B to a primitive p^t-th root of unity. Let $V = \mathrm{Ind}_B^P(\chi)$; this is a p^s-dimensional complex representation of P. The action of P on the associated sphere $S(V)$ restricts to a free B-action; this can be checked using Mackey's formula (it suffices to show that the unique and central subgroup of order p in B acts freely). Now let the quotient group $A = P/B$ act freely on $\mathbb{S}^1$ through multiplication by a primitive p^s-th root of unity. From this we can define a diagonal P action on $\mathbb{S}^{2p^s-1} \times \mathbb{S}^1$ which is evidently free. $\qquad\square$

Obviously the action described above is a *product* action. An interesting fact is that for $p > 3$, any exceptional rank two p-group does not admit a free product action; indeed we have

Proposition 3.9. *If $p > 3$ and P is a rank two p-group which is not metacyclic, then it cannot act freely on $X = \mathbb{S}^m \times \mathbb{S}^n$ via a product action.*

Proof. By the results in [14], every action of a finite p-group on a sphere can be modeled using a linear action; in particular given such a product action there exist representations V, W such that for any $g \in P$ we have $X^{\langle g \rangle} \neq \emptyset$ if and only if $V^{\langle g \rangle} \neq \{0\}$ and $W^{\langle g \rangle} \neq \{0\}$. In particular this implies that if P acts freely on X, then it acts freely on $S(V) \times S(W)$. However it is shown in [26], page 486 that this is impossible for a non-metacyclic p-group if $p > 3$. $\qquad\square$

We can now state and prove a geometric characterization of rank two p-groups (for $p > 3$) which naturally extends the classical rank one situation, and which

rather surprisingly has been a conjecture until now:

Theorem 3.10. *Let $p > 3$ be a prime number. Then a p-group P acts freely and smoothly on $M = \mathbb{S}^m \times \mathbb{S}^n$ for some $m, n > 0$ if and only if P does not contain $\mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$ as a subgroup.*

Proof. We have known for decades that $(\mathbb{Z}/p)^3$ does not act freely on a product of two spheres (see [16]). All we need to do is construct the actions. By Proposition 3.7, we know that P is either metacyclic, or of exceptional type and having fixity equal to one. By Lemma 3.8, the metacyclic case is taken care of; by Theorems 2.11 and 3.1, the rank two p-groups of exceptional type will all act freely and smoothly on $\mathbb{S}^{2p-1} \times \mathbb{S}^{4p-5}$. $\qquad\square$

The situation at the 'small primes' $p = 2$ and $p = 3$ is still unresolved. It can be shown that for $n > 4$, there exist exactly two 3-groups of order 3^n which are neither of fixity one, nor act freely on a product of two linear spheres. At the prime 2 the group theory is considerably more complicated. We briefly summarize recent work by Ö. Ünlü in this direction (see [31]). The smallest example of a rank two 2-group of fixity greater than one that does not act freely on a product of two linear spheres is the extraspecial 2-group $Q_8 * D_8 = Q_8 \times D_8/\Delta$, where Δ is the diagonal central subgroup of order 2. Note that $Q_8 * D_8$ is of order 32 and does not contain a maximal subgroup which is abelian. In fact one can show that a rank two 2-group P does not contain $Q_8 * D_8$ if and only if it either has fixity equal to one or acts freely on a product of two linear spheres. Hence all such groups act freely and smoothly on a product of two spheres. In order to handle this particular group, we need the notion of *quaternionic fixity*, expressed in terms of representations $G \subset Sp(n)$. In the case of $Q_8 * D_8$, it embeds in $Sp(2)$, and twice the associated bundle is seen to split, hence producing a free and smooth action of this group on $\mathbb{S}^7 \times \mathbb{S}^7$. Using these techniques, Ünlü has shown that among the 396 groups of rank equal to two, and having order which divides 256, there is exactly **one** group which is not known to act freely and smoothly on a product of two spheres. Verifying this requires using the computer algebra program MAGMA.

Remark 3.11. It seems worthwhile to compare our results with those obtained in [20] and [35] for free p-group actions on a product of two equidimensional spheres. From their work we know for example that the extraspecial p-group of order p^3 and exponent p (p an odd prime) cannot act freely on any $\mathbb{S}^n \times \mathbb{S}^n$. An interesting related problem is determining general restrictions on n and m imposed by the existence of a free P-action on $\mathbb{S}^n \times \mathbb{S}^m$. Our construction produces a *homogeneous system of parameters* $\{\alpha, \beta\}$ for $H^*(P, \mathbb{F}_p)$ such that $|\alpha| = n + 1$ and $|\beta| = m + 1$ (see [2], page 419). This is a significant cohomological restriction, which deserves further consideration.

For fixity equal to two we can apply the results in [2] to obtain a free action on a finite complex, although not necessarily a manifold. As a consequence we will obtain new examples of free actions for rank three groups.

Theorem 3.12. *If $G \subset U(n)$ is of fixity equal to two, then G acts freely on a finite complex $X \simeq \mathbb{S}^{2n-1} \times \mathbb{S}^{4n-5} \times \mathbb{S}^M$ for some $M > 0$.*

Proof. Consider the G-action on $Y = U(n)/U(n-2)$; by our hypothesis it has rank one isotropy. Using the same approach as in Theorem 3.1 we obtain a G-action on $Y' = \mathbb{S}^{2n-1} \times \mathbb{S}^{4n-5}$ with the same isotropy, hence of rank equal to one. Applying 1.4 in [2] we obtain that G acts freely on a finite complex $X \simeq Y' \times \mathbb{S}^M$ for some $M > 0$, which completes the proof. $\square$

Corollary 3.13. *If $G \subset SU(4)$ then G acts freely on a finite complex $X \simeq \mathbb{S}^7 \times \mathbb{S}^{11} \times \mathbb{S}^M$ for some $M > 0$.*

Proof. It suffices to observe that if $G \subset SU(4)$, then it will act on $SU(4)/SU(2)$ with isotropy of rank at most one. $\square$

Example 3.14. We are of course interested in groups of rank equal to three and of fixity equal to two. The list of finite linear subgroups in [15] yields two interesting examples in $SU(4)$. Let $T = \mathrm{SL}_2(\mathbb{F}_3) * \mathrm{SL}_2(\mathbb{F}_3) = \mathrm{SL}_2(\mathbb{F}_3) \times \mathrm{SL}_2(\mathbb{F}_3)/\Delta$, where $\Delta \cong \mathbb{Z}/2$ is the diagonal subgroup of order two. Note that $|T| = 2^5 \cdot 3^2$ and that its 2-Sylow subgroup is the central product $Q_8 * Q_8$, which is an extra-special 2-group of order 32 and has rank equal to three. The second example is $G = \mathrm{Sp}_4(\mathbb{F}_3)$, in this case $|G| = 2^6 \cdot 3^4 \cdot 5$. It has 2-rank and 3-rank both equal to three. Hence we obtain

Corollary 3.15. *The groups $\mathrm{SL}_2(\mathbb{F}_3) * \mathrm{SL}_2(\mathbb{F}_3)$ and $\mathrm{Sp}_4(\mathbb{F}_3)$ act freely on a finite complex of the form $X \simeq \mathbb{S}^7 \times \mathbb{S}^{11} \times \mathbb{S}^M$ for some $M > 0$.*

Example 3.16. From the Atlas [10] we see that there is an embedding of $G = \mathrm{PSL}_2(\mathbb{F}_8)$ in $SU(7)$. Note that $|G| = 2^3 \cdot 3^2 \cdot 7$ and that the 3-Sylow subgroup is cyclic while the 2-Sylow subgroup S is elementary abelian. From the character table in the Atlas we can infer that this 7-dimensional representation V restricts to the reduced regular representation on the subgroup S. Hence on any rank two subgroup $E \subset S$, $V_{|_E} \cong I \oplus I \oplus \mathbb{C}$, where I is the 3-dimensional reduced regular representation. This representation clearly cannot be conjugated into the subgroup $SU(5) \subset SU(7)$ defined as before by extending matrices in $SU(5)$ by the 2×2 identity matrix. Hence E acts with cyclic isotropy on $SU(7)/SU(5)$. Applying the techniques outlined above, we infer that $G = \mathrm{PSL}_2(\mathbb{F}_8)$ acts freely on a finite complex $X \simeq \mathbb{S}^{13} \times \mathbb{S}^{23} \times \mathbb{S}^M$ for some $M > 0$. Note that in this example G acts freely on $SU(7)/SU(3)$ but not on $SU(7)/SU(4)$, hence $G \subset SU(7)$ has fixity equal to three; nevertheless our methods can be applied to this representation.

More generally if G acts on $U(n)/U(n-f)$ with periodic isotropy, the results in [2] establish the existence of a free action on a finite complex $X \simeq U(n)/U(n-f) \times \mathbb{S}^M$ for some large M.

In the next section we use surgery theory (propagation of group actions) to construct a free action of a subgroup $G \subset U(n)$ of fixity f on a product of $f + 1$ spheres provided the order of G is prime to $(n-1)!$. If G is such a group and has fixity one, these techniques provide a stronger result than Theorem 3.1 in the sense that the action is on a lower-dimensional product of spheres.

4. Propagating group actions

In this section we will show that a free action of a group of order prime to $(n-1)!$ on a Stiefel manifold $U(n)/U(k)$ propagates to a free action on a product of spheres. To begin we recall the integral cohomology of Stiefel manifolds.

Proposition 4.1. *The Stiefel manifold $U(n)/U(k)$ has the same integral cohomology as a product $\mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1}$ of $n - k$ spheres.*

Proof. We will prove this by downward induction on k; clearly it is true for k equal to $n - 1$. Consider the fibration $\mathbb{S}^{2k+1} \to U(n)/U(k) \to U(n)/U(k + 1)$; the Euler class must be zero since, using the inductive hypothesis, it sits in a zero group. Hence the associated Gysin sequence breaks into a sequence of short exact sequences. We infer that the cohomology of $U(n)/U(k)$ is an exterior algebra on generators in the desired dimensions. $\qquad\square$

One would not expect the Stiefel manifold to have the same homotopy type as a product of spheres, since Bott periodicity forces a regularity on the homotopy groups of Stiefel manifolds which is absent from the homotopy groups of a product of spheres. However, the differences only involve primes less than n. The next several results illustrate these phenomena.

Proposition 4.2. $\pi_{2n}(U(n)) \cong \mathbb{Z}/n!$ *and the generator is given by the characteristic element of the fibration $U(n) \to U(n+1) \to S^{2n+1}$.*

Proof. This result is due to Bott [5], however we will sketch a proof which uses only stable Bott periodicity. Note that $\pi_{2n}(U(n+1)) = \pi_{2n}(U) = 0$ and $\pi_{2n+1}(U(n+1)) = \pi_{2n+1}(U) = \mathbb{Z}$ (these groups are in the stable range), hence we have an exact sequence

$$\pi_{2n+1}(U(n+1)) \xrightarrow{p_*} \pi_{2n+1}(\mathbb{S}^{2n+1}) \xrightarrow{\Delta_*} \pi_{2n}(U(n)) \to 0.$$

Recall (see [34], page 206) that the characteristic element of the fibration above is given by $\Delta_*(\iota_{2n+1}) \in \pi_{2n}(U(n))$, where ι_{2n+1} is the canonical generator for $\pi_{2n+1}(\mathbb{S}^{2n+1})$. Hence $\pi_{2n}(U(n))$ is cyclic with generator the characteristic element.

The homomorphism p_* can be identified with taking the Euler class of the corresponding $(n+1)$-dimensional complex bundle over $\mathbb{S}^{2n+2}$, and hence with the top Chern class c_{n+1}. But the Chern classes which arise from bundles over $\mathbb{S}^{2n+2}$ are precisely the multiples of $n!$ (see [19], page 280). The result follows. $\square$

The following technical proposition will be a key ingredient in our construction of group actions.

Proposition 4.3. *Given integers $n > k$, there is a map*

$$\mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1} \to U(n)/U(k)$$

which induces an isomorphism in homology with coefficients in $\mathbb{Z}[\frac{1}{(n-1)!}]$.

Proof. Consider the fiber bundle

$$U(n-1)/U(k) \to U(n)/U(k) \to \mathbb{S}^{2n-1}$$

with structure group $U(n-1)$. By Proposition 4.1 (or a Wang sequence)

$$H^*(U(n)/U(k); \mathbb{Z}) = H^*(\mathbb{S}^{2n-1}; \mathbb{Z}) \otimes H^*(U(n-1)/U(k); \mathbb{Z}).$$

By Proposition 4.2 the fiber bundle is classified by the homotopy class $\mathbb{S}^{2n-1} \to BU(n-1)$ representing the generator of $\pi_{2n-1}(BU(n-1)) = \mathbb{Z}/(n-1)!$. Hence if we take a map $g : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ of degree $(n-1)!$ then the induced bundle with fiber $U(n-1)/U(k)$ will be trivial, with total space E homeomorphic to $\mathbb{S}^{2n-1} \times U(n-1)/U(k)$; note that it comes equipped with a map $E \to U(n)/U(k)$ which induces a $\mathbb{Z}[\frac{1}{(n-1)!}]$ homology equivalence. Using downward induction on n we can easily obtain the desired map. $\square$

Hence the Stiefel manifold $U(n)/U(k)$ has the same integral cohomology as a product of spheres, and after inverting $(n-1)!$, we can realize this isomorphism by a map from the product of spheres to the Stiefel manifold. Under these conditions, if we have a free action of a group on $U(n)/U(k)$ and if the order of the group is prime to $(n-1)!$, then the action will "propagate" to a free action on an actual product of spheres.

Theorem 4.4.   *Let G denote a finite subgroup of $U(n)$ which acts freely on $U(n)/U(k)$ with $k \geq 1$. If the order of G is prime to $(n-1)!$, then G acts freely and smoothly on $\mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1}$.*

Remark 4.5. In the proof of the theorem we use the fact that $U(n)/U(k)$ is simply-connected when $k \geq 1$. It seems likely that Theorem 4.4 is true in the non-simply-connected case $G \subset U(n)$ ($k = 0$), but the surgery theoretic complications are considerable, and we will not consider them here. However our methods will still allow us to construct a free G-action on a finite complex with the homotopy type of a product of n spheres (see Corollary 4.13).

The proof of Theorem 4.4 uses the method of propagation of group actions. We refer to [8] for the basic technique, to [1] for a short survey, to [12] for a nice application, to [11] for a key lemma, and to [13] for definitive statements of results. The philosophy is that if two manifolds (e.g. the Stiefel manifold and the product of spheres) resemble each other homologically at the order of a finite group G, then their behavior with respect to G-actions should be similar. To get actions on manifolds, the methods are surgery theoretic and the technical details can be formidable.

Definition 4.6. A G-action on Y *propagates across a map* $f : X \to Y$ if there is a G-action on X and an equivariant map homotopic to f.

Definition 4.7. A G-action on a space X is *homologically trivial* if the induced action on $H_*(X; \mathbb{Z}[1/q])$ is trivial, where q is the order of G.

The main theorem involving propagation is stated below. We will spend the rest of the section defining terms in its statement, outlining the proof, and applying it to the case of interest (Theorem 4.4). Most of the theorem below is due to Cappell–Weinberger [8], however the general case is due to Davis–Löffler [11]. We outline the proof since that will make application of the theorem easier, and also because the full statement is not easy to find in the literature.

Theorem 4.8. *Let $f : X \to Y$ be a map between simply-connected spaces having the homotopy type of CW-complexes. Let G be a group of order q acting freely and homologically trivially on Y. Consider the following conditions:*
(1) *f is a $\mathbb{Z}_{(q)}$-equivalence:*
$$f_* : H_*(X; \mathbb{Z}_{(q)}) \xrightarrow{\cong} H_*(Y; \mathbb{Z}_{(q)}).$$
(2) *The Swan obstruction vanishes:*
$$\sigma(\chi^{tor}(f)) = 0 \in \widetilde{K}_0(\mathbb{Z}G).$$
(3) *X and Y are closed smooth manifolds of dimension greater than four and the action of G on Y is smooth.*
(4) *The normal invariant of the $\mathbb{Z}_{(q)}$-local homotopy equivalence f is in the image of the transfer map:*
$$\nu_{(q)}(f) \in \operatorname{im}\left(p^* : [Y/G, F/O]_{(q)} \to [Y, F/O]_{(q)}\right).$$

Then
(a) *If (1) holds, there is a CW-complex X' and a homotopy equivalence $h : X' \to X$ so that the G-action on Y propagates across $f \circ h$, with a cellular G-action on X'. Furthermore, the homotopy type of X'/G is uniquely determined.*
(b) *If (1) holds and Y/G and X have the homotopy type of finite-dimensional CW-complexes, there is a finite-dimensional CW-complex X' satisfying the conclusion of (a).*

(c) *If* (1) *and* (2) *hold and* Y/G *and* X *have the homotopy type of finite CW-complexes, there is a finite CW-complex* X' *satisfying the conclusion of* (a).

(d) *If* q *is odd and* (1), (2) *and* (3) *hold, then* X' *can be taken to be a closed smooth manifold.*

(e) *If* q *is odd and* (1), (2) *and* (3) *and* (4) *hold, then the* G-*action on* Y *propagates across* f, *with a smooth action on* X.

Before sketching a proof of Theorem 4.8 we recall some basic background material. The homotopy aspects of propagation of group actions depend on localizing topological spaces (see [17], [6], and [7] for details). Let $R \subset \mathbb{Q}$ be a subring of the rationals. For an abelian group A, let $A_R = A \otimes R$. A homomorphism $A \to B$ is an R-*equivalence* if the induced map $A_R \to B_R$ is an isomorphism. An abelian group A is R-*local* if the map $A \to A_R$ is an isomorphism.

We assume any space discussed below has the homotopy type of a connected CW-complex. A map $X \to Y$ is an R-*equivalence* if the induced map on the fundamental group is an isomorphism and the induced maps on higher homotopy groups $\pi_n, n > 1$ are R-local equivalences. Equivalently, the map is an isomorphism on the fundamental group and the induced map on the homology of the universal covers $H_*(\widetilde{X}) \to H_*(\widetilde{Y})$ is an R-equivalence. A space X is R-local if $\pi_n(X)$ is R-local for all $n > 1$, equivalently $H_*(\widetilde{X})$ is R-local. An R-*localization of* X is a R-equivalence $X \to Y$ where Y is R-local. There are existence and uniqueness theorems for R-localizations of X. Their existence follows by applying the fiberwise localization theorem of Bousfield–Kan [7, p. 40] to $X \to B\pi_1 X$. Their R-localization is functorial on the geometric realizations of simplicial sets and maps. If $f : X \to Y$ and $g : X \to Z$ are two R-localizations of X, then there is a homotopy equivalence $h : Y \to Z$ so that $h \circ f \simeq g$. We will write $X \to X_R$ to denote an R-localization of X.

Let q be a nonzero integer. Let

$$\mathbb{Z}_{(q)} = \mathbb{Z}\left[\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}, \dots\right] \subset \mathbb{Q},$$

where $\{p_1, p_2, p_3, \dots\}$ is the set of primes which do *not* divide q. For a space X, let $X \to X_{(q)}$ and $X \to X[1/q]$ denote the R-localizations of X where R equals $\mathbb{Z}_{(q)}$ and $\mathbb{Z}[1/q]$ respectively. Then X is the homotopy pullback of

$$
\begin{array}{ccc}
 & & X[1/q] \\
 & & \downarrow \\
X_{(q)} & \longrightarrow & X_{\mathbb{Q}},
\end{array}
$$

(i.e. X is homotopy equivalent to what results after converting the vertical map to a fibration and taking the pullback).

S. Weinberger [33] made the following key observation.

Lemma 4.9. *A free cellular* G-*action on a simply-connected CW-complex* X *is*

homologically trivial if and only if $(X/G)[1/q] \simeq X[1/q] \times BG$.

Proof. Suppose a finite group G of order q acts freely on a simply-connected space X. Then $X/G \subset Y = (X/G) \cup e_1^2 \cup \cdots \cup e_k^2$ with Y simply-connected (add 2-cells to kill the fundamental group). Since $H_1(X/G) = G/[G,G]$ is q-torsion, the map α below is an epimorphism:

$$\pi_2 Y \cong H_2 Y \to H_2(Y; \mathbb{Z}[1/q]) \xrightarrow{\alpha} H_2(Y, X/G; \mathbb{Z}[1/q]) \cong \mathbb{Z}[1/q]^k.$$

Let $Z = Y \cup e_1^3 \cup \cdots \cup e_k^3$ where the attaching maps of the 3-cells represent a $\mathbb{Z}[1/q]$-basis of $H_2(Y, X/G; \mathbb{Z}[1/q])$. By means of this "plus" construction we have constructed a map $i : X/G \to Z$ to a simply-connected space inducing an isomorphism on $H_*(\ ; \mathbb{Z}[1/q])$.

Now suppose, in addition, that G acts homologically trivially on X. Then a transfer argument shows that the covering map $\pi : X \to X/G$ induces an isomorphism on $H_*(\ ; \mathbb{Z}[1/q])$, hence so does $i \circ \pi : X \to Z$. Thus $Z[1/q] \simeq X[1/q]$, so we have $\mathbb{Z}[1/q]$-equivalence $X/G \to X[1/q] \times BG$ to a $\mathbb{Z}[1/q]$-space. By uniqueness of localization, there is a homotopy equivalence $(X/G)[1/q] \to X[1/q] \times BG$ as desired.

Conversely if $(X/G)[1/q] \simeq X[1/q] \times BG$, then the G-action on X is homologically trivial since the G-action on $X[1/q] \times EG$ clearly is. $\square$

One could also prove the above lemma by using obstruction theory to show that the fibration

$$X[1/q] \to X[1/q] \times_G EG \to BG$$

is fiber homotopically trivial and noting $X[1/q] \times_G EG \simeq X/G[1/q]$.

Proof of Theorem 4.8 (a). Let X'/G be a CW-complex having the homotopy type of the homotopy pullback of

$$
\begin{array}{ccc}
 & & X[1/q] \times BG \\
 & & \downarrow \\
(Y/G)_{(q)} & \longrightarrow & Y_{\mathbb{Q}} \times BG
\end{array}
$$

where the vertical map is given by applying $f[1/q]$ and then $\mathbb{Q}$-localization, and the horizontal map is provided by applying $\mathbb{Q}$-localization and then Weinberger's Lemma. $\square$

Proof of Theorem 4.8 (b). We will use the criterion of Wall [32, Thm. E] which says that a CW-complex Z has the homotopy type of an complex of dimension N if and only if $H^i(\widetilde{Z}; \mathbb{Z}) = 0$ for all $i > N$ and $H^{N+1}(Z; M) = 0$ for all local coefficient systems M.

Recall that both Y/G and X are assumed to have the homotopy type of finite dimensional CW-complexes; let N be greater than or equal to both dimensions.

Let X'/G be a CW-complex produced by Theorem 4.8 (a). Then for $i > N$, $H^i(\widetilde{X'/G}; \mathbb{Z}) = H^i(X; \mathbb{Z}) = 0$. Let M be a local coefficient system for X'/G. For a prime p dividing the order of G, $H^{N+1}(X'/G; M)_{(p)}$ is isomorphic to

$$H^{N+1}((X'/G)_{(p)}; M_{(p)}) \cong H^{N+1}((Y/G)_{(p)}; M_{(p)}) \cong H^{N+1}(Y/G; M)_{(p)} = 0.$$

Likewise for a prime p not dividing the order of G, a transfer argument shows

$$
\begin{aligned}
H^{N+1}(X'/G; M)_{(p)} &\cong H^{N+1}((X'/G)_{(p)}; M_{(p)}) \\
&\cong H^{N+1}(X'_{(p)}; M_{(p)}) \cong H^{N+1}(X; M)_{(p)} = 0.
\end{aligned}
$$

Hence $H^{N+1}(X'/G; M) = 0$. Thus X'/G has the homotopy type of an N-complex. $\qquad\square$

This leads to a nontrivial result.

Corollary 4.10. *Let G denote a finite subgroup of $U(n)$ which acts freely on $U(n)/U(k)$ with $k \geq 1$. Suppose the order of G is prime to $(n-1)!$. Then G acts freely on a finite dimensional CW-complex having the homotopy type of $\mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1}$.*

It is traditional in the study of actions on products of spheres to be satisfied with an action on a finite-dimensional complex which has the homotopy type of a product of spheres. But that is not good enough for us; we won't stop until we have constructed an action on the product of spheres themselves! Next we will try and improve from a finite-dimensional complex to a finite complex.

Wall [32] showed that a connected CW-complex W has the homotopy type of a finite CW-complex if and only if $\pi_1(W)$ is finitely presented and $C_*(\widetilde{W})$ is chain homotopy equivalent to finite free $\mathbb{Z}[\pi_1 W]$-chain complex. If $C_*(\widetilde{W})$ is finitely dominated this is equivalent to the vanishing of the finiteness obstruction $[C_*(\widetilde{W})] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(W)])$.

We recall the definition of the Swan homomorphism appearing in Theorem 4.8. For an integer r relatively prime to $q = |G|$, define the projective $\mathbb{Z}G$-module $P_r = \ker \epsilon : \mathbb{Z}G \to \mathbb{Z}/r$. One can show that $P_r \oplus P_s \cong P_{rs} \oplus \mathbb{Z}G$ and that $P_{1+kq} \cong \mathbb{Z}G$. Hence there is a well-defined *Swan homomorphism* $\sigma : (\mathbb{Z}/q)^\times \to \widetilde{K}_0(\mathbb{Z}G)$, defined by $\sigma([r]) = [P_r]$. Now given a rational homology equivalence $f : X \to Y$, define $\chi^{\mathrm{tor}}(f)$ to be the rational number $\prod_i |H_i(C_f)|^{(-1)^i}$, where C_f is the mapping cone of f. If f is a $\mathbb{Z}_{(q)}$-equivalence, then $\chi^{\mathrm{tor}}(f) \in \mathbb{Z}_{(q)}^\times$ and hence determines an element $[\chi^{\mathrm{tor}}(f)] \in (\mathbb{Z}/q)^\times$.

Proof of Theorem 4.8 (c). By Theorem 4.8 (a) there is a CW-complex Z, a homotopy equivalence $h_Z : Z \to X$, a free, cellular G-action on Z, and an equivariant map $f_Z : Z \to Y$ so that $f_Z \simeq f \circ h_Z$. There is a short exact sequence of chain

complexes

$$0 \to C_*(Y) \to C_*(C_{f_Z}) \to \Sigma C_*(Z) \to 0.$$

Our goal is to show that $C_*(Z)$ is finitely dominated and that the finiteness obstruction $[C_*(Z)] \in \widetilde{K}_0(\mathbb{Z}G)$ vanishes. Since $\oplus H_i(C_{f_Z})$ is finitely generated and each $H_i(C_{f_Z})$ has a finite resolution over $\mathbb{Z}G$, $C_*(C_{f_Z})$ is finitely dominated, as is $C_*(Y)$. Hence $\Sigma C_*(Z)$ is finitely dominated and

$$[C_*(Z)] = -[\Sigma C_*(Z)] = [C_*(Y)] - [C_*(C_{f_Z})] = -\sigma([\chi^{\mathrm{tor}}(f)]).$$

Thus $C_*(Z)$ is chain homotopy equivalent to a finite free $\mathbb{Z}G$ chain complex (equivalently Z/G has the homotopy type of a finite CW-complex) if and only if $\sigma([\chi^{\mathrm{tor}}(f)]) = 0$. $\qquad\square$

Corollary 4.11. *Let G be a subgroup of $U(n)$ which acts freely on $U(n)/U(k)$ with $k \geq 1$ and whose order is prime to $(n-1)!$. Then G acts freely on a finite CW-complex having the homotopy type of $\mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1}$.*

Proof. Here are two properties of χ^{tor}, whose verification is left to the reader.

- $\chi^{\mathrm{tor}}(f \circ g) = \chi^{\mathrm{tor}}(f)\chi^{\mathrm{tor}}(g)$.
- If A_*, B_*, and C_* are graded abelian groups with C_* free, and if $f_* : A_* \to B_*$ is a graded map, then $\chi^{\mathrm{tor}}(f_* \otimes \mathrm{Id}_{C_*}) = \chi^{\mathrm{tor}}(f_*)^{\chi(C_*)}$.

In the proof of Proposition 4.3 we constructed a $\mathbb{Z}_{(q)}$-equivalence

$$f_{n,k} : \mathbb{S}^{2n-1} \times U(n-1)/U(k) \to U(n)/U(k)$$

so that on homology

$$\begin{aligned} f_{n,k_*} = g_* \otimes \mathrm{Id}_* : H_*(\mathbb{S}^{2n-1}) \otimes H_*(U(n-1)/U(k)) \\ \to H_*(\mathbb{S}^{2n-1}) \otimes H_*(U(n-1)/U(k)), \end{aligned}$$

where $g : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ is a map of degree $(n-1)!$. Thus $\chi^{\mathrm{tor}}(f_{n,k}) = ((n-1)!)^0 = 1$.

The $\mathbb{Z}_{(q)}$-equivalence produced by Proposition 4.3

$$f : \mathbb{S}^{2n-1} \times \cdots \times \mathbb{S}^{2k+1} \to U(n)/U(k)$$

is the composite

$$f = (\mathrm{Id}_{\mathbb{S}^{2n-1} \times \cdots \times \mathbb{S}^{2k+5}} \times f_{k+2,k}) \circ \cdots \circ (\mathrm{Id}_{\mathbb{S}^{2n-1}} \times f_{n-1,k}) \circ f_{n,k}$$

and hence $\chi^{\mathrm{tor}}(f) = 1$, and so the result follows from Theorem 4.8(c). $\qquad\square$

Corollary 4.12. *Let G be a subgroup of $U(n)$ which acts on $U(n)/U(k)$ with $k > 0$, having rank one isotropy and whose order is prime to $(n-1)!$. Then G acts freely on a finite CW-complex having the homotopy type of $\mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1} \times \mathbb{S}^M$ for some $M > 1$.*

Proof. By [2], G acts freely on a finite complex X having the homotopy type of $U(n)/U(k) \times \mathbb{S}^M$ for some $M > 1$. One then propagates across the map

$$f \times \mathrm{Id} : \mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1} \times \mathbb{S}^M \to U(n)/U(k) \times \mathbb{S}^M \simeq X. \qquad \square$$

The main application is the following intermediate result in the non-simply connected case:

Corollary 4.13. *If $G \subset U(n)$ and $|G|$ is prime to $(n-1)!$, then G acts freely on a finite complex $X \simeq \mathbb{S}^{2n-1} \times \cdots \times \mathbb{S}^3 \times \mathbb{S}^M$ for some $M > 1$.*

Proof. It suffices to observe that G will act on $U(n)/U(1)$ with periodic isotropy, whence we can apply the previous corollary. $\qquad \square$

We do not give the outline of the surgery theoretic proof of Theorem 4.8(d) and (e), but instead refer to the original source [8] which proves the theorem for p-groups and to [11] which proves the key fact needed to prove the theorem for general groups G.

We are now ready to prove our main theorem by checking the propagation hypotheses.

Proof of Theorem 4.4. Let $X = \mathbb{S}^{2n-1} \times \mathbb{S}^{2n-3} \times \cdots \times \mathbb{S}^{2k+1}$ and $Y = U(n)/U(k)$ and let $f : X \to Y$ be the map produced by Proposition 4.3. We have seen that conditions (1), (2), and (3) of Theorem 4.4 hold; we only need to verify condition (4).

The definition of the local normal invariant $\nu_{(q)}(f) \in [Y, F/O]_{(q)}$ is quite subtle (see [13, p. 12] for a definition), but we will sidestep the subtleties by noting that $[Y, F/O]_{(q)} \to [Y, F/O]_{(0)}$ is injective, since $H_*(Y; \mathbb{Z})$ is torsion-free. Since $F/O \to BO$ is a rational equivalence [22], it suffices to show that the image of the local normal invariant in $[Y, BO]_{(0)}$ is trivial. Its image in $[Y, BO]_{(0)}$ is given by the difference of $((\deg f)(f_{(0)}^{-1})^* \tau_X) - \tau_Y$. But the tangent bundle τ_X is stably trivial, and, according to [29], τ_Y is trivial. $\qquad \square$

Given a group of rank equal to r, a challenging open problem is to construct a free action of G on a finite complex with the homotopy type of a product of r spheres (see [2]). Constructing free actions on actual products of spheres is of course the final goal, and progress on this beyond spherical space forms has been very scant. Indeed, we still cannot verify if every finite group acts freely and homologically trivially on some product of spheres. The following application of Theorem 4.4 provides an infinite number of new examples.

Corollary 4.14. *Let P denote a finite non-abelian p-group with cyclic center and an abelian maximal subgroup. If the rank of P is r and $r < p$, then there exists a*

free, smooth and homologically trivial action of P on $M = \mathbb{S}^{2p-1} \times \cdots \times \mathbb{S}^{2(p-r)+1}$, a product of r spheres.

Proof. Given a group of this type, we know from the results in §2 that there is a faithful irreducible representation $P \subset U(p)$ of fixity equal to $r - 1$. Hence P acts freely on $U(p)/U(p - r)$. Using the propagation theorem, we infer that in fact P must act freely and smoothly on the stated product of spheres. □

Example 4.15. Let P denote any p-group of fixity equal to one, where p is an odd prime. Then P has a faithful p-dimensional representation such that P acts freely on $U(p)/U(p-2)$; hence it will act freely and smoothly on $M = \mathbb{S}^{2p-1} \times \mathbb{S}^{2p-3}$. Note that this provides an action on a lower dimensional manifold than that provided by Theorem 3.1.

Example 4.16. Let $P = \mathbb{Z}/p \wr \mathbb{Z}/p$ (the wreath product); this is a group of order p^{p+1}, with cyclic center and having a maximal subgroup which is abelian, $(\mathbb{Z}/p)^p$. Indeed, there is a natural embedding $P \subset U(p)$, where the elementary abelian subgroup is mapped to the diagonal matrices and the $\mathbb{Z}/p$ action is represented by a permutation matrix. Applying Corollary 4.13, we obtain that $P = \mathbb{Z}/p \wr \mathbb{Z}/p$ acts freely on a finite complex with the homotopy type of a product of p spheres, namely $X \simeq \mathbb{S}^{2p-1} \times \cdots \times \mathbb{S}^3 \times \mathbb{S}^M$ for some $M > 1$.

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