Comment. Math. Helv. 80 (2005), 1–27 **Commentarii Mathematici Helvetici**

Quiver varieties of type A

Andrea Maffei

Abstract. We prove a conjecture of Nakajima describing the relation between quiver varieties of type A and the geometry of partial flag varieties and of the nilpotent variety.

The kind of quiver varieties we are interested in were introduced by Nakajima as a generalization of the description of the moduli space of anti-self-dual connections on ALE spaces constructed by Kronheimer and Nakajima [4]. They turn out to have a rich and interesting geometry and they were used by Nakajima to give a geometric construction of the representations of Kac–Moody algebras [7], [8]. A similar construction had already been made in the case of $\mathfrak{sl}_n$ by Ginzburg (see [1]) using partial flag varieties. A precise conjecture of Nakajima [7, §8] describes the relation between quiver varieties and the geometry of the nilpotent cone and of partial flag varieties: in particular Slodowy's varieties and Slodowy's transversal slices (see Theorem 8 for the precise statement). In this paper this conjecture is proved.

In the first section we state the conjecture and we recall the definition of quiver variety and Slodowy's variety. In order to treat conveniently the coordinate ring of the affine quiver variety M_0, in the second section we introduce a modification of the path algebra that we call the algebra of admissible polynomials. In the third section we define a map between a quiver variety and the related Slodowy's variety and in the fourth section we prove that it is an isomorphism.

I wish to thank Corrado De Concini who introduced me to quiver varieties and Hiraku Nakajima who pointed out an error in a previous version of this paper and the solution to it. Finally I am grateful to Nicoletta Cantarini and Rocco Chirivì for their helpful suggestions on the exposition of the paper.

1. The conjecture of Nakajima describing quiver varieties of type A

1.1. Slodowy's varieties. In this section we recall some definitions related to the nilpotent variety and the partial flag variety. Standard references for the material contained in this section are [1, Sections 3.2, 3.5, 3.7, 4.1 and 4.4] and [9, Sections 5.3 and 7.4].

Definition 1. If E is a vector space of dimension N the *nilpotent cone* $\mathcal{N} = \mathcal{N}_E$ is the variety of nilpotent elements in $\mathfrak{gl}(E)$. Following the standard procedure, we parameterize the $\mathrm{GL}(E)$ orbits in $\mathcal{N}$ through the partitions of N by associating to a nilpotent endomorphism the dimensions of its Jordan blocks, and we denote by the orbit associated to the partition λ by O_λ. Given $x \in \mathcal{N}$ and x, y, h an $\mathfrak{sl}_2$-triple in $\mathfrak{gl}(E)$, we define the *Slodowy's transversal slice* of the orbit of x in the point x as the variety:

$$\mathcal{S}_x = \{u \in \mathcal{N} \text{ such that } [u - x, y] = 0\}.$$

From now on, using a nonstandard convention, we admit $0, 0, 0$ as an $\mathfrak{sl}_2$-triple, so that when $x = 0$ we have $\mathcal{S}_0 = \mathcal{N}$. Notice also that up to isomorphism of algebraic varieties $\mathcal{S}_x$ does not depend on the choice of y, h.

Definition 2. If E is a vector space of dimension N and $r = (r_1, \dots, r_n)$ is a vector of nonnegative integers such that $r_1 + \cdots + r_n = N$, a *partial flag* of E of type r is an increasing sequence $F : \{0\} = F_0 \subset F_1 \subset \cdots \subset F_n = E$ of subspaces of E such that $\dim F_i - \dim F_{i-1} = r_i$. The $\mathrm{GL}(E)$-homogeneous variety of partial flags of type r will be denoted by $\mathcal{F}_r$. Let us define

$$\widetilde{\mathcal{N}}_r = T^*\mathcal{F}_r \cong \{(u, F) \in \mathfrak{gl}(E) \times \mathcal{F}_r \text{ such that } u(F_i) \subset F_{i-1}\},$$
$$\mu_r : \widetilde{\mathcal{N}}_r \longrightarrow \mathcal{N} \text{ the projection onto the first factor.}$$

For N, r, E as above let $\rho = (\rho_1, \rho_2, \dots, \rho_n)$ be the permutation of r such that $\rho_1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_n$ and consider the corresponding partition of N, $\lambda_r := 1^{\rho_1-\rho_2} 2^{\rho_2-\rho_3} \dots n^{\rho_n}$. It is known that if (u, F) is in $\widetilde{\mathcal{N}}_r$ then u is in the closure of O_{λ_r}. Moreover the map

$$\mu_r : \widetilde{\mathcal{N}}_r \longrightarrow \bar{O}_{\lambda_r}$$

is a resolution of singularity and it is an isomorphism over O_{λ_r}. Let us define

$$\mathcal{S}_{r,x} = \mathcal{S}_x \cap \bar{O}_{\lambda_r}, \quad \widetilde{\mathcal{S}}_{r,x} = \mu_r^{-1}(\mathcal{S}_{r,x}).$$

We call $\widetilde{\mathcal{S}}_{r,x}$ the *Slodowy's variety* associated to r and x.

It is convenient for our purposes to define these varieties also in the case of vectors r allowing negative coefficients. Therefore we set $\mathcal{S}_{r,x} = \widetilde{\mathcal{S}}_{r,x} = \varnothing$ when $r_i < 0$ for some i.

The following result is well known (see [1, Corollaries 3.5.9 and 3.7.15] or the original proofs in [9, Sections 5.3 and 7.4]).

Proposition 3. *Let $x \in \mathcal{N}_E$ be a nilpotent endomorphism with Jordan decomposition of type $1^{d_1} 2^{d_2} \dots (n-1)^{d_{n-1}}$ and let $r = (r_1, \dots, r_n)$ be a partition of N. Then:*

1) $\widetilde{\mathcal{S}}_{r,x} \neq \varnothing \iff x \in \bar{O}_{\lambda_r} \iff$ *for all* $1 \leqslant k \leqslant n$ *and for all* $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ *the following inequality holds:*

$$d_1 + 2d_2 + \cdots + (k-1)d_{k-1} + kd_k + kd_{k+1} \cdots + kd_{n-1} \geqslant r_{i_1} + \cdots + r_{i_k}. \quad (1)$$

2) *Any* $\widetilde{\mathcal{S}}_{r,x} \neq \varnothing$ *is a smooth variety of dimension* $\dim Z_{\mathfrak{gl}}(x) - \dim Z_{\mathfrak{gl}}(u_r)$*, where* u_r *is an element of* O_{λ_r}*.*

1.2. Notation on root system and dimension vectors. In the following we will denote by $v = (v_1, \dots, v_{n-1})$ and $d = (d_1, \dots, d_{n-1})$ vectors of integers. When v and d are vectors of nonnegative integers V_i and D_i will be complex vector spaces of dimension v_i and d_i for $i = 1, \dots, n-1$.

Some formulas become simpler and more familiar if we identify the two vectors d and v with elements of the weight and root lattices.

Let P be the weight lattice for the root system of type A_{n-1}, $R \subset P$ be the root lattice and $\boldsymbol{\kappa}(\cdot\,;\cdot)\colon P \times P \longrightarrow \mathbb{Q}$ be the Killing form. Let us fix a basis $\alpha_1, \dots, \alpha_{n-1} \in R$ of simple roots with the usual order and let $\omega_1, \dots, \omega_{n-1}$ be the corresponding fundamental weights.

In the following we will identify the vector $d = (d_1, \dots, d_{n-1})$ (or $\tilde{d}$, d', etc.) with the element $\sum_{i=1}^{n-1} d_i\omega_i$ of the weight lattice and the vector $v = (v_1, \dots, v_{n-1})$ (or $\tilde{v}$, v', etc.) with the element $\sum_{i=1}^{n-1} v_i\alpha_i$ of the root lattice.

Observe that these identifications allow us to introduce the *dominant order* $\preccurlyeq$ on the set of vectors d, v and to define an action of the Weyl group S_n of the Dynkin diagram of type A_{n-1} on the dimension vectors d, v.

1.3. Quiver varieties of type A_{n-1}. Let us fix two dimension vector d and v as in 1.2 and assume that $d_i, v_i \geqslant 0$ for all i. Let us choose also vector spaces D_i, V_i such that $\dim D_i = d_i$ and $\dim V_i = v_i$.

The space of "double free representation" of type (D, V) of the quiver of type A_{n-1} is the vector space

$$S(D, V) = \bigoplus_{i=1}^{n-2} \mathrm{Hom}(V_i, V_{i+1}) \oplus \bigoplus_{i=1}^{n-2} \mathrm{Hom}(V_{i+1}, V_i) \oplus \bigoplus_{i=1}^{n-1} \mathrm{Hom}(D_i, V_i) \oplus \bigoplus_{i=1}^{n-1} \mathrm{Hom}(V_i, D_i).$$

Since in what follows we will give explicit constructions it is convenient to fix some notation: for $i = 1, \dots, n-2$ we will denote by A_i an element of $\mathrm{Hom}(V_i, V_{i+1})$ and by B_i an element of $\mathrm{Hom}(V_{i+1}, V_i)$. Besides, for $i = 1, \dots, n-1$ we will also denote by Γ_i an element of $\mathrm{Hom}(D_i, V_i)$ and by Δ_i an element of $\mathrm{Hom}(V_i, D_i)$. Finally we set $A = (A_1, \dots, A_{n-2})$, $B = (B_1, \dots, B_{n-2})$, $\Gamma = (\Gamma_1, \dots, \Gamma_{n-1})$ and

$\Delta = (\Delta_1, \dots, \Delta_{n-1})$. Our notation can be summarized in the following diagram:

$$\begin{array}{ccccccccc}
D_1 & & D_2 & & & & D_{n-2} & & D_{n-1} \\
\downarrow{\scriptstyle \Gamma_1} & & \downarrow{\scriptstyle \Gamma_2} & & & & \downarrow{\scriptstyle \Gamma_{n-2}} & & \downarrow{\scriptstyle \Gamma_{n-1}} \\
V_1 & \underset{A_1}{\overset{B_1}{\rightleftarrows}} & V_2 & \underset{A_2}{\overset{B_2}{\rightleftarrows}} & \cdots & \underset{A_{n-3}}{\overset{B_{n-3}}{\rightleftarrows}} & V_{n-2} & \underset{A_{n-2}}{\overset{B_{n-2}}{\rightleftarrows}} & V_{n-1} \\
\downarrow{\scriptstyle \Delta_1} & & \downarrow{\scriptstyle \Delta_2} & & & & \downarrow{\scriptstyle \Delta_{n-2}} & & \downarrow{\scriptstyle \Delta_{n-1}} \\
D_1 & & D_2 & & & & D_{n-2} & & D_{n-1}\,.
\end{array}$$

According to this notation we will refer to an element of $S(D, V)$ as a quadruple (A, B, Γ, Δ).

Definition 4 (Nakajima, [7], [8]). A quadruple (A, B, Γ, Δ) of $S(D, V)$ is said to satisfy the ADHM equations or, equivalently, is said to be *admissible* if it satisfies the following relations:

$$\begin{aligned}
B_1 A_1 &= \Gamma_1 \Delta_1, \\
B_i A_i &= A_{i-1} B_{i-1} + \Gamma_i \Delta_i \quad \text{for } 2 \leqslant i \leqslant n-2, \\
0 &= A_{n-2} B_{n-2} + \Gamma_{n-1} \Delta_{n-1}.
\end{aligned}$$

We denote by $\Lambda(D, V)$ the set of all admissible elements. An admissible element is said to be *stable* if each collection $U = (U_1, \dots, U_{n-1})$ of subspaces of V (i.e. U_i is a linear subspace of V_i for every i) and containing $\operatorname{Im}\Gamma$ (i.e. $\operatorname{Im}\Gamma_i \subset U_i$ for every i) and invariant by the action of A and B (i.e. $A_i(U_i) \subset U_{i+1}$ and $B_i(U_{i+1}) \subset U_i$ for every i) must be equal to V (i.e. $U_i = V_i$ for every i). We denote by $\Lambda^+(D, V)$ the set of stable admissible elements.

Notice that the two groups $\mathrm{GL}(V) = \prod \mathrm{GL}(V_i)$ and $\mathrm{GL}(D) = \prod \mathrm{GL}(D_i)$ act naturally on S: if $g = (g_i) \in \mathrm{GL}(V)$ and $h = (h_i) \in \mathrm{GL}(D)$ then

$$\begin{aligned}
g((A_i), (B_i), (\Gamma_i), (\Delta_i)) &= ((g_{i+1} A_i g_i^{-1}), (g_i B_i g_{i+1}^{-1}), (g_i \Gamma_i), (\Delta_i g_i^{-1})); \\
h((A_i), (B_i), (\Gamma_i), (\Delta_i)) &= ((A_i), (B_i), (\Gamma_i h_i^{-1}), (h_i \Delta_i)).
\end{aligned}$$

Definition 5 (Nakajima, [7], [8], p. 521–522). Observe that $\Lambda(D, V)$ and $\Lambda^+(D, V)$ are invariant with respect to the action of $\mathrm{GL}(V)$ so, following Nakajima, we can define quiver varieties as the categorical quotients (see [6, Definition 0.5]) under the action of the group $\mathrm{GL}(V)$ of the two varieties $\Lambda(D, V)$ and $\Lambda^+(D, V)$:

$$\begin{aligned}
M_0(D, V) &= \Lambda(D, V) /\!/ \mathrm{GL}(V), \ \ p_0 \text{ is the projection from } \Lambda(D, V) \text{ to } M_0(D, V), \\
M(D, V) &= \Lambda^+(D, V) /\!/ \mathrm{GL}(V), \ \ p \text{ is the projection from } \Lambda^+(D, V) \text{ to } M(D, V).
\end{aligned}$$

The existence of the quotients M and M_0 can be obtained, using standard arguments of GIT, (see [6, Ch. 1 Theorem 1.1 and Theorem 1.10] and [8, p. 521–522]). The construction shows also that the map $\pi : M(D, V) \longrightarrow M_0(D, V)$ induced by the inclusion $\Lambda^+(D, V) \subset \Lambda(D, V)$ is projective. Observe that the actions of the two groups $\mathrm{GL}(D)$ and $\mathrm{GL}(V)$ commute, hence the group $\mathrm{GL}(D)$ acts on $M(D, V)$ and $M_0(D, V)$ and the map π is equivariant.

Let us also notice that the quadruple $0 = (0, 0, 0, 0)$ is always admissible and we set $0 := p_0(0, 0, 0, 0) \in M_0(D, V)$. Finally let us denote by $M_1(D, V)$ the image of π with the reduced structure: $M_1(D, V)$ is a closed subvariety of $M_0(D, V)$ since π is projective.

The vector space $S(D, V)$ and the varieties $\Lambda(D, V)$, $\Lambda^+(D, V)$, $M(D, V)$, $M_0(D, V)$ and $M_1(D, V)$ do not depend, up to isomorphism, on the choice of the vector spaces D_i, V_i, hence we will denote them also by $S(d, v)$, $\Lambda(d, v)$, $\Lambda^+(d, v)$, $M(d, v)$, $M_0(d, v)$ and $M_1(d, v)$ (or simply S, Λ, Λ^+, M, M_0, M_1 when d, v are clear from the context).

Moreover, as in the case of Slodowy's varieties it will turn out to be useful to extend the definition of $M(d, v)$, $M_0(d, v)$ and $M_1(d, v)$ also to the case of arbitrary $(n-1)$-tuples of integers: we set $M(d, v) = M_0(d, v) = M_1(d, v) = \varnothing$ if there exists i such that $v_i < 0$ or $d_i < 0$.

Remark 6. In [8] a condition of stability dual to the one given in Definition 4 above was used: an admissible element is called $*$-*stable* if each collection $U = (U_1, \dots, U_{n-1})$ contained in $\ker \Delta$ and invariant under the action of A and B is trivial. The isomorphism between the quiver varieties constructed using this stability condition and the one used in this paper is given by

$$((A_i), (B_i), (\Gamma_i), (\Delta_i)) \longmapsto ((B_i^t), (A_i^t), (\Delta_i^t), (\Gamma_i^t)).$$

As for the varieties $\mathcal{S}_{r,x}$ and $\widetilde{\mathcal{S}}_{r,x}$ we need a criterion in order to understand when $M(d, v)$ is not empty: this is given by Nakajima's construction of the irreducible representation of $\mathfrak{sl}_n$ (see [8, §10]). In order to state it we recall that we identify d (resp. v) with elements of the weight (resp. root) lattice (see 1.2) and we observe that if $\sigma \in S_n$ then $\sigma(v - d) + d$ is in the root lattice.

Lemma 7. *If $\sigma \in S_n$ is such that $\sigma(d - v)$ is dominant and $v' = \sigma(v - d) + d$ then*

$$M(d, v) \neq \varnothing \iff M(d, v') \neq \varnothing \iff v'_i \geqslant 0 \ \textit{for } i = 1, \dots, n-1.$$

Proof. If $v \in \mathbb{N}_{\geqslant 0}^{n-1}$ the result immediately follows by Nakajima's main theorem (see [8, §10 Theorem 10.2]) applied to the case $x = 0$. Indeed Nakajima's theorem implies that $H_{\mathrm{top}}(M(d, v))$ is isomorphic to the weight space of weight $d - v$ of the irreducible representation of $\mathfrak{sl}(n)$ of highest weight d. It is well known (see, for example, Humphreys [2], Chapter 6, §21.3) that this weight space is not zero if and only if $v'_i \geqslant 0$ for every i.

Now suppose that there exists i such that $v_i < 0$. It is enough to prove that there exists j such that $v'_j < 0$. Indeed if $v' \succcurlyeq 0$ (i.e. $v' \in \sum \mathbb{N}_{\geqslant 0}\alpha_i$) we have $v = \sigma^{-1}(v' - d) + d = v' + (d - v') - \sigma^{-1}(d - v') \succcurlyeq 0$ since $u \succcurlyeq \tau u$ for all dominant u and all τ in the Weyl group S_n. □

1.4. Nakajima's conjecture. If $d = (d_1, \dots, d_{n-1})$ and $v = (v_1, \dots, v_{n-1})$ are two $(n-1)$-tuples of integers we define the n-tuple $r = r(d, v) = (r_1, \dots, r_n)$ by setting:

$$r_1 = d_1 + \dots + d_{n-1} - v_1, \quad r_n = v_{n-1}, \text{ and}$$
$$r_i = d_i + \dots + d_{n-1} - v_i + v_{i-1} \quad \text{for } i = 2, \dots, n-1.$$

We observe that $\sum_{i=1}^{n} r_i = N = \sum_{i=1}^{n-1} i d_i$. Moreover we notice that once d is fixed the map r gives a bijection between $(n-1)$-tuples of integers v and n-tuples of integers r such that $\sum r_i = N$. Indeed we have that

$$v_{n-1} = r_n, \quad v_i = r_n + \dots + r_{i+1} - d_{i+1} - 2d_{i+2} \dots - (n-i-1)d_{n-1}$$

for $i = 1, \dots, n-2$. Now we can state the main result of this paper. We recall that we have settled $M(d, v) = M_1(d, v) = \varnothing$ if $v_i < 0$ for some i and $\widetilde{\mathcal{S}}_{r,x} = \mathcal{S}_{r,x} = \varnothing$ if $r_i < 0$ for some i. The following theorem was conjectured by Nakajima in [7].

Theorem 8. *Let $v, d, N, r = r(d, v)$ as above. Let $x \in \mathcal{N}$ be a nilpotent element of type $1^{d_1} \dots (n-1)^{d_{n-1}}$. Then there exist two isomorphisms of algebraic varieties, $\widetilde{\varphi}$ between $M(d, v)$ and $\widetilde{\mathcal{S}}_{r,x}$, and φ_1 between $M_1(d, v)$ and $\mathcal{S}_{r,x}$, such that the following diagram commutes:*

$$\begin{array}{ccc} M(d,v) & \xrightarrow{\widetilde{\varphi}} & \widetilde{\mathcal{S}}_{r,x} \\ \pi\downarrow & & \mu_r\downarrow \\ M_1(d,v) & \xrightarrow{\varphi_1} & \mathcal{S}_{r,x} . \end{array} \tag{2}$$

Moreover φ_1 maps $0 \in M_1(d, v)$ to $x \in \mathcal{S}_{r,x}$.

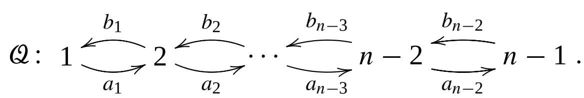
Remark 9. If $M(d, v) \neq \varnothing$ then it is easy to see that $0 \in M_1(d, v)$. This will be also obtained as a consequence of the proof of Theorem 8.

Remark 10. Let Λ_{reg} be the open (and possibly empty) subset of Λ consisting of elements with closed orbit and trivial $\mathrm{GL}(V)$-stabilizer. If $\Lambda_{\text{reg}} \neq \varnothing$ we know by [7, Theorem 4.1] that π is a resolution of singularities so that $M_0 = M_1$ and Theorem 8 above reduces to the conjecture as stated in [7, §8]. We observe also that by [8, Proposition 10.5 and Theorem 10.2] $\Lambda_{\text{reg}} \neq \varnothing$ if and only if $v \succcurlyeq 0$ and $d - v$ is dominant.

2. Path algebra and admissible polynomials

In this section we define the path algebra. Our path algebra will be a modification of the path algebra of the double quiver of type A which takes into account the presence of the extra vector spaces D_i. This algebra will play an important role in our proof. It will be used as an “universal coordinate ring” for quiver varieties which is independent of the dimension vectors d, v.

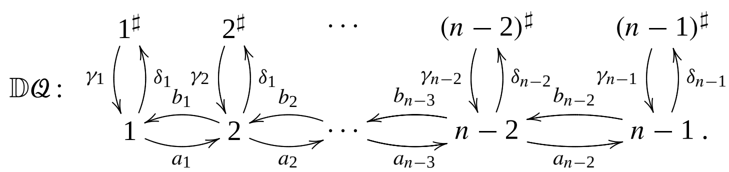
2.1. Notation on quivers and paths. The vertices and the arrows of the *double quiver* $\mathcal{Q}$ of type A_{n-1} will be denoted as indicated in the following diagram:



(3)

In particular, $I = \{1, \dots, n-1\}$ is the set of vertices and $H = \{a_1, \dots, a_{n-2}, b_1, \dots, b_{n-2}\}$ the set of arrows.

It is convenient to consider also the following double version $\mathbb{D}\mathcal{Q}$ of the quiver $\mathcal{Q}$:



(4)

Let us define $I^\sharp = \{1^\sharp, \dots, (n-1)^\sharp\}$ and $H^\sharp = \{\gamma_1, \dots, \gamma_{n-1}, \delta_1, \dots, \delta_{n-1}\}$ and let us denote by $\mathbb{I} = I \cup I^\sharp$ the set of vertices and $\mathbb{H} = H \cup H^\sharp$ the set of arrows of this quiver.

Given an arrow $h \in \mathbb{H}$ we call h_0 its source and h_1 its target. A *path* α in a quiver is a sequence $h^{(m)} \dots h^{(1)}$ of arrows such that $h_1^{(i)} = h_0^{(i+1)}$ for $i = 1, \dots, m-1$. We call $h_0^{(1)}$ the *source* of α and we denote it by α_0 and we call $h_1^{(m)}$ the *target* of α and we denote it by α_1. Moreover we say that the *degree* of α is m and we denote it by $\mathrm{degree}(\alpha)$. If $\alpha_0 = \alpha_1$ we say that α is a closed path. If i is a vertex we define the i-empty path $\varnothing_i$ whose source $(\varnothing_i)_0$ and target $(\varnothing_i)_1$ are equal to i and whose degree is equal to 0. The composition of paths is defined in the obvious way.

We call a path for the quiver $\mathcal{Q}$ a $\mathcal{Q}$-path and a path for the quiver $\mathbb{D}\mathcal{Q}$ a $\mathbb{D}\mathcal{Q}$-path. A $\mathbb{D}\mathcal{Q}$-path α is said to be an *admissible path* if $\alpha_0, \alpha_1 \in I^\sharp$.

2.2. The algebra of admissible polynomials. The *path algebra* of a quiver is the vector space spanned by all paths with the product induced by composition. It is an associative algebra graded by the degree of paths. Consider now the path algebra $\widetilde{\mathcal{R}}$

of the quiver $\mathbb{D}\mathcal{Q}$. For $i \in I$ let θ_i be the following element of $\widetilde{\mathcal{R}}$:

$$\theta_i = \begin{cases} \gamma_1\delta_1 - b_1a_1 & \text{if } i = 1, \\ \gamma_i\delta_i + a_{i-1}b_{i-1} - b_ia_i & \text{if } i = 2, \dots, n-2, \\ \gamma_{n-1}\delta_{n-1} + a_{n-2}b_{n-2} & \text{if } i = n-1. \end{cases}$$

Define $\mathcal{I}$ to be the bilateral ideal of $\widetilde{\mathcal{R}}$ generated by these elements, and $\mathcal{R}'$ to be the quotient algebra $\widetilde{\mathcal{R}}/\mathcal{I}$. If α is a $\mathbb{D}\mathcal{Q}$-path then we denote by $[\alpha]$ its image in $\mathcal{R}'$.

Definition 11. The algebra of *admissible polynomials* is the subalgebra $\mathcal{R}$ of $\mathcal{R}'$ generated by the elements $[\alpha]$ with α an admissible path. Since the ideal $\mathcal{I}$ is homogeneous, $\mathcal{R}'$ and $\mathcal{R}$ can be graded using the degree of paths and we define $\mathcal{R}_m$ as the subspace of $\mathcal{R}$ of homogeneous admissible polynomials of degree m and $\mathcal{R}_+$ as $\bigoplus_{m>0} \mathcal{R}_m$.

If $i, j \in I$ we also set $\mathcal{R}_{i,j} := [\varnothing_{j^\sharp}]\mathcal{R}[\varnothing_{i^\sharp}]$ and we say that an element of $\mathcal{R}_{i,j}$ is an admissible polynomial of type (i, j).

Finally we define some special paths: for $1 \leqslant i \leqslant j \leqslant n-1$ let $\gamma_{j\to i} = b_i \dots b_{j-1}\gamma_j$ and $\delta_{i\to j} = \delta_j a_{j-1} \dots a_i$, and observe that $[\delta_{l\to j}\gamma_{i\to l}] \in \mathcal{R}_{i,j}$.

Lemma 12. *The algebra of admissible paths is generated by the elements* $[\varnothing_{i^\sharp}]$ *for* $i^\sharp \in I^\sharp$ *and by the admissible polynomials in the following set:*

$$\mathcal{P} = \{[\delta_{l\to j}\gamma_{i\to l}] : i, j \in I \text{ and } l \leqslant \min(i, j)\}. \tag{5}$$

Proof. Let $\mathcal{A}$ be the subalgebra of $\mathcal{R}$ generated by $\mathcal{P}$ and by the elements$[\varnothing_{i^\sharp}]$ for $i^\sharp \in I^\sharp$.

Observe first that the algebra $\mathcal{R}$ is generated by the admissible polynomials $[\varnothing_{i^\sharp}]$ with $i^\sharp \in I^\sharp$ and by the admissible polynomials β of the form $[\delta_j\alpha\gamma_i]$ with α a (possibly empty) $\mathcal{Q}$-path. In particular it is enough to show that the admissible polynomials of this form belong to $\mathcal{A}$.

Notice that $[\delta_{i\to i}\gamma_{i\to i}] = [\delta_i\gamma_i]$ hence we can restrict ourself to study the case in which α is not empty. Let $\alpha = h^{(m)} \dots h^{(1)}$ with $h^{(t)} \in H$ for all t. We say that ℓ is a corner of α if there exists t such that $h^{(t)} = a_{\ell-1}$ and $h^{(t+1)} = b_{\ell-1}$. Observe that if α has no corner then β is an element of $\mathcal{P}$. We define $\mathrm{MC}(\alpha)$ to be the maximal corner of α if α has a corner, and 0 if α has no corner. If $\ell = \mathrm{MC}(\alpha)$ we define also $c(\alpha) = \mathrm{card}\{s : h^{(s)} = a_{\ell-1} \text{ and } h^{(s+1)} = b_{\ell-1}\}$.

Now we prove $\beta \in \mathcal{A}$ by induction on $N = \mathrm{degree}(\alpha) + \mathrm{MC}(\alpha) + c(\alpha)$.

Observe that if $N = 0$ then $\alpha = \varnothing_i$ for some i and we have already examined this case.

If $N > 0$ and α has no corner then β is an element of $\mathcal{P}$.

So we can suppose $N > 0$ and α has a corner. Let $\ell = \mathrm{MC}(\alpha)$ and notice that α can be written in the form $\alpha'' b_{\ell-1} a_{\ell-1} \alpha'$. Now using the definition of the ideal $\mathcal{I}$, if $\ell > 1$ we have

$$\begin{aligned}\beta = [\delta_j \alpha \gamma_i] &= [\delta_j \alpha'' b_{\ell-1} a_{\ell-1} \alpha' \gamma_i] \\ &= [\delta_j \alpha'' b_{\ell-2} a_{\ell-2} \alpha' \gamma_i] + [\delta_j \alpha'' \gamma_\ell \delta_\ell \alpha' \gamma_i] \\ &= [\delta_j \alpha'' b_{\ell-2} a_{\ell-2} \alpha' \gamma_i] + [\delta_j \alpha'' \gamma_\ell] \cdot [\delta_\ell \alpha' \gamma_i].\end{aligned}$$

We observe now that the admissible paths $[\delta_j \alpha'' b_{\ell-2} a_{\ell-2} \alpha' \gamma_i]$, $[\delta_j \alpha'' \gamma_\ell]$ and $[\delta_\ell \alpha' \gamma_i]$ belong to $\mathcal{A}$ by the inductive hypothesis.

In the case $\ell = 1$ the argument is the same, but the summand $[\delta_j \alpha'' b_{\ell-2} a_{\ell-2} \alpha' \gamma_i]$ on the right hand side of the formula above is zero. □

We define now an evaluation of paths on S: if $h \in \mathbb{H}$ is an arrow (or an empty path) we define an evaluation of h on an element $s = (A, B, \Gamma, \Delta) \in S(D, V)$ in the following way:

$$h(s) := \begin{cases} \mathrm{id}_{V_i} & \text{if } h = \varnothing_i \text{ and } i \in I, \\ \mathrm{id}_{D_i} & \text{if } h = \varnothing_{i^\sharp} \text{ and } i^\sharp \in I^\sharp, \\ A_i & \text{if } h = a_i, \\ B_i & \text{if } h = b_i, \\ \Gamma_i & \text{if } h = \gamma_i, \\ \Delta_i & \text{if } h = \delta_i. \end{cases}$$

Let us extend this evaluation to all paths using composition: if $\alpha = h^{(m)} \dots h^{(1)}$ then

$$\alpha(s) = h^{(m)}(s) \circ \dots \circ h^{(1)}(s).$$

In particular if $i \leqslant j$ we set $\Gamma_{j \to i} := \gamma_{j \to i}(s)$ and $\Delta_{i \to j} := \delta_{i \to j}(s)$.

Finally observe that if $s \in \Lambda(D, V)$, this evaluation is well defined also on the vector spaces $\mathcal{R}_{i,j}$. In particular if $f \in \mathcal{R}_{i,j}$ then $f(s) \in \mathrm{Hom}(D_i, D_j)$. Moreover we observe that, if $f \in \mathcal{R}_{i,j}$ then $f(g \cdot s) = f(s)$ for all $s \in \Lambda(D, V)$ and for all $g \in \mathrm{GL}(V)$, hence they are well-defined regular functions on the varieties M and M_0.

2.3. The coordinate ring of M_0. The following theorem describes the relation between the path algebra and the coordinate ring of M_0 in the case of the quiver of type A.

Theorem 13 (Lusztig [5], Theorem 1.3). *The ring $\mathbb{C}[S(D, V)]^{\mathrm{GL}(V)}$ is generated by the polynomials*

$$s \longmapsto \varphi(\alpha(s)) \qquad \text{for } \alpha \text{ an admissible path and } \varphi \in (\mathrm{Hom}(D_{\alpha_0}, D_{\alpha_1}))^* \tag{6}$$

and

$$s \longmapsto \mathrm{Tr}(\alpha(s)) \qquad \text{for } \alpha \text{ a closed } \mathcal{Q}\text{-path.} \tag{7}$$

As a consequence the coordinate ring $\mathbb{C}[\Lambda(D,V)]^{\mathrm{GL}(V)}$ of the affine variety M_0 is generated by the same polynomials restricted to $\Lambda(D,V)$. In the case of a quiver of type A one can see that the second type of polynomials are not necessary: indeed, as in proof of Lemma 12, we can show by induction that we can express the second type of polynomials in terms of the first type. The following lemma describes a finite set of generators of the coordinate ring of M_0.

Lemma 14. 1) $\mathbb{C}[\Lambda(D,V)]^{\mathrm{GL}(V)}$ *is generated by the polynomials*

$$s \longmapsto \varphi(\beta(s)) \quad \textit{for } \beta \in \mathcal{P} \textit{ and } \varphi \in (\mathrm{Hom}(D_{\beta_0}, D_{\beta_1}))^*.$$

2) *If* $(A, B, \Gamma, \Delta) \in \Lambda(D,V)$ *then it is an element of* $\Lambda^+(D,V)$ *if and only if for all* $1 \leqslant i \leqslant n-1$ *we have*

$$\mathrm{Im}\, A_{i-1} + \sum_{j=i}^{n-1} \mathrm{Im}\, \Gamma_{j\to i} = V_i.$$

Proof. 1) is a consequence of Lemma 12 and Theorem 13 above. In order to prove 2) let us notice first that the condition of stability is equivalent to

$$\sum_{\substack{\alpha \text{ a } \mathcal{Q}\text{-path} \\ \text{and } \alpha_1 = i}} \mathrm{Im}(\alpha(s)\gamma_{\alpha_0}(s)) = V_i \quad \text{for } i = 0, \dots, n-1.$$

Indeed if U_i is the vector space on the left of the formula then $(U_1, \dots, U_{n-1})$ is the minimal subspace of V containing $\mathrm{Im}\,\Gamma$ and invariant by the action of A and B. The proof can now be completed following the line of the proof of Lemma 12. □

3. Construction of the isomorphism

In this section we will define the maps φ_1 and $\widetilde{\varphi}$ in the case $v_i, d_i \geqslant 0$ for each i. We examine first a simple and already known case of Theorem 8:

Lemma 15 (Nakajima [7]). *If* $N \geqslant v_1 \geqslant \cdots \geqslant v_{n-1}$ *and if* $d = (N, 0, \dots, 0)$ *then the conjecture is true. In this case we have* $M(d,v) \simeq \widetilde{\mathcal{N}_r}$ *and* $M_1(d,v) \simeq \overline{O}_{\lambda_r}$.

Proof. The proof is given in [7], Theorem 7.2, but there Nakajima considers the inverse condition of stability so we remind the definition of the isomorphism in our case. Observe that in this case we have $D = (D_1, 0, \dots, 0)$ and that we can choose the vector space E of Section 1.1 to be D_1. The isomorphism $\widetilde{\varphi}$ between $M(d,v)$ and $T^*\mathcal{F}_r$ is given by:

$$p(A, B, \Gamma, \Delta) \longmapsto (\Delta_1\Gamma_1, \{0\} \subset \ker \Gamma_1 \subset \ker A_1\Gamma_1 \subset \cdots \subset \ker A_{n-1} \dots A_1\Gamma_1).$$

It is easy to check that this map is well defined and that it is bijective so that, since M and $T^*\mathcal{F}$ are complex smooth varieties, it is an algebraic isomorphism.

The map φ_0 between $M_1(d, v)$ and $\bar{O}_{\lambda_r}$ or the map φ_1 between $M_0(d, v)$ and $\mathcal{N}$ is given by $p_0(A, B, \Gamma, \Delta) \mapsto \Delta_1\Gamma_1$. We observe first that $\varphi_1\pi = \mu_r\widetilde{\varphi}$ so that $\varphi_1(M_1) = \bar{O}_{\lambda_r}$. By Lemma 14 we see that the coordinate ring of M_0, hence the coordinate ring of its closed subvariety M_1, is generated by the matrix coefficients of the matrix $\Delta_1\Gamma_1$ hence the map between M_1 and $\mathcal{S}_{r,x} \subset \mathcal{N}$ is a closed immersion. Thus φ_1 is a surjective closed immersion between two affine (reduced) varieties (over an algebraically closed field), hence it is an isomorphism. □

Before giving the proof of the general case I will explain the main steps of the proof:

(1) Given $(n-1)$-tuples d, v of natural numbers and vector spaces D_i, V_i of the dimension prescribed by these natural numbers, we construct new $(n-1)$-tuples of natural number $\tilde{d}$, $\tilde{v}$ and new vector spaces $\widetilde{D}_i$, $\widetilde{V}_i$. In particular the $(n-1)$-tuple $\tilde{d}$ will be of the form of the Lemma 15 above: $\tilde{d} = (N, 0, \dots, 0)$.
(2) We use the Lemma 15 above to give a description of the varieties $\mathcal{S}_{r,x}$ and $\widetilde{\mathcal{S}}_{r,x}$ as subvarieties of $M_1(\tilde{d}, \tilde{v})$ and $M(\tilde{d}, \tilde{v})$.
(3) We construct a subvariety $\mathfrak{T}$ of the variety $\Lambda(\tilde{d}, \tilde{v})$, that we call the set of transversal elements, and we introduce also its open subset $\mathfrak{T}^+ = \Lambda^+(\tilde{d}, \tilde{v}) \cap \mathfrak{T}$. We observe also that the image of $\mathfrak{T}$ (resp. of $\mathfrak{T}^+$) in $M_1(\tilde{d}, \tilde{v})$ (resp. $M(\tilde{d}, \tilde{v})$) is contained in $\mathcal{S}_{r,x}$ (resp. $\widetilde{\mathcal{S}}_{r,x}$).
(4) The main point of the proof is now to prove that this set $\mathfrak{T}$ is isomorphic to $\Lambda(d, v)$: Lemma 18 allows to construct a map from $\Lambda(d, v)$ to $\mathfrak{T}$ and in Lemma 19 we prove that it is an isomorphism. We also observe that this isomorphism sends stable elements in stable elements, so that at the end of this section we are able to introduce maps

$$\phi\colon \Lambda(d, v) \longrightarrow \mathfrak{T} \quad \text{and} \quad \phi^+\colon \Lambda^+(d, v) \longrightarrow \mathfrak{T}^+.$$

(5) We observe that ϕ and ϕ^+ define maps at the level of quiver varieties:

$$\varphi_1\colon M_1(d, v) \longrightarrow \mathcal{S}_{r,x} \subset M_1(\tilde{d}, \tilde{v})$$

and

$$\widetilde{\varphi}\colon M(d, v) \longrightarrow \widetilde{\mathcal{S}}_{r,x} \subset M(\tilde{d}, \tilde{v}).$$

In Lemma 23 we prove that φ_1 is a closed immersion and this allows us to show that the map $\widetilde{\varphi}$ is proper. The injectivity of the map $\widetilde{\varphi}$ follows from Lemma 22.
(6) We conclude the proof as follows: $\widetilde{\varphi}$ is a proper injective map between two smooth complex varieties of the same dimension, moreover it is known that the variety $\widetilde{\mathcal{S}}_{r,x}$ is connected (this can be deduced for example by the Zariski Main Theorem and the normality of the closures of nilpotent orbits proved by Kraft

and Procesi [3], see for example [1], Lemma 4.1.3), so the map $\widetilde{\varphi}$ has to be also surjective, hence it is an isomorphism of algebraic varieties. Finally, since $\widetilde{\varphi}$ is surjective, also φ_1 has to be surjective hence it is a surjective closed immersion between two affine varieties, which implies that φ_1 is an isomorphism.

3.1. Notation and description of Slodowy's varieties as subvarieties of a quiver variety. Let d, v, r, λ_r be as in Theorem 8 and let us define $\tilde{d}_i = 0$ if $i > 1$ and $\tilde{d}_1 = N = \sum_{j=1}^{n-1} jd_j$, $\tilde{v}_i = v_i + \sum_{j=i+1}^{n-1}(j-i)d_j$. We will construct some vector spaces $\widetilde{V}_i, \widetilde{D}_i$ of dimension $\tilde{d}_i$ and $\tilde{v}_i$ in terms of the vector spaces V_i, D_i. Let $D_i^{(j)}$ be an isomorphic copy of D_i and define:

$$E = \widetilde{D}_1 = \bigoplus_{1\leqslant k\leqslant j\leqslant n-1} D_j^{(k)} \quad \text{and} \quad \widetilde{D}_i = 0 \quad \text{for } i = 2,\dots,n-1, \tag{8a}$$

$$\widetilde{V}_i = V_i \oplus \bigoplus_{1\leqslant k\leqslant j-i\leqslant n-i-1} D_j^{(k)} \quad \text{for } i = 2,\dots,n-1. \tag{8b}$$

It will be convenient to set $\widetilde{V}_0 = \widetilde{D}_1$, and, if $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta})$ is an element of $\Lambda(\widetilde{D}, \widetilde{V})$, $\widetilde{A}_0 = \widetilde{\Gamma}_1$, and $\widetilde{B}_0 = \widetilde{\Delta}_1$. We will always consider the maps $\widetilde{A}_i, \widetilde{B}_i$ as block-matrices with respect to the decomposition of $\widetilde{V}_i, \widetilde{D}_1$ given in (8a), (8b) and by a projection on to one of these subspaces, we will always mean a projection with respect to the same decomposition. If $\tilde{s} = (\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) \in S(\widetilde{D}, \widetilde{V})$ we fix the following notation for the blocks of the maps $\widetilde{A}_i$ and $\widetilde{B}_i$ for $i = 0,\dots,n-1$:

$$\begin{aligned}
\pi_{D_j^{(h)}} \widetilde{A}_i\big|_{D_{j'}^{(h')}} &= \mathbb{T}_{i,j,h}^{j',h'} & \pi_{D_j^{(h)}} \widetilde{B}_i\big|_{D_{j'}^{(h')}} &= \mathbb{S}_{i,j,h}^{j',h'}\\
\pi_{D_j^{(h)}} \widetilde{A}_i\big|_{V_i} &= \mathbb{T}_{i,j,h}^{V} & \pi_{D_j^{(h)}} \widetilde{B}_i\big|_{V_{i+1}} &= \mathbb{S}_{i,j,h}^{V}\\
\pi_{V_{i+1}} \widetilde{A}_i\big|_{D_{j'}^{(h')}} &= \mathbb{T}_{i,V}^{j',h'} & \pi_{V_i} \widetilde{B}_i\big|_{D_{j'}^{(h')}} &= \mathbb{S}_{i,V}^{j',h'}\\
\pi_{V_{i+1}} \widetilde{A}_i\big|_{V_i} &= \mathbb{A}_i & \pi_{V_i} \widetilde{B}_i\big|_{V_{i+1}} &= \mathbb{B}_i.
\end{aligned} \tag{9}$$

Whenever we want to stress the dependence on $\tilde{s}$ we will write $\mathbb{T}_*^*(\tilde{s}), \mathbb{S}_*^*(\tilde{s})$, etc.

Let us define the subspaces $D_i' = \bigoplus_{1\leqslant k\leqslant j-i\leqslant n-i-1} D_j^{(k)}$ and observe that for $i = 1,\dots,n-1$ we have $\widetilde{V}_i = V_i \oplus D_i'$ and for $i = 0$ we have $\widetilde{V}_0 = \widetilde{D}_1 = D_0'$. We consider the group $\mathrm{GL}(V)$ as the subgroup of $\mathrm{GL}(\widetilde{V})$ acting as the identity map on D_i' and mapping V_i into V_i.

Now let us now choose the following special $\mathfrak{sl}(2)$-triple $(x_i, y_i, [x_i, y_i])$ of $\mathfrak{sl}(D_i')$:

$$\begin{aligned}
x_i\big|_{D_j^{(1)}} &= 0, & x_i\big|_{D_j^{(h)}} &= \mathrm{id}_{D_j} : D_j^{(h)} \to D_j^{(h-1)},\\
y_i\big|_{D_j^{(j-i)}} &= 0, & y_i\big|_{D_j^{(h)}} &= h(j-i-h)\mathrm{id}_{D_j} : D_j^{(h)} \to D_j^{(h+1)},
\end{aligned} \tag{10}$$

and let us notice that $x = x_0$, $y = y_0$, and $[x, y]$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}(\widetilde{D}_1)$ of the type required in Theorem 8.

Observe that, by Lemma 15, $M(\widetilde{D}, \widetilde{V}) = T^*\mathcal{F}_r$ and $M_1(\widetilde{D}, \widetilde{V}) = \bar{O}_{\lambda_r}$. Hence we can describe $\mathcal{S}_{r,x}$ and $\widetilde{\mathcal{S}}_{r,x}$ as subvarieties of $M_1(\widetilde{D}, \widetilde{V})$ and $M(\widetilde{D}, \widetilde{V})$:

$$\mathcal{S}_{r,x} = p_0\big(\big\{(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) \in \Lambda(\widetilde{D}, \widetilde{V}) \,:\, [\widetilde{\Delta}_1\widetilde{\Gamma}_1 - x, y] = 0\big\}\big) \cap M_1(\widetilde{D}, \widetilde{V}),$$
$$\widetilde{\mathcal{S}}_{r,x} = p\big(\big\{(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) \in \Lambda^+(\widetilde{D}, \widetilde{V}) \,:\, [\widetilde{\Delta}_1\widetilde{\Gamma}_1 - x, y] = 0\big\}\big).$$

3.2. The transversal subvariety. This subsection is devoted to the description of a special subvariety of $\Lambda(\widetilde{D}, \widetilde{V})$. We will first introduce a formal degree of the blocks of our matrices. More precisely we will define two different kinds of degrees, the *grado* and the *degree* and we denote them by grad and by deg respectively:

$$\begin{aligned}
\operatorname{grad}\big(\mathbb{T}^{j',h'}_{i,j,h}\big) &= \min(h - h' + 1, h - h' + 1 + j' - j),\\
\deg\big(\mathbb{T}^{j',h'}_{i,j,h}\big) &= 2h - 2h' + 2 + j' - j,\\
\operatorname{grad}\big(\mathbb{S}^{j',h'}_{i,j,h}\big) &= \min(h - h', h - h' + j' - j),\\
\deg\big(\mathbb{S}^{j',h'}_{i,j,h}\big) &= 2h - 2h' + j' - j.
\end{aligned}$$

Let us recall that for $i = 0, \dots, n-2$ we have defined $D'_i = \bigoplus_{1 \leqslant k \leqslant j-i \leqslant n-i-1} D^{(k)}_j$.

Definition 16. An element $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta})$ of $\Lambda(\widetilde{D}, \widetilde{V})$ is called *transversal* if it satisfies the following relations for $0 \leqslant i \leqslant n-2$:

$$\begin{aligned}
&\mathbb{T}^{j',h'}_{i,j,h} = 0 && \text{if } \operatorname{grad}\big(\mathbb{T}^{j',h'}_{i,j,h}\big) < 0\\
&\mathbb{T}^{j',h'}_{i,j,h} = 0 && \text{if } \operatorname{grad}\big(\mathbb{T}^{j',h'}_{i,j,h}\big) = 0 \text{ and } (j',h') \neq (j,h+1)\\
&\mathbb{T}^{j',h'}_{i,j,h} = \mathrm{id}_{D_j} && \text{if } \operatorname{grad}\big(\mathbb{T}^{j',h'}_{i,j,h}\big) = 0 \text{ and } (j',h') = (j,h+1)\\
&\mathbb{T}^{V}_{i,j,h} = 0 &&\\
&\mathbb{T}^{j',h'}_{i,V} = 0 && \text{if } h' \neq 1\\
&\mathbb{S}^{j',h'}_{i,j,h} = 0 && \text{if } \operatorname{grad}\big(\mathbb{S}^{j',h'}_{i,j,h}\big) < 0\\
&\mathbb{S}^{j',h'}_{i,j,h} = 0 && \text{if } \operatorname{grad}\big(\mathbb{S}^{j',h'}_{i,j,h}\big) = 0 \text{ and } (j',h') \neq (j,h)\\
&\mathbb{S}^{j',h'}_{i,j,h} = \mathrm{id}_{D_j} && \text{if } \operatorname{grad}\big(\mathbb{S}^{j',h'}_{i,j,h}\big) = 0 \text{ and } (j',h') = (j,h)\\
&\mathbb{S}^{V}_{i,j,h} = 0 && \text{if } h \neq j - i\\
&\mathbb{S}^{j',h'}_{i,V} = 0 &&
\end{aligned} \tag{11}$$

and, finally,

$$\left[\pi_{D_i'}\widetilde{B}_i\widetilde{A}_i\big|_{D_i'} - x_i\,,\, y_i\right] = 0. \tag{12}$$

We denote by $\mathfrak{T}$ the set of transversal elements and by $\mathfrak{T}^+$ the set of transversal elements which are additionally stable

Observe that $p(\mathfrak{T}^+) \subset \widetilde{\mathcal{S}}_{r,x}$ and that $p_0(\mathfrak{T}) \cap M_1(\widetilde{D}, \widetilde{V}) \subset \mathcal{S}_{r,x}$. Observe also that $\mathfrak{T}$ and $\mathfrak{T}^+$ are closed $\mathrm{GL}(V)$-invariant subset of $\Lambda(\widetilde{D}, \widetilde{V})$ and $\Lambda^+(\widetilde{D}, \widetilde{V})$ respectively (but they are not $\mathrm{GL}(\widetilde{V})$ invariant).

Before giving the construction of the maps $\varphi, \widetilde{\varphi}$, let us give an example in order to explain the notation and the definitions introduced hitherto. We consider $\widetilde{A}_{n-4} \colon \widetilde{V}_{n-4} \longrightarrow \widetilde{V}_{n-3}$. We have:

$$\begin{aligned}
\widetilde{V}_{n-4} &= V_{n-4} \oplus D^{(1)}_{n-3} \oplus D^{(1)}_{n-2} \oplus D^{(2)}_{n-2} \oplus D^{(1)}_{n-1} \oplus D^{(2)}_{n-1} \oplus D^{(3)}_{n-1};\\
\widetilde{V}_{n-3} &= V_{n-3} \oplus D^{(1)}_{n-2} \oplus D^{(1)}_{n-1} \oplus D^{(2)}_{n-1}.
\end{aligned}$$

Now let us write $\widetilde{A}_{n-4}$ as a block matrix and let us write down the blocks introduced in (9) explicitly:

	V_{n-4}	$D^{(1)}_{n-3}$	$D^{(1)}_{n-2}$	$D^{(2)}_{n-2}$	$D^{(1)}_{n-1}$	$D^{(2)}_{n-1}$	$D^{(3)}_{n-1}$
V_{n-3}	$\mathbb{A}_{n-4}$	$\mathbb{T}^{n-3,1}_{n-4,V}$	$\mathbb{T}^{n-2,1}_{n-4,V}$	$\mathbb{T}^{n-2,2}_{n-4,V}$	$\mathbb{T}^{n-1,1}_{n-4,V}$	$\mathbb{T}^{n-1,2}_{n-4,V}$	$\mathbb{T}^{n-1,3}_{n-4,V}$
$D^{(1)}_{n-2}$	$\mathbb{T}^{V}_{n-4,n-2,1}$	$\mathbb{T}^{n-3,1}_{n-4,n-2,1}$	$\mathbb{T}^{n-2,1}_{n-4,n-2,1}$	$\mathbb{T}^{n-2,2}_{n-4,n-2,1}$	$\mathbb{T}^{n-1,1}_{n-4,n-2,1}$	$\mathbb{T}^{n-1,2}_{n-4,n-2,1}$	$\mathbb{T}^{n-1,3}_{n-4,n-2,1}$
$D^{(1)}_{n-1}$	$\mathbb{T}^{V}_{n-4,n-1,1}$	$\mathbb{T}^{n-3,1}_{n-4,n-1,1}$	$\mathbb{T}^{n-2,1}_{n-4,n-1,1}$	$\mathbb{T}^{n-2,2}_{n-4,n-1,1}$	$\mathbb{T}^{n-1,1}_{n-4,n-1,1}$	$\mathbb{T}^{n-1,2}_{n-4,n-1,1}$	$\mathbb{T}^{n-1,3}_{n-4,n-1,1}$
$D^{(2)}_{n-1}$	$\mathbb{T}^{V}_{n-4,n-1,2}$	$\mathbb{T}^{n-3,1}_{n-4,n-1,2}$	$\mathbb{T}^{n-2,1}_{n-4,n-1,2}$	$\mathbb{T}^{n-2,2}_{n-4,n-1,2}$	$\mathbb{T}^{n-1,1}_{n-4,n-1,2}$	$\mathbb{T}^{n-1,2}_{n-4,n-1,2}$	$\mathbb{T}^{n-1,3}_{n-4,n-1,2}$

(In the matrix above we indicated on the boundary the domain and the codomain of each block). In the following matrix we list the degree and the grado of each block (observe that we have not defined these numbers for the first row and the first column)

grad ; deg	V_{n-4}	$D^{(1)}_{n-3}$	$D^{(1)}_{n-2}$	$D^{(2)}_{n-2}$	$D^{(1)}_{n-1}$	$D^{(2)}_{n-1}$	$D^{(3)}_{n-1}$
V_{n-3}	-	-	-	-	-	-	-
$D^{(1)}_{n-2}$	-	$0\,;\,-1$	$1\,;\,2$	$0\,;\,0$	$1\,;\,3$	$0\,;\,1$	$-1\,;\,-1$
$D^{(1)}_{n-1}$	-	$-1\,;\,-4$	$0\,;\,-1$	$-1\,;\,-3$	$1\,;\,2$	$0\,;\,0$	$-1\,;\,-2$
$D^{(2)}_{n-1}$	-	$0\,;\,-2$	$1\,;\,1$	$0\,;\,-1$	$2\,;\,4$	$1\,;\,2$	$0\,;\,0$

.

Finally we write a matrix satisfying conditions (11) in the definition of a transversal element

	V_{n-4}	$D^{(1)}_{n-3}$	$D^{(1)}_{n-2}$	$D^{(2)}_{n-2}$	$D^{(1)}_{n-1}$	$D^{(2)}_{n-1}$	$D^{(3)}_{n-1}$
V_{n-3}	$*$	$*$	$*$	0	$*$	0	0
$D^{(1)}_{n-2}$	0	0	$*$	$\mathrm{id}_{D_{n-2}}$	$*$	0	0
$D^{(1)}_{n-1}$	0	0	0	0	$*$	$\mathrm{id}_{D_{n-1}}$	0
$D^{(2)}_{n-1}$	0	0	$*$	0	$*$	$*$	$\mathrm{id}_{D_{n-1}}$

3.3. The main construction. We will define the maps $\widetilde{\varphi}$, φ by giving a $\mathrm{GL}(V)$-equivariant map Φ from $\Lambda(D, V)$ to $\mathfrak{T}$. The following two lemmas are the main ingredient in this construction. The proofs of the two lemmas are very similar but in the second lemma we consider a more complicated situation in which the definitions of degree and grado given above play an essential role.

We recall that in case $s = (A, B, \Gamma, \Delta) \in S(D, V)$ and $i \leqslant j$ we have settled $\Gamma_{j \to i} := \gamma_{j \to i}(s)$ and $\Delta_{i \to j} := \delta_{i \to j}(s)$.

Lemma 17. *Let $s = (A, B, \Gamma, \Delta) \in \Lambda(D, V)$ and let $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) \in \mathfrak{T}$ such that*

$$\mathbb{A}_i = A_i, \qquad \mathbb{B}_i = B_i, \tag{13a}$$

$$\mathbb{T}^{i+1,1}_{i,V} = \Gamma_{i+1}, \qquad \mathbb{S}^{V}_{i,i+1,1} = \Delta_{i+1}, \tag{13b}$$

for all $i = 0, \dots, n-2$. Then for all $i = 0, \dots, n-2$ and for all $j > i$ we have

$$\mathbb{T}^{j,1}_{i,V} = \gamma_{j \to i+1}(s), \qquad \mathbb{S}^{V}_{i,j,j-i} = \delta_{i+1 \to j}(s). \tag{13c}$$

Proof. We prove this claim by decreasing induction on i. If $i = n-2$ we have nothing to prove. Let now $0 \leqslant i \leqslant n-3$ and assume that formula (13c) holds for $i+1, \dots, n-2$. Consider the ADHM equation in Definition 5: $\widetilde{A}_i \widetilde{B}_i = \widetilde{B}_{i+1} \widetilde{A}_{i+1}$. Now using the equality $\pi_{V_{i+1}} \widetilde{A}_i \widetilde{B}_i\big|_{D^{(1)}_j} = \pi_{V_{i+1}} \widetilde{B}_{i+1} \widetilde{A}_{i+1}\big|_{D^{(1)}_j}$, relations (11) and induction we obtain

$$\mathbb{T}^{j,1}_{i,V} = B_{i+1} \Gamma_{j \to i+2} = \Gamma_{j \to i+1}.$$

Besides, using the equality $\pi_{D^{(j-i)}_j} \widetilde{A}_i \widetilde{B}_i\big|_{V_{i+1}} = \pi_{D^{(j-i)}_j} \widetilde{B}_{i+1} \widetilde{A}_{i+1}\big|_{V_{i+1}}$, relations (11) and induction we obtain

$$\mathbb{S}^{V}_{i,j,j-i} = \Delta_{i+2 \to j} A_{i+1} = \Delta_{i+1 \to j}$$

proving the thesis. □

Lemma 18. *For a given* $s = (A, B, \Gamma, \Delta) \in \Lambda(D, V)$ *there exists a unique* $\tilde{s} = (\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) \in \mathfrak{T}$ *such that* (13a) *and* (13b) *are satisfied. Moreover there exist homogeneous admissible polynomials* $t^{j',h'}_{i,j,h}$ *and* $s^{j',h'}_{i,j,h}$ *in* $\mathcal{R}_{j',j}$ *such that*

$$\mathbb{T}^{j',h'}_{i,j,h}(\tilde{s}) = t^{j',h'}_{i,j,h}(s) \quad \text{and} \quad \mathbb{S}^{j',h'}_{i,j,h}(\tilde{s}) = s^{j',h'}_{i,j,h}(s) \tag{14}$$

for all $s \in \Lambda(D, V)$. *Finally when the grado of the corresponding block is positive* (*i.e. when* $\operatorname{grad}(\mathbb{T}^{j',h'}_{i,j,h}) > 0$ *or when* $\operatorname{grad}(\mathbb{S}^{j',h'}_{i,j,h}) > 0$) *these polynomials satisfy the following properties:*

(i) $\operatorname{degree}(t^{j',h'}_{i,j,h}) = \deg(\mathbb{T}^{j',h'}_{i,j,h})$ *and* $\operatorname{degree}(s^{j',h'}_{i,j,h}) = \deg(\mathbb{S}^{j',h'}_{i,j,h})$*;*

(ii) $t^{j',h'}_{i,j,h}$ *and* $s^{j',h'}_{i,j,h}$ *can be written in the following form:*

$$t^{j',h'}_{i,j,h} = \lambda^{j',h'}_{i,j,h}[\delta_{\ell\to j}\gamma_{j'\to\ell}] + q^{j',h'}_{i,j,h} \quad \text{and} \quad s^{j',h'}_{i,j,h} = \mu^{j',h'}_{i,j,h}[\delta_{\ell\to j}\gamma_{j'\to\ell}] + p^{j',h'}_{i,j,h}$$

where $\ell = j + h' - h$, q^*_* *and* p^*_* *are homogeneous admissible polynomials in the subalgebra* $\mathcal{R}_\sharp$ *of* $\mathcal{R}$ *generated by* $\mathcal{R}_+ \cdot \mathcal{R}_+$ *and* $\lambda^{j',h'}_{i,j,h}$, $\mu^{j',h'}_{i,j,h}$ *are rational numbers;*

(iii) *if* $h' = 1$, $i+2 \leqslant j' \leqslant n-1$ *and* $1 \leqslant h \leqslant j-i-1 \leqslant n-i-2$ *then* $\lambda^{j',h'}_{i,j,h} > 0$*;*

(iv) *if* $1 < h' \leqslant j' - i - 1 \leqslant n - i - 2$ *and* $1 \leqslant h \leqslant j - i - 1 \leqslant n - i - 2$ *then* $\lambda^{j',h'}_{i,j,h} + \mu^{j',h'-1}_{i,j,h} > 0$*;*

(v) *if* $1 \leqslant h' \leqslant j' - i - 1 \leqslant n - i - 2$, $h = j - i$ *and* $i + 1 \leqslant j \leqslant n - 1$ *then* $\mu^{j',h'}_{i,j,h} > 0$.

Proof. We shall prove that all the blocks of $\widetilde{A}_i$ and $\widetilde{B}_i$ for $i = 0, \dots, n-2$ are uniquely determined and have the required form. We observe first that by Lemma 17 and relations (11) the following relations hold for all i, j, h

$$\mathbb{T}^{V}_{i,j,h} = 0; \qquad \mathbb{S}^{j',h'}_{i,V} = 0;$$

$$\mathbb{T}^{j',h'}_{i,V} = \begin{cases} \gamma_{j'\to i+1}(s) & \text{if } h' = 1, \\ 0 & \text{otherwise;} \end{cases} \qquad \mathbb{S}^{V}_{i,j,h} = \begin{cases} \delta_{i+1\to j}(s) & \text{if } h = j - i, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover if $\operatorname{grad}(\mathbb{T}^{j',h'}_{i,j,h}) \leqslant 0$ (resp. if $\operatorname{grad}(\mathbb{S}^{j',h'}_{i,j,h}) \leqslant 0$) then $\mathbb{T}^{j',h'}_{i,j,h}$ (resp. $\mathbb{S}^{j',h'}_{i,j,h}$) is completely determined by relations (11). We prove the lemma for the remaining cases by decreasing induction on i, giving an inductive formula for the computation of the blocks $\mathbb{T}^{j',h'}_{i,j,h}$, $\mathbb{S}^{j',h'}_{i,j,h}$ in these cases.

Notice that $\widetilde{A}_{n-2}$ and $\widetilde{B}_{n-2}$ are already completely defined by relations (13) and they verify the relation $\widetilde{A}_{n-2}\widetilde{B}_{n-2} = 0$ and $[\pi_{D'_{n-2}}\widetilde{B}_{n-2}\widetilde{A}_{n-2}|_{D'_{n-2}} - x_{n-2}, y_{n-2}] = 0$. Now we assume to have constructed $t^{*,*}_{j,*,*}$ and $s^{*,*}_{j,*,*}$, for $j \geqslant i + 1$, satisfying the properties stated in the lemma and such that the ADHM equations $\widetilde{A}_i\widetilde{B}_i = \widetilde{B}_{i+1}\widetilde{A}_{i+1}, \ldots, \widetilde{A}_{n-3}\widetilde{B}_{n-3} = \widetilde{B}_{n-2}\widetilde{A}_{n-2}$ and relations (12) for $i+1, \ldots, n-2$ are satisfied. We need to prove that there exist unique $\mathbb{T}^{*,*}_{i,*,*}$ and $\mathbb{S}^{*,*}_{i,*,*}$ such that:

$$[\pi_{D'_i}\widetilde{B}_i\widetilde{A}_i\big|_{D'_i} - x_i, y_i] = 0 \quad \text{and} \quad \widetilde{A}_i\widetilde{B}_i = \widetilde{B}_{i+1}\widetilde{A}_{i+1}, \tag{15}$$

and that they have the required form. First we observe that relations (11) and (13) imply the following equations:

$$\pi_{V_{i+1}}\widetilde{A}_i\widetilde{B}_i\big|_{V_{i+1}} = A_iB_i + \Gamma_{i+1}\Delta_{i+1} = B_{i+1}A_{i+1} = \pi_{V_{i+1}}\widetilde{B}_{i+1}\widetilde{A}_{i+1}\big|_{V_{i+1}}$$
$$\pi_{V_{i+1}}\widetilde{A}_i\widetilde{B}_i\big|_{D^{(h)}_j} = \mathfrak{d}_{h,1}\Gamma_{j\to i+1} = \mathfrak{d}_{h,1}B_{i+1}\Gamma_{j\to i+2} = \pi_{V_{i+1}}\widetilde{B}_{i+1}\widetilde{A}_{i+1}\big|_{D^{(h)}_j}$$
$$\pi_{D^{(h)}_j}\widetilde{A}_i\widetilde{B}_i\big|_{V_{i+1}} = \mathfrak{d}_{h,j-i-1}\Delta_{i+1\to j} = \mathfrak{d}_{h,j-i-1}\Delta_{i+2\to j}A_{i+1} = \pi_{D^{(h)}_j}\widetilde{B}_{i+1}\widetilde{A}_{i+1}\big|_{V_{i+1}}$$

($\mathfrak{d}$ is Kronecker's delta). Now we express equations (15) in a more suitable form. We introduce the linear maps $L := \widetilde{B}_{i+1}\widetilde{A}_{i+1}$, $M := \widetilde{A}_i\widetilde{B}_i$ and $N := \pi_{D'_i}\widetilde{B}_i\widetilde{A}_i\big|_{D'_i} - x_i$ and let us define the blocks $L^{j',h'}_{j,h}$, $M^{j',h'}_{j,h}$ and $N^{j',h'}_{j,h}$ analogously to (9):

$$L^{j',h'}_{j,h} := \pi_{D^{(h)}_j}L\big|_{D^{(h')}_{j'}} = \mathbb{S}^V_{i+1,j,h}\mathbb{T}^{j',h'}_{i+1,V} + \sum_{l,m}\mathbb{S}^{l,m}_{i+1,j,h}\mathbb{T}^{j',h'}_{i+1,l,m} \tag{16}$$

and similarly for $M^{j',h'}_{j,h}$ and $N^{j',h'}_{j,h}$. Hence equations (15) can be formulated as follows:

$$M^{j',h'}_{j,h} = L^{j',h'}_{j,h} \tag{17a}$$

for $1 \leqslant h' \leqslant j'-i-1 \leqslant n-i-2$ and $1 \leqslant h \leqslant j-i-1 \leqslant n-i-2$,

$$N^{j',j'-i}_{j,h} = 0 \tag{17b}$$

for $1+i \leqslant j' \leqslant n-1$ and $1 \leqslant h \leqslant j-i-1 \leqslant n-i-2$,

$$N^{j',h'}_{j,1} = 0 \tag{17c}$$

for $1+i \leqslant j \leqslant n-1$ and $2 \leqslant h' \leqslant j'-i \leqslant n-i-1$, and

$$h'(j'-i-h')N^{j',h'+1}_{j,h+1} = h(j-i-h)N^{j',h'}_{j,h} \tag{17d}$$

for $1 \leqslant h' \leqslant j'-i-1 \leqslant n-i-2$ and $1 \leqslant h \leqslant j-i-1 \leqslant n-i-2$.

In order to study these equations we introduce two kinds of degrees, the *grado* that we denote with grad and the *degree* that we denote with deg, as we have done for $\mathbb{T}$ and $\mathbb{S}$:

$$\operatorname{grad}(L^{j',h'}_{j,h}) = \operatorname{grad}(M^{j',h'}_{j,h}) = \operatorname{grad}(N^{j',h'}_{j,h}) = \min(h-h'+1, h-h'+1+j'-j),$$
$$\deg(L^{j',h'}_{j,h}) = \deg(M^{j',h'}_{j,h}) = \deg(N^{j',h'}_{j,h}) = 2h-2h'+2+j'-j.$$

Equations (17) are homogeneous and we will call grado (resp. degree) of each of these equations the grado (resp. the degree) of the blocks involved in the equation.

Observe now that $\min(m-h'+1, m-h'+1+j'-j) + \min(h-m, h-m+l-j) \leqslant \min(h-h'+1, h-h'+1+j'-j)$ and $\min(m-h', m-h'+j'-j) + \min(h-m+1, h-m+1+l-j) \leqslant \min(h-h'+1, h-h'+1+j'-j)$, hence grad and deg behave well under composition:

$$\begin{aligned} \operatorname{grad}(S^{l,m}_{i+1,j,h}) + \operatorname{grad}(T^{j',h'}_{i+1,l,m}) &\leqslant \operatorname{grad}(L^{j',h'}_{j,h}) \\ \deg(S^{l,m}_{i+1,j,h}) + \deg(T^{j',h'}_{i+1,l,m}) &= \deg(L^{j',h'}_{j,h}) \end{aligned} \tag{18a}$$

$$\begin{aligned} \operatorname{grad}(T^{l,m}_{i,j,h}) + \operatorname{grad}(S^{j',h'}_{i,l,m}) &\leqslant \operatorname{grad}(M^{j',h'}_{j,h}) \\ \deg(T^{l,m}_{i,j,h}) + \deg(S^{j',h'}_{i,l,m}) &= \deg(M^{j',h'}_{j,h}) \end{aligned} \tag{18b}$$

$$\begin{aligned} \operatorname{grad}(S^{l,m}_{i,j,h}) + \operatorname{grad}(T^{j',h'}_{i,l,m}) &\leqslant \operatorname{grad}(N^{j',h'}_{j,h}) \\ \deg(S^{l,m}_{i,j,h}) + \deg(T^{j',h'}_{i,l,m}) &= \deg(N^{j',h'}_{j,h}). \end{aligned} \tag{18c}$$

One can check that when the grado of the a block is less than or equal to 0 equations (17a), (17b), (17c) and (17d) are always satisfied independently of the choice of $\mathbb{T}^{*,*}_{i,*,*}$ and $\mathbb{S}^{*,*}_{i,*,*}$. Here we consider just the case of equation (17a). In this case observe first that if $h'=1$ and $h=j-i$ then $\operatorname{grad}(L^{j',h'}_{j,h}) = \min(j-i, j'-i) \geqslant 2$ in the case of equation (17a). Hence if grad $\leqslant 0$ then $\mathbb{S}^{V}_{i+1,j,h}\mathbb{T}^{j',h'}_{i+1,V}$ in the right hand side of formula (16) is always 0.

Now if grad < 0, by relations (18), at least one of the two factors of the summand $\mathbb{S}^{l,m}_{i+1,j,h}\mathbb{T}^{j',h'}_{i+1,l,m}$ in the right hand side of formula (16) has grad less than 0, hence vanishes by relations (11). The same argument applies to the block $M^{j',h'}_{j,h}$ hence if grad < 0 equation (17a) reduces to the identity $0=0$.

In the case grad $= 0$ the same argument together with relations (11) and equation (17a) is equivalent to the following:

$$\sum_{\substack{\operatorname{grad}\mathbb{T}=0 \text{ and } (m,l)=(j',h'-1) \\ \operatorname{grad}\mathbb{S}=0 \text{ and } (m,l)=(j,h)}} \mathbb{S}^{l,m}_{i+1,j,h}\mathbb{T}^{j',h'}_{i+1,l,m} = \sum_{\substack{\operatorname{grad}\mathbb{T}=0 \text{ and } (m,l)=(j,h+1) \\ \operatorname{grad}\mathbb{S}=0 \text{ and } (m,l)=(j',h')}} \mathbb{T}^{l,m}_{i,j,h}\mathbb{S}^{j',h'}_{i,l,m}$$

which reduces to $0 = 0$ if $(j', h') \neq (j, h+1)$ and to $\mathrm{id}_{D_j} = \mathrm{id}_{D_j}$ if $(j', h') = (j, h+1)$.

In particular we observe that in the case of equations (17b) and (17c) the grado is always less than or equal to 0, hence these equations are always satisfied.

Now we study the remaining equations arguing by induction on $d = \mathrm{grad} > 0$. We assume to have constructed $\mathbb{T}^{j',h'}_{i,j,h}$ and $\mathbb{S}^{j',h'}_{i,j,h}$ for the blocks with grad $< d$ such that all the equations (17a) and (17d) with grad $< d$ are satisfied, and we prove that $\mathbb{T}^{j',h'}_{i,j,h}$ and $\mathbb{S}^{j',h'}_{i,j,h}$ for blocks of grad $= d$ are uniquely determined by equations (17a) and (17d) of grad $= d$. We need to solve the following equations:

$$M^{j',h'}_{j,h} = L^{j',h'}_{j,h} \quad \text{and} \quad h'(j'-i-h')N^{j',h'+1}_{j,h+1} = h(j-i-h)N^{j',h'}_{j,h} \qquad (*_d)$$

for $1 \leqslant h' \leqslant j'-i-1 \leqslant n-i-2$ and $1 \leqslant h \leqslant j-i-1 \leqslant n-i-2$ and $\min(h-h'+1, h-h'+1+j'-j) = d > 0$. (The shape of the equation is the same as in (17a) and (17d); what is changed is the range of the indices involved in these equations). By the inductive hypothesis, under this assumptions on j, j', h, h', d the following formulas for the blocks of L, M, N hold:

$$\begin{aligned}
L^{j',h'}_{j,h} &= \nu^{j',h'}_{j,h}\Delta_{\ell\to j}\Gamma_{j'\to\ell} + c^{j',h'}_{j,h}(s),\\
M^{j',h'}_{j,h} &= \mathbb{S}^{j',h'}_{i,j,h+1} + \mathbb{T}^{j',h'}_{i,j,h} + d^{j',h'}_{j,h}(s),\\
N^{j',h'}_{j,h} &= e^{j',h'}_{j,h}(s) + \begin{cases} \mathbb{T}^{j',h'}_{i,j,h} & \text{if } h'=1\\ \mathbb{T}^{j',h'}_{i,j,h} + \mathbb{S}^{j',h'-1}_{i,j,h} & \text{if } 1 < h' \leqslant j'-i-1,\end{cases}\\
N^{j',h'+1}_{j,h+1} &= f^{j',h'}_{j,h}(s) + \begin{cases} \mathbb{S}^{j',h'}_{i,j,h+1} & \text{if } h = j-i-1\\ \mathbb{T}^{j',h'+1}_{i,j,h+1} + \mathbb{S}^{j',h'}_{i,j,h+1} & \text{if } 1 \leqslant h < j-i-1,\end{cases}
\end{aligned}$$

where $\ell = j + h' - h$ and $c^{j',h'}_{j,h}$, $d^{j',h'}_{j,h}$, $e^{j',h'}_{j,h}$, $f^{j',h'}_{j,h}$ are homogeneous admissible polynomials that we already know by induction. In particular these polynomials belong to the subalgebra $\mathcal{R}_\sharp$ of $\mathcal{R}$ generated by $\mathcal{R}_+ \cdot \mathcal{R}_+$ and their degree is equal to the degree (deg) of the corresponding block. Finally ν^*_* is the following coefficient:

$$\nu^{j',h'}_{j,h} = \begin{cases} 1 & \text{if } h'=1 \text{ and } h = j-i-1,\\ \lambda^{j',h'}_{i+1,j,h} & \text{if } h'=1 \text{ and } h < j-i-1,\\ \mu^{j',h'-1}_{i+1,j,h} & \text{if } h = j-i-1 \text{ and } h' > 1,\\ \lambda^{j',h'}_{i+1,j,h} + \mu^{j',h'-1}_{i+1,j,h} & \text{if } h' > 1 \text{ and } h < j-i-1.\end{cases}$$

By the inductive hypothesis we see that ν^*_* is always a positive rational number.

Now we group all the equations with the same j and the same j' together and we solve them. Once we have fixed j and j' we can organize the indeces in a more convenient form. Let $h_1 = j - i - 1$ and

$$h_0 = \begin{cases} d & \text{if } j' \geqslant j \\ d + j - j' & \text{if } j' < j, \end{cases} \qquad k = \begin{cases} 1 - d & \text{if } j' \geqslant j \\ 1 + j' - j - d & \text{if } j' < j. \end{cases}$$

For fixed j, j' (and d) let us also introduce the positive rational numbers $\alpha_h = (h+k)(j' - i - k - h)$, $\beta_h = h(j - i - h)$ and $\nu_h = \nu_{j,h}^{j',h+k}$. Then the equations $(*_d)$ can be written as follows:

$$M_{j,h}^{j',h'} = L_{j,h}^{j',h'} \quad \text{and} \quad \beta_h N_{j,h+1}^{j',h'+1} = \alpha_h N_{j,h}^{j',h'} \\ \text{for } h' = k + h \text{ and } h_0 \leqslant h \leqslant h_1. \tag{$*_{djj'}$}$$

Now let us introduce the following variables: $X_h = \mathbb{T}_{i,j,h}^{j',h'}$ and $Y_h = \mathbb{S}_{i,j,h+1}^{j',h'}$. and let us write system $(*_{djj'})$ in the following way:

$$\begin{aligned} X_{h_0} + Y_{h_0} &= \nu_{h_0} \Delta_{\ell \to j} \Gamma_{j' \to \ell} + p_{1,h_0}(s) \\ &\vdots \\ X_{h_1} + Y_{h_1} &= \nu_{h_1} \Delta_{\ell \to j} \Gamma_{j' \to \ell} + p_{1,h_1}(s) \end{aligned} \tag{19}$$

and

$$\begin{aligned} \alpha_{h_0}(Y_{h_0} + X_{h_0+1}) &= \beta_{h_0} X_{h_0} + p_{2,h_0}(s) \\ \alpha_{h_0+1}(Y_{h_0+1} + X_{h_0+2}) &= \beta_{h_0+1}(Y_{h_0} + X_{h_0+1}) + p_{2,h_0+1}(s) \\ &\vdots \\ \alpha_{h_1-1}(Y_{h_1-1} + X_{h_1}) &= \beta_{h_1-1}(Y_{h_1-2} + X_{h_1-1}) + p_{2,h_1-1}(s) \\ \alpha_{h_1} Y_{h_1} &= \beta_{h_1}(Y_{h_1-1} + X_{h_1}) + p_{2,h_1}(s) \end{aligned} \tag{20}$$

where $\ell = j + k$ and $p_{*,*}$ are known homogeneous admissible polynomials of degree equal to the degree (deg) of the corresponding block and that are elements of $\mathcal{R}_\sharp$.

Observe that the system of equations (20) can be rewritten in the following form:

$$\begin{aligned} Y_{h_0} + X_{h_0+1} &= \rho_{h_0} X_{h_0} + p_{3,h_0}(s) \\ Y_{h_0+1} + X_{h_0+2} &= \rho_{h_0+1} X_{h_0} + p_{3,h_0+1}(s) \\ &\vdots \\ Y_{h_1-1} + X_{h_1} &= \rho_{h_1-1} X_{h_0} + p_{3,h_1-1}(s) \\ Y_{h_1} &= \rho_{h_1} X_{h_0} + p_{3,h_1}(s) \end{aligned} \tag{21}$$

where the ρ_i are (strictly) positive rational numbers and the $p_{3,i}$ are a linear combination of the $p_{2,i}$. Now the system (19,21) is a linear system in as many variables as equations and has a unique solution. Indeed this system can be written as

$$\mathsf{M}\,\mathsf{x} = \mathsf{v}$$

where

$$\begin{aligned}
\mathsf{x} &= (X_{h_0}, \dots, X_{h_1}, Y_{h_0}, \dots, Y_{h_1});\\
\mathsf{v} &= (\nu_{h_0}\Delta_{\ell\to j}\Gamma_{j'\to\ell} + p_{1,h_0}(s), \dots, \nu_{h_1}\Delta_{\ell\to j}\Gamma_{j'\to\ell}\\
&\qquad + p_{1,h_1}(s)), p_{3,h_0}(s), \dots, p_{3,h_1}(s));\\
\mathsf{M} &= \begin{pmatrix} \mathrm{id}_{h_1-h_0+1} & \mathrm{id}_{h_1-h_0+1} \\ \mathsf{N} & \mathrm{id}_{h_1-h_0+1} \end{pmatrix}
\end{aligned}$$

and

$$\mathsf{N} = \begin{pmatrix} -\rho_{h_0} & 1 & 0 & \dots & 0 \\ -\rho_{h_0+1} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -\rho_{h_1-1} & 0 & 0 & \dots & 1 \\ -\rho_{h_1} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Now if we subtract the i-th column to the $i + (h_1 - h_0 + 1)$-th column we obtain that $\det \mathsf{M} = \det(\mathrm{id}_{h_1-h_0+1} - \mathsf{N}) = p_{\mathsf{N}}(1)$, the characteristic polynomial of N evaluated in 1. Observe that $p_{\mathsf{N}}(t) = t^{h_1-h_0} + \rho_{h_0} t^{h_1-h_0} + \dots + \rho_{h_1}$ is a polynomial with strictly positive coefficients, hence $\det \mathsf{M} = p_{\mathsf{N}}(1) \neq 0$. So M is invertible and $\mathsf{x} = \mathsf{M}^{-1}\mathsf{v}$. In particular X_h and Y_h can be expressed as in equations (14) by homogeneous admissible polynomials satisfying properties (i) and (ii).

In order to prove that properties (iii), (iv) and (v) hold we observe that we can use equations (19) and (21) to give an inductive formula for the coefficients $\lambda^{j',h'}_{i,j,h}$ and $\mu^{j',h'}_{i,j,h}$. Indeed they are the coefficients of the term $\Delta_{\ell\to j}\Gamma_{j'\to\ell}$ in the expression of $X_h = \mathbb{T}^{j',h'}_{i,j,h}$ and $Y_h = \mathbb{S}^{j',h'}_{i,j,h}$ above. Hence they solve the systems (19) and (21) but with the “constant coefficients” $p_{*,*}$ equal to zero. Therefore if we use the variables $x_h = \lambda^{j',h'}_{i,j,h}$ and $y_h = \mu^{j',h'}_{i,j,h}$ we obtain from systems (19) and (21) that they are the solutions of the following systems:

$$\begin{aligned}
x_{h_0} + y_{h_0} &= \nu_{h_0}\\
&\vdots\\
x_{h_1} + y_{h_1} &= \nu_{h_1}
\end{aligned} \tag{22}$$

and

$$\begin{aligned} y_{h_0} + x_{h_0+1} &= \rho_{h_0} x_{h_0} \\ &\vdots \\ y_{h_1-1} + x_{h_1} &= \rho_{h_1-1} x_{h_0} \\ y_{h_1} &= \rho_{h_1} x_{h_0} \end{aligned} \tag{23}$$

Notice that x_{h_0}, $(y_{h_0}+x_{h_0+1}), \dots, y_{h_1}$ are exactly the coefficients appearing in points (iii), (iv) and (v) of the lemma, hence, by system (23) it is enough to prove that $x_{h_0} > 0$. If we sum the equations in system (22) we obtain:

$$x_{h_0} = \frac{\nu_{h_0} + \dots + \nu_{h1}}{1 + \rho_{h_0} + \dots + \rho_{h_1}}$$

which is a positive rational number. The lemma is proved. □

Using this lemma we can define the map $\Phi\colon \Lambda(D,V) \longrightarrow \mathfrak{T}$ by $\Phi(s) = \tilde{s}$. By formulas (13) and (14) this is a $\mathrm{GL}(V)$-equivariant algebraic morphism. The next lemma shows that it is an isomorphism.

Lemma 19. 1) $\Phi\colon \Lambda(D,V) \longrightarrow \mathfrak{T}$ *is a* $\mathrm{GL}(V)$*-equivariant isomorphism.*
2) $\Phi(s) \in \mathfrak{T}^+ \iff s \in \Lambda^+(D,V)$ *and* $\Phi\big|_{\Lambda^+}\colon \Lambda^+(D,V) \longrightarrow \mathfrak{T}^+$ *is a* $\mathrm{GL}(V)$*-equivariant isomorphism*

Proof. We prove 1) writing the explicit formula for the inverse of Φ:

$$\Phi^{-1}(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) = ((\mathbb{A}_i), (\mathbb{B}_i), (\mathbb{T}^{i+1,1}_{i,V}), (\mathbb{S}^V_{i,i+1,1})).$$

The equation $\Phi^{-1} \circ \Phi = \mathrm{id}_{\Lambda(D,v)}$ follows now by (13a) and (13b), while the relation $\Phi \circ \Phi^{-1} = \mathrm{id}_{\mathfrak{T}}$ follows by the unicity of the element $\tilde{s}$ proved in Lemma 18.

In order to prove 2) we first notice that for $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) \in S(\widetilde{D}, \widetilde{V})$ the stability condition is equivalent to the surjectivity of $\widetilde{A}_i$ for $i = 0, \dots, n-2$. For $i = 0, \dots, n-2$ consider the subspace D'_i of $\widetilde{V}_i$ and the subspace D^+_i of $\widetilde{V}_i$ defined by $D^+_i = \bigoplus_{2 \leqslant k \leqslant j-i \leqslant n-i-1} D^{(k)}_j$. Observe that for $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) \in \mathfrak{T}$ we have by relations (11) that $\widetilde{A}_i\big|_{D^+_i}$ is an isomorphism onto D'_{i+1}. Notice also that $V^+_i := V_i \oplus D^{(1)}_{i+1} \oplus \dots \oplus D^{(1)}_{n-1}$ is a complementary subspace of D^+_i in $\widetilde{V}_i$ and that V_{i+1} is a complementary subspace of D'_{i+1} in $\widetilde{V}_{i+1}$, hence in this case the stability condition is equivalent to $\pi_{V_{i+1}} \widetilde{A}_i\big|_{V^+_i}$ is surjective for $i = 0, \dots, n-2$ and by equations (13) this is equivalent to $A_i \oplus \Gamma_{i+1} \oplus \dots \oplus \Gamma_{n-1 \to i+1}\colon V_i \oplus D^{(1)}_{i+1} \oplus \dots \oplus D^{(1)}_{n-1} \longrightarrow V_{i+1}$ is surjective for $i = 0, \dots, n-2$; which is exactly the condition of Lemma 14, assertion 2) for the stability of (A, B, Γ, Δ). □

Definition 20. As observed Φ is a $\mathrm{GL}(V)$-equivariant morphism, so we can define φ_0 and $\widetilde{\varphi}$ as the maps making the following diagrams commute:

$$\begin{array}{ccc} \Lambda(d,v) & \xrightarrow{\Phi} & \mathfrak{T} \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle p_0}\downarrow \\ M_0(d,v) & \xrightarrow{\varphi_0} & M_0(\tilde{d},\tilde{v}) \end{array} \qquad \begin{array}{ccc} \Lambda^+(d,v) & \xrightarrow{\Phi} & \mathfrak{T}^+ \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ M(d,v) & \xrightarrow{\widetilde{\varphi}} & \widetilde{\mathcal{S}}_{r,x} \end{array}$$

and if we set $\varphi_1 = \varphi_0\big|_{M_1(d,v)}$ we observe that by definition the diagram (2) commutes, and that $\mathrm{Im}\,\varphi_1 \subset \mu_d(\widetilde{\mathcal{S}}_{r,x}) = \mathcal{S}_{r,x}$.

4. Proof of Theorem 8

We begin the proof of the theorem with some remarks on the degenerate cases and on the dimension of the varieties $M(d,v)$ and $\widetilde{\mathcal{S}}_{r,x}$.

Lemma 21. *Let r, d, v, N be as in Section* 1.4*. Then the following holds:*

1) *If there exists i such that $r_i < 0$ then $M(d,v) = \varnothing$.*
2) *If there exists i such that $v_i < 0$ then $\widetilde{\mathcal{S}}_{r,x} = \varnothing$.*
3) *$\widetilde{\mathcal{S}}_{r,x} \neq \varnothing$ if and only if $M(d,v) \neq \varnothing$ and in this case they are two smooth varieties of the same dimension.*

Proof. 1) This is an easy consequence of 2) in Lemma 14 and the definition of r.

2) If $v_i < 0$ then

$$\begin{aligned} N - (r_1 + \cdots + r_i) &= r_n + \cdots + r_{i+1} \\ &< d_{i+1} + 2d_{i+2} \cdots + (n-i-1)d_{n-1} \\ &= N - (d_1 + \cdots + (i-1)d_{i-1} + id_i + id_{i+1} + \cdots + id_n). \end{aligned}$$

So $r_1 + \cdots + r_i > d_1 + \cdots + id_i + \cdots + id_n$ and $\widetilde{\mathcal{S}}_{r,x}$ is empty by Proposition 3.

3) Since we have constructed a map from $M(d,v)$ to $\widetilde{\mathcal{S}}_{r,x}$ it is clear that $M(d,v) \neq \varnothing$ implies $\widetilde{\mathcal{S}}_{r,x} \neq \varnothing$. To show the converse we observe that the Weyl group S_n acts by permutation on the n-tuple r and that if $\sigma \in S_n$:

(1) $\widetilde{\mathcal{S}}_{\sigma(r),x} \neq \varnothing \iff \widetilde{\mathcal{S}}_{r,x} \neq \varnothing$,
(2) $r(d, \sigma(v-d)+d) = \sigma(r(d,v))$.

The first property is clear from Proposition 3 (indeed with a little more effort one could check that $\widetilde{\mathcal{S}}_{\sigma(r),x} \simeq \widetilde{\mathcal{S}}_{r,x}$ but we do not need this result). The second property is a computation that can easily be checked for $\sigma = (i, i+1)$. So by Lemma 8 it is enough to prove that $\widetilde{\mathcal{S}}_{r,x} \neq \varnothing \Rightarrow M(d,v) \neq \varnothing$ when $d - v$ is dominant. If we set

$i_1 = 1, \dots, i_k = k$ in the inequality (1) we obtain $v_k \geqslant 0$ for $k = 1, \dots, n-1$ and $M(d, v) \neq \varnothing$ by Lemma 7.

Finally by a result of Nakajima (see [8, Corollary 3.12]) if $M(d, v)$ is not empty then it is a smooth variety of dimension $\boldsymbol{\kappa}(2d - v\,;\, v)$ and the equality of dimensions is an easy consequence of Proposition 3. □

In the following two lemmas we study the injectivity of the maps φ_0 and $\widetilde{\varphi}$.

Lemma 22. *Let $\tilde{s} \in \mathfrak{T}$, $\tilde{g} \in \mathrm{GL}(\widetilde{V})$ and assume that $\tilde{g}(\tilde{s}) \in \mathfrak{T}$. Then there exists $g \in \mathrm{GL}(V)$ such that $\tilde{g}(\tilde{s}) = g(\tilde{s})$.*

Proof. We prove first that $\tilde{g}_i(V_i) = V_i$ and $\tilde{g}_i(D'_i) = D'_i$. In order to prove them we introduce for $i = 0, \dots, n-2$, $l = 0, \dots, n-2-i$ and $h = 0, \dots, n-2-i-l$ the following subspaces of $\widetilde{V}_i$:

$$W_i^{l,(h)} = \bigoplus_{\substack{0 \leqslant h' \leqslant h \\ i+1+l+h' \leqslant j \leqslant n-1}} D_j^{(j-i-h')} \quad \text{and} \quad D_i^- = \bigoplus_{\substack{i+2 \leqslant j \leqslant n-1 \\ 1 \leqslant k \leqslant j-i-1}} D_j^{(k)}.$$

and observe that $W_i^{0,(n-i-2)} = D'_i$ and that $W_i^{0,(0)} \oplus D_i^- = D'_i$. Notice that by relations (11) for all $(\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) \in \mathfrak{T}$ the following properties hold:

(1) $\widetilde{A}_i\big|_{W_i^{l,(h)}}$ is an isomorphism onto $W_{i+1}^{l-1,(h)}$ for $l \geqslant 1$,

(2) $\widetilde{B}_i\big|_{D'_{i+1}}$ is an isomorphism onto D_i^-,

(3) $\widetilde{B}_i(V_{i+1}) \subset W_i^{0,(0)} \oplus V_i$.

Now we prove that $\tilde{g}_i(W_i^{l,(h)}) = W_i^{l,(h)}$ by induction on i. Let $\tilde{s} = (\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta})$ and $\tilde{s}' = \tilde{g}(\tilde{s}) = (\widetilde{A}', \widetilde{B}', \widetilde{\Gamma}', \widetilde{\Delta}')$. In the case $i = 0$ there is nothing to prove since $\tilde{g}_i$ does not act on $\widetilde{V}_0 = \widetilde{D}_1$. If $i+1 > 0$ using the inductive hypothesis and property (1) above, we obtain

$$\begin{aligned} \tilde{g}_{i+1}(W_{i+1}^{l-1,(h)}) &= \tilde{g}_{i+1}(\widetilde{A}_i(W_i^{l,(h)})) \\ &= \tilde{g}_{i+1}(\widetilde{A}_i(\tilde{g}_i^{-1}(W_i^{l,(h)}))) = \widetilde{A}'_i(W_i^{l,(h)}) = W_{i+1}^{l-1,(h)} \end{aligned}$$

proving the claim.

In particular for $l = 0$ and $h = n - i - 2$ we obtain $\tilde{g}_i(D'_i) = D'_i$.

In order to prove $\tilde{g}_i(V_i) = V_i$ we also argue by induction on i. For $i = 0$ again there is nothing to prove. If $i + 1 > 0$ using induction and property (3) above, we obtain

$$\widetilde{B}'_i\, \tilde{g}_{i+1}(V_{i+1}) = \tilde{g}_i\, \widetilde{B}_i(V_{i+1}) \subset \tilde{g}_i(W_i^{0,(0)} \oplus V_i) = W_i^{0,(0)} \oplus V_i.$$

Hence by property (2) above and the fact that $\widetilde{V}_i = V_i \oplus W_i^{0,(0)} \oplus D_i^-$ we obtain $\tilde{g}_{i+1}(V_{i+1}) = V_{i+1}$.

Now we consider $g_i = \tilde{g}_i\big|_{V_i}$ and we prove that $\tilde{g}(\tilde{s}) = g(\tilde{s})$. Let $\tilde{s}'' = g(\tilde{s})$ and observe that $\tilde{s}'$ and $\tilde{s}''$ are elements of $\mathfrak{T}$. Hence, by the unicity proved in Lemma 18, in order to show that they are equal it is enough to show that for all i:

$$\mathbb{A}_i(\tilde{s}') = \mathbb{A}_i(\tilde{s}''), \qquad \mathbb{B}_i(\tilde{s}') = \mathbb{B}_i(\tilde{s}''),$$
$$\mathbb{T}^{i+1,1}_{i,V}(\tilde{s}') = \mathbb{T}^{i+1,1}_{i,V}(\tilde{s}''), \qquad \mathbb{S}^{V}_{i,i+1,1}(\tilde{s}') = \mathbb{S}^{V}_{i,i+1,1}(\tilde{s}'').$$

By construction we have already proved the equality of the $\mathbb{A}_i$ and $\mathbb{B}_i$ blocks. Now for the remaining blocks we observe that it is enough to prove $\tilde{g}_i\big|_{D^{(1)}_{i+1}} = \mathrm{id}_{D^{(1)}_{i+1}}$. Indeed we observe that $\widetilde{A}_i\big|_{W^{l,(0)}_i}$ is the identity map from $W^{l,(0)}_i$ to $W^{l-1,(0)}_{i+1}$. Arguing by induction as above we conclude that $\tilde{g}_i\big|_{W^{l,(0)}_i}$ is the identity map. Finally we observe that $D^{(1)}_{i+1} \subset W^{0,(0)}_i$. □

Lemma 23. *φ_0 and φ_1 are closed immersions.*

Proof. It is enough to prove that φ_0 is a closed immersion, hence to show that the associate map ${\varphi_0}^\sharp$ between the coordinate rings of the two affine varieties $M_0(d, v)$ and $M_0(\tilde{d}, \tilde{v})$ is surjective.

Observe that by Lemma 18 there exist homogeneous admissible polynomials $n^{j',h'}_{j,h}$ of type (j', j) such that for all $\tilde{s} = (\widetilde{A}, \widetilde{B}, \widetilde{\Gamma}, \widetilde{\Delta}) = \Phi(s)$ we have $\pi_{D^{(h)}_j} \widetilde{D}_1 \widetilde{\Gamma}_1\big|_{D^{(h')}_{j'}} = n^{j',h'}_{j,h}(s)$. Let $\mathcal{R}n$ be the subalgebra of $\mathcal{R}$ generated by these polynomials. By Theorem 13, the surjectivity of ${\varphi_0}^\sharp$ follows if we show that for all $[\alpha] \in \mathcal{P}$ there exists $\tilde{f} \in \mathcal{R}n$ of type (α_0, α_1) such that $[\alpha](s) = \tilde{f}(s)$ for all $s \in \Lambda(D, V)$. We prove this claim by induction on the degree d of the polynomial $\beta = [\alpha] \in \mathcal{P}$.

If $d \leqslant 0$ there is nothing to prove since there are not polynomials in $\mathcal{P}$ in this case.

In order to study the case $d > 0$ we observe first that if $\mathcal{R}(m)$ is the subalgebra of $\mathcal{R}$ generated by the polynomials in $\mathcal{P}$ of degree less than or equal to m then, $f \in \mathcal{R}_\sharp$ and degree $f \leqslant m + 1$ implies $f \in \mathcal{R}(m)$. Now we study the case $d > 0$. Let $\beta = [\delta_{\ell\to j}\gamma_{j'\to\ell}] \in \mathcal{P}$ of degree d. By relations (11) and the definition of the map Φ we have:

$$n^{j',h'}_{j,h}(s) = \pi_{D^{(h)}_j}(\widetilde{\Delta_1}\widetilde{\Gamma}_1)\big|_{D^{(h')}_{j'}} = f(s) + \begin{cases} 1 \cdot \beta(s) & \text{if } \ell = 1, \\ \lambda^{j',1}_{0,j,h} \cdot \beta(s) & \text{if } \ell > 1, \end{cases}$$

where, by Lemma 18, f is an homogeneous admissible polynomials of type (j', j) of degree d and it is also an element of $\mathcal{R}_\sharp$. In particular, by what we have noticed above, $f \in \mathcal{R}(d-1)$ and by the inductive hypothesis, there exists $\tilde{f} \in \mathcal{R}n$ of type (j', j) such that $f(s) = \tilde{f}(s)$ for all $s \in \Lambda(D, V)$. Finally we observe that if ν is the coefficient of β in the formula above, by property (iii) in Lemma 18 we always have $\nu \neq 0$. Hence $\beta(s) = \nu^{-1}(n_{j,h}^{j',h'}(s) - \tilde{f}(s))$ for all s proving the claim. □

Proof of Theorem 8. By Lemma 23 and the fact that μ_d and π are projective we see that $\widetilde{\varphi}$ is proper. Since by a result of Nakajima (see [8, Corollary 3.12]) all orbits in $\Lambda^+(d, v)$ and $\Lambda^+(\tilde{d}, \tilde{v})$ are closed, Lemmas 19 and 22 imply that $\widetilde{\varphi}$ is also injective. Since by Lemma 21 $M(d, v)$ and $\widetilde{\mathcal{S}}_{r,x}$ are smooth varieties of the same dimension and $\widetilde{\mathcal{S}}_{r,x}$ is connected we have proved that it is an isomorphism of holomorphic varieties and by consequence is also an isomorphism of algebraic varieties. In particular $\widetilde{\varphi}$ is surjective and this together with the surjectivity of the map μ_d implies the surjectivity of φ_1. So φ_1 is a surjective closed immersion of affine varieties, hence it must be an isomorphism of algebraic varieties. Finally $\varphi_0(0) = x \in \mathcal{S}_{r,x}$, and x is in the image of φ_1, hence by the injectivity of φ_0 proved in the previous lemma we have $0 \in M_1(d, v)$ and $\varphi_1(0) = x$. □

Remark 24. In Nakajima's theory an essential role is played by the variety $\Gamma(d, v) := \pi^{-1}(0) \subset M(d, v)$. We observe that the map $\widetilde{\varphi}$ restricted to $\Gamma(d, v)$ take a more explicit and simple form. Indeed it is easy to see that in this case the maps Δ_i vanish, hence $t_{i,j,h}^{j',h'}(s) = s_{i,j,h}^{j',h'}(s) = 0$ and (11) and (13) give an explicit formula for the map $\widetilde{\varphi}$.

Remark 25. As it is noticed in [7], Nakajima's conjecture does not generalize to diagrams of type E and D. However we observe that, in general, interesting subvarieties of a quiver variety can be described as quiver varieties themselves (see, for example, the stratification of quiver varieties constructed by Nakajima in [7], [8]) From this point of view let us remark that it is possible to explicitly give a pair of injective maps $\widetilde{\psi}$ and ψ from $M(d, v)$ to $M(\tilde{d}, \tilde{v})$, and from $M_0(d, v)$ to $M_0(\tilde{d}, \tilde{v})$ respectively, such that diagram (2) commutes and $\psi(0) = x$. As we have already noted their definition is simpler than that of $\widetilde{\varphi}$ and φ_1 but, on the other hand, their image is not contained in $\widetilde{\mathcal{S}}_{r,x}$ and $\mathcal{S}_{r,x}$ respectively, hence they "describe" different transversal slices.

References

[1] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*. Birkhäuser, Boston, MA, 1997. Zbl 0879.22001 MR 1433132

[2] J. Humphreys, *Introduction to Lie algebras and representation theory*. Grad. Texts in Math. 9, Springer-Verlag, New York, Berlin 1978. Zbl 0447.17001 MR 0499562

[3] H. Kraft and C. Procesi, Closures of conjugacy classes of matrices are normal. *Invent. Math.* **53** (1979), 227–247. Zbl 0434.14026 MR 0549399

[4] P. Kronheimer and H. Nakajima, Yang-Mills instantons on ALE gravitational instantons. *Math. Ann.* **288** (1990), 263–307. Zbl 0694.53025 MR 1075769

[5] G. Lusztig, On quiver varieties. *Adv. Math.* **136** (1998), 141–182. Zbl 0915.17008 MR 1623674

[6] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*. Third ed., Ergeb. Math. Grenzgeb. 34, Springer-Verlag, Berlin 1994. Zbl 0797.14004 MR 1304906

[7] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. *Duke Math. J.* **76** (1994), 365–416. Zbl 0826.17026 MR 1302318

[8] —, Quiver varieties and Kac-Moody algebras. *Duke Math. J.* **91** (1998), 515–560. Zbl 0970.17017 MR 1604167

[9] P. Slodowy, *Simple singularities and simple algebraic groups*. Lecture Notes in Math. 815, Springer-Verlag, Berlin 1980. Zbl 0441.14002 MR 0584445

Received January 13, 2003

Andrea Maffei, Dipartimento di Matematica "Guido Castenuovo", Università di Roma "La Sapienza", P.le A. Moro 5, 00185 Roma, Italy

E-mail: amaffei@mat.uniroma1.it