The macroscopic spectrum of nilmanifolds with an emphasis on the Heisenberg groups

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Abstract. Take a Riemannian nilmanifold, lift its metric on its universal cover. In that way one obtains a metric invariant under the action of some co-compact subgroup. We use it to define metric balls and then study the spectrum of the Dirichlet Laplacian. Using homogenization techniques we describe the asymptotic behavior of the spectrum when the radius of these balls goes to infinity. This involves the spectrum, which we call macroscopic spectrum, of a so called homogenized operator on a specific domain. Furthermore we show that the first macroscopic eigenvalue is bounded from above, by a universal constant in the case of the three dimensional Heisenberg group, and by a constant depending on the Albanese torus for the other nilmanifolds. We also show that the Heisenberg groups belong to a family of nilmanifolds, where the equality characterizes some pseudo-left-invariant metrics.

1. Introduction and statement of the results

This article deals with geometric properties of large balls in periodic Riemannian manifolds. A Riemannian manifold (N^n, g) is periodic if it possesses a discrete group Γ of isometries with a compact fundamental domain. Given $x_0 \in N^n$, we are interested in the asymptotic behavior of two geometric invariants of the metric ball, $B_g(x_0, \rho)$, with radius ρ and centered at x_0, as ρ tends to ∞:

- the Riemannian volume $\mathrm{Vol}_g(B_g(x_0, \rho))$;
- the spectrum of the Dirichlet Laplacian on $B_g(x_0, \rho)$.

Our approach consists in rescaling the metric, i.e., replacing the original Riemannian metric g on N with $g_\rho = 1/\rho^2 g$, so that $B_g(x_0, \rho)$ becomes $B_{g_\rho}(x_0, 1)$, and applying homogenization techniques to the family of Riemannian manifolds with

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boundary $N_\rho = (B_{g_\rho}(x_0, 1), g_\rho)$. There are several notions of convergence of metric spaces (see [Gro81b], [Gro93]). It turns out that the balls N_ρ converge, in the Gromov–Hausdorff sense, to a compact metric space if and only if the group Γ contains a finite index subgroup Γ' that is nilpotent, torsion-free, and finitely generated. This follows from a celebrated result of M. Gromov [Gro81a], characterizing finitely generated groups of polynomial growth, completed by P. Pansu [Pan83] and Van den Dries–Wilkie [vdDW84]. Therefore, actions of nilpotent groups seem to provide the proper setting for application of homogenization techniques in Riemannian geometry. According to Malcev, such a group uniquely embeds into a simply connected nilpotent Lie group G, and G/Γ is called a nilmanifold. In the sequel, we assume that $N = G$ is equipped with a Γ-invariant Riemannian metric. The manifold N^n can be viewed as the Riemannian universal covering of $M^n = G/\Gamma$ equipped with the quotient metric.

Although the results presented here are geometric in nature, we use homogenization techniques. Hence this article can be read under two different lights.

1.1. From the geometric point of view: The Riemannian volume and the Dirichlet spectrum of $B_g(x_0, \rho)$ are linked by Weyl's asymptotic formula, which states that if $\lambda_k(\rho)$ is the kth eigenvalue of the ball of radius ρ and $\text{Vol}(\rho)$ is its volume, then as $k \to \infty$, there exists a universal constant $c(n)$ such that

$$\lambda_k(\rho) \sim c(n) \frac{k^{2/n}}{\text{Vol}^{2/n}(\rho)}$$

One could expect that the asymptotic behaviors of the volume and the Dirichlet spectrum when the radius of the balls increases would be related. This is not the case; we shall see that they are described by two different limit metrics.

Problem 1. Make the asymptotic behavior of the volume of a ball with respect to its radius, precise, and extract geometric information from it.

In the case of nilmanifolds there is a precise equivalent to the volume of balls given by P. Pansu [Pan83], which depends on the algebraic structure. Let $G^1 = G$, and $G^{i+1} = [G^i, G]$; then $d_h = \sum_{i=1}^{\infty} \dim G^i$ is called the *homogeneous dimension* of G, and

$$\text{Vol}(\rho) \sim \text{Asvol}(g)\rho^{d_h}.$$

The constant $\text{Asvol}(g)$ is usually called the *asymptotic volume*. In the particular case of tori, D. Burago and S. Ivanov [BI95] gave a lower bound on the asymptotic volume, which is achieved if and only if the metric is flat (see also [Ver04] for an alternate proof in dimension 2 using homogenization theory and [Bab91] for the first proof in dimension 2).

To the nilpotent Lie group G, we can associate its limit group at infinity, G_∞, which is nilpotent and graded. Furthermore, thanks to a theorem of K. Nomizu

[Nom54], $H_1(M, \mathbb{R})$ can be identified with a subspace of the Lie algebra of G_∞, hence to a left invariant distribution $\mathcal{H}$ of vector fields over G_∞. Thus to any norm on $H_1(M, \mathbb{R})$ we can associate a left-invariant sub-finslerian structure on G_∞, hence, thanks to Chow's theorem on accessibility, a left-invariant distance on G_∞. For a Euclidean norm we obtain a left-invariant sub-Riemannian structure. The metric g induces two important norms on $H_1(M, \mathbb{R})$. The first one, called *the stable norm*, comes from the sup norm on the 1-forms over M, which induces a norm (usually *not* Euclidean) on $H^1(M, \mathbb{R})$, and by duality on $H_1(M, \mathbb{R})$. The second one, called the *Albanese metric*, comes from the L^2 normalized norm on 1-forms, which induces a Euclidean norm on $H^1(M, \mathbb{R})$, and by duality on $H_1(M, \mathbb{R})$. The two distances induced by these two metrics on G_∞ are often said to be of *Carnot–Carathéodory type*. We call them, respectively, the stable distance and the Albanese distance.

The following inequality, if not the best one, gives a hint of what we might expect for all nilmanifolds.

Theorem 1. *Let* (M^n, g) *be a nilmanifold. Let* G_∞ *be the limit group at infinity associated to the universal covering of* M^n. *Then the asymptotic volume of* M^n *satisfies the following:*

1. $\mathrm{Asvol}(g) \geq \mathrm{Vol}_g(M^n) \dfrac{\mu(B_{\mathrm{al}}(1))}{\mu(D_M)};$

2. *in case of equality the stable norm coincides with the Albanese metric.*

Here, μ *is a Haar measure on* G_∞, $B_{\mathrm{al}}(1)$ *is the unit ball of the Albanese distance centered at the unit element, and* D_M *is the image in* G_∞ *of a fundamental domain on the universal covering of* M^n, *by the canonical projection.*

Concerning the spectrum of the Laplacian on balls, a theorem of R. Brooks [Bro85] (see also Sunada [Sun89]) states that the bottom of the spectrum on the universal cover is zero if and only if the fundamental group is amenable. The first eigenvalue goes to the bottom of the spectrum as the radius of the ball goes to infinity. R. Brooks's theorem implies, in our case, as the fundamental group is nilpotent hence amenable, that the first eigenvalue goes to zero as the radius goes to infinity.

Problem 2. Make the speed of convergence to the bottom of the spectrum on the universal cover with respect to the radius, precise, and extract more geometric information from the spectrum of large balls.

To state our results to that problem, let us remark that to the Albanese metric we can also associate a kind of Laplacian Δ_∞ on G_∞. Δ_∞ is usually called the *Kohn Laplacian*. It is a dilation invariant hypoelliptic second order differential operator, which is symmetric and without a constant term.

Theorem 2. *Let (M^n, g) be a nilmanifold, with universal cover $\widetilde{M}$, and let $x \in \widetilde{M}$. Let $B_g(x, \rho)$ be the corresponding Riemannian ball of radius ρ and center $x \in \widetilde{M}$, and let $\lambda_1(B_g(x, \rho))$ be the first eigenvalue of the Laplacian for the Dirichlet problem on $B_g(x, \rho)$. Then*

1. $\lim_{\rho \to +\infty} \rho^2 \lambda_1(B_g(x, \rho)) = \lambda_1^\infty \leq \lambda_1(g, \text{Alb})$*;*

2. *in case of equality, the stable norm coincides with the Albanese metric, hence all harmonic 1-forms are of constant length.*

Here, $\lambda_1(g, \text{Alb})$ is the first eigenvalue of the Kohn Laplacian arising from the Albanese metric for the Dirichlet problem on $B_{\text{al}}(1)$, the unit ball of the Albanese distance centered at the unit element. Furthermore, for tori this is a constant dependent only on the dimension, and for the 3-dimensional Heisenberg group it is also independent of the metric.

In the case of a 2-step nilmanifold with a 1-dimensional center, we can determine for which metrics equality holds. We call these metrics pseudo-left-invariant (see Section 6 for the definition). One of their main properties being that they arise as fiber metrics over a flat torus (i.e. the nilmanifold submerges onto a flat torus).

Theorem 3. *In the case of a 2-step nilmanifold whose center is one dimensional, the Albanese metric and the stable norm coincide if and only if the metric is pseudo-left-invariant.*

Actually this behavior is shared by all the eigenvalues, and Theorem 2 is partially a consequence of the following:

Theorem 4. *Let (M^n, g) be a nilmanifold, with universal cover $\widetilde{M}$, and let $x \in \widetilde{M}$. Let $B_g(x, \rho)$ be the corresponding Riemannian ball of radius ρ, and center $x \in \widetilde{M}$ and let $\lambda_i(B_g(x, \rho))$ be the ith eigenvalue of the Laplacian for the Dirichlet problem on $B_g(x, \rho)$.*

Then there exists an hypoelliptic operator Δ_∞ (the Kohn Laplacian of the Albanese metric), whose i^{th} eigenvalue for the Dirichlet problem on the unit ball of the stable distance (centered at the unit element) is λ_i^∞, and such that

$$\lim_{\rho \to \infty} \rho^2 \lambda_i(B_g(\rho)) = \lambda_i^\infty.$$

*We call $(\lambda_i^\infty)_{i \in \mathbb{N}}$ **the macroscopic spectrum**.*

1.2.   From the point of view of analysis, let

$$L = -\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_i},$$

be a uniformly elliptic differential operator on $\mathbb{R}^n$, and assume that the coefficients a_{ij} are periodic, i.e., $a_{ij}(x+k) = a_{ij}(x)$ for any $k \in \mathbb{Z}^n$ and C^∞. After rescaling we get a family of operators

$$0 < \epsilon \le 1, \quad L_\epsilon = -\frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\epsilon}\right) \frac{\partial}{\partial x_i}.$$

We can associate, to this family of operators, a so-called homogenized operator,

$$L_0 = -q_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Now, if D is a domain of $\mathbb{R}^n$, then we can consider the Dirichlet problem for this family of operators, and hence we have eigenvalues and eigenfunctions, denoted respectively by

$$\lambda_0^\epsilon \le \lambda_1^\epsilon \le \lambda_2^\epsilon \le \cdots \quad \text{and} \quad \phi_0^\epsilon, \phi_1^\epsilon, \phi_2^\epsilon, \ldots .$$

We now have the following problem.

Problem 1′.   Study the convergence of λ_i^ϵ and ϕ_i^ϵ to λ_i^0 and ϕ_i^0, respectively, as $\epsilon \to 0$.

In the case of $\mathbb{R}^n$, this is the subject of Chapter III of [OSY92], and Chapter 11 of [JKO94], and of [CD99].

The operators L_ϵ define Riemannian distances d_ϵ. So another related problem is the following:

Problem 2′.   Study the relationship between the distance d_ϵ and the distance d_0.

The papers [Dav93], [Nor94] and [Nor97] are related to this problem in conjunction with the existence of bounds on the heat kernel (see also [KS00] for a probabilistic approach).

The present paper is concerned with the problem above, when $\mathbb{R}^n$ is replaced by a nilpotent Lie group N and $\mathbb{Z}^n$ by a uniform lattice Γ of N. Homogenization in this context, when N is stratified (graded), has been the subject of [BBJR95], [BMT96] and [BMT97].

However our paper differs in three ways from the previous work. First of all, we are not dealing with a stratified group, hence we must not only homogenize the operator, but also the space, by using its associated graded Lie group. Secondly we begin by studying a family of elliptic operators, which happens to have an hypoelliptic homogenized operator. And finally, our domain moves with the operator. The relationship with the long time asymptotics of the heat kernel is shortly studied in Section 8.

2. Geometry of nilmanifolds

2.1. Nilpotent Lie algebras. Let $\mathfrak{u}$ be a Lie algebra. One says that it is nilpotent if the sequence defined by

$$\mathfrak{u}^1 = \mathfrak{u}, \quad \mathfrak{u}^{i+1} = [\mathfrak{u}^i, \mathfrak{u}],$$

is such that for some $k \in \mathbb{N}$, $\mathfrak{u}^{k+1} = \{0\}$. Let r be the smallest of such k; then we say that $\mathfrak{u}$ is an r-step nilpotent Lie algebra.

A distinguished family of nilpotent Lie algebras consists of the graded ones. A nilpotent Lie Algebra $\mathfrak{u}$ is graded if it admits a decomposition:

$$\mathfrak{u} = V_1 \oplus \cdots \oplus V_r, \tag{1}$$

such that

1. V_i is a complement of $\mathfrak{u}^{i+1}$ in $\mathfrak{u}^i$;

2. $[V_i, V_j] \subset V_{i+j}$.

It is quite important in our work that to such a graduation one can attach a one-parameter group of automorphisms $(\tau_\rho)_{\rho \in \mathbb{R}^+}$ called dilations such that:

$$\tau_\rho(x) = \rho^i x \quad \text{for all } x \in V_i.$$

In fact, the existence of such a family of dilations is equivalent to the existence of a graduation. These dilations play the same role as the dilations in Euclidean space.

Not all nilpotent Lie algebras are graded. But to each nilpotent Lie algebra, we can associate a graded nilpotent one in the following way:

$$\mathfrak{u}_\infty = \sum_{i=1}^{r} \mathfrak{u}_i / \mathfrak{u}_{i+1},$$

the Lie bracket being induced. We will denote by $\tilde{\pi} : \mathfrak{u} \to \mathfrak{u}_\infty$ the induced projection and by $\tilde{\tau}_\rho$ the dilations in $\mathfrak{u}_\infty$.

The *Homogeneous dimension of* $\mathfrak{u}$ is the number

$$d_h = \sum_{i=1}^{r} i \dim(\mathfrak{u}^i / \mathfrak{u}^{i+1}).$$

There is another way to make that graded Lie algebra appear: start with a nilpotent Lie algebra $\mathfrak{u}$, remark that for all i, $\mathfrak{u}^{i+1} \subset \mathfrak{u}^i$, and build a basis $(X_i)_i$ of $\mathfrak{u}$ by taking independent vectors $X_{d_1+\cdots+d_{i-1}+1}, \ldots, X_{d_1+\cdots+d_{i-1}+d_i}$ such that the vector space V_i that they span is a complement of $\mathfrak{u}^{i+1}$ in $\mathfrak{u}^i$. Hence the direct sum (1) holds. We shall denote by pr_{V_i} the projection induced on V_i by this direct sum. Now we define a function $\tau_\rho : \mathfrak{u} \to \mathfrak{u}$ by

$$\tau_\rho(X_p) = \rho^{\alpha(p)} X_p,$$

with $\alpha(p) = i$ if $d_{i-1} < p \le d_i$, and $d_0 = 0$.

We obtain a new Lie algebra $\mathfrak{u}_\rho$ by modifying the Lie bracket in the following way: for any X and Y in $\mathfrak{u}_\rho$, $[X, Y]_\rho = \tau_{1/\rho}[\tau_\rho X, \tau_\rho Y]$. Thus τ_ρ becomes a Lie algebra isomorphism from $\mathfrak{u}_\rho = (\mathfrak{u}, [\,\cdot\,, \cdot\,]_\rho)$ to $(\mathfrak{u}, [\,\cdot\,, \cdot\,])$.

Now as ρ goes to infinity, $\mathfrak{u}_\rho$ goes to $\mathfrak{u}_\infty$, in the sense that for $i, j = 1, \ldots, n$, we have

$$[X_i, X_j]_\infty = \mathrm{pr}_{V_{\alpha(i)+\alpha(j)}}[X_i, X_j].$$

Notice that all $\mathfrak{u}_\rho$ have the same graded Lie algebra. We will denote by $\tilde{\pi}_\rho$ the projection from $\mathfrak{u}_\rho$ to $\mathfrak{u}_\infty$ (in fact we could avoid the subscript in $\tilde{\pi}_\rho$, because we can identify the Lie algebras as linear spaces).

Notice that if the Lie algebra is graded, then $[X, Y]_\rho = [X, Y]$, and τ_ρ is a Lie algebra automorphism. Otherwise, remark that for all $X \in \mathfrak{u}_\rho$

$$\tilde{\pi}(\tau_\rho(X)) = \tilde{\tau}_\rho(\tilde{\pi}_\rho(X)).$$

2.2. Remarks on exponential coordinates. Let G be the simply connected Lie group associated with the nilpotent Lie algebra $\mathfrak{u}$. For nilpotent Lie groups, the exponential is a diffeomorphism between the Lie algebra and the Lie group, hence thanks to the exponential coordinates, we can identify G, as a differential manifold, with some $\mathbb{R}^n$:

$$\phi : \mathbb{R}^n \to G, \quad \phi : x = (x_1, \ldots, x_n) \mapsto \exp x_1 X_1 \ldots \exp x_n X_n.$$

Let X_i^* be the dual form of X_i.

Moreover, we denote by δ_ρ the following family of dilations:

$$\delta_\rho(x_1, \ldots, x_n) = (\rho^{\alpha(1)} x_1, \ldots, \rho^{\alpha(n)} x_n).$$

Notice also that $d\delta_\rho = \tau_\rho$. We define a family of group products $*_\rho$ by setting

$$x *_\rho y = \delta_{1/\rho}[\delta_\rho(x)\delta_\rho(y)].$$

Finally

$$x *_\infty y = \lim_{\rho \to \infty} x *_\rho y.$$

Thus we get a family of nilpotent Lie groups $G_\rho = (G, *_\rho)$, $0 < \rho \le \infty$, whose Lie algebras are isomorphic, respectively, to the algebras $\mathfrak{u}_\rho$, $0 < \rho \le \infty$. We also denote by $\pi_\rho : G_\rho \to G_\infty$ the function which sends $x \in G_\rho$ to $x \in G_\infty$, i.e., $\pi_\rho = \phi_\infty \circ \phi_\rho^{-1}$ (and to simplify $\pi_1 = \pi$).

Observe that for $1 \le j \le d_1$, the x_j live on $G/[G, G]$.

If $e \in G$ is the unit element and $X \in \mathfrak{u}$, then for $\rho \in \overline{\mathbb{R}}$, X^ρ will be the $*_\rho$ left invariant field in G_ρ such that $X^\rho(e) = X(e)$. Thus to the basis (X_i) defined in 2.1, we will associate the $*_\rho$ left invariant fields (X_i^ρ). Notice also that

$$d\delta_\rho(X_i^\rho) = \tau_\rho(X_i^\rho) = \rho^{\alpha(i)} X_i.$$

We also define $\nabla_{\mathcal{H}}$ by

$$\nabla_{\mathcal{H}} f = (X_1^\infty \cdot f, \dots, X_{d_1}^\infty \cdot f).$$

3. Asymptotic behavior of the distance

3.1. The stable norm

3.1.a.   Recall that (M^n, g) is a manifold whose universal covering is a simply connected nilpotent Lie group G. We shall denote by $\tilde{g}$ the lifted metric on G. On the graded nilpotent Lie group G_∞ associated to G, we obtain a natural distribution by left multiplication of $V_1 = \mathfrak{u}_1/\mathfrak{u}_2 \subset \mathfrak{u}_\infty$. We shall call that distribution *horizontal* and denote it by $\mathcal{H}$.

Let us remark that since the Lie algebra $\mathfrak{u}_\infty$ is generated by V_1, a basis of V_1 satisfies the so called Chow (or Hörmander) condition in the Lie group G_∞. Let us recall what the *stable norm* is:

Definition 5. Let $\| \cdot \|_\infty^*$ be the quotient of the sup norm on 1-forms, arising from the metric g, on the cohomology $H^1(M^n, \mathbb{R})$. Then its dual norm on the homology $H_1(M^n, \mathbb{R})$, is called the stable norm and we denote it by $\| \cdot \|_\infty$.

By a theorem of K. Nomizu [Nom54], $H_1(M^n, \mathbb{R}) \equiv V_1$, thus we can transport the stable norm on $\mathcal{H}$. Now the Rashevsky–Chow theorem (see Theorem 2.4, p. 15, in [BR96]) asserts that two points of G_∞ can be joined by a curve tangent to $\mathcal{H}$ (usually called an admissible curve). For an admissible curve $\gamma : [a, b] \to G_\infty$, we consider its stable length $l_\infty(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_\infty dt$. Hence we can define a distance d_∞, which we call the stable distance, between two points of G_∞, by taking the infimum of the stable lengths of admissible curves between these points. This kind of distance is usually said to be of Carnot–Carathéodory type. We shall call the unit ball for the stable distance centered at 0 the stable ball and denote it by $B_\infty(1)$.

3.1.b.   For any $x, y \in G_\rho$, let us introduce $d_\rho(x, y) = \dfrac{d_g(\delta_\rho x, \delta_\rho y)}{\rho}$.   Then the work of P. Pansu [Pan83], implies that for any $x, y \in G_\rho$

$$\lim_{\rho \to \infty} \frac{d_\infty(\pi_\rho(x), \pi_\rho(y))}{d_\rho(x, y)} = \lim_{\rho \to \infty} \frac{d_\infty(\pi \circ \delta_\rho(x), \pi \circ \delta_\rho(y))}{d_g(\delta_\rho x, \delta_\rho y)} = 1.$$

This implies the simple convergence of the functionals $x \mapsto d_\rho(0, \pi_\rho^{-1}(x))$ toward $x \mapsto d_\infty(0, x)$ on $B_\infty(1) \backslash \partial B_\infty(1)$.

3.1.c.   Remark that the distance d_ρ is also given by the metric g_ρ on G_ρ, obtained by rescaling the pull back of the metric $\tilde{g}$ on G in the following way:

$$g_\rho = \frac{1}{\rho^2}(\delta_\rho)^*\tilde{g}.$$

3.2. Gromov–Hausdorff convergence of balls

3.2.a.   Recall that a family of spaces X_n, endowed with metrics d_n and measures μ_n, is said to converge in the Gromov–Hausdorff Measured sense toward (X, d, μ) if and only if there is a family $(f_n)_{n\in\mathbb{N}}$, where for all n, f_n is an μ_n measurable function from X_n to X, and there is a sequence (ϵ_n), decreasing to 0, such that

1. the ϵ_n neighborhood of $f_n(X_n)$ in X is X;

2. for any $x, y \in X_n$, $|d_n(x, y) - d(f_n(x), f_n(y))| \leq \epsilon_n$;

3. for any continuous function $u : X \to \mathbb{R}$ we have

$$\int_{X_n} u \circ f_n d\mu_n \to \int_X u d\mu.$$

3.2.b.   Let μ_ρ (resp. μ_g) denote the Riemannian volume associated to g_ρ (resp. g), and let μ_∞ be defined as follows. Let D_Γ be a fundamental domain in G and μ a Haar measure on G_∞, then (recall that π is the canonical projection from $G \to G_\infty$)

$$\mu_\infty = \frac{\mu_g(D_\Gamma)}{\mu(\pi(D_\Gamma))} \mu.$$

Adding to this that for any compact domain A in G_∞, whose boundary is of Haar measure 0, and any function $f \in L^1(A, \mu_\infty)$, we have

$$\lim_{\rho\to\infty} \int_{\pi_\rho^{-1}(A)} f(\pi_\rho(x))d\mu_\rho(x) = \int_A f d\mu_\infty. \tag{2}$$

Theorem 6. *The family of metric spaces* $(B_\rho(1), d_\rho, \mu_\rho)$ *converges in the Gromov–Hausdorff measure topology to* $(B_\infty(1), d_\infty, \mu_\infty)$ *as ρ goes to infinity.*

To prove the convergence (2) let A be a domain in G_∞, then $\pi_\rho^{-1}(A)$ belongs to G_ρ and $\delta_\rho \circ \pi_\rho^{-1}(A)$ belongs to G. We will denote by $*$ the law group of G. Let $z_1, \ldots, z_k$ and $\zeta_1, \ldots, \zeta_l$ be elements of Γ such that $\zeta_j * D_\Gamma \cap \delta_\rho \circ \pi_\rho^{-1}(A) \neq \emptyset$ for any j, and

$$\bigcup_i z_i * D_\Gamma \subset \delta_\rho \circ \pi_\rho^{-1}(A) \subset \bigcup_j \zeta_j * D_\Gamma.$$

Let us notice that

$$\mu_g(D_\Gamma) = \frac{\mu_g(D_\Gamma)}{\mu(\pi(D_\Gamma))}\mu(\pi(D_\Gamma)) = \mu_\infty(\pi(D_\Gamma)).$$

Then we get

$$\sum_i \inf_{\delta_\rho \circ \pi_\rho^{-1}(x) \in z_i * D_\Gamma} f(x)\mu_\infty(\pi(D_\Gamma)) \le \int_{\delta_\rho \circ \pi_\rho^{-1}(A)} f(\tilde{\delta}_{1/\rho} \circ \pi(x))d\mu_g(x)$$

$$\le \sum_j \sup_{\delta_\rho \circ \pi_\rho^{-1}(x) \in \zeta_j * D_\Gamma} f(x)\mu_\infty(\pi(D_\Gamma)).$$

Dividing both sides by ρ^{d_h} (see 2.1), we get:

$$\sum_i \inf_{x \in \pi_\rho \circ \delta_{1/\rho}(z_i * D_\Gamma)} f(x)\mu_\infty(\tilde{\delta}_{1/\rho} \circ \pi(D_\Gamma)) \le \int_{\pi_\rho^{-1}(A)} f(\pi_\rho(x))d\mu_\rho$$

$$\le \sum_j \sup_{x \in \pi_\rho \circ \delta_{1/\rho}(\zeta_j * D_\Gamma)} f(x)\mu_\infty(\tilde{\delta}_{1/\rho} \circ \pi(D_\Gamma)).$$

Then the extremal terms are Riemann sums that converge toward $\int_A f d\mu_\infty$.

3.2.c.   We are now able to define and identify the asymptotic volume by

$$\mathrm{Asvol}(g) = \lim_{\rho \to \infty} \frac{\mu_g(B_g(\rho))}{\rho^{d_h}} = \mu_\infty(B_\infty(1)).$$

3.3. Convergence of the elements of the set $\mathcal{L}^2$

3.3.a.   For $\rho \in \overline{\mathbb{R}}$, $L_\rho^2 = L^2(B_\rho(1), d\mu_\rho)$ will be the space of square integrable functions over the ball $B_\rho(1)$, which is a Hilbert space with the scalar product

$$(u, v)_\rho = \int_{B_\rho(1)} u\bar{v} \, d\mu_\rho.$$

Its norm will be denoted by $|\cdot|_\rho$.

   Let $\mathcal{L}^2$ be the set of nets $(u_\rho)_{\rho \in \mathbb{R}^+}$ such that for $1 \le \rho \le \infty$, $u_\rho \in L_\rho^2$. Thanks to the Gromov–Hausdorff measured convergence of balls, we can give a meaning to the sentence "the net $(u_\rho)_{\rho \in \mathbb{R}^+}$ converges" in the following way.

Definition 7. Let $(u_\rho)_{\rho \in \mathbb{R}^+}$ be an element of $\mathcal{L}^2$, we say that it strongly converges to u_∞ if and only if there exists a net (v_α) in $C^0(B_\infty(1))$ strongly converging to u_∞ in L_∞^2, and such that

$$\lim_\alpha \limsup_\rho |v_\alpha \circ \pi_\rho - u_\rho|_\rho = 0.$$

This allows us to introduce the weak convergence as follows.

Definition 8. Let $(u_\rho)_{\rho \in \mathbb{R}^+}$ be an element of $\mathcal{L}^2$. We say that it converges weakly to u_∞, if for every strongly convergent net $(v_\rho)_{\rho \in \mathbb{R}^+}$ of $\mathcal{L}^2$, we have

$$\lim_{\rho \to +\infty} (u_\rho, v_\rho)_\rho = (u_\infty, v_\infty)_\infty.$$

For the properties of these convergences see our previous work [Ver04] and [Ver01]. It suffices to say that they satisfy the usual properties of weak and strong convergence in L^2.

3.3.b. We shall say that a function f is periodic with respect to Γ (the co-compact subgroup) if for every $\gamma \in \Gamma$ and $x \in G$ we have $f(\gamma * x) = f(x)$. Thus the metric $\tilde{g}$ lifted from M^n to G is periodic with respect to Γ.

To finish this section remark, that it is not difficult to adapt the proof of the limit (2) to obtain (see [BBJR95] page 431).

Lemma 9. *Let h be a function that is periodic with respect to Γ on G. Let h_ρ be defined on G_ρ by $h_\rho(x) = h(\delta_\rho x)$. Then $(h_\rho)_{\rho \in \mathbb{R}^+}$ weakly converges in $\mathcal{L}^2$ toward*

$$h_\infty = \frac{1}{\mu_g(D_\Gamma)} \int_{D_\Gamma} h \, d\mu_g.$$

I.e. for any $u_\rho \to u_\infty$ strongly in $\mathcal{L}^2$, we have

$$\int_{B_\rho(1)} u_\rho h_\rho d\mu_\rho \to h_\infty \int_{B_\infty(1)} u_\infty d\mu_\infty.$$

4. Behavior of the eigenvalues: setting

4.1. The Albanese metric

4.1.a. Let D_Γ be a fundamental domain for the action of Γ on G. Let χ^i be the unique solution (up to an additive constant) of

$$\Delta \chi^i = \Delta x_i \text{ on } D_\Gamma, \quad \text{for } 1 \le i \le r,$$

that is periodic with respect to Γ.

Let us define the operator Δ_∞ by

$$\Delta_\infty f = -\frac{1}{\mathrm{Vol}_g(M)} \sum_{1 \le i,j \le d_1} \left(\int_{D_\Gamma} g^{ij} - \sum_{k=1}^n g^{ik} X_k \cdot \chi^j \, d\mu_g \right) X_i^\infty \cdot X_j^\infty f \quad (3)$$

$$= -\sum_{1 \le i,j \le d_1} q^{ij} X_i^\infty \cdot X_j^\infty f. \quad (4)$$

Remark that $\eta_j(x) = \chi^j(x) - x_j$ is a harmonic function on G, and by construction so are the 1-forms $d\eta_i$ on the nilmanifold. It is not difficult to show the following.

Proposition 10. *Let $\langle \cdot, \cdot \rangle_2$ be the scalar product induced on 1-forms by the Riemannian metric g. Then*

$$q^{ij} = \frac{1}{\mathrm{Vol}(g)} \langle d\eta_i, d\eta_j \rangle_2 = q^{ji}.$$

Thus Δ_∞ is an Hypoelliptic operator.

4.1.b.   Recall that thanks to Nomizu's work [Nom54], $H_1(M^n, \mathbb{R}) \equiv V_1$, hence by duality we get that the dimension of $H^1(M^n, \mathbb{R})$ is d_1. Remark that (q^{ij}) is the matrix of the L^2 normalized scalar product on harmonic 1-forms, written in the basis $(d\eta_i)$, hence on $H^1(M^n, \mathbb{R})$ by Hodge's theorem (whose norm will be written $\|\cdot\|_2$). By duality it gives a scalar product on $H_1(M^n, \mathbb{R})$ (whose norms will be written $\|\cdot\|_2^*$).

The norm $\|\cdot\|_2^*$ induces another Carnot–Carathéodory metric, which we shall call the Albanese metric and denote by d_{al}, as follows. Take on $\mathcal{H}_e \equiv H_1(M, R)$ (the horizontal subspace of the tangent space at the unit element) an orthonormal basis $Y_1(e), \dots, Y_{d_1}(e)$ for $\|\cdot\|_2^*$. It induces a left-invariant orthonormal frame field on $\mathcal{H}$, and for any admissible curve $\gamma: [a, b] \to G_\infty$, we have that $\dot\gamma(t) = \sum_{i=1}^{d_1} \alpha_i(t) Y_i(\gamma(t))$. Then the Albanese length of γ is $l_{\mathrm{al}}(\gamma) = \int_a^b \left(\sum_{i=1}^{d_1} \alpha_i^2(t) \right)^{1/2} dt$, and the Albanese distance between two points is the shortest Albanese length among all admissible curves joining them. A comparison of the L^2 norm and the L^∞ norm gives the following

Proposition 11. *For every 1-form α and $\gamma \in H_1(M^n, \mathbb{R})$ we have*

$$\|\alpha\|_2 \leq \|\alpha\|_\infty^* \quad \text{and} \quad \|\gamma\|_\infty \leq \|\gamma\|_2^*. \tag{5}$$

In other words the unit ball $B_{\mathrm{al}}(1)$ of the Albanese metric d_{al} is included in $B_\infty(1)$.

Proof. For α a 1-form we have

$$\|\alpha\|_2 = \left(\frac{1}{\mathrm{Vol}_g(M)} \int_M |\alpha|^2 d\mu_g \right)^{1/2} \leq \sup_{x \in M} |\alpha(x)| = \|\alpha\|_\infty^*.$$

Hence our proposition follows, first by passing to the quotient and by duality, and finally by integrating over admissible paths.   □

4.2. The eigenvalues, at last! All the balls considered here, will be centered at a fixed point x_0 of the universal covering of $M^n = (G/\Gamma, g)$. We study the eigenvalues of the Dirichlet problem on $B_g(\rho)$, the geodesic ball of radius ρ:

$$\begin{cases} \Delta\phi = \lambda\phi & \text{on } B_g(\rho); \\ \phi = 0 & \text{on } \partial B_g(\rho). \end{cases}$$

It is well known that the eigenvalues are a discrete family accumulating at infinity. We shall denote them by $\lambda_1(\rho) \le \lambda_2(\rho) \le \cdots \le \lambda_i(\rho) \ldots$.

R. Brooks's theorem [Bro85] on the first eigenvalue of the whole group implies that as ρ goes to infinity, $\lambda_1(\rho) \to 0$. We are going to estimate how fast it converges in our case.

4.2.a.　On each G_ρ (see 3.1.c), we pulled back the lifted metric of M^n on G, $\tilde{g}$, and rescaled it in the following way

$$g_\rho = \frac{1}{\rho^2}(\delta_\rho)^*\tilde{g}.$$

This gives a net of Riemannian manifolds $(G_\rho, g_\rho)_{\rho \in \mathbb{R}^+}$. Let $B_\rho(1)$ be the unit geodesic ball for the metric g_ρ, and consider the Dirichlet problem for Δ_ρ the Laplacian associated to g_ρ, i.e.,

$$\begin{cases} \Delta_\rho \phi = \psi & \text{on } B_\rho(1); \\ \phi = 0 & \text{on } \partial B_\rho(1). \end{cases}$$

To a function f from $B_g(\rho)$ to $\mathbb{R}$ let us associate a function f_ρ on $B_\rho(1)$ by $f_\rho(x) = f(\delta_\rho \cdot x)$. Then it is an easy calculation to see that for any $x \in B_\rho(1)$,

$$\rho^2(\Delta f)(\delta_\rho \cdot x) = (\Delta_\rho f_\rho)(x).$$

This implies that the eigenvalues of Δ_ρ on $B_\rho(1)$ are exactly the eigenvalues of Δ on $B_g(\rho)$ multiplied by ρ^2.

Enlightened by what happens on tori we would like to show that the net of resolvents of the Laplacians $(\Delta_\rho)_{\rho \in \mathbb{R}^+}$ compactly converges towards the resolvent of Δ_∞, which implies the convergence of the spectrum towards the spectrum of Δ_∞ for the Dirichlet problem on $B_\infty(1)$ (see Theorem 15, 17 and 21 of [Ver04]).

4.3. Upper bound on the eigenvalues, lower bound on the asymptotic volume and the equality cases.　Recall that $B_{al}(1)$ is the unit ball for the Albanese metric on G_∞, centered at the unit element. Let D be a bounded domain of G_∞, and denote by $\lambda_i^\infty(D)$ the ith eigenvalue of Δ_∞ on D for the Dirichlet problem. Then by Proposition 11, we have $B_\infty(1) \supset B_{al}(1)$. Thus by the min-max property, for any i, we obtain

$$\lambda_i^\infty(B_\infty(1)) \le \lambda_i^\infty(B_{al}(1)). \tag{6}$$

Following the maximum principle (see J.-M. Bony [Bon69]), equality holds if and only if the two balls coincide, and thus the norms in Proposition 11 also coincide. The same argument also shows that we have equality in the following estimate if and only if the stable norm and the Albanese metric coincide.

Proposition 12. *Let (M^n, g) be a nilmanifold. Let G_∞ be the limit group at infinity associated to the universal covering of M^n. Then the asymptotic volume of M^n satisfies the following inequality:*

$$\text{Asvol}(g) \geq \mu_g(M^n) \frac{\mu(B_{\text{al}}(1))}{\mu(\pi(D_\Gamma))}.$$

Here, μ is a Haar measure on G_∞, $B_{\text{al}}(1)$ is the unit ball for the Albanese distance and D_Γ a fundamental domain on the universal covering of M^n.

Proof. From 11, for any Haar measure μ, one gets the following inequality:

$$\mu(B_{\text{al}}(1)) \leq \mu(B_\infty(1)).$$

We can conclude by taking the Haar measure μ_∞ for μ (see Section 3.2), giving the asymptotic volume. $\qquad\square$

5. Homogenization and proof of Theorem 4

The first step consists in showing the convergence of the metric geodesic balls with respect to the Gromov–Hausdorff measure topology (completed in 3.2).

5.1. Asymptotic compactness

5.1.a. Let us now define the various functional spaces involved. Recall (see 3.3.a) that for $\rho \in \overline{\mathbb{R}}^+$, $L_\rho^2 = L^2(B_\rho(1), d\mu_\rho)$ is the Hilbert space of square integrable functions over the ball $B_\rho(1)$ with the norm $|\cdot|_\rho$.

5.1.b. Following the usual nomenclature, we will be interested in the following spaces, for an r-step nilmanifold (see Section 2.1):

$$H_\rho^1(B_\rho(1)) = \left\{ v \mid v, X_i^\rho \cdot v \in L^2(B_\rho(1), d\mu_\rho), \ 1 \leq \alpha(i) \leq r \right\} \tag{7}$$

$$\text{(resp. } H_\infty^1(B_\infty(1)) = \left\{ v \mid v, X_i^\infty \cdot v \in L^2(B_\infty(1), d\mu_\infty), \ 1 \leq i \leq d_1 \right\}). \tag{8}$$

These spaces become Hilbert spaces when endowed with the quadratic forms $\|\cdot\|_\rho$, defined by

$$\|v\|_\rho^2 = |v|_\rho^2 + \sum_{1 \leq \alpha(i) \leq r} \|X_i^\rho \cdot v\|_\rho^2 \tag{9}$$

$$\text{(resp. } \|v\|_\infty^2 = |v|_\infty^2 + \sum_{1 \leq i \leq d_1} \|X_i^\infty \cdot v\|_\infty^2). \tag{10}$$

We will denote by $H^1_{\rho,0}(B_\rho(1))$ the closure in $H^1_\rho(B_\rho(1))$, with respect to the norm $\|\cdot\|_\rho$, of the space of $C^\infty(B_\rho(1))$ functions with compact support in $B_\rho(1)$.

5.1.c.   We can define a self adjoint operator on L^2_ρ, whose resolvent will be R^ρ_λ for $\lambda \in \mathbb{R}$, thanks to the Friedrichs extension of the Laplacian (sub-Laplacian for Δ_∞) defined on $H^1_{\rho,0}(B_\rho(1))$, endowed with the following quadratic form

$$\|v\|^2_{\rho,0} = |v|^2_\rho + (v, \Delta_\rho v)_\rho.$$

Now for a bounded net in $\left(H^1_{\rho,0}(B_\rho(1))\right)_{\rho\in\mathbb{R}^+}$ with respect to these quadratic forms we have the following Lemma.

Lemma 13. *Let $(u_\rho)_{\rho\in\mathbb{R}^+}$ be a net with $u_\rho \in H^1_{\rho,0}(B_\rho(1))$ for every $\rho \geq 1$, and assume the existence of a constant C such that for every $\rho \geq 1$, we have*

$$\|u_\rho\|_{\rho,0} \leq C.$$

Then there is sub-net which is strongly convergent in $\mathcal{L}^2$.

Proof.   Let B be a compact set such that $\bigcup_{\rho\in\mathbb{R}^+} \pi_\rho(B_\rho(1)) \subset B \subset G_\infty$. We are going to show that the strong convergence in $L^2(B, \mu_\infty)$ implies the strong convergence in $\mathcal{L}^2$. Then the compact embedding of $H^1_\infty(B)$ in $L^2(B, \mu_\infty)$ will conclude the proof.

Let us first notice that the periodicity with respect to Γ, and the co-compactness of Γ gives the existence of two constants α and β such that (we suppose the norms are defined on B, and identify B and $\pi_\rho^{-1}B$)

$$\alpha|v|_\infty \leq |v|_\rho \leq \beta|v|_\infty.$$

Let us start by taking a net (v_ρ), strongly converging in $L^2(B, \mu_\infty)$ to v_∞. We also assume $v_\rho \circ \pi_\rho \in H^1_{\rho,0}(B_\rho(1))$ for every ρ and is zero outside $B_\rho(1)$ (because it is all we need).

First we will prove that $v_\infty \in L^2_\infty$ (we mean that, outside $B_\infty(1)$, v_∞ can be considered equal to zero), indeed, the strong L^2 convergence implies the existence of a subnet of (v_ρ) which simply converges almost everywhere to v_∞. Hence the Gromov–Hausdorff convergence implies that v_∞ is zero almost everywhere on $B\setminus B_\infty$.

Thus, let us take $c_p \in C^\infty_0(B_\infty(1))$, $p \in \mathbb{N}$, such that $(c_p)_{p\in\mathbb{N}}$ is a sequence of functions strongly converging to v_∞ in L^2_∞. We have

$$|c_p \circ \pi_\rho - v_\rho \circ \pi_\rho|_\rho \leq \beta|c_p - v_\infty|_\infty + \gamma|v_\infty - v_\rho|_\infty.$$

Now let $\varepsilon > 0$. Then for p large enough, $\beta|c_p - v_\infty|_\infty \leq \varepsilon$. We fix p large enough, and take ρ large enough for the second term to converge to 0, which gives us the strong convergence we needed (see Definition 7).

Now to conclude, observe that from the assumptions, the net $(u_\rho \circ \pi_\rho^{-1})$ (if need be we extend this function by zero outside $B_\rho(1)$) is bounded in $H_\infty^1(B)$, hence using the compact embedding of $H_\infty^1(B)$ in $L^2(B, \mu_\infty)$ (with the right regularity assumption on the boundary of B), we can extract a strongly converging net in $L^2(B, \mu_\infty)$ and by what we just did in $\mathcal{L}^2$. $\qquad\square$

5.2. Compact convergence of the resolvents. For $\lambda > 0$ and $\rho > 1$, let $a_\lambda^\rho(u, v) = (\Delta_\rho u, v)_\rho + \lambda(u, v)_\rho$ and G_λ^ρ be the operator from L_ρ^2 to $H_{\rho,0}^1 \subset L_\rho^2$ such that

$$a_\lambda^\rho(G_\lambda^\rho f, \phi) = (f, \phi)_\rho \quad \text{for all } \phi \in H_{\rho,0}^1. \tag{11}$$

For any $u, v \in H_{\infty,0}^1$, let

$$a_\lambda^\infty(u, v) = \int_{B_\infty(1)} q^{ij} \, X_i^\infty u \, X_j^\infty v \, d\mu_\infty + \lambda(u, v)_\infty.$$

Then we define $G_\lambda \colon L_\infty^2 \to H_{\infty,0}^1$, by

$$a_\lambda^\infty(G_\lambda F, \Phi) = (F, \Phi)_\infty \quad \text{for all } \Phi \in H_{\infty,0}^1. \tag{12}$$

The aim of this part is the following theorem, after noticing that $R_\lambda^\rho = -G_{-\lambda}^\rho$ and $R_\lambda^\infty = -G_{-\lambda}$.

Theorem 14. *For every $\lambda < 0$, the net of resolvents $(R_\lambda^\rho)_{\rho \in \mathbb{R}^+}$ of the net of Laplacians $(\Delta_\rho)_{\rho \in \mathbb{R}^+}$ converges compactly to R_λ^∞, the resolvent of Δ_∞ from the homogenized problem, i.e, for any net $(u_\rho)_{\rho \in \mathbb{R}^+}$ of $\mathcal{L}^2$ weakly converging, the net $(R_\lambda^\rho \cdot u_\rho)_{\rho \in \mathbb{R}^+}$ of $\mathcal{L}^2$ strongly converges to $R_\lambda^\infty \cdot u_\infty$.*

The proof is an adaptation of Tartar's method of oscillating test functions (see Chapter 8 of [CD99] for the classical method).

Proof. First step: Let f_ρ be a weakly convergent net to f in $\mathcal{L}^2$. Then up to subnets

$$u_\rho = G_\lambda^\rho f_\rho \quad \to \quad \tilde{u}_\lambda \text{ strongly in } \mathcal{L}^2; \tag{13}$$

$$P_\rho = (g_\rho^{ij}) \nabla G_\lambda^\rho f_\rho \quad \to \quad \tilde{P}_\lambda \text{ weakly in } \mathcal{L}^2. \tag{14}$$

One obtains (13) because the net $(f_\rho)_{\rho \in \mathbb{R}^+}$ is uniformly bounded in $\mathcal{L}^2$, and for all $\rho \in \mathbb{R}$, f_ρ is also bounded in H_ρ^{-1}, the dual space of $H_{\rho,0}^1$. Thus thanks to equality (11) and Lemma 13, we can extract a strongly converging net in $\mathcal{L}^2$ from the uniformly bounded net $(G_\lambda^\rho f_\rho)_{\rho \in \mathbb{R}^+}$ (with respect to the norms $(\|\cdot\|_{\rho,0})_{\rho \in \mathbb{R}^+}$). To get (14), simply remark that $(P_\rho)_{\rho \in \mathbb{R}^+}$ is also bounded in $\mathcal{L}^2$.

Now for any $\phi_\infty \in L_\infty^2$, by passing to the limit in equation (11), we obtain

$$\int_{B_\infty(1)} \tilde{P}_\lambda \cdot \nabla_{\mathcal{H}} \phi_\infty \, d\mu_\infty + \lambda(u_\lambda^*, \phi_\infty)_\infty = (f, \phi_\infty)_\infty. \tag{15}$$

Before passing to the next step, remark that $\tilde{P}_\lambda$ is horizontal. Indeed denoting by P_ρ^i and $\tilde{P}_\lambda^i$ the coordinates of P_ρ and $\tilde{P}_\lambda$, we have

$$P_\rho^i = (g_\rho^{ij}) \nabla G_\lambda^\rho f_\rho = \rho^{2-\alpha(i)-\alpha(j)} (g^{ij}(\delta_\rho x)) \nabla G_\lambda^\rho f_\rho.$$

So if $\alpha(i) \geq 2$, then this net of coordinates strongly converges to 0 in $\mathcal{L}^2$, because $(g^{ij}(\delta_\rho x)) \nabla G_\lambda^\rho f_\rho$ is also bounded for any $\rho \in \mathbb{R}^+$.

Second step: This step consists in showing $\tilde{P}_\lambda = (q^{ij}) \nabla_{\mathcal{H}} \tilde{u}_\lambda$ on $B_\infty(1)$, as it induces $\tilde{u}_\lambda = G_\lambda f$.

We just give the ingredient needed to copy the classical proof (see also [Ver04], section 4.3).

Consider $\chi^k(y)$ (see 4.1.a) such that its mean value on a fundamental domain is zero, and for every $k = 1, \ldots, d_1$, define the oscillating function

$$w_\rho^k(x) = x_k - \frac{1}{\rho} \chi^k(\delta_\rho x). \tag{16}$$

Then we have

$$w_\rho^k \to x_k \quad \text{strongly in } \mathcal{L}^2. \tag{17}$$

Using the usual trick in Tartar's method, we obtain for every $\varphi \in C_0^\infty(B_\infty(1))$ and for ρ large enough, for the support of φ to be in $\pi_\rho(B_\rho(1))$:

$$\int_{B_\rho(1)} g_\rho^{ij} \left(X_j^\rho u_\rho (X_i^\rho(\varphi \circ \pi_\rho)) w_\rho - X_j^\rho w_\rho (X_i^\rho(\varphi \circ \pi_\rho)) u_\rho \right) d\mu_\rho$$
$$= \int_{B_\rho(1)} f_\rho \, w_\rho \, \varphi \circ \pi_\rho \, d\mu_\rho - \lambda \int_{B_\rho(1)} \varphi \circ \pi_\rho \, u_\rho \, w_\rho \, d\mu_\rho. \tag{18}$$

To pass to the limit in this identity, we use the following facts:

Fact 1. $(X_i^\rho(\varphi \circ \pi_\rho)) w_\rho^k$ strongly converges to $(X_i^\infty \varphi) x_k$ in $\mathcal{L}^2$ because, writing the left multiplication by x in G_ρ as l_x^ρ, we have

$$X_i^\rho(\varphi \circ \pi_\rho)_{|x} = d\varphi_{\pi_\rho \circ l_x^\rho(e)} \circ d\pi_{\rho|l_x^\rho(e)} \circ dl_x^\rho \cdot X_i^\rho(e).$$

Now by definition $l_x^\rho \to l_x^\infty$ and $\pi_\rho \to \text{id}_{G_\infty}$, which explains why

$$X_i^\rho(\varphi \circ \pi_\rho) \to X_i^\infty \varphi$$

pointwise (and weakly $\mathcal{L}^2$ from the claim in the proof of Section 3.2).

Fact 2. For $1 \leq i, j \leq d_1$, $g_\rho^{ij} X_i^\rho w_\rho^k$ is periodic with respect to $\delta_{1/\rho}\Gamma$ and weakly converges in $\mathcal{L}^2$, by Lemma 9, towards its mean value

$$q^{jk} = \frac{1}{\mu_g(D_\Gamma)} \int_{D_\Gamma} \left(g^{ij}(y)(\delta_{ik} - X_i\chi^k(y))\right)d\mu_g.$$

Fact 3. For $\alpha(i)+\alpha(j) > 2$, $g_\rho^{ij} X_i^\rho w_\rho^k = \rho^{2-\alpha(i)-\alpha(j)} g^{ij}(\delta_\rho x) X_i^\rho w_\rho^k$, thus this term weakly converges in $\mathcal{L}^2$ towards 0.

Hence the identity (18) becomes

$$\int_{B_\infty(1)} (\tilde{P}_\lambda^j x_k - q^{jk}\tilde{u}_\lambda)X_j^\infty \varphi d\mu_\infty = \int_{B_\infty(1)} f\, x_k\, \varphi\, d\mu_\infty - \lambda \int_{B_\infty(1)} \varphi\, \tilde{u}_\lambda\, x_k d\mu_\infty.$$
$$(19)$$

Furthermore, if we put $\phi_\infty = \varphi x_k$ into equation (15) and subtract the result from the equality (19), then we obtain the following identity in terms of distribution.

$$-\sum_{j=1}^{d_1} X_j^\infty(\tilde{P}_\lambda^j x_k - q^{jk}\tilde{u}_\lambda) = -\sum_{j=1}^{d_1} X_j^\infty \tilde{P}_\lambda^j\, x_k \iff \tilde{P}_\lambda^k = \sum_{j=1}^{d_1} q^{jk}\, X_j^\infty \tilde{u}_\lambda. \quad \square$$

5.3. Conclusion. Theorem 14 gives the compact convergence of the resolvents. Hence we can use Theorem 21 in [Ver04], which states that if the resolvents are compact, and they converge compactly, then the net of kth eigenvalues converges to the kth eigenvalue of the limit operator.

6. Emphasis on the Heisenberg Groups in the equality case

The aim of this part is to characterize metrics for which the inequality (6) is an equality (see also Theorem 2) for a class of nilmanifolds that contains the Heisenberg nilmanifolds. The first thing to remark, which is always true, is that equality holds if and only if the stable norm and the Albanese metric are equal. In that case, all harmonic 1-forms are of constant pointwise norm (same proof as in [Ver04]). Now let us introduce the pseudo-left-invariant metrics.

Definition 15. Let $N^{n+1} = \Gamma\backslash G$ be a nilmanifold such that G is 2-step nilpotent with one dimensional kernel. Let p be a submersion of N^{n+1} onto a flat torus $\mathbb{T}^n$. Let $(\alpha_1, \ldots, \alpha_n)$ be the lift of an orthonormal basis of harmonic 1-forms over the torus, and choose a 1-form ϑ of N^{n+1} such that $d\vartheta = p^*b$, where b is a closed 2-form

over the torus (in other words we chose a connection). Let g_ϑ be the Riemannian metric such that the dual basis of $(\alpha_1, \ldots, \alpha_n, \vartheta)$ is orthonormal. Thus p becomes a Riemannian submersion. We will call such a metric pseudo-left-invariant or bundle-like.

The idea is that if the 2-form b has constant coefficients, then ϑ can be chosen so that the above construction gives a left invariant metric. Thus this pseudo-left-invariant metric can be seen as a perturbation of a left invariant metric, obtained by perturbing a left invariant basis of vector fields.

We are now able to give our precise claim.

Lemma 16. *Let $(\mathbb{H}_{2n+1}, g)$ be the $2n + 1$-dimensional Heisenberg group, equipped with a periodic metric. Then its stable norm coincides with its Albanese metric if and only if g is pseudo-left-invariant.*

Remark also that in the case of the 3-dimensional Heisenberg group, the function $\lambda_1(g, \text{Alb})$ in Theorem 2 is actually a constant that does not depend on the metric. This is due to the fact that, up to isometries, there is only one left-invariant sub-Riemannian metric in that case (see chapter IV of [Ver01]). Hence in that case, the theorem has a similar form as the theorem for tori (see [Ver04]), for which the function is also constant because up to isometries there is only one Euclidean metric on $\mathbb{R}^n$.

Actually, we have a result that is slightly more general than Lemma 16. We focus on 2-step nilmanifolds, whose Lie algebras have a 1-dimensional center.

Lemma 17. *Let (M^{n+1}, g) be a 2-step nilmanifold whose center is of dimension 1. Then its stable norm and its Albanese metric coincide if and only if the metric is pseudo-left-invariant.*

As the Albanese metric and the stable norm coincide if and only if all harmonic 1-forms are of constant norm, Lemma 17 is a consequence of the main theorem in [NV04]:

Theorem 18 (P.-A. Nagy, C. Vernicos [NV04]). *Let (M^{n+1}, g) be a Riemannian manifold with first Betti number equal to n, all of whose harmonic 1-forms are of constant norm. Then (M^{n+1}, g) is a 2-step nilmanifold whose center is of dimension 1, and g is pseudo-left-invariant.*

7. Graded nilmanifolds with totally geodesic fibers over a torus

There is one last particular case we would like to study, the case where the nilmanifold is graded (i.e. its algebra is nilpotent and graded as defined in Section 2.1), and the

metric on (M^n, g) is as follows. We suppose that the first Betti number $b_1(M^n) = k$, and we recall that $\mathcal{H}$ is the horizontal distribution coming from V_1 (see Sections 3.1.a and 2.1). Moreover we assume that we have the following Riemannian submersion, with totally geodesics fibers and with a metric equivariant on the fibers:

$$[M, M] \hookrightarrow (M^n, g) \xrightarrow{\;p\;} (\mathbb{T}^k, \check{g}),$$

where dp_x is an isometry (we write $\hat{g} = g_{|\mathcal{H}}$) from $(\mathcal{H}_x, \hat{g}_x)$ to $(T_{p(x)}\mathbb{T}^k, \check{g}_{p(x)})$.

Then, in the case of equality in Theorems 1 and 2, the Albanese map is a Riemannian submersion, which implies that $\check{g}$ is flat. Which in turn, using our assumptions implies that the metric g is left invariant (indeed see Chapter 9 Section F in [Bes87]). In other words:

Proposition 19. *Let (M, g) satisfy the above assumptions. The Albanese metric and the stable norm coincides if and only if the metric is left invariant.*

In other words, we could say heuristically that for sub-Riemannian metrics the equality case in Theorem 2 (which holds in that context too, see [Ver01] for the convergence of the spectrum) characterizes the left-invariant sub-Riemannian metrics.

8. On the long time asymptotics of the heat kernel

Let $(G/\Gamma, g)$ be a nilmanifold and $(G, \tilde{g})$ its universal cover with the lifted metrics. Recall that we associated to this Lie group the net (G_ρ, g_ρ) of Riemannian manifolds. Let us focus on the heat kernel:

$$\begin{cases} \frac{\partial u}{\partial t} + \Delta u = 0 & \text{in }]0, +\infty[\times G; \\ u(0, x) = u_o(x). \end{cases} \tag{20}$$

Let us introduce the rescaled functions on G_ρ,

$$u_\rho(t, x) = \rho^{d_h} u(\rho^2 t, \delta_\rho x), \quad \rho > 0.$$

Then an easy computation shows that u is a solution of (20) if and only if u_ρ is a solution of

$$\begin{cases} \frac{\partial u_\rho}{\partial t} + \Delta_\rho u_\rho = 0 & \text{in }]0, +\infty[\times G_\rho; \\ u_\rho(0, x) = \rho^{d_H} u_0(\delta_\rho x). \end{cases} \tag{21}$$

Thus the study of $u(t, \cdot)$ as t goes to infinity is related to the study of $u_\rho(1, \cdot)$ as $\rho \to \infty$. We can imitate the proof of Theorem 14 to obtain:

Theorem 20. *The net of resolvent (R_λ^ρ) weakly converges to the resolvent (R_λ^∞) of Δ_∞ on G_∞.*

Imitating the proof of Theorems 4 and 6 in [ZKON79], as in [BBJR95], we get the following theorem (let $d_{\mathrm{al}}(e, x) = |x|$ be the Albanese distance between the unit element and x).

Theorem 21. *The fundamental solution $k(t, x, y)$ of* (20) *has the following asymptotic expansion*

$$k(t, x, y) = k_\infty(t, \pi(x), \pi(y)) + t^{-\frac{d_h}{2}} \theta(t, x, y).$$

Here $k_\infty(t, x, y)$ is the fundamental solution of

$$\frac{\partial u_\infty}{\partial t} + \Delta_\infty u_\infty = 0 \ in \ ]0, +\infty[\times G_\infty,$$

and $\theta(t, x, y) \to 0$ uniformly as $t \to \infty$ on $|x|^2 + |y|^2 \le at$, for any fixed constant $a > 0$.

The next theorem follows by integrating the previous one.

Theorem 22. *Let $u_0 \in L^1(G) \cap L^\infty(G)$. Then $u(t, x)$, the solution of* (20)*, has the following asymptotic expansion:*

$$u(t, x) = c_0 t^{-\frac{d_h}{2}} \int_G u_0(y) dy + t^{-\frac{d_h}{2}} \theta(t, x),$$

and $\theta(t, x)$ converges uniformly to 0 for $|x| < R$, where R is a positive constant, and c_0 depends on Δ_∞.

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