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On the cohomology of generalized triangle groups

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Abstract. We describe a general approach to constructing small free $\mathbb{Z}\Gamma$-resolutions for certain infinite isometry groups Γ. We apply the method to a class of generalized triangle groups and use the resolution to compute the integral homology of these groups. In illustrating the method we also obtain resolutions for the classical triangle groups and for their infinite cyclic central extensions, considered previously by Strebel.

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1. Introduction

It was explained in [4] how, for an arbitrary finite group H, one can construct a small free $\mathbb{Z}H$-resolution of $\mathbb{Z}$ by first choosing a faithful linear representation $\alpha\colon H \to \mathrm{GL}(\mathbb{R}^n)$ and then considering the cellular chain complex of the convex hull of the orbit of a suitable point $v \in \mathbb{R}^n$. Our present aim is to show that the method can also be applied to infinite groups Γ for which one has an appropriate representation as a group of isometries of some suitable space X. For such groups our first approximation to a free resolution is the cellular chain complex of X corresponding to a tessellation arising from the action of Γ.

As an application, we calculate resolutions for most of the groups defined by the following presentation:

$$G(l,m,n) = \langle\, x, y \mid x^l = y^m = [x,y]^n = 1 \,\rangle, \tag{1}$$

where $[x,y] = xyx^{-1}y^{-1}$ and, as throughout this paper, l, m, n denote integers with $|l|, |m|, |n| \geq 2$. The groups $G(l,m,n)$ have a long history, dating back to Coxeter and Sinkov [3], [18]. More recently they have been of interest in the context of generalized triangle groups [5], [9], [10], [14], [16]. (A generalized triangle

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group is a group that can be presented in the form $\langle\, x, y \mid x^l = y^m = w(x, y)^n = 1\,\rangle$, where $w(x, y)$ is an element of the free product $\langle\, x, y \mid x^l = y^m = 1\,\rangle$.) Generalized triangle groups have been studied for a variety of algebraic, geometric and topological reasons [1], [5], [6], [9], [10], [13], [15], [20]; Euler characteristics have been calculated [7], [23], but the cohomology of generalized triangle groups has not yet been considered. Our results can therefore be regarded as a first step towards such investigations.

This paper is structured as follows. In Section 2 we describe the method we use to obtain the resolutions. In Section 5 we use the method to provide a free $\mathbb{Z}G$-resolution for the generalized triangle group $G = G(l, m, n)$ (with some restrictions on l, m, n), from which we calculate the integral homology. In the construction of this resolution we will require resolutions for the classical triangle groups

$$T(l, m, n) = \langle\, a, b \mid a^l = b^m = (ab^{-1})^n = 1\,\rangle. \tag{2}$$

For this reason, in Section 3 we use the triangle groups to illustrate our method and to provide the necessary resolutions. As a slight digression, in Section 4 we also construct a resolution for the following cyclic central extensions of the classical triangle groups, whenever these are infinite:

$$S(l, m, n) = \langle\, a, b \mid a^l = b^m = (ab^{-1})^n\,\rangle. \tag{3}$$

A different resolution for these extensions has previously been derived by Strebel [19] using alternative methods.

2. General method

Let Y denote either euclidean space E^n or hyperbolic space H^n, and let Γ be a group with a representation as isometries of Y. We shall make the following assumptions:

(1) Y can be embedded into some contractible CW-space X such that the action of Γ on Y extends to a cellular action on X.

(2) For the stabilizer group Γ_e of each cell e of X we have some free $\mathbb{Z}\Gamma_e$-resolution

$$R_*^{\Gamma_e}\colon \quad \cdots \longrightarrow R_2^{\Gamma_e} \longrightarrow R_1^{\Gamma_e} \longrightarrow R_0^{\Gamma_e}$$

of the integers.

The CW-space X can often be constructed as follows. Suppose $v \in Y$ is a vector such that the orbit $v^\Gamma = \{gv \mid g \in \Gamma\}$ is a discrete subset of Y. The *Dirichlet–Voronoi region* centred at v is the set

$$D_\Gamma(v) = \{x \in Y \mid \|x - v\| \le \|x - gv\| \text{ for all } 1 \ne g \in \Gamma\}.$$

This region is a convex polyhedron which tessellates Y under the action of Γ. If the region is bounded and an intersection of only finitely many half planes then the tessellation induces a CW-structure on Y and we set $X = Y$. If the region is unbounded but still an intersection of only finitely many half planes, then the tessellation induces a CW-structure on a contractible space X formed by suitably adjoining a discrete set of points to Y.

The cellular chain complex $C_*(X)$ is our first approximation to a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$. The action of Γ on X induces an action of Γ on $C_*(X)$. Since X is contractible the chain complex $C_*(X)$ is certainly a $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$, but in general it is not free. However, by adapting a technique of Wall [21], the resolutions R^{Γ_e} can be combined with $C_*(X)$ to obtain a free $\mathbb{Z}\Gamma$-resolution of the integers, as we now describe.

Let $[e]$ denote the orbit of a cell e in X under the action of Γ, and let $\mathrm{Orb}(k)$ denote the orbits of the k-cells. The module $C_p(X)$ is a direct sum of $\mathbb{Z}\Gamma$-modules

$$C_p(X) = \bigoplus_{[e]\in \mathrm{Orb}(p)} \big(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma_e} \mathbb{Z}\big).$$

By defining

$$F_{p,q} := \bigoplus_{[e]\in \mathrm{Orb}(p)} \big(\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma_e} R_q^{\Gamma_e}\big)$$

we obtain a free $\mathbb{Z}\Gamma$-resolution

$$F_{p,*}\colon \quad \cdots \longrightarrow F_{p,q} \longrightarrow \cdots \longrightarrow F_{p,2} \longrightarrow F_{p,1} \longrightarrow F_{p,0}$$

of the module $C_p(X)$.

The boundary maps in $C_*(X)$ induce chain maps $\partial^h\colon F_{p,q} \to F_{p-1,q}$ in the system $F_{*,*}$ of free $\mathbb{Z}\Gamma$-modules depicted in Figure 1. By construction, the 'vertical' maps $\partial^v\colon F_{p,q} \to F_{p,q-1}$ satisfy $\partial^v\partial^v = 0$; however $F_{*,*}$ is not in general a bicomplex because the 'horizontal' maps ∂^h do not necessarily square to zero. Nevertheless, we can construct a free $\mathbb{Z}\Gamma$-chain complex R_*^Γ with $R_n^\Gamma = \bigoplus_{p+q=n} F_{p,q}$ and differential d given by $d|_{F_{p,q}} = \partial^v + (-1)^p\partial^h + \varepsilon$, where $\varepsilon\colon R_n^\Gamma \to R_{n-1}^\Gamma$ is a 'perturbation'. The perturbation is an infinite sum of module homomorphisms $\varepsilon = \varepsilon_2 + \varepsilon_3 + \cdots$ where $\varepsilon_k\colon F_{*,*} \to F_{*-k,*+k-1}$. (The maps ε_2 and ε_3 are indicated in Figure 1 by dotted and dashed arrows, respectively.) On any given summand $F_{p,q}$ only finitely many terms ε_k are non-zero. The existence of such a perturbation follows from an easy generalization of a theorem of Wall [21] (see also [4]), but for our examples we show its existence by defining the map explicitly. The filtration on R_*^Γ arising from the filtration by columns of $F_{*,*}$ yields a spectral sequence with $E^1_{p,q} = H_q(F_{p,*})$. This spectral sequence implies, as in [21], that R_*^Γ is a resolution of $\mathbb{Z}$. We shall refer to the free $\mathbb{Z}\Gamma$-resolution R_*^Γ as the *total complex* of the system $F_{*,*}$. We shall use the notation $f_i^{p,q}$ to denote free generators of a summand $F_{p,q}$.

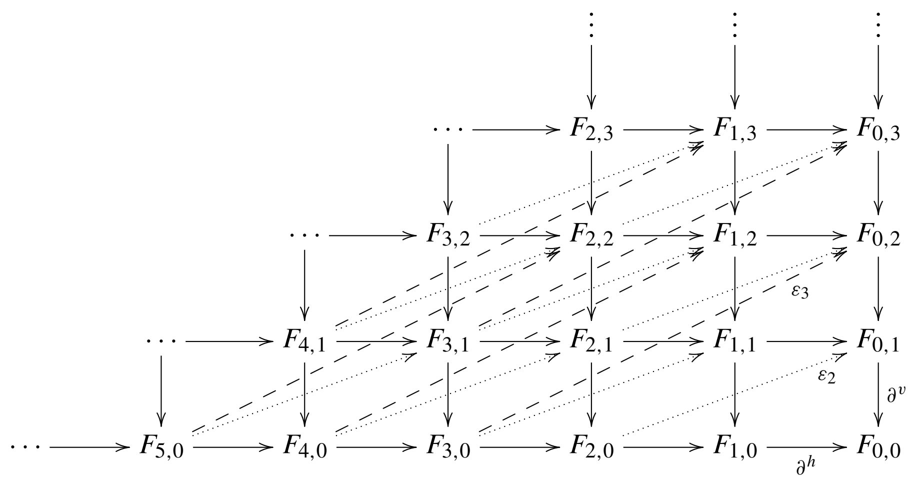

Figure 1. System $F_{*,*}$ of free $\mathbb{Z}\Gamma$-modules.

3. Triangle groups

We illustrate the method on the infinite triangle groups $T = T(l, m, n)$. Let Y denote the hyperbolic plane H^2 if $1/|l| + 1/|m| + 1/|n|$ is less than 1, or the euclidean plane E^2 if $1/|l| + 1/|m| + 1/|n|$ is equal to 1. We define an action of T on Y as follows. Let v_1, v_2 be distinct points in Y, and let the generator a of T act as clockwise rotation about v_1 through an angle $2\pi/|l|$, and let b act as clockwise rotation about v_2 through an angle $2\pi/|m|$. It follows that (ab^{-1}) acts as anticlockwise rotation about some point $v_3 \in Y$ by an angle $2\pi/|n|$. Let $\triangle$ denote the triangle with vertices v_1, v_2, v_3 and let σ denote reflection in the side v_1v_2. Then the region

$$D := \triangle \cup \sigma(\triangle)$$

is a fundamental region for T. (In fact it is a Dirichlet–Voronoi region.)

Since D is a finite polyhedron (a quadrilateral) the tessellation of Y by D under the action of T yields a CW-structure, and so we set $X = Y$. The cellular action of T on X is summarized in Table 1. In this table the 0-cells e_1^0, e_2^0, e_3^0 are the points v_1, v_2, v_3; the boundary of a j-dimensional cell e_i^j is an element of the (not necessarily free) $\mathbb{Z}T$-module $C_{j-1}(X)$. (We use the notation $K^g = g^{-1}Kg$.) The cellular chain complex $C_*(X)$ is 2-dimensional and of the form

$$0 \longrightarrow \mathbb{Z}T \longrightarrow \mathbb{Z}T \oplus \mathbb{Z}T \longrightarrow \Big(\mathbb{Z}T \underset{\mathbb{Z}\langle a\rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}T \underset{\mathbb{Z}\langle b\rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}T \underset{\mathbb{Z}\langle ab^{-1}\rangle}{\otimes} \mathbb{Z}\Big).$$

Table 1. Cellular action of $T(l, m, n)$.

Cell	Boundary		Stabilizer group
e^2	∂e^2	$\partial e^2 = (1-a^{-1})e_1^1 + (1-b^{-1})e_2^1$	1
e_1^1	∂e_1^1	$\partial e_1^1 = -e_1^0 + e_3^0$	1
$a^{-1}e_1^1$	$a^{-1}\partial e_1^1$		1
e_2^1	∂e_2^1	$\partial e_2^1 = e_2^0 - e_3^0$	1
$b^{-1}e_2^1$	$b^{-1}\partial e_2^1$		1
e_1^0	—		$\langle a\rangle$
e_2^0	—		$\langle b\rangle$
e_3^0	—		$\langle ab^{-1}\rangle$
$a^{-1}e_3^0$	—		$\langle ab^{-1}\rangle^a$

The stabilizer groups of the 0-cells are the cyclic groups $\langle a\rangle$, $\langle b\rangle$, $\langle ab^{-1}\rangle$, so we can use the free resolutions

$$R_*^{\langle a\rangle}:\quad \cdots \xrightarrow{\sum_{i=0}^{l-1} a^i} \mathbb{Z}\langle a\rangle \xrightarrow{a-1} \mathbb{Z}\langle a\rangle \xrightarrow{\sum_{i=0}^{l-1} a^i} \mathbb{Z}\langle a\rangle \xrightarrow{a-1} \mathbb{Z}\langle a\rangle,$$

$$R_*^{\langle b\rangle}:\quad \cdots \xrightarrow{\sum_{i=0}^{m-1} b^i} \mathbb{Z}\langle b\rangle \xrightarrow{b-1} \mathbb{Z}\langle b\rangle \xrightarrow{\sum_{i=0}^{m-1} b^i} \mathbb{Z}\langle b\rangle \xrightarrow{b-1} \mathbb{Z}\langle b\rangle,$$

$$R_*^{\langle ab^{-1}\rangle}:\quad \cdots \xrightarrow{\sum_{i=0}^{n-1} (ab^{-1})^i} \mathbb{Z}\langle ab^{-1}\rangle \xrightarrow{ab^{-1}-1} \mathbb{Z}\langle ab^{-1}\rangle$$
$$\xrightarrow{\sum_{i=0}^{n-1} (ab^{-1})^i} \mathbb{Z}\langle ab^{-1}\rangle \xrightarrow{ab^{-1}-1} \mathbb{Z}\langle ab^{-1}\rangle$$

of the integers to obtain the free resolution

$$F_{0,*}:\quad \cdots \xrightarrow{\overline{\alpha}} (\mathbb{Z}T)^3 \xrightarrow{\alpha} (\mathbb{Z}T)^3 \xrightarrow{\overline{\alpha}} (\mathbb{Z}T)^3 \xrightarrow{\alpha} (\mathbb{Z}T)^3$$

of $C_0(X)$, where $\alpha, \overline{\alpha}$ are given in Table 2. The stabilizers of the cells e_1^1, e_2^1, e^2 are all trivial so we immediately have the free resolutions

$$F_{1,*}:\quad 0 \longrightarrow (\mathbb{Z}T)^2,$$
$$F_{2,*}:\quad 0 \longrightarrow (\mathbb{Z}T),$$

of $C_1(X)$ and $C_2(X)$ respectively. We thus obtain the system of free $\mathbb{Z}T$-modules in Figure 2, where the vertical maps $\alpha, \overline{\alpha}$, the induced horizontal maps δ_2, δ_1, and the perturbation map ξ are given in Table 2. The total complex of this system is then the

Table 2. Boundary maps for system of free $\mathbb{Z}T$-modules.

Summand	Boundary map
$F_{2,0}$	$\delta_2(f^{2,0}) = (a^{-1}-1)f_1^{1,0} + (b^{-1}-1)f_2^{1,0}$ $\xi(f^{2,0}) = a^{-1}f_1^{0,1} - b^{-1}f_2^{0,1} - a^{-1}f_3^{0,1}$
$F_{1,0}$	$\delta_1(f_1^{1,0}) = f_1^{0,0} - f_3^{0,0}$ $\delta_1(f_2^{1,0}) = -f_2^{0,0} + f_3^{0,0}$
$F_{0,2q}$ $(q \geq 1)$	$\overline{\alpha}(f_1^{0,2q}) = \left(\sum_{i=0}^{l-1} a^i\right) f_1^{0,2q-1}$ $\overline{\alpha}(f_2^{0,2q}) = \left(\sum_{i=0}^{m-1} b^i\right) f_2^{0,2q-1}$ $\overline{\alpha}(f_3^{0,2q}) = \left(\sum_{i=0}^{n-1} (ab^{-1})^i\right) f_3^{0,2q-1}$
$F_{0,2q+1}$ $(q \geq 0)$	$\alpha(f_1^{0,2q+1}) = (a-1)f_1^{0,2q}$ $\alpha(f_2^{0,2q+1}) = (b-1)f_2^{0,2q}$ $\alpha(f_3^{0,2q+1}) = (ab^{-1}-1)f_3^{0,2q}$

following free $\mathbb{Z}T$-resolution of the integers:

$$\begin{aligned} \cdots \xrightarrow{\overline{\alpha}} (\mathbb{Z}T)^3 \xrightarrow{\alpha} (\mathbb{Z}T)^3 \xrightarrow{\overline{\alpha}} (\mathbb{Z}T)^3 \xrightarrow{\alpha} (\mathbb{Z}T)^3 \\ \xrightarrow{\overline{\alpha}} (\mathbb{Z}T)^3 \xrightarrow{\alpha} (\mathbb{Z}T)^4 \xrightarrow{d_2} (\mathbb{Z}T)^5 \xrightarrow{d_1} (\mathbb{Z}T)^3 \end{aligned} \tag{4}$$

where d_1, d_2 are given by

$$\begin{aligned} d_1 &= \alpha + \delta_1, \\ d_2 &= \overline{\alpha} + \delta_2 + \xi. \end{aligned}$$

The resolution can be used to make calculations. For instance, in dimensions $k \geq 3$ we have the known result that

$$H_k(T, A) \cong H_k(\mathbb{Z}_l, A) \oplus H_k(\mathbb{Z}_m, A) \oplus H_k(\mathbb{Z}_n, A)$$

for any $\mathbb{Z}T$-module A. Also $H_2(T, \mathbb{Z}) \cong \mathbb{Z}$.

Free resolutions for finite triangle groups T can be obtained using the methods of [4]. In Section 5 we will require free resolutions of the integers for the group $T(3, 3, 2)$ (isomorphic to $T(3, 2, 3)$ and the alternating group A_4) and for the group $T(2, 2, n)$ (isomorphic to the dihedral group of order $|2n|$, $D_{|2n|}$). We sketch the derivation of a resolution for $T = T(3, 2, 3)$.

Suppose $\psi : T \to A_4$ is an isomorphism. Assume that $g \in T$ acts on a vector $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ by

$$g \cdot \alpha = (\alpha_{\psi(g^{-1})(1)}, \alpha_{\psi(g^{-1})(2)}, \alpha_{\psi(g^{-1})(3)}, \alpha_{\psi(g^{-1})(4)}).$$

$$\begin{tikzcd}
 & & \vdots \arrow[d, "\overline{\alpha}"] \\
 & & (\mathbb{Z}T)^3 \arrow[d, "\alpha"] \\
 & & (\mathbb{Z}T)^3 \arrow[d, "\overline{\alpha}"] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & (\mathbb{Z}T)^3 \arrow[d, "\alpha"] \\
(\mathbb{Z}T) \arrow[r, "\delta_2"'] \arrow[urr, dotted, "\xi"'] & (\mathbb{Z}T)^2 \arrow[r, "\delta_1"'] & (\mathbb{Z}T)^3
\end{tikzcd}$$

Figure 2. System of free $\mathbb{Z}T$-modules.

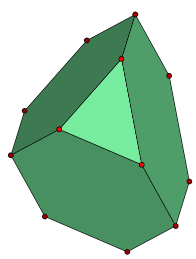

Figure 3. Polytope P.

Under this action the orbit of the vector $v = (1, 2, 3, 1)$ is a collection of 12 vectors whose convex hull is the 3-dimensional polytope P pictured in Figure 3. Here the triangular faces have boundary label a^3 and the hexagonal faces have boundary label $(ab^{-1})^3$. The triangles have stabilizer group $\langle\, a \mid a^3 \,\rangle$, the hexagons have stabilizer group $\langle\, (ab^{-1}) \mid (ab^{-1})^3 \,\rangle$, the edges labelled a have trivial stabilizer group, and the edges labelled b have stabilizer group $\langle\, b \mid b^2 \,\rangle$.

The action of T on $\mathbb{R}^4$ induces a cellular action of T on P. The cellular chain complex $C_*(P)$ can be regarded as a 3-dimensional chain complex of $\mathbb{Z}T$-modules. Now P is contractible so $H_k(P, \mathbb{Z}) = 0$ for $k \geq 1$ and $H_0(P, \mathbb{Z}) = \mathbb{Z}$. Since $C_3(P) = \mathbb{Z}$ we can splice together copies of $C_*(P)$ to form a $\mathbb{Z}G$-resolution of the integers

$$\begin{aligned} C'_*(P)\colon\quad \cdots \longrightarrow C_1(P) \longrightarrow C_0(P) \longrightarrow C_2(P) \longrightarrow C_1(P) \\ \longrightarrow C_0(P) \longrightarrow C_2(P) \longrightarrow C_1(P) \longrightarrow C_0(P). \end{aligned}$$

As $\mathbb{Z}T$-modules we have

$$C_0(P) = \big(\mathbb{Z}T\big),$$
$$C_1(P) = \big(\mathbb{Z}T\big) \oplus \Big(\mathbb{Z}T \underset{\mathbb{Z}\langle b\rangle}{\otimes} \mathbb{Z}\Big),$$
$$C_2(P) = \Big(\mathbb{Z}T \underset{\mathbb{Z}\langle a\rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}T \underset{\mathbb{Z}\langle ab^{-1}\rangle}{\otimes} \mathbb{Z}\Big).$$

Using Wall's perturbation technique to combine standard free resolutions for the cyclic groups $\langle\, a \mid a^3 \,\rangle$, $\langle\, b \mid b^2 \,\rangle$, $\langle\, (ab^{-1}) \mid (ab^{-1})^3 \,\rangle$ with the (non-free) resolution $C'_*(P)$ we obtain a free $\mathbb{Z}T$-resolution of the integers

$$R_*\colon \quad \cdots \longrightarrow R_6 \longrightarrow R_5 \longrightarrow R_4 \longrightarrow R_3 \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0 \qquad (5)$$

where for $p \geq 0$, $R_p = (\mathbb{Z}T)^{p+1}$.

The same method can be applied in the case $T \cong D_{|2n|}$. If the standard action of $D_{|2n|}$ on $\mathbb{R}^2$ is used, then the resolution obtained is again of the form (5) where for $p \geq 0$, $R_p = (\mathbb{Z}T)^{p+1}$.

4. Central extensions of triangle groups

We now turn our attention to the groups $S = S(l,m,n)$ defined in (3). The element $(ab^{-1})^n$ of $S(l,m,n)$ is a power of each generator, so is central. Since $S(l,m,n)/\langle (ab^{-1})^n\rangle \cong T(l,m,n)$ we have that the groups $S(l,m,n)$ are cyclic central extensions of the triangle groups $T(l,m,n)$.

The group $T(l,m,n)$ is infinite if and only if $1/|l| + 1/|m| + 1/|n| \leq 1$. Since $S(l,m,n)$ maps homomorphically onto $T(l,m,n)$, if $1/|l|+1/|m|+1/|n| \leq 1$ then $S(l,m,n)$ is infinite. For the converse, suppose that $1/|l| + 1/|m| + 1/|n| > 1$, i.e. $\{|l|,|m|,|n|\} = \{3,3,2\}, \{3,4,2\}, \{3,5,2\}$ or $\{2,2,q\}$ where $q \geq 2$. In the first three cases, calculations in GAP [8] show that $S(l,m,n)$ is finite. In the case $\{|l|,|m|,|n|\} = \{2,2,q\}$, the group $S(l,m,n)$ is isomorphic to one of $S(q,2,2)$, $S(q,-2,2)$, $S(2,-q,2)$, $S(-2,-q,2)$. It is easy to show that the abelianizations of these groups are finite of orders $4, 4(q-1), 4, 4(q+1)$, respectively. The central extension $\langle (ab^{-1})^n\rangle \triangleleft S(l,m,n) \to T(l,m,n)$ yields the exact sequence

$$H_2(S,\mathbb{Z}) \longrightarrow H_2(T,\mathbb{Z}) \longrightarrow \langle (ab^{-1})^n\rangle \longrightarrow S^{\mathrm{ab}} \longrightarrow T^{\mathrm{ab}} \longrightarrow 0. \qquad (6)$$

Since T is finite so is $H_2(T,\mathbb{Z})$, and (6) then shows that $\langle (ab^{-1})^n\rangle$, and hence S, is finite. Thus we have shown that $S(l,m,n)$ is infinite if and only if $1/|l| + 1/|m| + 1/|n| \leq 1$. For a more detailed analysis of the finite groups $S(l,m,n)$ see [19].

We will calculate a free $\mathbb{Z}S$-resolution of the integers whenever S is infinite. A different resolution was obtained in [19] using alternative methods. We will require the following technical result.

Lemma 1. *If $S = S(l,m,n)$ is infinite then the subgroup $\langle (ab^{-1})^n \rangle$ is infinite.*

Proof. If $1/l - 1/m = 1/n$ then there exists a representation $S(l,m,n) \to \mathbb{Z}$ given by $a \mapsto mn$, $b \mapsto ln$. Under this map the element $(ab^{-1})^n$ of $S(l,m,n)$ maps to $lmn \in \mathbb{Z}$. This is of infinite order, so the subgroup $\langle (ab^{-1})^n \rangle$ of $S(l,m,n)$ is infinite, as required. Assume then that $1/l - 1/m \neq 1/n$.

Let $R_1 = (ab^{-1})^n a^{-l}$, $R_2 = (ab^{-1})^n b^{-m}$. As explained in [2], [17] the presentation $S(l,m,n) = \langle a, b \mid R_1, R_2 \rangle$ yields a resolution

$$\cdots \longrightarrow C_4 \longrightarrow C_3 \longrightarrow (\mathbb{Z}S)^2 \xrightarrow{\delta_2} (\mathbb{Z}S)^2 \longrightarrow (\mathbb{Z}S)$$

of the integers. The map δ_2 is given by

$$\begin{aligned} \delta_2 \colon e_1^2 &\mapsto \frac{\partial R_1}{\partial a} e_1^1 + \frac{\partial R_1}{\partial b} e_2^2, \\ \delta_2 \colon e_2^2 &\mapsto \frac{\partial R_2}{\partial a} e_1^1 + \frac{\partial R_2}{\partial b} e_2^2, \end{aligned}$$

where $\partial/\partial a$, $\partial/\partial b$ denote Fox derivatives with respect to a and b, and where e_1^2, e_2^2 are the free generators of C_2, and e_1^1, e_2^1 are the free generators of C_1. Tensoring with $\mathbb{Z}$ over $\mathbb{Z}S$ yields a chain complex

$$\cdots \longrightarrow C_4 \underset{\mathbb{Z}S}{\otimes} \mathbb{Z} \longrightarrow C_3 \underset{\mathbb{Z}S}{\otimes} \mathbb{Z} \longrightarrow \mathbb{Z}^2 \xrightarrow{\delta_2 \otimes_{\mathbb{Z}S} \mathbb{Z}} \mathbb{Z}^2 \longrightarrow \mathbb{Z}.$$

The required Fox derivatives are given by

$$\begin{aligned} \partial R_1/\partial a &= \operatorname{sign}(n)(ab^{-1})^{(n-|n|)/2}(1 + (ab^{-1}) + \cdots + (ab^{-1})^{|n|-1}) \\ &\quad - \operatorname{sign}(l)(ab^{-1})^n a^{-(l+|l|)/2}(1 + a + \cdots + a^{|l|-1}), \\ \partial R_1/\partial b &= \operatorname{sign}(n)(ab^{-1})^{(n-|n|)/2}(1 + (ab^{-1}) + \cdots + (ab^{-1})^{|n|-1})(-ab^{-1}), \\ \partial R_2/\partial a &= \operatorname{sign}(n)(ab^{-1})^{(n-|n|)/2}(1 + (ab^{-1}) + \cdots + (ab^{-1})^{|n|-1}), \\ \partial R_2/\partial b &= \operatorname{sign}(n)(ab^{-1})^{(n-|n|)/2}(1 + (ab^{-1}) + \cdots + (ab^{-1})^{|n|-1})(-ab^{-1}) \\ &\quad - \operatorname{sign}(m)(ab^{-1})^n b^{-(m+|m|)/2}(1 + b + \cdots + b^{|m|-1}), \end{aligned}$$

so the map $\delta_2 \otimes_{\mathbb{Z}S} \mathbb{Z}$ is given by

$$\begin{aligned} \delta_2 \otimes_{\mathbb{Z}S} \mathbb{Z} \colon \bar{e}_1^2 &\mapsto (n - l)\bar{e}_1^1 - n\bar{e}_2^2, \\ \delta_2 \otimes_{\mathbb{Z}S} \mathbb{Z} \colon \bar{e}_2^2 &\mapsto n\bar{e}_1^1 - (n + m)\bar{e}_2^2. \end{aligned}$$

Since $1/l - 1/m \neq 1/n$ this is an injective map, and $H_2(S, \mathbb{Z}) = 0$. The five term exact sequence (6) now implies that $H_2(T, \mathbb{Z})$ injects into $\langle (ab^{-1})^n \rangle$. Using the resolution obtained in Section 3 we have that $H_2(T, \mathbb{Z}) \cong \mathbb{Z}$ and thus $\langle (ab^{-1})^n \rangle$ is infinite. $\square$

Theorem 2. *Let $S = S(l, m, n)$ where $1/|l| + 1/|m| + 1/|n| \leq 1$. Then there is a free 3-dimensional $\mathbb{Z}S$-resolution of the integers*

$$0 \longrightarrow (\mathbb{Z}S) \longrightarrow (\mathbb{Z}S)^3 \longrightarrow (\mathbb{Z}S)^5 \longrightarrow (\mathbb{Z}S)^3$$

obtained as the total complex of the system of free $\mathbb{Z}S$-modules in Figure 4.

Proof. An action of S on the euclidean or hyperbolic plane Y can be defined in the same way as the action of T on Y was defined in Section 3. As before, this yields a CW-structure so we set $X = Y$. The cellular action of S on X is the same as the action of T on X (summarized in Table 1), except that the stabilizer group of 2-cell e^2 is the group $\langle a^l \rangle$, and the stabilizer groups of the 1-cells e^1_1, e^1_2 are the groups $\langle a^l \rangle, \langle b^m \rangle$, respectively. (Note that since the 2 and 3-dimensional cells have non-trivial stabilizers the action of S is not faithful.) The cellular chain complex $C_*(X)$ is 2-dimensional of the form

$$0 \longrightarrow \Big(\mathbb{Z}S \underset{\mathbb{Z}\langle a^l \rangle}{\otimes} \mathbb{Z}\Big) \longrightarrow \Big(\mathbb{Z}S \underset{\mathbb{Z}\langle a^l \rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}S \underset{\mathbb{Z}\langle b^m \rangle}{\otimes} \mathbb{Z}\Big) \longrightarrow \Big(\mathbb{Z}S \underset{\mathbb{Z}\langle a \rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}S \underset{\mathbb{Z}\langle b \rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}S \underset{\mathbb{Z}\langle ab^{-1} \rangle}{\otimes} \mathbb{Z}\Big).$$

By Lemma 1 each of the stabilizer groups S_e is isomorphic to the infinite cyclic group, so for each cell $e \in C_*(X)$ we can use resolutions $R^{S_e}_*$ of the form

$$R^{S_e}_*: \quad 0 \longrightarrow \mathbb{Z}S_e \longrightarrow \mathbb{Z}S_e.$$

We then have the free resolutions

$$F_{0,*}: \quad 0 \longrightarrow (\mathbb{Z}S)^3 \xrightarrow{\alpha} (\mathbb{Z}S)^3,$$
$$F_{1,*}: \quad 0 \longrightarrow (\mathbb{Z}S)^2 \xrightarrow{\beta} (\mathbb{Z}S)^2,$$
$$F_{2,*}: \quad 0 \longrightarrow (\mathbb{Z}S) \xrightarrow{\gamma} (\mathbb{Z}S),$$

of $C_0(X), C_1(X), C_2(X)$ respectively, where α, β, γ are given in Table 3. We thus obtain the system of free $\mathbb{Z}S$-modules in Figure 4, where the vertical maps α, β, γ, the induced horizontal maps $\delta_1, \overline{\delta}_1, \delta_2$, and the pertubation map ξ are given in Table 3. (Note that the maps are defined in such a way that the squares in this system anti-

$$\begin{tikzcd}
(\mathbb{Z}S) \arrow[r, "-\delta_2"] \arrow[d, "\gamma"'] & (\mathbb{Z}S)^2 \arrow[r, "\overline{\delta}_1"] \arrow[d, "\beta"'] & (\mathbb{Z}S)^3 \arrow[d, "\alpha"] \\
(\mathbb{Z}S) \arrow[r, "\delta_2"'] \arrow[urr, dotted, "\xi"] & (\mathbb{Z}S)^2 \arrow[r, "\delta_1"'] & (\mathbb{Z}S)^3
\end{tikzcd}$$

Figure 4. System of free $\mathbb{Z}S$-modules.

Table 3. Boundary maps for system of free $\mathbb{Z}S$-modules.

Summand	Boundary map
$F_{2,q}$ $(q = 0, 1)$	$\delta_2(f^{2,q}) = (a^{-1} - 1) f_1^{1,q} + (b^{-1} - 1) f_2^{1,q}$
$F_{2,1}$	$\gamma(f^{2,1}) = (a^l - 1) f_1^{2,0}$
$F_{2,0}$	$\xi(f_1^{2,0}) = a^{-1} f_1^{0,1} - b^{-1} f_2^{0,1} - a^{-1} f_3^{0,1}$
$F_{1,1}$	$\beta(f_1^{1,1}) = (a^l - 1) f_1^{1,0}$ $\beta(f_2^{1,1}) = (b^m - 1) f_2^{1,0}$ $\overline{\delta}_1(f_1^{1,1}) = -\big(\sum_{i=0}^{l-1} a^i\big) f_1^{0,1} + \big(\sum_{i=0}^{n-1} (ab^{-1})^i\big) f_3^{0,1}$ $\overline{\delta}_1(f_2^{1,1}) = \big(\sum_{i=0}^{m-1} b^i\big) f_2^{0,1} - \big(\sum_{i=0}^{n-1} (ab^{-1})^i\big) f_3^{0,1}$
$F_{1,0}$	$\delta_1(f_1^{1,0}) = f_1^{0,0} - f_3^{0,0}$ $\delta_1(f_2^{1,0}) = -f_2^{0,0} + f_3^{0,0}$
$F_{0,1}$	$\alpha(f_1^{0,1}) = (a - 1) f_1^{0,0}$ $\alpha(f_2^{0,1}) = (b - 1) f_2^{0,0}$ $\alpha(f_3^{0,1}) = (ab^{-1} - 1) f_3^{0,0}$

commute.) The total complex of this system is then the following free $\mathbb{Z}S$-resolution of the integers:

$$0 \longrightarrow (\mathbb{Z}S) \xrightarrow{d_3} (\mathbb{Z}S)^3 \xrightarrow{d_2} (\mathbb{Z}S)^5 \xrightarrow{d_1} (\mathbb{Z}S)^3$$

where d_1, d_2, d_3 are given by

$$\begin{aligned} d_1 &= \alpha + \delta_1, \\ d_2 &= \overline{\delta}_1 + \beta + \delta_2 + \xi, \\ d_3 &= -\delta_2 + \gamma. \end{aligned}$$ □

This resolution can be used to make calculations such as the following. If A is any $\mathbb{Z}S$-module then $H_k(S, A) = 0$ when $k \geq 4$. Also $H_3(S, \mathbb{Z}) = \mathbb{Z}$; $H_2(S, \mathbb{Z}) = \mathbb{Z}$ if $1/l - 1/m = 1/n$ and $H_2(S, \mathbb{Z}) = 0$ otherwise.

5. Generalized triangle groups

We now consider the groups $G = G(l, m, n)$. If $\{|l|, |m|\} = \{2, 2\}$ then $G \cong D_{|4n|}$; if $(\{|l|, |m|\}, |n|) = (\{2, 3\}, 2)$ then $G \cong A_4 \times \mathbb{Z}_2$; if $(\{|l|, |m|\}, |n|) = (\{3, 3\}, 2)$ then G is finite of order 288; if $(\{|l|, |m|\}, |n|) = (\{2, 4\}, 2)$ or $(\{2, 3\}, 3)$ then G is infinite and soluble [14]. In all other cases there exists a faithful action of G on hyperbolic 3-space H^3 [9], [10]. The Euler characteristics of generalized triangle groups admitting such an action were calculated in [23] and we have

$$\chi(G) = \frac{\min\{0, 2/|l| + 1/|n| - 1\} + \min\{0, 2/|m| + 1/|n| - 1\}}{2}.$$

The orbifold corresponding to the faithful representation of G is non-compact in every case, and it has finite volume precisely in the cases $(\{3, 3\}, 3)$, $(\{3, 4\}, 2)$, $(\{4, 4\}, 2)$. In these cases G has a faithful representation as an arithmetic Kleinian group; the $(\{4, 4\}, 2)$ and $(\{3, 4\}, 2)$ groups are commensurable with the Picard group $\mathrm{PSL}(2, \mathcal{O}_1)$ and the $(\{3, 3\}, 3)$ group is commensurable with the Bianchi group $\mathrm{PSL}(2, \mathcal{O}_3)$ (where $\mathcal{O}_d$ denotes the ring of integers in $\mathbb{Q}(\sqrt{-d})$) [10].

We apply our method to obtain free $\mathbb{Z}G$-resolutions of the integers in all cases where there is a faithful action on H^3.

Theorem 3. *Let $G = G(l, m, n)$ where $2/|l| + 1/|n| \leq 1$ and $2/|m| + 1/|n| \leq 1$. Then there is a free $\mathbb{Z}G$-resolution of the integers*

$$\begin{aligned} \cdots \longrightarrow (\mathbb{Z}G)^9 \longrightarrow (\mathbb{Z}G)^9 \longrightarrow (\mathbb{Z}G)^9 \\ \longrightarrow (\mathbb{Z}G)^{10} \longrightarrow (\mathbb{Z}G)^{13} \longrightarrow (\mathbb{Z}G)^{13} \longrightarrow (\mathbb{Z}G)^6 \end{aligned}$$

obtained as the total complex of the system of free $\mathbb{Z}G$-modules in Figure 5.

Proof. Let Y denote hyperbolic 3-space H^3. It was shown in [9], [10] that there exists a faithful action of G on Y, which we now describe. There exist skew axes L_1, L_2 in Y such that x acts as rotation through an angle $2\pi/l$ about L_1 and y acts as rotation through an angle $2\pi/m$ about L_2. The triangle subgroups $\langle x, yxy^{-1}\rangle$, $\langle x, yxy^{-1}\rangle^y$ of G (isomorphic to $T(l, l, n)$) act on Y with 'fixed points' v_1, $y^{-1}v_1$, respectively. Using the language of [20], v_1, $y^{-1}v_1$ are ordinary points in H^3, infinite points on the boundary ∂H^3, or ideal points outside H^3 depending on whether the value of $2/|l| + 1/|n| - 1$ is positive, zero, or negative, respectively. Similarly the triangle subgroups $\langle y, xyx^{-1}\rangle$, $\langle y, xyx^{-1}\rangle^x$ of G (isomorphic to $T(m, m, n)$) act on Y with fixed points v_2, $x^{-1}v_2$ respectively. These points are ordinary, infinite, or ideal depending on the value of $2/|m| + 1/|n| - 1$. The four points v_1, $y^{-1}v_1$, v_2, $x^{-1}v_2$ form a 'tetrahedron' in Y; this tetrahedron is a fundamental domain for G.

Under the hypotheses of the theorem v_1, v_2 are actually either infinite or ideal points. We form the contractible space X by adjoining 0-cells to Y in one to one correspondence with points in the orbits of v_1, v_2. The tessellation of Y under the action of G induces a CW-structure on X. The cellular action of G on X is summarized in Table 4. In this table the 0-cells e_1^0, e_2^0 correspond to the points v_1, v_2; the boundary

Table 4. Cellular action of $G(l, m, n)$.

Cell	Boundary		Stabilizer group
e^3	∂e^3	$\partial e^3 = (1-x^{-1})e_1^2 + (1-y^{-1})e_2^2$	1
e_1^2	∂e_1^2	$\partial e_1^2 = e_1^1 + (y^{-1}-1)e_3^1$	1
$x^{-1}e_1^2$	$x^{-1}\partial e_1^2$		1
e_2^2	∂e_2^2	$\partial e_2^2 = e_2^1 + (1-x^{-1})e_3^1$	1
$y^{-1}e_2^2$	$y^{-1}\partial e_2^2$		1
e_1^1	∂e_1^1	$\partial e_1^1 = (1-y^{-1})e_1^0$	$\langle x\rangle$
e_2^1	∂e_2^1	$\partial e_2^1 = (1-x^{-1})e_2^0$	$\langle y\rangle$
e_3^1	∂e_3^1	$\partial e_3^1 = e_1^0 - e_2^0$	$\langle [x, y]\rangle$
$y^{-1}e_3^1$	$y^{-1}\partial e_3^1$		$\langle [x, y]\rangle^y$
$x^{-1}y^{-1}e_3^1$	$x^{-1}y^{-1}\partial e_3^1$		$\langle [x, y]\rangle^{yx}$
$x^{-1}e_3^1$	$x^{-1}\partial e_3^1$		$\langle [x, y]\rangle^x$
e_1^0	—		$\langle x, yxy^{-1}\rangle$
$y^{-1}e_1^0$	—		$\langle x, yxy^{-1}\rangle^y$
e_2^0	—		$\langle y, xyx^{-1}\rangle$
$x^{-1}e_2^0$	—		$\langle y, xyx^{-1}\rangle^x$

of a j-dimensional cell e_i^j is an element of the (not necessarily free) $\mathbb{Z}G$-module $C_{j-1}(X)$. The (3-dimensional) cellular chain complex $C_*(X)$ is of the form

$$0 \longrightarrow C_3(X) \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X)$$

where

$$\begin{aligned}
C_3(X) &= (\mathbb{Z}G),\\
C_2(X) &= (\mathbb{Z}G)^2,\\
C_1(X) &= \Big(\mathbb{Z}G \underset{\mathbb{Z}\langle x\rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}G \underset{\mathbb{Z}\langle y\rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}G \underset{\mathbb{Z}\langle [x,y]\rangle}{\otimes} \mathbb{Z}\Big),\\
C_0(X) &= \Big(\mathbb{Z}G \underset{\mathbb{Z}\langle x, yxy^{-1}\rangle}{\otimes} \mathbb{Z}\Big) \oplus \Big(\mathbb{Z}G \underset{\mathbb{Z}\langle y, xyx^{-1}\rangle}{\otimes} \mathbb{Z}\Big).
\end{aligned}$$

The stabilizer groups of the 0-cells are triangle groups, so we can use free resolutions of the form given in Section 3 to obtain the free resolution

$$\begin{aligned}F_{0,*}:\quad \cdots &\xrightarrow{\overline{\alpha}} (\mathbb{Z}G)^6 \xrightarrow{\alpha} (\mathbb{Z}G)^6\\ &\xrightarrow{\overline{\alpha}} (\mathbb{Z}G)^6 \xrightarrow{\sigma} (\mathbb{Z}G)^8 \xrightarrow{\nu} (\mathbb{Z}G)^{10} \xrightarrow{\mu} (\mathbb{Z}G)^6\end{aligned}$$

of $C_0(X)$, where $\alpha, \overline{\alpha}, \sigma, \nu, \mu$ are given in Table 5 (see p. 586). The stabilizer groups of the 1-cells are the cyclic groups $\langle x\rangle$, $\langle y\rangle$, $\langle [x,y]\rangle$, so we can use the free resolutions

$$\begin{aligned}R_*^{\langle x\rangle}:\quad &\cdots \xrightarrow{\sum_{i=0}^{l-1} x^i} \mathbb{Z}\langle x\rangle \xrightarrow{x-1} \mathbb{Z}\langle x\rangle \xrightarrow{\sum_{i=0}^{l-1} x^i} \mathbb{Z}\langle x\rangle \xrightarrow{x-1} \mathbb{Z}\langle x\rangle,\\ R_*^{\langle y\rangle}:\quad &\cdots \xrightarrow{\sum_{i=0}^{m-1} y^i} \mathbb{Z}\langle y\rangle \xrightarrow{y-1} \mathbb{Z}\langle y\rangle \xrightarrow{\sum_{i=0}^{m-1} y^i} \mathbb{Z}\langle y\rangle \xrightarrow{y-1} \mathbb{Z}\langle y\rangle,\\ R_*^{\langle [x,y]\rangle}:\quad &\cdots \xrightarrow{\sum_{i=0}^{n-1} [x,y]^i} \mathbb{Z}\langle [x,y]\rangle \xrightarrow{[x,y]-1} \mathbb{Z}\langle [x,y]\rangle\\ &\qquad \xrightarrow{\sum_{i=0}^{n-1} [x,y]^i} \mathbb{Z}\langle [x,y]\rangle \xrightarrow{[x,y]-1} \mathbb{Z}\langle [x,y]\rangle,\end{aligned}$$

to obtain the free resolution

$$F_{1,*}:\quad \cdots \xrightarrow{\bar{\beta}} (\mathbb{Z}G)^3 \xrightarrow{\beta} (\mathbb{Z}G)^3 \xrightarrow{\bar{\beta}} (\mathbb{Z}G)^3 \xrightarrow{\beta} (\mathbb{Z}G)^3$$

of $C_1(X)$, where $\beta, \bar{\beta}$ are given in Table 5. The stabilizers of the cells e_1^2, e_2^2, e^3 are all trivial so we immediately have the free resolutions

$$\begin{aligned}F_{2,*}:&\quad 0 \longrightarrow (\mathbb{Z}G)^2,\\ F_{3,*}:&\quad 0 \longrightarrow (\mathbb{Z}G),\end{aligned}$$

of $C_2(X)$ and $C_3(X)$ respectively.

We thus obtain the system of free $\mathbb{Z}G$-modules depicted in Figure 5, where the vertical maps $\mu, \nu, \sigma, \alpha, \overline{\alpha}, \beta, \overline{\beta}$, the induced horizontal maps $\delta_1, \overline{\delta}_1, \tilde{\delta}_1, \delta_2, \delta_3$, and the perturbation maps ϕ, θ, ρ are given in Table 5. (Note that the maps are defined in such a way that the squares in this system anti-commute.) The total complex of this system is then the following free $\mathbb{Z}G$-resolution of the integers:

$$\begin{aligned}\cdots &\xrightarrow{d_4} (\mathbb{Z}G)^9 \xrightarrow{d_5} (\mathbb{Z}G)^9 \xrightarrow{d_4} (\mathbb{Z}G)^9 \xrightarrow{d_5} (\mathbb{Z}G)^9\\ &\xrightarrow{d_4} (\mathbb{Z}G)^{10} \xrightarrow{d_3} (\mathbb{Z}G)^{13} \xrightarrow{d_2} (\mathbb{Z}G)^{13} \xrightarrow{d_1} (\mathbb{Z}G)^6\end{aligned}$$

$$
\begin{tikzcd}
 & & \vdots \arrow[d, "\bar{\beta}"'] & \vdots \arrow[d, "\overline{\alpha}"] \\
 & & (\mathbb{Z}G)^3 \arrow[r, "-\delta_1"] \arrow[d, "\beta"'] & (\mathbb{Z}G)^6 \arrow[d, "\alpha"] \\
 & & (\mathbb{Z}G)^3 \arrow[r, "\delta_1"] \arrow[d, "\bar{\beta}"'] & (\mathbb{Z}G)^6 \arrow[d, "\overline{\alpha}"] \\
 & & (\mathbb{Z}G)^3 \arrow[r, "-\delta_1"] \arrow[d, "\beta"'] & (\mathbb{Z}G)^6 \arrow[d, "\sigma"] \\
 & & (\mathbb{Z}G)^3 \arrow[r, "\tilde{\delta}_1"] \arrow[d, "\bar{\beta}"'] & (\mathbb{Z}G)^8 \arrow[d, "\nu"] \\
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & (\mathbb{Z}G)^3 \arrow[r, "-\overline{\delta}_1"] \arrow[d, "\beta"'] & (\mathbb{Z}G)^{10} \arrow[d, "\mu"] \\
(\mathbb{Z}G) \arrow[r, "\delta_3"'] \arrow[uurrr, dashed, "\rho"] \arrow[urr, dotted, "\theta"'] & (\mathbb{Z}G)^2 \arrow[r, "\delta_2"'] \arrow[urr, dotted, "\phi"'] & (\mathbb{Z}G)^3 \arrow[r, "\delta_1"'] & (\mathbb{Z}G)^6
\end{tikzcd}
$$

Figure 5. System of free $\mathbb{Z}G$-modules.

where d_1, d_2, d_3, d_4, d_5 are given by

$$
\begin{aligned}
d_1 &= \mu + \delta_1, \\
d_2 &= \nu - \overline{\delta}_1 + \beta + \phi + \delta_2, \\
d_3 &= \sigma + \tilde{\delta}_1 + \overline{\beta} + \rho + \theta + \delta_3, \\
d_4 &= \overline{\alpha} - \delta_1 + \beta, \\
d_5 &= \alpha + \delta_1 + \overline{\beta}.
\end{aligned}
$$

□

Corollary 4. *Let $G = G(l, m, n)$ where $2/|l| + 1/|n| \leq 1$ and $2/|m| + 1/|n| \leq 1$. Then*

$$
H_k(G, \mathbb{Z}) \quad = \quad \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}_l \oplus \mathbb{Z}_m & k = 1 \\ \mathbb{Z} & k = 2 \\ \mathbb{Z}_l \oplus \mathbb{Z}_m \oplus \mathbb{Z}_n & k \geq 3,\ k \text{ odd} \\ 0 & k \geq 4,\ k \text{ even.} \end{cases}
$$

Table 5. Boundary maps for system of free $\mathbb{Z}G$-modules.

Summand	Boundary map
$F_{3,0}$	$\delta_3(f^{3,0}) = (x^{-1}-1)f_1^{2,0} + (y^{-1}-1)f_2^{2,0}$ $\theta(f^{3,0}) = -x^{-1}f_1^{1,1} - y^{-1}f_2^{1,1} + y^{-1}x^{-1}f_3^{1,1}$ $\rho(f^{3,0}) = y^{-1}f_1^{0,2} + x^{-1}f_5^{0,2}$
$F_{2,0}$	$\delta_2(f_1^{2,0}) = -f_1^{1,0} + (1-y^{-1})f_3^{1,0}$ $\delta_2(f_2^{2,0}) = -f_2^{1,0} + (x^{-1}-1)f_3^{1,0}$ $\phi(f_1^{2,0}) = -f_1^{0,1} - y^{-1}f_2^{0,1} + (1-y^{-1})f_6^{0,1} - y^{-1}f_8^{0,1}$ $\phi(f_2^{2,0}) = (1-x^{-1})f_1^{0,1} - x^{-1}f_3^{0,1} - f_6^{0,1} - x^{-1}f_7^{0,1}$
$F_{1,2q+1}$ $(q \geq 0)$	$\beta(f_1^{1,2q+1}) = (x-1)f_1^{1,2q}$ $\beta(f_2^{1,2q+1}) = (y-1)f_2^{1,2q}$ $\beta(f_3^{1,2q+1}) = ([x,y]-1)f_3^{1,2q}$
$F_{1,2q}$ $(q \geq 1)$	$\bar{\beta}(f_1^{1,2q}) = \sum_{i=0}^{l-1} x^i f_1^{1,2q-1}$ $\bar{\beta}(f_2^{1,2q}) = \sum_{i=0}^{m-1} y^i f_2^{1,2q-1}$ $\bar{\beta}(f_3^{1,2q}) = \sum_{i=0}^{n-1} [x,y]^i f_3^{1,2q-1}$
$F_{1,q}$ $(q \neq 1,2)$	$\delta_1(f_1^{1,q}) = -f_1^{0,q} + y^{-1}f_2^{0,q}$ $\delta_1(f_2^{1,q}) = -f_4^{0,q} + x^{-1}f_5^{0,q}$ $\delta_1(f_3^{1,q}) = -f_3^{0,q} + f_6^{0,q}$
$F_{1,2}$	$\tilde{\delta}_1(f_1^{1,2}) = -f_2^{0,2} + y^{-1}f_3^{0,2}$ $\tilde{\delta}_1(f_2^{1,2}) = -f_6^{0,2} + x^{-1}f_7^{0,2}$ $\tilde{\delta}_1(f_3^{1,2}) = -f_4^{0,2} + f_8^{0,2}$
$F_{1,1}$	$\bar{\delta}_1(f_1^{1,1}) = -f_3^{0,1} + y^{-1}f_4^{0,1}$ $\bar{\delta}_1(f_2^{1,1}) = -f_8^{0,1} + x^{-1}f_9^{0,1}$ $\bar{\delta}_1(f_3^{1,1}) = -f_5^{0,1} + f_{10}^{0,1}$

$F_{0,2q+1}$ $(q \geq 2)$	$\alpha(f_1^{0,2q+1}) = (x-1)\,f_1^{0,2q}$ $\alpha(f_2^{0,2q+1}) = (yxy^{-1}-1)\,f_2^{0,2q}$ $\alpha(f_3^{0,2q+1}) = ([x,y]-1)\,f_3^{0,2q}$ $\alpha(f_4^{0,2q+1}) = (y-1)\,f_4^{0,2q}$ $\alpha(f_5^{0,2q+1}) = (xyx^{-1}-1)\,f_5^{0,2q}$ $\alpha(f_6^{0,2q+1}) = ([x,y]-1)\,f_6^{0,2q}$
$F_{0,2q}$ $(q \geq 2)$	$\overline{\alpha}(f_1^{0,2q}) = \sum_{i=0}^{l-1} x^i\, f_1^{0,2q-1}$ $\overline{\alpha}(f_2^{0,2q}) = \sum_{i=0}^{l-1} (yxy^{-1})^i\, f_2^{0,2q-1}$ $\overline{\alpha}(f_3^{0,2q}) = \sum_{i=0}^{n-1} [x,y]^i\, f_3^{0,2q-1}$ $\overline{\alpha}(f_4^{0,2q}) = \sum_{i=0}^{m-1} y^i\, f_4^{0,2q-1}$ $\overline{\alpha}(f_5^{0,2q}) = \sum_{i=0}^{m-1} (xyx^{-1})^i\, f_5^{0,2q-1}$ $\overline{\alpha}(f_6^{0,2q}) = \sum_{i=0}^{n-1} [x,y]^i\, f_6^{0,2q-1}$
$F_{0,3}$	$\sigma(f_1^{0,3}) = (x-1)\,f_2^{0,2}$ $\sigma(f_2^{0,3}) = (yxy^{-1}-1)\,f_3^{0,2}$ $\sigma(f_3^{0,3}) = ([x,y]-1)\,f_4^{0,2}$ $\sigma(f_4^{0,3}) = (y-1)\,f_6^{0,2}$ $\sigma(f_5^{0,3}) = (xyx^{-1}-1)\,f_7^{0,2}$ $\sigma(f_6^{0,3}) = ([x,y]-1)\,f_8^{0,2}$
$F_{0,2}$	$\nu(f_1^{0,2}) = (x^{-1}-1)\,f_1^{0,1} + (yx^{-1}y^{-1}-1)\,f_2^{0,1} + x^{-1}f_3^{0,1}$ $\quad -yx^{-1}y^{-1}f_4^{0,1} - x^{-1}f_5^{0,1}$ $\nu(f_2^{0,2}) = \sum_{i=0}^{l-1} x^i\, f_3^{0,1}$ $\nu(f_3^{0,2}) = \sum_{i=0}^{l-1} (yxy^{-1})^i\, f_4^{0,1}$ $\nu(f_4^{0,2}) = \sum_{i=0}^{n-1} [x,y]^i\, f_5^{0,1}$ $\nu(f_5^{0,2}) = (y^{-1}-1)\,f_6^{0,1} + (xy^{-1}x^{-1}-1)\,f_7^{0,1} + y^{-1}f_8^{0,1}$ $\quad -xy^{-1}x^{-1}f_9^{0,1} + xy^{-1}x^{-1}f_{10}^{0,1}$

	$\nu(f_6^{0,2}) = \sum_{i=0}^{m-1} y^i f_8^{0,1}$ $\nu(f_7^{0,2}) = \sum_{i=0}^{m-1} (xyx^{-1})^i f_9^{0,1}$ $\nu(f_8^{0,2}) = \sum_{i=0}^{n-1} [x, y]^i f_{10}^{0,1}$
$F_{0,1}$	$\mu(f_1^{0,1}) = f_1^{0,0} - f_3^{0,0}$ $\mu(f_2^{0,1}) = -f_2^{0,0} + f_3^{0,0}$ $\mu(f_3^{0,1}) = (x-1) f_1^{0,0}$ $\mu(f_4^{0,1}) = (yxy^{-1} - 1) f_2^{0,0}$ $\mu(f_5^{0,1}) = ([x, y] - 1) f_3^{0,0}$ $\mu(f_6^{0,1}) = f_4^{0,0} - f_6^{0,0}$ $\mu(f_7^{0,1}) = -f_5^{0,0} + f_6^{0,0}$ $\mu(f_8^{0,1}) = (y-1) f_4^{0,0}$ $\mu(f_9^{0,1}) = (xyx^{-1} - 1) f_5^{0,0}$ $\mu(f_{10}^{0,1}) = ([x, y] - 1) f_6^{0,0}$

If, under the hypotheses of Theorem 3, we additionally have $|n| \geq 4$ then the homology groups $H_k(G, \mathbb{Z})$ ($k \geq 3$) can be obtained immediately from results of Howie (see Corollary D of [11], [12]). These results were obtained using algebraic techniques and they apply in the more general setting of one-relator products of groups.

For the remaining cases we provide only the form of a free $\mathbb{Z}G$-resolution over the integers. The method can of course be used to obtain the maps as well, if required.

Theorem 5. *Let $G = G(l, m, n)$ where $2/|l| + 1/|n| > 1$ or $2/|m| + 1/|n| > 1$ and where $(\{|l|, |m|\}, |n|) \neq (\{2, 2\}, |n|)$, $(\{2, 3\}, 2)$, $(\{3, 3\}, 2)$, $(\{2, 4\}, 2)$, $(\{2, 3\}, 3)$. Then there is a free $\mathbb{Z}G$-resolution of the integers*

$$\cdots \longrightarrow R_6 \longrightarrow R_5 \longrightarrow R_4 \longrightarrow (\mathbb{Z}G)^{11} \longrightarrow (\mathbb{Z}G)^{12} \longrightarrow (\mathbb{Z}G)^{10} \longrightarrow (\mathbb{Z}G)^4$$

where for $p \geq 4$, $R_p = (\mathbb{Z}G)^{p+7}$.

Proof. Without loss of generality, suppose $2/|l|+1/|n| > 1$. Then $(|l|, |n|) = (2, |n|)$ or $(3, 2)$ and $2/|m| + 1/|n| \leq 1$. As described in the proof of Theorem 3 there is a faithful action of G on H^3. In these cases v_1, $y^{-1}v_1$ are ordinary points in H^3 and the groups $\langle x, yxy^{-1}\rangle$, $\langle x, yxy^{-1}\rangle^y$ are isomorphic to the finite triangle group $T = T(l, l, n)$. Note that T is either the dihedral group $D_{|2n|}$ or the alternating group A_4. The points $v_2, x^{-1}v_2$ are again infinite or ideal points and the groups $\langle y, xyx^{-1}\rangle$, $\langle y, xyx^{-1}\rangle^x$ are isomorphic to the infinite triangle group $T' = T(m, m, n)$.

As explained in Section 3, there is a free $\mathbb{Z}T$-resolution of the integers of the form

$$S_*\colon\quad \cdots \longrightarrow S_5 \longrightarrow S_4 \longrightarrow S_3 \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0$$

where for $p \geq 0$, $S_p = (\mathbb{Z}T)^{p+1}$. Also, there is a free $\mathbb{Z}T'$-resolution of the integers of the form

$$S'_*\colon\quad \cdots \longrightarrow S'_5 \longrightarrow S'_4 \longrightarrow S'_3 \longrightarrow (\mathbb{Z}T')^4 \longrightarrow (\mathbb{Z}T')^5 \longrightarrow (\mathbb{Z}T')^3$$

where for $p \geq 3$, $S'_p = (\mathbb{Z}T')^3$. Combining these resolutions we get the following free resolution of $C_0(X)$

$$F_{0,*}\colon\quad \cdots \longrightarrow F_{0,5} \longrightarrow F_{0,4} \longrightarrow F_{0,3} \longrightarrow (\mathbb{Z}G)^7 \longrightarrow (\mathbb{Z}G)^7 \longrightarrow (\mathbb{Z}G)^4$$

where for $p \geq 3$, $F_{0,p} = (\mathbb{Z}G)^{p+4}$. The free resolutions $F_{1,*}$, $F_{2,*}$, $F_{3,*}$ of $C_1(X)$, $C_2(X)$, $C_3(X)$ are as given in the proof of Theorem 3. Combining these four free resolutions we obtain a system of free $\mathbb{Z}G$-modules whose total complex is of the required form. □

In addition to the groups $G(l, m, n)$, there are other classes of generalized triangle groups which are known to admit faithful representations as groups of isometries of hyperbolic 3-space. The second author has shown [22] that (except in the cases $(\{|l|, |m|\}, |n|) = (\{2, 3\}, 2)$, $(\{2, 2\}, n)$) the groups

$$\tilde{G} = \langle\, x, y \mid x^l = y^m = ((xy)^2(x^{-1}y^{-1})^2)^n = 1 \,\rangle$$

admit such a representation. The method of proof follows [9] in explicitly constructing a fundamental domain. The stabilizer groups of the edges of this domain are cyclic groups and the stabilizer groups of the vertices are triangle groups. A free $\mathbb{Z}\tilde{G}$-resolution of the integers can therefore be constructed exactly as in Section 5. More generally, Jones and Reid [13] consider a class of generalized triangle groups which arise as the fundamental groups of 3-dimensional orbifolds whose singular sets are obtained by adding an unknotting tunnel to a 2-bridge knot or link. They show that in most cases such a group has a faithful representation as a group of isometries of hyperbolic 3-space. It seems likely that our method can also be applied in this setting.

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