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Lorentzian Kleinian groups

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Abstract. In this article we introduce some basic tools for the study of Lorentzian Kleinian groups. These groups are discrete subgroups of the Lorentzian Möbius group $O(2, n)$, acting properly discontinuously on some nonempty open subset of Einstein's universe, the Lorentzian analogue of the conformal sphere.

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1. Introduction

To understand a hyperbolic manifold $\mathbb{H}^{n+1}/\Gamma$ ($\mathbb{H}^{n+1}$ denotes here the $(n+1)$-hyperbolic space and $\Gamma \subset O(1, n+1)$ is a discrete group of hyperbolic isometries), a nice and powerful tool is the dynamical study of the conformal action of Γ on the sphere $\mathbb{S}^n$. This deep relationship between hyperbolic and conformally flat geometry has a counterpart in Lorentzian geometry, often quoted by physicists as *AdS/CFT correspondence*. Let us first recall what is the Lorentzian analogue of the pair $(\mathbb{H}^{n+1}, \mathbb{S}^n)$. The $(n+1)$-dimensional Lorentzian model space of constant curvature -1 is called *anti-de Sitter space*, denoted AdS_{n+1} (precisely, we are speaking here of the quotient of the simply connected model $\widetilde{\mathrm{AdS}}_{n+1}$ by the center of its isometry group, see [O'N], [Wo]). This space, like the hyperbolic space, has a conformal boundary. It is called Einstein's universe, denoted Ein_n, and it can be defined, up to a two-sheeted covering, as the product $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ endowed with the conformal class of the metric $-dt^2 \times g_{S^{n-1}}$. From the conformal viewpoint, Einstein's universe has a lot of properties reminiscent of those of the sphere. In particular, the group $O(2, n)$ of isometries of AdS_{n+1} turns out to be also the group of conformal transformations of Ein_n. The understanding of an anti-de Sitter manifold $\mathrm{AdS}_{n+1} / \Gamma$ thanks to the conformal dynamics of Γ on Ein_n is one of the motivations for studying *Lorentzian Kleinian groups*, which we define by analogy with the classical theory as discrete subgroups of $O(2, n)$ acting freely and properly discontinuously on some nonempty open subset of Ein_n.

Since the works of Poincaré and Klein at the end of the nineteenth century, the classical theory of Kleinian groups has generated a great amount of works and progressed very far (we refer the reader to [A], [Ka], [Ma], [MK] for a historical account and good expositions on the subject).

Other notions of Kleinian groups also appeared in other geometric contexts, such as complex hyperbolic and projective geometry (see for example [Go], [SV]).

To our knowledge, nothing systematic has been done for studying Lorentzian Kleinian groups, so that the aim of this article is to lay some basis for the theory. In particular, our first task is to build and study nontrivial examples of such groups.

The first part of the paper (Sections 3 and 4) is devoted to what could be called *Lorentzian Möbius dynamics*, namely the dynamical study of divergent sequences of $O(2,n)$ acting on Ein_n. This dynamics appears richer than that of classical Möbius transformations on the sphere. This is essentially due to the fact that $O(2,n)$ has rank two, and the different ways to reach infinity in $O(2,n)$ induce different dynamical patterns for the action on Ein_n. These patterns, which are essentially three, are described in Section 3, Propositions 3, 4 and 5. Let us mention here two new phenomena (with respect to the Riemannian context) illustrating the dynamical complications we are confronted with. Firstly, the Lorentzian Möbius group $O(2,n)$ is not a convergence group for its action on Ein_n (roughly speaking, a group G acting by homeomorphisms on a manifold X is a convergence group if any sequence (g_i) of G tending to infinity admits a subsequence with a "north–south" dynamics, i.e. a dynamics with an attracting pole p^+ and a repelling one p^-. See for example [A], p. 40, for a precise definition). Secondly, a discrete subgroup $\Gamma \subset O(2,n)$ does not always act properly on AdS_{n+1}.

In spite of these differences with respect to the classical theory it is still possible to define the *limit set* of a discrete subgroup $\Gamma \subset O(2,n)$ (see Section 4). This is a closed Γ-invariant subset $\Lambda_\Gamma \subset \mathrm{Ein}_n$, such that the action on the complement Ω_Γ is proper. Moreover it is a union of lightlike geodesics, so that it defines naturally a Γ-invariant closed subset $\hat{\Lambda}_\Gamma$ of $\mathbb{L}_n$, the space of lightlike geodesics of Ein_n (this space is described in Section 2.5). Unfortunately, the nice properties of the limit set in the classical case of groups of conformal transformations of the sphere are generally no longer satisfied in our situation. For example, the limit set that we define is not, in general, a minimal set for the action of Γ on Ein_n (although $\hat{\Lambda}_\Gamma$ is sometimes minimal for the action of Γ on $\mathbb{L}_n$, see Theorem 1 below). The groups $\Gamma \subset O(2,n)$ acting properly on AdS_{n+1} are those whose behaviour is closest to that of classical Kleinian groups. They will be called *groups of the first type*. For them we get nice properties for the limit set.

Theorem 1. *Let Γ be a Kleinian group of the first type and Λ_Γ its limit set.*

(i) *The action of Γ is proper on $\Omega_\Gamma \cup \mathrm{AdS}_{n+1} \subset \mathrm{Ein}_{n+1}$.*

(ii) Ω_Γ *is the unique maximal element among the open sets* $\Omega \subset \mathrm{Ein}_n$ *such that* Γ *acts properly on* $\Omega \cup \mathrm{AdS}_{n+1}$.

(iii) *If moreover* Γ *is Zariski dense in* $\mathrm{O}(2, n)$*, then* Ω_Γ *is the unique maximal open subset of* Ein_n *on which* Γ *acts properly, and* $\hat{\Lambda}_\Gamma$ *is a minimal set for the action of* Γ *on* $\mathbb{L}_n$.

In Section 5, we give several examples of families of Lorentzian Kleinian groups. These basic examples being constructed, it is natural to try to combine two of them to get other more complicated examples. This is the aim of Section 6, where we prove the following result (an analogue of the celebrated Klein's combination theorem):

Theorem 2. *Let* Γ_1 *and* Γ_2 *be two cocompact Lorentzian Kleinian groups with fundamental domains* D_1 *and* D_2*. Suppose that both* D_1 *and* D_2 *contain a lightlike geodesic. Then one can construct from* Γ_1 *and* Γ_2 *another cocompact Kleinian group, isomorphic to the free product* $\Gamma_1 * \Gamma_2$.

By a *cocompact Kleinian group* we mean a group acting properly on some open subset of Ein_n with compact quotient.

We then use Theorem 2 in Section 7 to construct *Lorentzian Schottky groups*. The study of such groups can be carried out quite completely. The limit set Λ_Γ and the topology of the conformally flat Lorentz manifold obtained as the quotient Ω_Γ / Γ of the domain of properness are made explicit in this case, and we get:

Theorem 3. *Let* $\Gamma = \langle s_1, \dots, s_g \rangle$ $(g \geq 2)$ *be a Lorentzian Schottky group.*

(i) *The group* Γ *is of the first type.*

(ii) *The limit set* Λ_Γ *is a lamination by lightlike geodesics. Topologically, it is a product of* $\mathbb{R}P^1$ *with a Cantor set.*

(iii) *The action of* Γ *is minimal on the set of lightlike geodesics of* Λ_Γ.

(iv) *The quotient manifold* Ω_Γ / Γ *is diffeomorphic to the product*

$$\mathbb{S}^1 \times (\mathbb{S}^1 \times \mathbb{S}^{n-1})^{(g-1)\sharp},$$

where $(\mathbb{S}^1 \times \mathbb{S}^{n-1})^{(g-1)\sharp}$ *is the connected sum of* $(g-1)$ *copies of* $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.

2. Geometry of Einstein's universe

A detailed description of the geometry of Einstein's universe can be found in [Fr1], [Fr2] and [CK]. Also, for the readers who are not very familiar with Lorentzian spacetimes of constant curvatures, good expositions can be found in [Wo], chapter 11, and [O'N], chapter 8. In this section we briefly recall (without any proof) the main properties which will be useful in this article.

2.1. Projective model for Einstein's universe. Let $\mathbb{R}^{2,n}$ be the space $\mathbb{R}^{n+2}$, endowed with the quadratic form $q^{2,n}(x) = -x_1^2 - x_2^2 + x_3^2 + \cdots + x_{n+2}^2$. The isotropic cone of $q^{2,n}$ is the subset of $\mathbb{R}^{2,n}$ on which $q^{2,n}$ vanishes. We call $C^{2,n}$ this isotropic cone, with the origin removed. Throughout this article we will denote by π the projection from $\mathbb{R}^{2,n}$ minus the origin on $\mathbb{R}P^{n+1}$. The set $\pi(C^{2,n})$ is a smooth hypersurface Σ of $\mathbb{R}P^{n+1}$. This hypersurface turns out to be endowed with a natural Lorentzian conformal structure. Indeed, for any $x \in C^{2,n}$, the restriction of $q^{2,n}$ to the tangent space $T_x C^{2,n}$, that we call $\hat{q}_x^{2,n}$, is degenerate. Its kernel is just the kernel of the tangent map $d_x\pi$. Thus, pushing $\hat{q}_x^{2,n}$ by $d_x\pi$, we get a well-defined Lorentzian metric on $T_{\pi(x)}\Sigma$. If $\pi(x) = \pi(y)$ the two Lorentzian metrics on $T_{\pi(x)}\Sigma$ obtained by pushing $\hat{q}_x^{2,n}$ and $\hat{q}_y^{2,n}$ are in the same conformal class. Thus the form $q^{2,n}$ determines a well-defined conformal class of Lorentzian metrics on Σ. One calls *Einstein's universe* the hypersurface Σ together with this canonical conformal structure.

The intersection of $C^{2,n}$ with the Euclidean sphere defined by $x_1^2 + x_2^2 + \cdots + x_{n+2}^2 = 1$ is a smooth hypersurface $\hat{\Sigma} \subset \mathbb{R}^{2,n}$. One can check that $q^{2,n}$ has Lorentzian signature when restricted to $\hat{\Sigma}$, and in fact, $\left(\hat{\Sigma}, q^{2,n}_{|\hat{\Sigma}}\right)$ is isometric to the product $(\mathbb{S}^1 \times \mathbb{S}^{n-1}, -dt^2 + g_{\mathbb{S}^{n-1}})$. Now Einstein's universe is conformally equivalent to the quotient of $(\mathbb{S}^1 \times \mathbb{S}^{n-1}, -dt^2 + g_{\mathbb{S}^{n-1}})$ by an involution (induced by the map $x \mapsto -x$ of $\mathbb{R}^{2,n}$).

2.2. Conformal group. In the previous projective model for Einstein's universe the subgroup $\mathrm{O}(2,n) \subset \mathrm{GL}_{n+2}(\mathbb{R})$ preserving $q^{2,n}$ acts conformally on Ein_n. In fact, the conformal group $\mathrm{Conf}(\mathrm{Ein}_n)$ of Ein_n is exactly $\mathrm{PO}(2,n)$. Let us now recall the following result, which is an extension to Einstein's universe of a classical theorem of Liouville in Euclidean conformal geometry (see for example [CK], [Fr3]):

Theorem 4. *Any conformal transformation between two open sets of* Ein_n *is the restriction of a unique element of* $\mathrm{PO}(2,n)$.

2.3. Lightlike geodesics and lightcones. It is a remarkable fact of pseudo-Riemannian geometry that all the metrics of a given conformal class have the same lightlike geodesics (as sets, but not as parametrized curves). In the case of Einstein's universe, the lightlike geodesics are the projections on Ein_n of 2-planes $P \subset \mathbb{R}^{2,n}$ such that $q^{2,n}_{|P} = 0$. Hence lightlike geodesics of Ein_n are copies of $\mathbb{R}P^1$.

Given a point p in Ein_n, the *lightcone with vertex* p, denoted by $C(p)$, is the set of lightlike geodesics containing p. In the projective model, if $p = \pi(u)$, with u some isotropic vector of $\mathbb{R}^{2,n}$, then $C(p)$ is just $\pi(P \cap C^{2,n})$, where P is the degenerate hyperplane $P = u^{\perp}$ (the orthogonal is taken for the form $q^{2,n}$). The lightcones are not smooth submanifolds of Ein_n. The only singular point of $C(p)$ is p, and $C(p)\backslash\{p\}$ is topologically $\mathbb{R} \times \mathbb{S}^{n-2}$.

2.4. Homogeneous open subsets. We will deal in this paper with several interesting open subsets of Ein_n, all obtained by removing from Ein_n the projectivization of peculiar linear subspaces of $\mathbb{R}^{2,n}$. We will be very brief here and refer to [Wo] for a more detailed study (especially concerning de Sitter and anti-de Sitter spaces).

Minkowski components. Given a point $p \in \mathrm{Ein}_n$, the complement of $C(p)$ in Ein_n is a homogeneous open subset of Ein_n, which is conformally equivalent to Minkowski space $\mathbb{R}^{1,n-1}$. We say that this is the Minkowski component associated to p. In fact, we have an explicit formula for the stereographic projection identifying $\mathrm{Ein}_n \setminus C(p)$ and $\mathbb{R}^{1,n-1}$ (see [CK], [Fr1]).

De Sitter and anti-de Sitter components. Just as Minkowski space arises by removing from Ein_n the projectivization of a lightlike hyperplane, one also gets interesting open subsets by removing the projectivization of other (i.e. nondegenerate) hyperplanes.

If P is some hyperplane of $\mathbb{R}^{2,n}$ with Lorentzian signature, then $\pi(P \cap C^{2,n})$ is a Riemannian sphere S of codimension one. The canonical conformal structure of Ein_n induces on this sphere the canonical Riemannian conformal structure. The stabilizer of S in $\mathrm{O}(2,n)$ is a group G isomorphic to $\mathrm{O}(1,n)$. The complement of S in Ein_n is a homogeneous open subset of Ein_n, conformally equivalent to the de Sitter space $\mathrm{d}\,\mathbb{S}_n$. Therefore $\mathbb{S}^{n-1}$, with its canonical conformal structure, appears as the conformal boundary of $\mathrm{d}\,\mathbb{S}_n$

If P is some hyperplane of $\mathbb{R}^{2,n}$ with signature $(2, n-1)$, then the projection $\pi(P \cap C^{2,n})$ is a codimension one Einstein universe E. The stabilizer of E in $\mathrm{O}(2,n)$ is a subgroup isomorphic to $\mathrm{O}(2, n-1)$. The complement of E is a homogeneous open subset of Ein_n, which is conformally equivalent to the anti-de Sitter space AdS_n. In this way we see Ein_{n-1} as the conformal boundary of AdS_n.

Complement of a lightlike geodesic. What do we get if we remove from Ein_n the projectivization of a maximal isotropic subspace of $\mathbb{R}^{2,n}$? Such subspaces are 2-planes, so that the resulting open set is the complement Ω_Δ of a lightlike geodesic $\Delta \subset \mathrm{Ein}_n$. Open sets like Ω_Δ admit a natural foliation by degenerate hypersurfaces, and this foliation $\mathcal{H}_\Delta$ is preserved by the whole conformal group of Ω_Δ. This foliation can be described as follows: given a point $p \in \Delta$, we consider the lightcone $C(p)$ with vertex p. Since Δ is a lightlike geodesic, we have $\Delta \subset C(p)$. Now the intersection of $C(p)$ with Ω_Δ is a degenerate hypersurface of Ω_Δ, diffeomorphic to $\mathbb{R}^{n-1}$. We call it $\mathcal{H}(p)$. If $p \neq p'$, $\mathcal{H}(p)$ and $\mathcal{H}(p')$ only intersect along Δ, and the leaves of the foliation $\mathcal{H}_\Delta$ are just the $\mathcal{H}(p)$ for $p \in \Delta$.

2.5. The space $\mathbb{L}_n$ of lightlike geodesics of Ein_n. Since this space will appear naturally when we will define the limit set of a Lorentzian Kleinian group, we briefly describe it.

The stabilizer of a lightlike geodesic in $\mathrm{O}(2,n)$ is a closed parabolic subgroup P, isomorphic to $(\mathbb{R} \times \mathrm{SL}(2,\mathbb{R}) \times \mathrm{O}(n-2)) \ltimes \mathrm{Heis}(2n-3)$, where $\mathrm{Heis}(2n-3)$

denotes the Heisenberg group of dimension $2n-3$. Thus $\mathbb{L}_n$ can be identified with the homogeneous space $\mathrm{O}(2,n)/P$, which has dimension $2n-3$.

Let $H \subset \mathbb{R}^{2,n}$ be a hyperplane with Lorentzian signature, and let Σ be the projection of $H \cap C^{2,n}$ on Ein_n. The hypersurface Σ is a codimension one Riemannian sphere of Ein_n. Now for any isotropic 2-plane $P \subset \mathbb{R}^{2,n}$, $P \cap H$ is 1-dimensional and isotropic. Equivalently, any lightlike geodesic of Ein_n intersects Σ in exactly one point. We get a well-defined submersion $p\colon \mathbb{L}_n \to \Sigma$. The fiber over $q \in \Sigma$ is the set of lightlike geodesics inside the lightcone $C(q)$. So, $\mathbb{L}_n$ is topologically an $\mathbb{S}^{n-2}$ fiber bundle over $\mathbb{S}^{n-1}$. Notice that $\mathrm{O}(2,n)$ does not preserve the bundle structure.

3. Conformal dynamics on Einstein's universe

3.1. Cartan decomposition of O(2, *n*). From now on it will be more convenient to work in a basis of $\mathbb{R}^{2,n}$ for which $q^{2,n}(x) = -2x_1x_{n+2} + 2x_2x_{n+1} + x_3^2 + \cdots + x_n^2$. We call $\mathrm{O}(2,n)$ the subgroup of $\mathrm{GL}_{n+2}(\mathbb{R})$ preserving the form $q^{2,n}$. Let A^+ be a the subgroup of diagonal matrices in $\mathrm{O}(2,n)$ of the form

$$\begin{pmatrix} e^{\lambda} & & & & & & \\ & e^{\mu} & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & e^{-\mu} & \\ & & & & & & e^{-\lambda} \end{pmatrix},$$

with $\lambda \geq \mu \geq 0$. Such an A^+ is usually called a *Weyl chamber*. The group $\mathrm{SO}(2,n)$ can be written as the product KA^+K where K is a maximal compact subgroup of $\mathrm{SO}(2,n)$. This decomposition is known as the *Cartan decomposition* of the group $\mathrm{SO}(2,n)$ (compare [B], [IW]). Such a decomposition also exists for $\mathrm{O}(2,n)$, with K a compact set of $\mathrm{O}(2,n)$. Moreover, for every $g \in \mathrm{O}(2,n)$, there is a unique $a(g) \in A^+$ such that $g \in Ka(g)K$. The element $a(g)$ is called the *Cartan projection* of g. As a matrix it is written

$$a(g) = \begin{pmatrix} e^{\lambda(g)} & & & & & & \\ & e^{\mu(g)} & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & e^{-\mu(g)} & \\ & & & & & & e^{-\lambda(g)} \end{pmatrix}.$$

The reals $\lambda(g) \geq \mu(g) \geq 0$ are called *the distortions* of the element g (associated with the given Cartan decomposition).

3.2. Qualitative dynamical description. We want to understand the possible dynamics for divergent sequences (g_k) of $\mathrm{O}(2,n)$ (i.e. sequences leaving every compact subset of $\mathrm{O}(2,n)$). Our approach considers sequences $g_k(x_k)$, where (x_k) is a converging sequence of Ein_n. It is important to consider arbitrary such convergent sequences, not only constant sequences, in order to characterize *proper actions*. Recall that given a subgroup Γ of homeomorphisms of a manifold X, one says that the action of Γ on X is *proper* if for all convergent sequences (x_k) of X and all divergent sequences (g_k) of Γ, the sequence $g_k(x_k)$ does not have any accumulation point in X. Notice that there exist actions for which $g_k(x)$ diverges for all divergent $(g_k) \in \Gamma$ and all $x \in X$, but which are not proper (look, for example, at the action of a hyperbolic linear transformation of $\mathrm{SL}(2,\mathbb{R})$ on the punctured plane $\mathbb{R}^2 \setminus \{0\}$).

Definition 1. Let (g_k) be a divergent sequence of homeomorphisms of a manifold X (i.e. (g_k) leaves any compact subset of $\mathrm{Homeo}(X)$). For any point $x \in X$, we define the set

$$D_{(g_k)}(x) = \bigcup_{x_k \to x} \{\text{accumulation points of } (g_k(x_k))\}.$$

The union is taken over all sequences converging to x.

Further, for any set $E \subset X$, $D_{(g_k)}(E) = \bigcup_{x \in E} D_{(g_k)}(x)$. Taking the union, over all divergent sequences $(g_k) \in \Gamma$, of the sets $D_{(g_k)}(E)$, we get a closed set $D_\Gamma(E) \subset X$ that we call the *dynamic set of E*.

Notice that for two points x and y in X, $y \in D_\Gamma(x)$ if and only if $x \in D_\Gamma(y)$. We say in this case that x and y are *dynamically related*.

The interest of this definition for the study of actions of discrete groups can be illustrated by the following: let Γ be a discrete group of $\mathrm{Homeo}(X)$ acting on some open subset $\Omega \subset X$. Then the next result is easily proved.

Proposition 1. *The group Γ acts properly on Ω iff no two points of Ω are dynamically related.*

Assuming that the action of Γ on Ω is proper, we also have:

Proposition 2. *If the action of Γ on Ω has compact quotient, then every $x \in \partial\Omega$ must be dynamically related to some point y of Ω (depending on x).*

Now let (g_k) be a divergent sequence of $\mathrm{O}(2,n)$. We define $\lambda_k = \lambda(g_k)$, $\mu_k = \mu(g_k)$ and $\delta_k = \lambda_k - \mu_k$. We say that the sequence (g_k) *tends simply to infinity* if

a) the three sequences (λ_k), (μ_k) and (δ_k) converge respectively to some λ_∞, μ_∞ and δ_∞ in $\overline{\mathbb{R}}$;

b) compact factors in the Cartan decomposition of (g_k) both admit a limit in K.

Of course, every sequence tending to infinity admits some subsequence tending simply to infinity, so that we will restrict our study to these last ones. The sequences tending simply to infinity split into three categories:

(i) *Sequences with balanced distortions.* This name denotes the sequences (g_k) for which $\lambda_\infty = \mu_\infty = +\infty$ and δ_∞ is finite.

(ii) *Sequences with bounded distortion.* This denotes the sequences (g_k) for which $\mu_\infty \neq +\infty$.

(iii) *Sequences with mixed distortions.* This denotes the sequences (g_k) for which $\lambda_\infty = \mu_\infty = \delta_\infty = +\infty$.

To each type corresponds, as we will see soon, distinct dynamical behaviours.

Notation. In the following we will use notations such as $C(p)$, $\mathcal{H}_\Delta, \ldots$. We invite the reader to look at Section 2, where these notation were introduced.

For any set E in $\mathbb{R}^{2,n}$, we use the notation $\check{\pi}(E)$ for $\pi(E \cap C^{2,n})$. If y and ε are two real numbers, we write $I_\varepsilon(y)$ for the closed interval $[y-\varepsilon, y+\varepsilon]$.

For every $x = (x_1, x_2, \ldots, x_{n+2})$ in $\mathbb{R}^{2,n}$, we define the ε-box centered at x as $B_\varepsilon(x) = I_\varepsilon(x_1) \times I_\varepsilon(x_2) \times \cdots \times I_\varepsilon(x_{n+2})$

For a sequence (g_k) of $\mathrm{O}(2,n)$ tending simply to infinity, we call $B_\varepsilon^\infty(x)$ the compact set obtained as the limit (for the Hausdorff topology) of the sequence of compact sets $g_k \circ \check{\pi}(B_\varepsilon(x))$ (this limit will always exist in the examples we will deal with).

Finally, we will often denote in the same way an element of $\mathrm{O}(2,n)$ and the conformal transformation of Ein_n that it induces.

3.2.1. Dynamics with balanced distortions

Proposition 3. *Let (g_k) be a sequence of $\mathrm{O}(2,n)$ with balanced distortions. Then we can associate to (g_k) two lightlike geodesics Δ^+ and Δ^-, called attracting and repelling circles of (g_k), and two submersions $\pi_+ \colon \mathrm{Ein}_n \backslash \Delta^- \to \Delta^+$ (resp. $\pi_- \colon \mathrm{Ein}_n \backslash \Delta^+ \to \Delta^-$), whose fibers are the leaves of $\mathcal{H}_{\Delta^-}$ (resp. $\mathcal{H}_{\Delta^+}$), such that the following holds.*

For every compact subset K of $\mathrm{Ein}_n \backslash \Delta^-$ (resp. $\mathrm{Ein}_n \backslash \Delta^+$), $D_{(g_k)}(K) = \pi_+(K)$ (resp. $D_{(g_k^{-1})}(K) = \pi_-(K)$).

Remark 1. Before beginning the proof, let us remark that if (g_k) has balanced distortions (resp. bounded distortion, resp. mixed distortions), it will be so for any compact perturbation of (g_k), i.e. any sequence $(l_k^{(1)} g_k l_k^{(2)})$ for $(l_k^{(1)})$ and $(l_k^{(2)})$ two converging sequences of $\mathrm{O}(2,n)$. In the same way the conclusions of the above proposition are not modified by a compact perturbation, even if of course $\pi_\pm$ and $\Delta^\pm$ are. So in the following (and also in Sections 3.2.2 and 3.2.3) we will restrict the proofs to the case where (g_k) is a sequence of A^+.

Proof. We restrict the proof to the case $\lambda_k = \mu_k$, so that $\delta_\infty = 0$.

We begin by defining $\Delta^\pm$ and $\pi^\pm$. Let us call P^+ (resp. P^-) the 2- plane spanned by e_1 and e_2 (resp. e_{n+1} and e_{n+2}), and Δ^+ (resp. Δ^-) the projection on Ein_n of these 2- planes. The space $\mathbb{R}^{2,n}$ splits as a direct sum $P_+ \oplus P_0 \oplus P_-$, where P_0 is the span of $e_3, \dots, e_n$. This splitting defines a projection $\tilde{\pi}_+$ (resp. $\tilde{\pi}_-$) from $\mathbb{R}^{2,n}$ to the plane P^+ (resp. P^-). The image $\tilde{\pi}_+(x)$ is nonzero as soon as x is an isotropic vector of $q^{2,n}$ which is not in P^-. Thus $\tilde{\pi}_+$ induces a projection π_+ of $\mathrm{Ein}_n \backslash \Delta^-$ on Δ^+ whose fibers are the projections on Ein_n of the fibers of $\tilde{\pi}^+$. These are degenerate hyperplanes of $\mathbb{R}^{2,n}$, defined as $q^{2,n}$-orthogonals of vectors of P^-. So, the fibers of π_+ are the intersections of $\mathrm{Ein}_n \backslash \Delta^-$ with the lightcones with vertex on Δ^-, i.e. the leaves of $\mathcal{H}_{\Delta^-}$.

Now let us choose x such that $\pi(x) \notin \Delta^-$. Since $g_k \circ \check{\pi}(B_\varepsilon(x)) = \check{\pi}(I_{e^{\lambda_k}\varepsilon}(e^{\lambda_k} x_1) \times I_{e^{\mu_k}\varepsilon}(e^{\mu_k} x_2) \times I_\varepsilon(x_3) \times \cdots \times I_\varepsilon(x_n) \times I_{e^{-\mu_k}\varepsilon}(e^{-\mu_k} x_{n+1}) \times I_{e^{-\lambda_k}\varepsilon}(e^{-\lambda_k} x_{n+2}))$, we obtain, for ε sufficiently small, that $B_\varepsilon^\infty(x) = \check{\pi}(I_\varepsilon(x_1) \times I_\varepsilon(x_2) \times \{0\} \times \cdots \times \{0\})$. We thus have $B_\varepsilon^\infty(x) \subset \Delta^+$. Since ε is arbitrarily close to 0, for any sequence (x_k) such that $\pi(x_k)$ tends to $\pi(x)$, we have $\lim_{k\to\infty} g_k \circ \pi(x_k) = \pi(x_1, x_2, 0, \dots, 0)$. This concludes the proof. □

3.2.2. Dynamics with bounded distortion

Proposition 4. *Let (g_k) be a sequence of $\mathrm{O}(2,n)$ with bounded distortions. Then we can associate to (g_k) two points p^+ and p^- of Ein_n, called attracting and repelling poles of (g_k), and a diffeomorphism $\hat{g}_\infty$ from the space of lightlike geodesics of $C^- = C(p^-)$ in the space of lightlike geodesics of $C^+ = C(p^+)$, conformal with respect to the natural conformal structure of these two spaces, such that we have:*

(i) *For all compact subset K inside $\mathrm{Ein}_n \backslash C^-$, we have $D_{(g_k)}(K) = \{p^+\}$.*

(ii) *For a lightlike geodesic $\Delta \subset C^-$ and a point x of Δ distinct from p^-, $D_{(g_k)}(x)$ is the lightlike geodesic $\hat{g}_\infty(\Delta)$.*

(iii) *The set $D_{(g_k)}(p^-)$ is the whole of Ein_n.*

The cones C^+ and C^- are called *attracting* and *repelling cones* of (g_k).

Remark 2. The dynamical pattern of the sequence (g_k^{-1}) is obtained by switching the $+$'s and the $-$'s in the statement. This remark holds also for Proposition 5.

Proof. Following Remark 1, we do the proof for a sequence (g_k) of A^+, with $\lim_{k\to\infty} \lambda_k = +\infty$.

Let $p^+ = \pi(e_1)$, $p^- = \pi(e_{n+2})$, $C^+ = \check{\pi}((e_1)^\perp)$, $C^- = \check{\pi}((e_{n+2})^\perp)$.

Let us first remark that if $x_1 \neq 0$, then clearly $B_\varepsilon^\infty(x) = p^+$. This proves (i), as well as (iii), passing to the complement.

If $\pi(x) \in C^-$ and if ε is sufficiently small, we get that $B_\varepsilon^\infty(x) = \check{\pi}(\mathbb{R} \times I_{e^{\mu_\infty}\varepsilon}(e^{\mu_\infty}x_2) \times I_\varepsilon(x_3) \times \cdots \times I_\varepsilon(x_n) \times I_{e^{-\mu_\infty}\varepsilon}(e^{-\mu_\infty}x_{n+1}) \times \{0\})$.

The lightlike geodesics of C^+ and C^- are parametrized by a sphere $\mathbb{S}^{n-2}$ corresponding to isotropic directions of the space spanned by $e_2, \dots, e_{n+1}$.

We define $\hat{g}_\infty$ as the element of $\mathrm{O}(1, n-1)$ given by

$$\hat{g}_\infty = \begin{pmatrix} e^{\mu_\infty} & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & e^{-\mu_\infty} \end{pmatrix}.$$

The spaces of lightlike geodesics of C^+ and C^- have a canonical conformal Riemannian structure, and we see that the map $\hat{g}_\infty$ is a conformal diffeomorphism between these two spaces.

By the above formula, if $\pi(x_k)$ converges to $\pi(x)$, the accumulation points of the sequence $g_k(\pi(x_k))$ are in every $B_\varepsilon^\infty(x)$, for arbitrary small ε. The intersection of all $B_\varepsilon^\infty(x)$ is $\check{\pi}(\mathbb{R} \times \{e^{\mu_\infty}x_2\} \times \{x_3\} \times \cdots \times \{x_n\} \times \{e^{-\mu_\infty}x_{n+1}\} \times \{0\})$, i.e. the image by $\hat{g}$ of the lightlike geodesic passing through p^- and $\pi(x)$. Conversely, every point $\pi(y)$ of this geodesic is in the Hausdorff limit of $g_k \circ \check{\pi}(B_\varepsilon(x))$. Hence, there exists a sequence x_k^ε of $B_\varepsilon(x)$ with $\lim_{k\to\infty} g_k \circ \pi(x_k^\varepsilon) = \pi(y)$. Let ε_k be a sequence tending to 0. Then $\lim_{k\to\infty} g_k \circ \pi(x_{n_k}^{\varepsilon_k}) = \pi(y)$ for some sequence of integers n_k, and $\pi(x_{n_k}^{\varepsilon_k})$ tends to $\pi(x)$. This concludes the proof of (ii). □

3.2.3. Mixed dynamics

Proposition 5. *Let (g_k) be a sequence of* $\mathrm{O}(2, n)$ *with mixed distortions. Then we can associate to (g_k) two points p^+ and p^-, called attracting and repelling poles of the sequence, as well as two lightlike geodesics Δ^+ et Δ^- (called attracting and repelling circles), with the inclusions $p^+ \in \Delta^+ \subset C^+ = C(p^+)$ and $p^- \in \Delta^- \subset C^- = C(p^-)$, such that the following properties hold:*

(i) *For every compact subset K inside* $\mathrm{Ein}_n \backslash C^-$*, the set $D_{(g_k)}(K)$ is $\{p^+\}$.*

(ii) *If x is a point of C^- not on Δ^-, then $D_{(g_k)}(x)$ is the lightlike geodesic Δ^+.*

(iii) *If x is a point of Δ^- distinct from p^-, then $D_{(g_k)}(x)$ is the attracting cone C^+.*

(iv) *The set $D_{(g_k)}p^-$ is the whole of* Ein_n.

The cones C^+ and C^- are called *attracting* and *repelling cones* of the sequence (g_k).

Proof. Once again we suppose that (g_k) is in A^+.

Let $p^+ = \pi(e_1)$, $p^- = \pi(e_{n+2})$, $C^+ = \check{\pi}((e_1)^\perp)$, $C^- = \check{\pi}((e_{n+2})^\perp)$. The circle Δ^+ (resp. Δ^-) is the projection of the 2-plane spanned by e_1 and e_2 (resp. e_{n+1} and e_{n+2}). We do not show (i) and (iv), the proof being exactly the same as for Proposition 4.

If $\pi(x) \in C^-$, but $\pi(x) \notin \Delta^-$, then $x_1 = 0$, but $x_2 \neq 0$. In this case we get $B_\varepsilon^\infty(x) = \check{\pi}(\mathbb{R} \times I_\varepsilon(x_2) \times \{0\} \times \cdots \times \{0\})$, that is to say Δ^+.

The intersection of all the $B_\varepsilon^\infty(x)$ is $\check{\pi}(\mathbb{R} \times \{x_2\} \times \{0\} \times \cdots \times \{0\})$, i.e. the lightlike geodesic Δ^+. The fact that $D_{(g_k)}(\pi(x)) = \Delta^+$ is proved exactly as in Proposition 4.

When $\pi(x) \in \Delta^-$, only x_{n+1} and x_{n+2} do not vanish and by the assumption $\pi(x) \neq p^+$, we get $x_{n+1} \neq 0$. Hence, we have that $B_\varepsilon^\infty(x)$ is $\check{\pi}(\mathbb{R} \times \cdots \times \mathbb{R} \times I_\varepsilon(x_{n+1}) \times \{0\})$, that is to say C^+.

As previously, we get $D_{(g_k)}(\pi(x)) = C^+$. □

Remark 3. Notice that different configurations for the dynamical elements described above can occur. For example, attracting and repelling circles of a dynamics with balanced or mixed distortions can intersect, or even be the same. In fact, all the possible configurations can occur.

4. About the limit set of a Lorentzian Kleinian group

4.1. Definition of the limit set. Given a Kleinian group Γ on a manifold X, it is quite natural to ask if there is in some sense a "canonical" open set $\Omega \subset X$ on which Γ acts properly. For example, any Kleinian group Γ on the sphere $\mathbb{S}^n$ admits a limit set Λ_Γ and the open set $\Omega_\Gamma = \mathbb{S}^n \backslash \Lambda_\Gamma$ is distinguished, since it is the only maximal open subset on which Γ acts properly. The nice properties of the limit set of a Kleinian group on $\mathbb{S}^n$ rest essentially on the fact that the Möbius group $\mathrm{O}(1, n+1)$ is a convergence group on $\mathbb{S}^n$. We just saw in the previous section that $\mathrm{O}(2, n)$ is quite far from being a convergence group on Ein_n, but we would nevertheless like to define a limit set Λ_Γ associated to a given discrete group $\Gamma \subset \mathrm{O}(2, n)$. We require that such a limit set have at least the two following properties:

(i) Λ_Γ is a Γ-invariant closed subset of Ein_n.

(ii) The action of Γ on $\Omega_\Gamma = \mathrm{Ein}_n \backslash \Lambda_\Gamma$ is properly discontinuous.

Definition 2. Given Γ discrete in $\mathrm{O}(2, n)$, we define $\mathcal{S}_\Gamma$ (resp. $\mathcal{T}_\Gamma$) the set of sequences (γ_k) of Γ, tending simply to infinity, with mixed or balanced distortions (resp. with bounded distortion). If (γ_k) is a sequence of $\mathcal{S}_\Gamma$ (resp. $\mathcal{T}_\Gamma$), we call $\Delta^+(\gamma_k)$ and $\Delta^-(\gamma_k)$ (resp. $C^+(\gamma_k)$ and $C^-(\gamma_k)$) its attracting and repelling circles (resp. attracting and repelling cones).

Definition 3. We define the limit set of a discrete $\Gamma \subset \mathrm{O}(2,n)$ as

$$\Lambda_\Gamma = \Lambda_\Gamma^{(1)} \cup \Lambda_\Gamma^{(2)},$$

where

$$\Lambda_\Gamma^{(1)} = \overline{\bigcup_{(\gamma_k)\in\mathcal{S}_\Gamma} \Delta^+(\gamma_k) \cup \Delta^-(\gamma_k)}$$

and

$$\Lambda_\Gamma^{(2)} = \overline{\bigcup_{(\gamma_k)\in\mathcal{T}_\Gamma} C^+(\gamma_k) \cup C^-(\gamma_k)}.$$

Notation. The complement of Λ_Γ in Ein_n is denoted by Ω_Γ.

It is clear that Λ_Γ is closed and Γ-invariant. Let us remark that Λ_Γ is a union of lightlike geodesics, so that it also defines a closed Γ-invariant subset $\hat{\Lambda}_\Gamma \subset \mathbb{L}_n$.

From the dynamical properties stated in the previous section, one checks easily that no pair of points in Ω_Γ can be dynamically related, so that the action of Γ on Ω_Γ is proper.

4.2. Lorentzian Kleinian groups of the first and the second type. Until now we did not focus on a fundamental difference between the action of $\mathrm{O}(1,n+1)$ on $\mathbb{S}^n$ and that of $\mathrm{O}(2,n)$ on Ein_n. Although any discrete group $\Gamma \subset \mathrm{O}(1,n+1)$ automatically acts properly on $\mathbb{H}^{n+1}$, it is not true in general that a discrete $\Gamma \subset \mathrm{O}(2,n)$ does so on AdS_{n+1}. This motivates the following distinction between subgroups of $\mathrm{O}(2,n)$.

Definition 4. A discrete group Γ of $\mathrm{O}(2,n)$ is of the *first type* if it acts properly on AdS_{n+1}. If not, it is said to be of the *second type*.

Notice that this terminology has no connection with the denomination of being of *first kind* and of *second kind* for the standard Kleinian groups on the sphere.

The previous dichotomy has a nice translation into dynamical terms due to the next result.

Proposition 6. *A Kleinian group Γ of $\mathrm{O}(2,n)$ is of the first type if and only if it does not admit any sequence (γ_k) with bounded distortion.*

Proof. We endow $\mathbb{R}^{2,n+1}$ with the quadratic form $q^{2,n+1}(x) = -2x_1x_{n+2} + 2x_2x_{n+1} + x_3^2 + \cdots + x_n^2 + x_{n+3}^2$ and call $e_1,\ldots,e_{n+3}$ the canonical basis. The subgroup of $\mathrm{O}(2,n+1)$ leaving invariant the subspace spanned by the first $n+2$ basis vectors can be canonically identified with $\mathrm{O}(2,n)$. This identification defines an embedding j from $\mathrm{O}(2,n)$ into $\mathrm{O}(2,n+1)$. The action of $j(\mathrm{O}(2,n))$ on Ein_{n+1} leaves invariant a codimension one Einstein universe that we call Ein_n. As we saw in the introduction, the complement of Ein_n in Ein_{n+1} is conformally equivalent to the anti-de Sitter space AdS_{n+1}.

Let us consider some g in $\mathrm{O}(2,n)$. In the basis $e_1,\dots,e_{n+3}$, $j(g) = \begin{pmatrix} g & \\ & 1 \end{pmatrix}$, so that when we perform the Cartan decomposition of $j(g)$, we find the same distortions as for g.

Suppose now that Γ admits some sequence (γ_k) with bounded distortion. By the remark above, $j(\gamma_k)$ has also bounded distortion as a sequence of $\mathrm{O}(2,n+1)$. We call C^+ and C^- its attracting and repelling cones in Ein_{n+1}. By Proposition 4, $D_{(g_k)}(C^- \cap \mathrm{AdS}_{n+1}) = C^+ \cap \mathrm{AdS}_{n+1}$. Therefore we can find two points of AdS_{n+1} which are dynamically related, so that the action of (γ_k) on AdS_{n+1} cannot be proper (Proposition 1).

Conversely, let us consider some sequence (γ_k) tending simply to infinity and with balanced or mixed distortions. Then the sequence $j(\gamma_k)$ has the same properties. Let us call Δ^+ and Δ^- the attracting and repelling circles of this latter sequence. Looking at the matrix expressions, it is clear that $\Delta^+ \subset \mathrm{Ein}_n$ and $\Delta^- \subset \mathrm{Ein}_n$. By Propositions 3 and 5, $D_{(g_k)}(x) \subset \mathrm{Ein}_n$ for any point $x \in \mathrm{AdS}_{n+1}$. So, if we assume that Γ has no sequence with bounded distortion, we get $D_\Gamma(x) \subset \mathrm{Ein}_n$ for any point $x \in \mathrm{AdS}_{n+1}$. Using Proposition 1, we get that Γ acts properly on AdS_{n+1}. □

4.3. Limit set of a group of the first type: proof of Theorem 1. Since Γ is of the first type, Λ_Γ is also the limit set of Γ, regarded as a subgroup of $\mathrm{O}(2,n+1)$ acting on Ein_{n+1}. The complement of this limit set in Ein_{n+1} is precisely $\Omega_\Gamma \cup \mathrm{AdS}_{n+1}$, so that (i) of the theorem is clear.

To prove (ii), let us suppose that Γ acts properly on some $\Omega \cup \mathrm{AdS}_{n+1}$ with Ω not included in Ω_Γ. Then there is a sequence (γ_k) of Γ (with balanced or mixed distortions) such that $\Delta^-(\gamma_k)$ meets Ω.

Lemma 1. *Let Γ be a discrete group of $\mathrm{O}(2,n)$ acting properly on some open set Ω. Then for any sequence (γ_k) of Γ with balanced distortions, neither $\Delta^+(\gamma_k)$ nor $\Delta^-(\gamma_k)$ meets Ω.*

Proof. Suppose on the contrary that for some (γ_k) with balanced distortions, we have $\Delta^+(\gamma_k) \cap \Omega \neq \emptyset$. From Proposition 3, we infer that the set $D_{(\gamma_k)}(\Delta^+(\gamma_k) \cap \Omega)$ contains a lightlike geodesic Δ in its interior. So, there is a tubular neighbourhood U of Δ contained in $\mathrm{Ext}(\Omega)$ ($\mathrm{Ext}(\Omega)$ denotes the complement of Ω in Ein_n). But we also infer from Proposition 3 that for any Δ not meeting $\Delta^-(\gamma_k)$, we have $\lim_{k\to+\infty} \gamma_k(\Delta) = \Delta^+(\gamma_k)$. As a consequence, any lightlike geodesic of $\mathrm{Ext}(\Omega)$ has to cut $\Delta^-(\gamma_k)$. Since all the lightlike geodesics included in U cannot all meet $\Delta^-(\gamma_k)$ we get a contradiction. □

The lemma above tells us that the sequence (γ_k) has mixed distortions. For any point $x \in \Delta^-(\gamma_k) \cap \Omega$, we have $D_{(\gamma_k)}(x) = C^+(\gamma_k)$. Since $C^+(\gamma_k)$ meets AdS_{n+1}, we get pairs of points in $\Omega \cup \mathrm{AdS}_{n+1}$ which are dynamically related, and the action cannot be proper by Proposition 1.

Remark 4. For Γ Kleinian of the first type, the manifold Ω_Γ / Γ appears as the conformal boundary of the complete anti-de Sitter manifold $\mathrm{AdS}_{n+1} / \Gamma$ (see [Fr4] for more details on this point).

To prove (iii), we begin by showing that $\hat{\Lambda}_\Gamma \subset \mathbb{L}_n$ is a minimal set. This is in fact a particular case of a general result of Benoist ([B]), but we give a simple proof.

Let $\hat{\Lambda}$ be a closed Γ-invariant subset of $\mathbb{L}_n$. Any sequence (γ_k) tending simply to infinity in Γ has either mixed or balanced distortions. As a simple consequence of Propositions 3 and 5, we get that if Δ is a lightlike geodesic of Ein_n which does not meet $\Delta^-(\gamma_k)$, then $\lim_{k\to+\infty} \gamma_k(\Delta) = \Delta^+(\gamma_k)$. So, if for any sequence (γ_k) as above, no geodesic of $\hat{\Lambda}$ meets $\Delta^-(\gamma_k)$, we have $\Lambda_\Gamma \subset \Lambda$, and we are done.

On the contrary, if for some (γ_k), all the geodesics of $\hat{\Lambda}$ meet $\Delta^-(\gamma_k)$, we claim that Γ cannot be Zariski dense. Indeed, by Zariski density, Γ cannot leave $\Delta^-(\gamma_k)$ invariant. So, let us choose $\gamma \in \Gamma$ such that $\gamma(\Delta^-(\gamma_k)) \neq \Delta^-(\gamma_k)$. If $\gamma(\Delta^-(\gamma_k))$ and $\Delta^-(\gamma_k)$ are disjoint, the set of lightlike geodesics meeting both $g(\Delta^-(\gamma_k))$ and $\Delta^-(\gamma_k)$ is contained in a 2-dimensional Einstein universe, which have to be fixed by Γ: a contradiction with the Zariski density of Γ.

If $g(\Delta^-(\gamma_k))$ and $\Delta^-(\gamma_k)$ meet in one point p, then any lightlike geodesic meeting both $g(\Delta^-(\gamma_k))$ and $\Delta^-(\gamma_k)$ has to contain p. Indeed, due to the fact that the quadratic form $q^{2,n}$ cannot have some 3-dimensional isotropic subspace, there is no nontrivial triangle of Ein_n, whose edges are pieces of lightlike geodesics. We infer that Γ has to fix the lightcone $C(p)$ and we get once again a contradiction.

We can now show that Ω_Γ is the maximal open set on which the action of Γ is proper. Suppose that Γ acts properly on Ω which is not included in Ω_Γ. We call Λ the complement of Ω in Ein_n. Since $\Lambda_\Gamma \not\subset \Lambda$, there is a sequence (γ_k) tending simply to infinity in Γ with $\Delta^+(\gamma_k) \cap \Omega \neq \emptyset$.

Lemma 2. *If an infinite Kleinian group $\Gamma \subset \mathrm{O}(2,n)$ acts properly on some open subset Ω, then the complement Λ of Ω in Ein_n contains a lightlike geodesic.*

Proof. Let us pick a sequence (γ_k) tending simply to infinity in Γ. Suppose first that (γ_k) has mixed dynamics. Suppose that $\Delta^-(\gamma_k)$ meets Ω at a point x (if $\Delta^-(\gamma_k) \cap \Omega = \emptyset$, we are done). By properness, $D_{(g_k)}(x) \cap \Omega = \emptyset$. But $D_{(g_k)}(x) = C^+(g_k)$, which contains infinitely many lightlike geodesics, and the conclusion holds.

Also, if (g_k) has balanced (resp. bounded) distorsions, the dynamic set $D_{(g_k)}x$ of $x \in \Delta^-(\gamma_k)$ (resp. $x \in C^-(\gamma_k)$) contains infinitely many lightlike geodesics. The proof works thus in the same way. □

Now let us look at the lightlike geodesics of Λ. Since by Zariski density, Γ cannot fix a finite family of lightlike geodesics, there are infinitely many lightlike geodesics in Λ. But all these geodesics have to meet $\Delta^-(\gamma_k)$, because if some Δ does not, $\lim_{k\to+\infty} \gamma_k(\Delta) = \Delta^+(\gamma_k)$. A contradiction with $\Delta^+(\gamma_k) \cap \Omega \neq \emptyset$. Now, we conclude as for proving the minimality property of $\hat{\Lambda}_\Gamma$: all the lightlike geodesics of Λ are in the same Γ-invariant Einstein torus, or the same Γ-invariant lightcone, and we get a contradiction with the Zariski density of Γ.

5. Some examples of Lorentzian Kleinian groups

5.1. Examples arising from structures with constant curvature. In Lorentzian geometry, a completeness result ensures that any compact Lorentzian manifold with constant sectional curvature is obtained as a quotient $\mathbb{R}^{1,n-1}/\,\Gamma$ or $\widetilde{\mathrm{AdS}}_n/\,\Gamma$, where Γ is a discrete group of Lorentzian isometries. This deep theorem was first proved for the case of curvature zero by Carrière in [Ca], and generalized by Klingler in [Kl] (note that compact Lorentzian manifolds cannot have curvature +1). Another result, known as *finiteness of level* (see [KR], [Ze]), ensures that any compact quotient $\widetilde{\mathrm{AdS}}_n/\widetilde{\Gamma}$ (where $\widetilde{\Gamma}$ is a discrete group of isometries) is in fact, up to finite cover, a quotient $\mathrm{AdS}_n\,/\,\Gamma$. Since $\mathbb{R}^{1,n-1}$ and AdS_n both embed conformally into Ein_n (see Section 2), by Theorem 4 we get that any compact Lorentzian structure with constant curvature is (up to finite cover) uniformized by a Lorentzian Kleinian groups. Moreover, in this case the structure of the groups involved is fairly well understood, due to [CaD], [Sa] and [Ze].

5.2. Examples arising from flat CR-geometry. Let us consider the complex vector space $\mathbb{C}^{n+1}$, endowed with the hermitian form $h^{1,n-1}(z) = -|z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots + |z_{n+1}|^2$. We consider $C^{1,n}_{\mathbb{C}}$, the lightcone defined as $\{z \in \mathbb{C}^{n+1} \mid h^{1,n}(z) = 0\}$, and call Ω^- the open set $\{z \in \mathbb{C}^{n+1} \mid h^{1,n}(z) < 0\}$. If we project Ω^- on the complex projective space $\mathbb{C}\mathrm{P}^n$, we get the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. If we project $C^{1,n}_{\mathbb{C}}$ minus the origin on $\mathbb{C}\mathrm{P}^n$, we get a sphere $\mathbb{S}^{2n-1}$, naturally endowed with a CR-structure. This CR-sphere can be seen at the infinity of $\mathbb{H}^n_{\mathbb{C}}$. If, instead of looking at the complex directions of $C^{1,n}_{\mathbb{C}}$, we consider the quotient $C^{1,n}_{\mathbb{C}}/\mathbb{R}^\star$ of $C^{1,n}_{\mathbb{C}}$ by the real homotheties, then the space that we get is Einstein's universe of dimension $2n$. In other words, there is a fibration $f\colon \mathrm{Ein}_{2n} \to \mathbb{S}^{2n-1}$ whose fibers are circles. The fibration is preserved by the group $\mathrm{U}(1,n)$, which acts on Ein_{2n} as a subgroup of $\mathrm{O}(2,2n)$. If Z denotes the center of $\mathrm{U}(1,n)$ (homotheties by complex numbers of modulus 1), then the fibers of f are exactly the orbits of Z on Ein_{2n}. These orbits are lightlike geodesics.

Proposition 7. *If $\Gamma \in \mathrm{U}(1,n)$ is a discrete group, whose projection $\hat{\Gamma}$ on $\mathrm{PU}(1,n)$ acts properly discontinuously on $\hat{\Omega} \subset \mathbb{S}^{2n-1}$, then Γ is a Kleinian group of Ein_{2n}*

and acts properly discontinuously on $\Omega = f^{-1}(\hat{\Omega})$. *If* $\hat{G}$ *acts with compact quotient on* $\hat{\Omega}$, *so does* Γ *on* Ω.

Remark 5. The group $\mathrm{PU}(1, n)$ acting on $\mathbb{S}^{2n-1}$ is a convergence group, and there is a good notion of limit set for a discrete group $\hat{G}$ as above (see for example [A]). In fact, it is not difficult to check that the Lorentzian Kleinian groups Γ built as in Proposition 7 are of the first type. Their limit set is just the preimage by f of the limit set $\hat{\Lambda}_{\hat{\Gamma}}$ of $\hat{\Gamma}$ on $\mathbb{S}^{2n-1}$.

To illustrate this case, let us mention the two following examples.

Example 1. We write each $z \in \mathbb{C}^{n+1}$ as $z = (x, y)$ with x and y in $\mathbb{R}^n$. We identify the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ with the set of points $(x, 0)$ with $-x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = -1$ and $x_1 > 0$. If (x, y) is moreover in the unit tangent bundle of $\mathbb{H}^n_{\mathbb{R}}$, it satisfies the following two extra equations:

$$-x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = 0,$$
$$-y_1^2 + y_2^2 + \cdots + y_{n+1}^2 = 1.$$

Projectivising, we get an open subset $\hat{\Omega} \subset \mathbb{S}^{2n-1}$. In fact $\hat{\Omega}$ is precisely $\mathbb{S}^{2n-1}$ minus an $(n-1)$-dimensional sphere Σ (the projection on $\mathbb{S}^{2n-1}$ of the set $\{z = (x, 0) \mid -x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 0\}$).

Now the subgroup $G = \mathrm{O}(1, n)$ of real matrices in $\mathrm{U}(1, n)$ acts on $\mathbb{S}^{2n-1}$ and preserves $\hat{\Omega}$. Identifying $\hat{\Omega}$ with $T^1\mathbb{H}^n_{\mathbb{R}}$, we get that G acts properly and transitively on $\hat{\Omega}$. As a consequence we have the following

Fact. *Any discrete group* Γ *in* $\mathrm{O}(1, n)$ *acts properly discontinuously on* $\hat{\Omega}$. *Considered as a subgroup of* $\mathrm{O}(2, 2n)$ *it yields a Kleinian group acting on* Ein_{2n}.

The Kleinian manifold Ω/Γ obtained in this way are circle bundles over $T^1(N)$, where N is the hyperbolic manifold $\mathbb{H}^n_{\mathbb{R}}/\Gamma$.

Example 2. Inside $\mathrm{U}(1, n)$ there is a group G isomorphic to the Heisenberg group of dimension $2n - 1$. The group G fixes a point p_∞ on $\mathbb{S}^{2n-1}$ and acts simply transitively on the complement of this point. By Proposition 7, any discrete group in G will yield a Lorentzian Kleinian group, acting properly on the complement of a lightlike geodesic. The Kleinian manifolds obtained in this way will be circle bundles over nilmanifolds.

5.3. Subgroups of $\mathrm{O}(1, r) \times \mathrm{O}(1, s)$. We still endow $\mathbb{R}^{2,n}$ with the quadratic form $q^{2,n}(x) = -2x_1x_{n+2} + 2x_2x_{n+1} + x_3^2 + \cdots + x_n^2$, and we consider an orthogonal splitting $\mathbb{R}^{2,n} = E_1 + E_2$ with E_1 and E_2 two spaces of signature $(1, r)$ and

$(1, s)$ respectively ($r \neq 0$, $s \neq 0$ and $r + s = n$). We suppose also $r \leq s$. For example, we take $E_1 = (e_1, e_3, \ldots, e_{3+r-2}, e_{n+2})$ and $E_2 = (e_2, e_{3+r-1}, \ldots, e_{n+1})$. The subgroup G of $\mathrm{O}(2, n)$ preserving this splitting is isomorphic to the product $\mathrm{O}(1, r) \times \mathrm{O}(1, s)$. Before describing some examples of Kleinian groups in G, let us say a few words about the geometric meaning of this splitting on Ein_n.

Lemma 3. *We can write* Ein_n *as a union* $\Omega_1 \cup \Omega_2 \cup \Sigma$. *The set* Ω_1 (*resp.* Ω_2) *is open, G-invariant, homogeneous under the action of G, and conformally equivalent to the product* $\mathrm{d}\,\mathbb{S}_r \times \mathbb{H}^s$ (*resp.* $\mathbb{H}^r \times \mathrm{d}\,\mathbb{S}_s$). Σ *is a singular, degenerate G-invariant hypersurface.*

Proof. We call π_1 and π_2 the projections of $\mathbb{R}^{2,n}$ on E_1 and E_2, respectively. The projection of vectors $u = (v, w)$ of $\mathbb{R}^{2,n}$, for which both $v = \pi_1(u)$ and $w = \pi_2(u)$ are isotropic, gives the hypersurface Σ. We will say more about it later.

The vectors $u = (v, w)$, for which neither v nor w is isotropic, are of two kinds.

Those for which $q^{2,n}(v) > 0$. Since we work projectively, we can suppose that $q^{2,n}(v) = 1$ and $q^{2,n}(w) = -1$. In a further quotient by $-\mathrm{Id}$ these vectors project on the product $\mathrm{d}\,\mathbb{S}_r \times \mathbb{H}^s$. They constitute the open set Ω_1.

Those for which $q^{2,n}(v) < 0$. These vectors project on a product $\mathbb{H}^r \times \mathrm{d}\,\mathbb{S}_s$ and constitute the open set Ω_2. □

The hypersurface Σ can be regarded as the conformal boundary of the spaces $\mathrm{d}\,\mathbb{S}_r \times \mathbb{H}^s$ and $\mathbb{H}^r \times \mathrm{d}\,\mathbb{S}_s$. Let us describe it more precisely. The isotropic vectors (v, w) of $\mathbb{R}^{2,n}$, for which v and w are isotropic, split themselves into two sets. Those for which either v or w is zero. Their projectivisation gives two Riemannian spheres Σ_1 and Σ_2 of dimension $(r - 1)$ and $(s - 1)$ respectively.

Those for which v and w are nonzero project on the product of the projectivisation of the lightcone of E_1 by the lightcone of E_2, namely $\mathbb{S}^{r-1} \times \mathbb{C}^{1,s}$. So Σ minus $\Sigma_1 \cup \Sigma_2$ has two connected components, each of which is diffeomorphic to $\mathbb{S}^{r-1} \times \mathbb{S}^{s-1} \times \mathbb{R}$. One can check that Σ is obtained as the union of the lightlike geodesics intersecting both Σ_1 and Σ_2.

We now give some examples of Kleinian groups in G.

Example 3. Let us take a discrete group $\hat{\Gamma}$ inside $\mathrm{O}(1, r)$ and any representation ρ of $\hat{\Gamma}$ inside $\mathrm{O}(1, s)$. We call $\Gamma_\rho = \mathrm{Graph}(\hat{\Gamma}, \rho) = \{(\hat{\gamma}, \rho(\hat{\gamma})) | \hat{\gamma} \in \hat{\Gamma}\}$. Then Γ_ρ is a Lorentzian Kleinian group of $\mathrm{O}(2, n)$. Indeed, its action on $\Omega_2 = \mathbb{H}^r \times \mathrm{d}\,\mathbb{S}_s$ is clearly proper. Let us say a little bit more about the limit set of these groups. We call $\Lambda_{\hat{\Gamma}}$ the limit set of the group $\hat{\Gamma}$ on the sphere Σ_1.

Case a): *ρ is injective with discrete image.* A sequence (γ_k) of Γ_ρ can be written as a matrix $\begin{pmatrix} \hat{\gamma}_k & \\ & \rho(\hat{\gamma}_k) \end{pmatrix}$. If (γ_k) tends simply to infinity, so does the sequence $(\hat{\gamma}_k)$ (resp. $\rho(\hat{\gamma}_k)$) in $\mathrm{O}(1, r)$ (resp. in $\mathrm{O}(1, s)$). We thus see that (γ_k) has either mixed or balanced distortions. In particular, the group Γ_ρ is always of the first type in this case.

The attracting and repelling circles of (γ_k) can be described as follows. Since the sequence $(\hat{\gamma}_k)$ (resp. $\rho(\hat{\gamma}_k)$) tends simply to infinity in $\mathrm{O}(1,r)$ (resp. $\mathrm{O}(1,s)$), it has two attracting and repelling poles $p^+(\hat{\gamma}_k)$ and $p^-(\hat{\gamma}_k)$ (resp. $p^+(\rho(\hat{\gamma}_k))$ and $p^-(\rho(\hat{\gamma}_k))$) on Σ_1 (resp. Σ_2). Then $\Delta^+(\gamma_k)$ (resp. $\Delta^-(\gamma_k)$) is simply the lightlike geodesic of Ein_n joining $p^+(\hat{\gamma}_k)$ and $p^+(\rho(\hat{\gamma}_k))$ (resp. $p^-(\hat{\gamma}_k)$ and $p^-(\rho(\hat{\gamma}_k))$). In particular, the limit set Λ_{Γ_ρ} is a closed subset of Σ (strictly included in Σ if $\Lambda_{\hat{\Gamma}} \neq \Sigma_1$). An interesting subcase arises when we take for $\hat{\Gamma}$ a cocompact lattice in $\mathrm{O}(1,2)$, and a quasi-fuchsian representation $\rho\colon \hat{\Gamma} \to \mathrm{O}(1,s)$ ($s \geq 2$). The limit set of $\rho(\hat{\Gamma})$ on Σ_2 is a topological circle, and we get for the limit set Λ_{Γ_ρ} a topological torus. One can prove moreover (which is omitted here) that the action of Γ_ρ is cocompact on the complement of its limit set.

Case b): *ρ is not injective with discrete image.* In this case there is a sequence (γ_k) tending simply to infinity in Γ_ρ such that $\rho(\hat{\gamma}_k)$ is bounded. Such a sequence (γ_k) has bounded distortion, and the group Γ_ρ is of the second type. The attracting and repelling poles $p^+(\gamma_k)$ and $p^-(\gamma_k)$ are both on Σ_1. In fact they are the attracting and repelling poles of $(\hat{\gamma}_k)$ (acting as a sequence of $\mathrm{O}(1,r)$ on Σ_1). In this case the limit set Λ_{Γ_ρ} is just the union of lightcones with vertex on $\Lambda_{\hat{\Gamma}}$.

6. About Klein's combination theorem

The examples of Kleinian groups given so far are not completely satisfactory, since they arise from geometrical contexts such as Lorentzian spaces with constant curvature or flat CR-geometry, and in some way are not "typical" of conformally flat Lorentzian geometry. For instance, we still do not have examples of Zariski dense Kleinian groups on Ein_n. One way to construct other classes of examples is to combine two existing Lorentzian Kleinian groups to get a third one. In the theory of Kleinian groups on the sphere this kind of construction is achieved on the basis of the celebrated Klein's combination theorem ([A], [Ma]). We now state a generalized version of this theorem. For this we need the following definition.

Definition 5. Let X be a manifold. A Kleinian group on X is a discrete subgroup of diffeomorphisms Γ acting properly discontinuously on some nonempty open set $\Omega \subset X$. We say that an open set $D \subset \Omega$ is a *fundamental domain* for the action of Γ on Ω if D does not contain two points of the same Γ-orbit and if moreover $\bigcup_{\gamma\in\Gamma} \gamma(\bar{D}) = \Omega$.

Notation. For any subset D of the manifold X, we call $\mathrm{Ext}(D)$ the complement of D in X.

Theorem 5 (Klein). *Let Γ_i $(i = 1, \dots, m)$ a finite family of Kleinian groups on a compact connected manifold X. We suppose that each Γ_i acts cocompactly on some open subset Ω_i of X with fundamental domain D_i. We assume moreover that for each $i \neq j$, $\mathrm{Ext}(D_i) \subset D_j$, and that $D = \bigcap_{i=1}^m D_i \neq \emptyset$. Then we have:*

(i) *The group Γ generated by the Γ_i's is isomorphic to the free product $\Gamma_1 * \cdots * \Gamma_m$.*

(ii) *The group Γ is Kleinian. More precisely, $\Omega = \bigcup_{\gamma\in\Gamma} \gamma(\overline{D})$ is an open subset of X, and Γ acts properly discontinuously and cocompactly on Ω, with fundamental domain D.*

Proof. We do the proof for two groups Γ_1 and Γ_2, the final result being then obtained by induction. Let $\gamma = \gamma_s\gamma_{s-1}\dots\gamma_2\gamma_1$ be a word of Γ such that $\gamma_i \in G_{j_i}$ $(j_i \in \{1, 2\})$ and $j_i \neq j_{i+1}$. Then the first condition on the fundamental domains yields the inclusions $\gamma_s\gamma_{s-1}\dots\gamma_2\gamma_1(D) \subset \gamma_s\gamma_{s-1}\dots\gamma_2(\mathrm{Ext}(\overline{D}_{j_1})) \subset \cdots \subset \gamma_s(\mathrm{Ext}(\overline{D}_{j_{s-1}})) \subset \mathrm{Ext}(\overline{D}_{j_s})$. So, for any nontrivial reduced g, $\gamma(D) \cap D = \emptyset$. This proves that γ cannot be the identity, and (i) follows. In the same way, we prove that $\gamma(\overline{D}) \cap \overline{D} = \emptyset$ as soon as $s > 1$. Since $\overline{D}$ is compact in Ω_1 and Ω_2 and the action of Γ_1 and Γ_2 is proper, we get

Lemma 4. *The intersection $\gamma(\overline{D}) \cap \overline{D}$ is empty for all but a finite number of γ's.*

Lemma 5. *There is a finite family $\gamma_1, \dots, \gamma_s$ of elements of Γ such that $\overline{D} \cup \gamma_1(\overline{D}) \cup \cdots \cup \gamma_m(\overline{D})$ contains $\overline{D}$ in its interior.*

Proof. We choose some open neighbourhood U_1 of ∂D_1 such that $U_1 \subset \Omega_1$ and $\overline{U}_1$ is a compact subset of Ω_1. Since D_1 is a fundamental domain of Γ_1, for each $x \in U_1$ there exists a $\gamma_x \in \Gamma_1$ such that $x \in \gamma_x(\overline{D}_1)$. But since the action of Γ_1 is proper $\gamma(\overline{D}_1) \cap U_1$ is nonempty only for a finite number of elements $\gamma_1^{(1)}, \dots, \gamma_s^{(1)}$ of Γ_1. Thus $\overline{D}_1 \cup U_1$ is included in $\overline{D}_1 \cup \gamma_1^{(1)}(\overline{D}_1) \cup \cdots \cup \gamma_s^{(1)}(\overline{D}_1)$, and $\overline{D}_1$ is contained in the interior of $\overline{D}_1 \cup \gamma_1^{(1)}(\overline{D}_1) \cup \cdots \cup \gamma_s^{(1)}(\overline{D}_1)$. But if $D_1' = D_1 \backslash K$, where K is a compact subset of D_1, then we also have $\overline{D}_1' \cup U_1 \subset \overline{D}_1' \cup \gamma_1^{(1)}(\overline{D}_1') \cup \cdots \cup \gamma_s^{(1)}(\overline{D}_1')$. In particular, when K is the exterior of D_2, we get that $\overline{D} \cup U_1 \subset \overline{D} \cup \gamma_1^{(1)}(\overline{D}) \cup \cdots \cup \gamma_s^{(1)}(\overline{D})$.

Now we can apply the same argument for a neighbourhood U_2 of ∂D_2 in Ω_2. We get a finite family $\gamma_1^{(2)}, \dots, \gamma_t^{(2)}$ of Γ_2 such that $\overline{D} \cup U_2 \subset \overline{D} \cup \gamma_1^{(2)}(\overline{D}) \cup \cdots \cup \gamma_t^{(2)}(\overline{D})$. Setting $m = s + t$, $\gamma_i = \gamma_i^{(1)}$ for $i = 1, \dots, s$ and $\gamma_{s+i} = \gamma_i^{(2)}$ for $i = 1, \dots, t$, we get the lemma. □

As a consequence of this lemma, we get that the set $\Omega = \bigcup_{\gamma\in\Gamma} \gamma(\overline{D})$ is an open set.

It remains to prove that the action of Γ on Ω is proper. Indeed, since Γ is not *a priori* a convergence group, the fact that Γ acts discontinuously on Ω no longer ensures that the action is proper. That is why our assumptions (in particular the assumption of cocompactness) are stronger as for the classical Klein's theorem on the sphere.

Suppose, on the contrary, that there is a sequence (x_i) of Ω converging to $x_\infty \in \Omega$, and a sequence (γ_i) tending to infinity in Γ, such that $y_i = \gamma_i(x_i)$ converges to $y_\infty \in \Omega$. We can assume that $x_\infty \in \bar{D}$. On the other hand, by definition of Ω, there is a γ_0 such that $y_\infty \in \gamma_0(\bar{D})$. Lemma 5 ensures that for i sufficiently large, x_i must be in $\bar{D} \cup \gamma_{i_1}(\bar{D}) \cup \cdots \cup \gamma_{i_m}(\bar{D})$, and y_i in $\gamma_0(\bar{D}) \cup \gamma_0\gamma_{i_1}(\bar{D}) \cup \cdots \cup \gamma_0\gamma_{i_m}(\bar{D}))$. But then, Lemma 4 implies that the sequence (γ_i) takes its values in a finite set, a contradiction with the fact that (γ_i) tends to infinity in Γ. □

We would like to apply the theorem above to combine Lorentzian Kleinian groups. Notice that for two Kleinian groups the condition $\mathrm{Ext}(D_1) \subset D_2$ implies $\partial\Omega_1 \subset D_2$ and $\partial\Omega_2 \subset D_1$. Together with Lemma 2, we get that if two cocompact Lorentzian Kleinian groups can be combined, then their fundamental domains have to contain a lightlike geodesic (in particular, no Kleinian group uniformizing a manifold with constant curvature can be combined with another Kleinian group). It turns out that this obstruction is the only one which forbids combining two Lorentzian Kleinian groups, as shown by Theorem 2, which we now prove.

6.1. Proof of Theorem 2. We choose $\Delta_1 \subset D_1$ and $\Delta_2 \subset D_2$, two lightlike geodesics. Since D_1 and D_2 are open, they contain not only one, but in fact infinitely many lightlike geodesics, so that we can moreover choose Δ_1 and Δ_2 disjoint. We begin with a useful lemma.

Lemma 6. *Given Δ_1 and Δ_2 two disjoint lightlike geodesics of* Ein_n*, there exists $g \in \mathrm{Conf}(\mathrm{Ein}_n)$ such that (g^k) has mixed distortions and admits Δ_1 and Δ_2 as attracting and repelling circles.*

Proof. The geodesic Δ_1 (resp. Δ_2) is the projection on Ein_n of a 2-plane (e'_1, e'_2) (resp. (e'_3, e'_4)) of $\mathbb{R}^{2,n}$. We choose moreover e'_3 and e'_4 such that $q^{2,n}(e'_1, e'_3) = -2$ and $q^{2,n}(e'_2, e'_4) = 2$. The $q^{2,n}$-orthogonal F to (e'_1, e'_2, e'_3, e'_4) has Riemannian signature and we denote by $e'_5, \dots, e'_{n+2}$ one of its orthonormal basis. Then we consider some element g of $\mathrm{O}(2, n)$, which writes in the base $(e'_1, \dots, e'_{n+2})$ as

$$g = \begin{pmatrix} e^{\lambda} & & & & & & \\ & e^{\mu} & & & & & \\ & & e^{-\lambda} & & & & \\ & & & e^{-\mu} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

If we choose $\lambda > \mu > 0$, then it is clear that (g^k) has mixed distortions with $\Delta^+ = \Delta_1$ and $\Delta^- = \Delta_2$. □

We now take some $g \in \mathrm{Conf}(\mathrm{Ein}_n)$ as in the lemma above. Let us choose V_1 (resp. V_2) an open tubular neighbourhood of Δ_1 (resp. Δ_2) such that $\overline{V_1} \subset D_1$ (resp. $\overline{V_2} \subset D_2$). The complement of V_i $(i = 1, 2)$ in Ein_n is denoted by $\mathrm{Ext}(V_i)$. It follows from Proposition 5 (i) and (ii) that the set dynamically associated to $\mathrm{Ext}(V_2)$ with respect to (g^k) is included in Δ^+. Since $\mathrm{Ext}(V_2)$ contains a lightlike geodesic, it is exactly Δ^+. Hence, for k_0 sufficiently large, $g^{k_0}(\mathrm{Ext}(V_2)) \subset V_1$. We call $\Gamma_2' = g^{k_0}\Gamma_2 g^{-k_0}$. The group Γ_2' is a cocompact Lorentzian Kleinian group with fundamental domain $D_2' = g^{k_0}(D_2)$. But $g^{k_0}(D_2)$ contains $g^{k_0}(\mathrm{Int}(V_2))$, and as we just saw, $\mathrm{Ext}(V_1) \subset g^{k_0}(\mathrm{Int}(V_2))$. So $\mathrm{Ext}(D_2') \subset D_1$. We can then apply Theorem 5, and we get that the group generated by Γ_2' and Γ_1 is still Kleinian, cocompact, and isomorphic to $\Gamma_1 * \Gamma_2'$, i.e. $\Gamma_1 * \Gamma_2$.

Example 4. All the cocompact Lorentzian Kleinian groups of the Examples 1 and 2 of Section 5 satisfy the hypothesis of Theorem 2. This is also the case of most instances of Example 3, when ρ is injective with discrete image. Thus such groups can be combined and give new examples. Notice that in the proof of Theorem 2, the gluing element g can be chosen in many ways. In particular, starting from two groups of the Examples 1, 2 or 3, suitable choices of g will give combined groups which are Zariski dense in $\mathrm{O}(2, n)$.

6.2. Lorentzian surgery. Theorem 2 reflects in fact the group theoretical aspect of a slightly more general process of conformal Lorentzian surgery.

Let M_1 and M_2 be two conformally flat Lorentzian manifolds (we do not make any compactness assumption). Suppose that M_1 contains a closed lightlike geodesic Δ_1 admitting some open neighbourhood U_1 which embeds conformally, via a certain embedding ϕ_1, into Ein_n. Suppose moreover that the same property is satisfied by M_2, for a closed lightlike geodesic Δ_2, an open neighbourhood U_2, and a conformal embedding ϕ_2. We can suppose that $\phi_1(\Delta_1)$ and $\phi_2(\Delta_2)$ are disjoint in Ein_n. By Lemma 6, $\phi_1(\Delta_1)$ and $\phi_2(\Delta_2)$ are the attracting and repelling circles of some element $g \in \mathrm{Conf}(\mathrm{Ein}_n)$. As in the proof of Theorem 2, there exist two open neighbourhoods V_1 and V_2 of Δ_1 and Δ_2 respectively, such that $V_1 \subset U_1$, $V_2 \subset U_2$, and $g(\mathrm{Ext}(\phi_2(V_2))) = \overline{\phi_1(V_1)}$. In particular $g(\partial(\phi_2(V_2))) = \partial(\phi_1(V_1))$ (recall that ∂ denotes the boundary). So the element g provides a gluing map f between ∂V_1 and ∂V_2. We denote by $\dot{M}_1$ (resp. $\dot{M}_2$) the manifold M_1 (resp. M_2) with V_1 (resp. V_2) removed. We call $M = \dot{M}_1 \sharp_f \dot{M}_2$ the manifold obtained from $\dot{M}_1 \cup \dot{M}_2$ after identification of ∂V_1 and ∂V_2 by means of the map f. Since $g \in \mathrm{Conf}(\mathrm{Ein}_n)$, the "surgered manifold" M is still endowed with a conformally flat Lorentzian structure. Theorem 2 ensures that if one starts with two compact Kleinian structures M_1 and M_2, the conformally flat structure on $\dot{M}_1 \sharp_f \dot{M}_2$ is still Kleinian.

Remark 6. This surgery process is reminiscent of Kulkarni's construction of a conformally flat Riemannian structure on the connected sum of two conformally flat Riemannian manifolds ([K1]). We do not know whether the connected sum of two conformally flat Lorentzian manifolds can still be endowed with a conformally flat Lorentzian structure.

7. Lorentzian Schottky groups

As an application of the former sections we study here the Lorentzian Schottky groups. These groups are interesting since we can completely determine their limit set and the Kleinian manifolds they uniformize. Moreover, they can be used to construct examples of conformally flat manifolds with some peculiar properties (see [Fr2]).

Let us consider a family $\{(\Delta_1^-, \Delta_1^+), \dots, (\Delta_g^-, \Delta_g^+)\}$ of pairs of lightlike geodesics in Ein_n. We suppose moreover that the $\Delta_i^\pm$ are all disjoint. By Lemma 6, there exists a family $s_1,\dots, s_g$ of elements of $\mathrm{Conf}(\mathrm{Ein}_n)$ with mixed dynamics such that the attracting and repelling circles of s_i are precisely Δ_i^+ and Δ_i^-. Looking if necessary at suitable powers $s_i^{k_i}$ of s_i, we can find open tubular neighbourhoods $U_i^\pm$ of the $\Delta_i^\pm$ with the following properties:

(i) The $\overline{U}_i^\pm$ are all disjoint.

(ii) $s_i(\mathrm{Ext}(U_i^-)) = \overline{U}_i{}^+$ for all $i = 1, \dots, g$.

Such a group $\Gamma = \langle s_1, \dots, s_g\rangle$ is called a *Lorentzian Schottky group*. Properties (i) and (ii) are classically known as *ping-pong dynamics* (see for example [dlH]). For each i, the group $\langle s_i\rangle$ acts properly cocompactly on the open set $\mathrm{Ein}_n\backslash\{\Delta_i^- \cup \Delta_i^+\}$, and a fundamental domain is just given by $D_i = \mathrm{Ein}_n\backslash\{\overline{U}_i^+ \cup \overline{U}_i^-\}$. Now, since the $\overline{U}_i^\pm$ are disjoint, we get that $\mathrm{Ext}(D_i) \subset D_j$ for all $i \neq j$. If we call $D = \bigcap_{i=1}^g D_i$, it is clear that $D \neq \emptyset$. We then apply Theorem 5 to obtain

Proposition 8. *A Lorentzian Schottky group $\Gamma = \langle s_1, \dots, s_g\rangle$ is a free group of* $\mathrm{Conf}(\mathrm{Ein}_n)$. *Moreover, Γ is Kleinian, it acts properly and cocompactly on $\Omega = \bigcup_{\gamma\in\Gamma} \gamma(\overline{D})$. A fundamental domain for this action is given by $D = \bigcap_{i=1}^g D_i$.*

We are now going to describe Ω and its complement $\Lambda \subset \mathrm{Ein}_n$ more precisely.

Let us recall that in a finitely generated free group each element γ can be written in an unique way as a reduced word in the generators. We denote by $|\gamma|$ the length of this word. Let us also recall that we can define the boundary $\partial\Gamma$ of Γ as the set of totally reduced words of infinite length. Hence the elements of the boundary can be written as $s_{i_1}^{\varepsilon_1} \dots .s_{i_k}^{\varepsilon_k} \dots$ with $\varepsilon_j \in \{\pm 1\}$ and $i_j\varepsilon_j \neq -i_{j+1}\varepsilon_{j+1}$ for all $j \geq 1$. Since we supposed that $g \geq 2$, the boundary $\partial\Gamma$ is a compact metrizable space, homeomorphic to a Cantor set (see [GdlH]).

For each $k \in \mathbb{N}$, we call $F_k = \bigcup_{|\gamma| \leq k} \gamma(\bar{D})$, with the convention $F_0 = \bar{D}$. It is not difficult to check that $F_{k-1} \subset F_k$, and $\Omega = \bigcup_{k\in\mathbb{N}} F_k$. So, $\Lambda = \bigcap_{k\in\mathbb{N}} \mathrm{Ext}(F_k)$. For each k, we set $\Lambda_k = \mathrm{Ext}(F_k)$, and thus, we also have $\Lambda = \bigcap_{k\in\mathbb{N}} \bar{\Lambda}_k$. The set $\bar{\Lambda}_k$ is a disjoint union of exactly $2g.(2g-1)^k$ connected components, in one to one correspondence with the words of length $k+1$ in Γ. For example, to the word $s_{i_1}^{\varepsilon_1} \dots s_{i_{k+1}}^{\varepsilon_{k+1}}$ corresponds the component $s_{i_1}^{\varepsilon_1} \dots s_{i_k}^{\varepsilon_k}(\bar{U}_{i_{k+1}}^{\varepsilon_{k+1}})$ of $\bar{\Lambda}_k$. We can now state the following result.

Lemma 7. *There is a homeomorphism K between the boundary* $\partial\Gamma$ *and the space of connected components of* Λ (*endowed with the Hausdorff topology for the compact subsets of* Ein_n).

Proof. Let $\gamma_\infty = s_{i_1}^{\varepsilon_1} \dots . s_{i_k}^{\varepsilon_k} \dots$ be an element of $\partial\Gamma$. We call $\gamma_k = s_{i_1}^{\varepsilon_1} \dots . s_{i_k}^{\varepsilon_k}$ and we look at the decreasing sequence of compact subsets $K(\gamma_k) = s_{i_1}^{\varepsilon_1} \dots . s_{i_{k-1}}^{\varepsilon_{k-1}}(\bar{U}_{i_k}^{\varepsilon_k})$. This decreasing sequence of compact sets tends to a limit compact set $K(\gamma_\infty)$ for the Hausdorff topology. Since the $U_i^{\pm}$ are connected, so are the $K(\gamma_k)$, and $K(\gamma_\infty)$ is itself connected. Let us remark that if γ_∞ and γ'_∞ are distinct in $\partial\Gamma$, then $K(\gamma_k)$ and $K(\gamma'_k)$ are disjoint for k large (they represent two distinct components of $\bar{\Lambda}_k$), so that $K(\gamma_\infty)$ and $K(\gamma'_\infty)$ are disjoint.

Reciprocally, choose $x_\infty \in \Lambda$. Since $\Lambda = \bigcap_{k\in\mathbb{N}} \bar{\Lambda}_k$ with $\bar{\Lambda}_{k+1} \subset \bar{\Lambda}_k$, x_∞ must be an element of some connected component $C_k \subset \bar{\Lambda}_k$ for each k. Moreover $C_{k+1} \subset C_k$. But C_k is then a decreasing sequence of compact subsets of the form $s_{i_1}^{\varepsilon_1} \dots . s_{i_{k-1}}^{\varepsilon_{k-1}}(\bar{U}_{i_k}^{\varepsilon_k})$ and thus converges to a limit compact set $K(\gamma_\infty)$ for $\gamma_\infty = s_{i_1}^{\varepsilon_1} \dots . s_{i_k}^{\varepsilon_k} \dots .$

We have proved that the mapping K between $\partial\Gamma$ and the set of connected components of Λ is a bijection. It remains to prove that it is a homeomorphism, and for this, it is sufficient to show that K is continuous. Let us consider a sequence $\gamma_\infty^{(n)}$ of elements of Γ, converging to some γ_∞. It means that there is a sequence (r_n) of integers which tends to infinity, such that $\gamma_\infty^{(n)}$ and γ_∞ have the same r_n first letters. For each $n \in \mathbb{N}$, $K(\gamma_\infty^{(n)})$ is a decreasing sequence of compact sets $C_k^{(n)}$, where each $C_k^{(n)}$ is a connected component of $\bar{\Lambda}_k$. On the other hand, $K(\gamma_\infty)$ is the limit of a decreasing sequence of C_k, where each C_k is a connected component of $\bar{\Lambda}_k$. Since $\gamma_\infty^{(n)}$ and γ_∞ have the same r_n first letters, we have $C_{r_n-1}^{(n)} = C_{r_n-1}$ for all n. Thus, the limit, as n tends to infinity, of $C_{r_n-1}^{(n)}$ is $K(\gamma_\infty)$. But since $K(\gamma_\infty^{(n)}) \subset C_{r_n-1}^{(n)}$, we get that $\lim_{n\to\infty} K(\gamma_\infty^{(n)}) = K(\gamma_\infty)$ and we are done. □

The next step is to show the following lemma.

Lemma 8. *The connected components of* Λ *are lightlike geodesics.*

Proof. Let us consider $\gamma_\infty = s_{i_1}^{\varepsilon_1} \dots s_{i_k}^{\varepsilon_k} \dots$ in the boundary of Γ. We know that $K(\gamma_\infty)$ is the limit of the sequence $s_{i_1}^{\varepsilon_1} \dots s_{i_{k-1}}^{\varepsilon_{k-1}}(\overline{U}_{i_k}^{\varepsilon_k})$. Since the sequence is decreasing, the limit remains the same if we consider a subsequence. Thus we can make the extra assumption that $K(\gamma_\infty)$ is the limit of a sequence $\gamma_k(\overline{U}_{j_0}^{\varepsilon_{j_0}})$, such that (γ_k) tends simply to infinity and the first and last letters of γ_k are always the same, namely $s_{i_1}^{\varepsilon_1}$ and $s_{j_1}^{\varepsilon_{j_1}}$. Observe that $j_1\varepsilon_{j_1} \neq -j_0\varepsilon_{j_0}$. We are going to discuss the different possible dynamics for (γ_k), and we first prove that (γ_k) cannot have bounded distortion.

Suppose that it is the case. We call p^+ (resp. p^-) and C^+ (resp. C^-) the attracting (resp. repelling) pole and cone of (γ_k). If x is a point of D, then for all $k \in \mathbb{N}$, $\gamma_k(x) \in U_{i_1}^{\varepsilon_1}$ and ${\gamma_k}^{-1}(x) \in U_{j_1}^{-\varepsilon_{j_1}}$. So we must have $p^+ \in U_{i_1}^{\varepsilon_1}$ and $p^- \in U_{j_1}^{-\varepsilon_{j_1}}$. In particular, p^- is not in $U_{j_0}^{\varepsilon_{j_0}}$. On the other hand, it is a general fact that in Ein_n any lightlike cone meets any lightlike geodesic (just because degenerate hyperplanes always meet null 2-planes in $\mathbb{R}^{2,n}$). In particular, the cone C^- meets $\Delta_{j_0}^{\varepsilon_{j_0}}$ and thus $U_{j_0}^{\varepsilon_{j_0}}$. We call $V_{j_0}^{\varepsilon_{j_0}} = C^- \cap U_{j_0}^{\varepsilon_{j_0}}$. Since $U_{j_0}^{\varepsilon_{j_0}}$ does not contain p^-, we infer from Proposition 4 (i) and (ii) that $K(\gamma_\infty) = D_{(\gamma_k)}(\overline{V}_{j_0}^{\varepsilon_{j_0}})$. More precisely, if $\hat{V}_{j_0}^{\varepsilon_{j_0}}$ is the set of lightlike geodesics of C^- meeting $V_{j_0}^{\varepsilon_{j_0}}$, then $K(\gamma_\infty)$ is the closure of the union the lightlike geodesics of $\hat{\gamma}_\infty(\hat{V}_{j_0}^{\varepsilon_{j_0}})$ (see Proposition 4 for the notation $\hat{\gamma}_\infty$). In particular, $K(\gamma_\infty)$ contains a lightlike geodesic. Now some lightlike geodesic of C^- does not meet $V_{j_0}^{\varepsilon_{j_0}}$. Indeed, if this is not the case, then Proposition 4 (ii) ensures that $K(\gamma_\infty) = C^-$. But if we take $\gamma'_\infty \neq \gamma_\infty$, then $K(\gamma'_\infty)$ contains some lightlike geodesic by the remark above. Since any lightlike geodesic meets C^-, we get a contradiction with the fact that $K(\gamma_\infty)$ and $K(\gamma_\infty)$ have to be disjoint.

Now let us perturb slightly the sets $U_{j_0}^{\varepsilon_{j_0}}$ and $U_{j_0}^{-\varepsilon_{j_0}}$ into some sets ${U'}_{j_0}^{\varepsilon_{j_0}}$ and ${U'}_{j_0}^{-\varepsilon_{j_0}}$, in order to get another fundamental domain D', very close to D. Since it is very near to D, $\overline{D}'$ is included in some F_k for k sufficiently large, and so $\bigcup_{\gamma\in\Gamma}\gamma(\overline{D}') = \bigcup_{\gamma\in\Gamma}\gamma(D)$. We prove as above that the limit of the compact sets $\gamma_k({U'}_{j_0}^{\varepsilon_{j_0}})$ is still a connected component of Λ and consequently of the form $K(\gamma'_\infty)$. We just saw that some lightlike geodesics of C^- do not meet $\overline{V}_{j_0}^{\varepsilon_{j_0}}$, so that $\hat{V}_{j_0}^{\varepsilon_{j_0}}$ is not the whole of $\mathbb{S}^{n-2}$. It is thus possible to choose ${U'}_{j_0}^{\varepsilon_{j_0}}$ in such a way that some points of $\hat{V}_{j_0}^{\prime\,\varepsilon_{j_0}}$ are not in $\hat{V}_{j_0}^{\varepsilon_{j_0}}$. But then $K(\gamma'_\infty)$ and $K(\gamma_\infty)$ will be two different components, hence disjoint. On the other hand, since the intersection of $U_{j_0}^{\varepsilon_{j_0}}$ and ${U'}_{j_0}^{\varepsilon_{j_0}}$ is not empty ($\Delta_{j_0}^{\varepsilon_{j_0}}$ is inside), $K(\gamma_\infty)$ and $K(\gamma'_\infty)$ must have some common points. We thus get a contradiction.

It remains to deal with the case where (γ_k) has mixed or balanced distortions. Once again, if x is a point of D then for all $k \in \mathbb{N}$, $\gamma_k(x) \in U_{i_1}^{\varepsilon_1}$ and ${\gamma_k}^{-1}(x) \in U_{j_1}^{-\varepsilon_{j_1}}$.

Hence the attracting circle Δ^+ is in $U_{i_1}^{\varepsilon_1}$ and the repelling one Δ^- is in $U_{j_1}^{-\varepsilon_{j_1}}$. In particular $\overline{U}_{j_0}^{\varepsilon_{j_0}}$ does not meet Δ^-. We infer from Proposition 5 and Proposition 3 that $\lim_{k\to\infty}\gamma_k(\overline{U}_{j_0}^{\varepsilon_{j_0}}) \subset \Delta^+$, but since $\overline{U}_{j_0}^{\varepsilon_{j_0}}$ contains a lightlike geodesic, we have the equality $\lim_{k\to\infty}\gamma_k(\overline{U}_{j_0}^{\varepsilon_{j_0}}) = \Delta^+$. We finally obtain that $K(\gamma_\infty) = \Delta^+$. □

7.1. Proof of Theorem 3. We begin by proving that the group Γ is of the first type. Suppose on the contrary that there is some sequence (γ_k) in Γ with bounded distortion. Then D meets the repelling cone C^-. Otherwise C^- would be included in some $U_i^{\pm}$, say U_1^+. But since Δ_1^- meets C^-, the intersection between Δ_1^- and U_1^+ would be nonempty, a contradiction. By Proposition 4 (ii), $\lim_{k\to\infty}\gamma_k(\overline{D})$ is a compact subset containing infinitely many lightlike geodesics. But $\lim_{k\to\infty}\gamma_k(\overline{D})$ is also a connected subset of Λ. This contradicts the fact that the connected components of Λ are lightlike geodesics.

We claim that the equality $\Lambda_\Gamma = \Lambda$ holds. Indeed, for any sequence (γ_k) of Γ tending simply to infinity, (γ_k) tends to $\Delta^+(\gamma_k)$. We thus see that $\Lambda_\Gamma \subset \Lambda$. Now it is a general fact that if a group Γ acts properly cocompactly on some open set Ω, then it cannot act properly on some open set Ω' strictly containing Ω. So Ω cannot be strictly contained in $\mathrm{Ein}_n \backslash \Lambda_\Gamma$, and we obtain $\Lambda_\Gamma = \Lambda$.

We now prove that Λ_Γ is the product of $\mathbb{R}P^1$ with a Cantor set. The space Ein_n is the quotient of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ by the product of antipodal maps, so that there is a fibration $f\colon \mathrm{Ein}_n \to \mathbb{R}P^1$. The fibers of f are conformal Riemannian spheres of codimension one. In the projective model they are obtained as the projection of the intersection between $C^{2,n}$ and some hyperplanes $P \subset \mathbb{R}^{2,n}$ of Lorentzian signature. As a consequence any lightlike geodesic is transverse to any fiber of f. Let us choose a fiber $\mathcal{F}_0$ above a point t_0 of $\mathbb{R}P^1$. From Lemmas 7 and 8, Λ (and thus Λ_Γ) is transverse to $\mathcal{F}_0$ and intersects it along a Cantor set $\mathcal{C}$. For each $x \in \mathcal{C}$, we call $x(t)$ the unique element of $f^{-1}(t) \cap \Lambda_\Gamma$ such that x and $x(t)$ are on the same lightlike geodesic of Λ. Then Lemma 7 ensures that the following mapping is a homeomorphism:

$$\mathbb{R}P^1 \times \mathcal{C} \to \Lambda,$$
$$(t, x) \mapsto x(t).$$

This proves (ii).

Due to the homeomorphism K we get that, since the action of Γ on its boundary is minimal (see for instance [GdlH]), the action of Γ on the space of lightlike geodesics of Λ_Γ is also minimal, which establishes (iii).

For the proof of (iv) we refer to Theorem 5 of [Fr2] (in fact, in [Fr2] we considered only particular cases of Schottky groups, but the proof of Theorem 5 includes the general case).

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