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Embeddings of Danielewski surfaces in affine spaces

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Abstract. We construct explicit embeddings of Danielewski surfaces [4] in affine spaces. The equations defining these embeddings are obtained from the 2×2 minors of a matrix attached to a weighted rooted tree γ. We characterize those surfaces S_γ with a trivial Makar-Limanov invariant in terms of their associated trees. We prove that every Danielewski surface S with a nontrivial Makar-Limanov invariant admits a closed embedding in an affine space $\mathbb{A}^n_k$ in such a way that every $\mathbb{G}_{a,k}$-action on S extends to an action on $\mathbb{A}^n$ defined by a triangular derivation. We show that a Danielewski surface S with a trivial Makar-Limanov invariant and non-isomorphic to a hypersurface with equation $xz - P(y) = 0$ in $\mathbb{A}^3_k$ admits nonconjugated algebraically independent $\mathbb{G}_{a,k}$-actions.

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Introduction

A *Danielewski surface* over a field k of characteristic zero is an integral affine surface S equipped with a morphism $\pi\colon S \to \mathbb{A}^1_k = \operatorname{Spec}(k[x])$ restricting to the trivial line bundle over $\mathbb{A}^1_k \setminus \{0\}$ and such that the fiber $\pi^{-1}(0)$ is nonempty and reduced, consisting of a disjoint union of affine lines $\mathbb{A}^1_k$. For instance, a surface $S_{P,n} \subset \operatorname{Spec}(k[x, y, z])$ with equation $x^n z - P(y) = 0$, where P is a nonconstant polynomial with $\deg(P)$ simple roots, is a Danielewski surface $\mathrm{pr}_x\colon S_{P,n} \to \operatorname{Spec}(k[x])$. Danielewski surfaces appear naturally as locally trivial fiber bundles $\rho\colon S \to \tilde{X}$ over an affine line with a multiple origin (see e.g. [5]). More precisely, see [4], every such bundle ρ is a principal homogeneous bundle under the action of a line bundle $p\colon L \to \tilde{X}$. These principal L-bundles are uniquely determined by data consisting of an invertible sheaf $\mathcal{L}$ on $\tilde{X}$ and a Čech 1-cocycle g with values in the dual $\mathcal{L}^\vee$ of $\mathcal{L}$ for a suitable covering $\mathcal{U}$ of $\tilde{X}$. In turn, the pair $(\mathcal{L}, g)$ is encoded in a combinatorial datum consisting of a rooted tree with weighted edges, which we call a *weighted tree* (see [4, Example 1.6 and Theorem 3.2] and 2.2 below). Here we use weighted trees in a different way to construct embeddings of Danielewski surfaces into affine spaces.

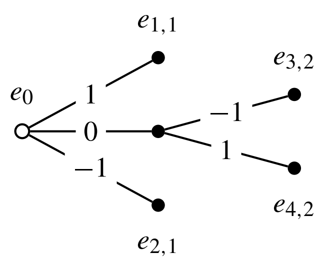
More precisely, starting with a suitable class of k-weighted trees γ, we construct explicit ideals of certain polynomial rings. In turn, these ideals define affine surfaces S_γ which are naturally Danielewski surfaces over the affine line $\mathbb{A}^1_k$.

The paper is divided as follows. In Section 1 we recall basic facts on weighted trees. We associate to every *fine k-weighted tree* $\gamma = (\Gamma, w)$ (see Definition 1.3 below) a polynomial ring $k[\Gamma]$ and a collection of polynomials in $k[\Gamma]$ defined recursively through the *weight function* w.

In Section 2, we review the construction of Danielewski surfaces as locally trivial bundles over the affine line with an n-fold origin given in [4]. Then we associate to every fine k-weighted tree γ a closed affine subscheme $S_\gamma = \operatorname{Spec}(B_\gamma)$ of $\mathbb{A}^1_k \times \operatorname{Spec}(k[\Gamma])$, and we prove the following result (Theorem 2.9).

Theorem. *For every fine k-weighted tree γ, the scheme S_γ is a Danielewski surface over $\mathbb{A}^1_k$ for the restriction of the projection* $\mathrm{pr}_1 : \mathbb{A}^1_k \times \operatorname{Spec}(k[\Gamma]) \to \mathbb{A}^1_k$.

For instance, the surface corresponding to the following fine k-weighted tree

$\gamma =$ 

is the Bandman and Makar-Limanov surface [1] $S \subset k[x][y, z, u]$ with equations

$$xz - y(y^2 - 1) = 0, \quad yu - z(z^2 - 1) = 0, \quad xu - (y^2 - 1)(z^2 - 1) = 0.$$

It is a Danielewski surface over $X = \operatorname{Spec}(k[x])$ via the projection morphism $\mathrm{pr}_x : S \to X$.

Then we show that every embedded Danielewski surface S_γ as above comes canonically equipped with actions of the additive group $\mathbb{G}_{a,k}$ which are the restrictions to S_γ of certain $\mathbb{G}_{a,k}$-actions on the ambient space $\mathbb{A}^1_k \times \operatorname{Spec}(k[\Gamma])$ defined by explicit locally nilpotent derivations $\tilde{\partial}_\gamma$ (see Proposition 2.15). In Section 3, we prove the following result (Corollary 3.8).

Theorem. *Every Danielewski surface $\pi : S \to X = \mathbb{A}^1_k$ is X-isomorphic to an embedded Danielewski surface $\pi_\gamma : S_\gamma = \operatorname{Spec}(B_\gamma) \to X$ for an appropriate fine k-weighted tree γ.*

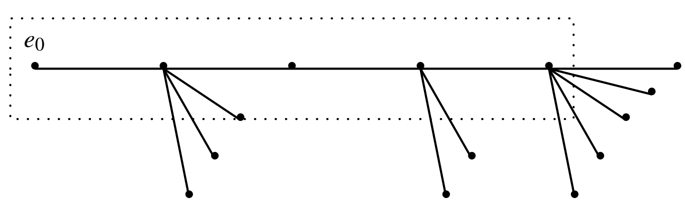
Moreover, we establish that every $\mathbb{G}_{a,X}$-action on $\pi : S_\gamma \to X$ is induced by a locally nilpotent derivation $\tilde{\partial}_\gamma$ as above. As a consequence of this description, we deduce that *every* Danielewski surface $\pi : S \to X = \mathbb{A}^1_k$ can be embedded in a relative affine space $\mathbb{A}^d_X$ in such a way that every $\mathbb{G}_{a,X}$-action on S extends to an action on $\mathbb{A}^d_X$ (Corollary 3.11). This generalizes a result obtained by Makar-Limanov ([8], [9]) for the Danielewski hypersurfaces $S_{P,n}$ above.

The *Makar-Limanov invariant* [6] of an affine k-scheme $X = \mathrm{Spec}(B)$ is defined as the sub-algebra $\mathrm{ML}(X) \subset B$ consisting of regular functions which are invariant under *all* $\mathbb{G}_{a,k}$-actions on X. If $\mathrm{ML}(X) = k$, then we say that X has a trivial Makar-Limanov invariant. For Danielewski surfaces with a nontrivial Makar-Limanov invariant, we prove the following result.

Theorem. *Every Danielewski surface with a nontrivial Makar-Limanov invariant can be embedded in an affine space* $\mathbb{A}^d_k = \mathrm{Spec}(k[x_1, \ldots, x_d])$ *in such a way that every* $\mathbb{G}_{a,k}$*-action on S extends to an action on* $\mathbb{A}^N_k$*. Furthermore, every such action is induced by a triangular locally nilpotent derivation of* $k[x_1, \ldots, x_d]$.

In Section 4, we study Danielewski surfaces with a trivial Makar-Limanov invariant, that is, Danielewski surfaces S which admits two nontrivial $\mathbb{G}_{a,k}$-actions with distinct general orbits. We obtain the following criterion which generalizes Theorem 5.4 in [4].

Theorem. *An embedded Danielewski surface* $\pi : S_\gamma = \mathrm{Spec}(B_\gamma) \to \mathbb{A}^1_k$ *defined by a fine k-weighted tree* γ *has a trivial Makar-Limanov invariant if and only if* γ *is a comb, i.e. a tree such that every element has at most one non-terminal direct descendant* (*see Definition* 4.1 *below*).



A comb rooted in e_0.

We obtain the following description (see 4.7 below). For every Danielewski surface S with a trivial Makar-Limanov invariant, there exists a collection of monic polynomials $P_0, \ldots, P_{h-1} \in k[t]$ with simple roots $a_{i,j} \in k^*$, $i = 0, \ldots, h-1$, $j = 1, \ldots, \deg_t(P_i)$, such that S is isomorphic to the nonsingular surface $S_{P_0,\ldots,P_{h-1}} \subset$

$\mathrm{Spec}(k[x][y_{-1},\ldots,y_{h-2}][z])$ defined by the equations

$$xz - y_{h-2}\prod_{l=0}^{h-1} P_l(y_{l-1}) = 0,$$

$$zy_{i-1} - y_i y_{h-2}\prod_{l=i+1}^{h-1} P_l(y_{l-1}) = 0, \quad xy_i - y_{i-1}\prod_{l=0}^{i} P_l(y_{l-1}) = 0, \quad 0 \le i \le h-2,$$

$$y_{i-1}y_j - y_i y_{j-1}\prod_{l=i+1}^{j} P_l(y_{l-1}) = 0, \quad 0 \le i < j \le h-2.$$

On an affine surface $S = \mathrm{Spec}(B)$, two $\mathbb{G}_{a,k}$-actions μ_1 and μ_2 with associated quotient fibrations $\pi_1\colon S \to \mathbb{A}^1_k$ and $\pi_2\colon S \to \mathbb{A}^1_k$ respectively are said to be *algebraically independent* if the general fibers of π_1 and π_2 do not coincide. In this situation, we say that μ_1 and μ_2 are *conjugated* if there exists an automorphism ϕ of S sending the fibers of π_1 onto the fibers of π_2. This means equivalently that there exists an automorphism ϕ^* of B such that $\mathrm{Ker}(\partial_2) = \phi^*(\mathrm{Ker}(\partial_1))$, where ∂_1 and ∂_2 denote the locally nilpotent derivations of B corresponding to the actions μ_1 and μ_2 respectively. Daigle [2] established that all the $\mathbb{G}_{a,k}$-actions on a Danielewski surface $S_{P,1} = \{xz - P(y) = 0\}$ are conjugated. From the explicit description above, we obtain the following result (Theorem 4.12).

Theorem. *If a Danielewski surface S non isomorphic to a surface $S_{P,1}$ admits two independent $\mathbb{G}_{a,k}$-actions, then it admits two algebraically independent* nonconjugated *$\mathbb{G}_{a,k}$-actions.*

We also deduce the following characterization (Corollary 4.13) of the Danielewski surfaces $S_{P,1}$, which generalizes the ones previously obtained by Bandman and Makar-Limanov [1] and Daigle [2].

Theorem. *For a Danielewski surface $\pi\colon S \to X = \mathbb{A}^1_k$ with a trivial Makar-Limanov invariant, the following are equivalent.*

1) *S admits a free $\mathbb{G}_{a,X}$-action.*
2) *The canonical sheaf ω_S of S is trivial.*
3) *S is isomorphic to a surface $S_{P,1} \subset \mathbb{A}^3_k = \mathrm{Spec}(k[x,y,z])$ with the equation $xz - P(y) = 0$ for a certain nonconstant polynomial P with $\deg(P)$ simple roots.*
4) *All $\mathbb{G}_{a,k}$-actions on S are conjugated.*

1. Preliminaries

Weighted rooted trees. A *poset* is a nonempty finite partially ordered set $G = (G, \leq)$. A totally ordered subset $C \subset G$ is called a *chain of length* $l(C) = \mathrm{Card}(C) - 1$. *A* chain which is maximal for the inclusion is called a *maximal chain.* For every $e \in G$, we let

$$(\uparrow e)_G = \{e' \in G, e \leq e'\} \quad \text{and} \quad (\downarrow e)_G = \{e' \in N, e' \leq e\}.$$

A subset $\overleftarrow{e'e}$ with two elements $e' < e$ such that $(\uparrow e')_G \cap (\downarrow e)_G = \{e' < e\}$ is called an *edge* of G. We denote the set of all edges in G by $E(G)$.

Definition 1.1. A (*rooted*) *tree* $\Gamma = (\Gamma, \leq)$ is a poset with a unique minimal element e_0 called the *root*, and such that $(\downarrow e)_\Gamma$ is a chain for every $e \in \Gamma$. A subposet $\Gamma' \subset \Gamma$ which is tree for the induced ordering is called a subtree of Γ. Given $e \in \Gamma$, the *maximal* (*rooted*) *subtree of* Γ *rooted in* e is the subtree $\Gamma(e) = (\uparrow e)_\Gamma$.

1.2. An element e such that $l(\downarrow e)_\Gamma = m$ is said to be *at level* m. The maximal elements $e_i = e_{i,m_i}$, where $m_i = l(\downarrow e_i)_\Gamma$, of a tree Γ are called the *leaves* of Γ. We denote the set of those elements by $L(\Gamma)$. The maximal chains of Γ are the chains

$$\Gamma_{e_{i,m_i}} = (\downarrow e_{i,m_i})_\Gamma = \{e_{i,0} = e_0 < e_{i,1} < \cdots < e_{i,m_i}\}, \quad e_{i,m_i} \in L(\Gamma). \tag{1.1}$$

We say that Γ has *height* $h(\Gamma) = \max(m_i)$. An element of $\Gamma \setminus L(\Gamma)$ is called a *parent*, and we denote the set of those elements by $\mathbf{P}(\Gamma)$. Given $e \in \Gamma \setminus \{e_0\}$, an element of the chain $\mathrm{Anc}(e) = (\downarrow e) \setminus \{e\}$ is called an *ancestor* of e. The *parent of* e is the maximal element $\mathrm{Par}(e)$ of $\mathrm{Anc}(e)$. More generally, the n-th *ancestor* of e is defined recursively by $\mathrm{Par}^n(e) = \mathrm{Par}(\mathrm{Par}^{n-1}(e)) \in \mathrm{Anc}(e)$. Given two different elements $e, e' \in \Gamma$, the *first common ancestor* of e *and* e' is the maximal element $\mathrm{Anc}(e, e')$ of the chain $\mathrm{Anc}(e) \cap \mathrm{Anc}(e)$. If e is not a leaf of Γ, then the minimal elements of $(\uparrow e)_\Gamma \setminus \{e\}$ are called the *children* of e, and we denote the set of those elements by $\mathrm{Ch}(e)$. The *degree* $\deg(e)$ of an element e is the number of its children.

Definition 1.3. Let Γ be a tree. A *fine weight function on* Γ, *with values in a field* k, is a function $w \colon E(\Gamma) \to k$, which assigns an element $a_{e',e} = w(\overleftarrow{e'e}) \in k$ to every edge $\overleftarrow{e'e}$ of Γ, in such a way that $a_{e',e_1} \neq a_{e',e_2}$ whenever e_1 and e_2 share the same parent e'. A tree Γ equipped with such a function w is referred to as *a fine* k*-weighted tree* $\gamma = (\Gamma, w)$.

Definition 1.4. An morphism of fine k-weighted trees $\tau \colon \gamma' = (\Gamma', w') \to \gamma = (\Gamma, w)$ is an order-preserving map $\tau \colon \Gamma' \to \Gamma$ satisfying the following properties.

a) The image of a maximal subchain of Γ' is a maximal subchain of Γ.

b) For every $e' \in \Gamma'$, $\tau^{-1}(\tau(e'))$ is either e' itself or a maximal subtree of Γ'.

c) For every edge $\overleftarrow{e'e}$ of Γ' such that $\tau(e) \neq \tau(e')$, we have $w'(\overleftarrow{e'e}) = w(\overleftarrow{\tau(e')\tau(e)})$.

Remark 1.5. A morphism of fine k-weighted trees maps the root e'_0 of Γ' on the root e_0 of Γ and a leaf e'_{i,m'_i} of Γ' at level m'_i onto a leaf $e_{j(i),m_{j(i)}}$ of Γ at level $m_{j(i)} \leq m'_i$. Then b) guarantees that $\tau(e'_{i,k}) = e_{j,\min(m_{j(i)},k)}$ for every $k = 0, \ldots, m'_i$, and so, condition c) above makes sense.

Genealogical matrix of a weighted tree. Here we associate to every fine k-weighted tree $\gamma = (\Gamma, w)$ a matrix with coefficients in a polynomial ring $k[\Gamma]$.

Definition 1.6. Given a tree Γ rooted in e_0, we associate to every parent $e \in \mathbf{P}(\Gamma)$ a symbol X_e. If $e' \in \mathbf{P}(\Gamma)$ is the parent of a given $e \in \mathbf{P}(\Gamma)$, then we will sometimes denote $X_{e'}$ as $X_{\mathrm{Par}(e)}$. We also extend this relationship between the X_e's by introducing the symbol $X_{e_{-1}} = X_{\mathrm{Par}(e_0)}$. We let $k[\Gamma] = k[(X_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}}]$ be the corresponding polynomial ring in $d(\Gamma) = \mathrm{Card}(\mathbf{P}(\Gamma)) + 1$ variables.

For every element $e \in \mathbf{P}(\Gamma)$ of a given fine k-weighted tree $\gamma = (\Gamma, w)$ rooted in e_0, we introduce below three polynomials $F_e(\gamma)$, $A_e(\gamma)$, $G_e(\gamma) \in k[\Gamma]$, defined recursively through the weight function $w\colon E(\Gamma) \to k$, $\overleftarrow{e'e} \mapsto a_{e',e} = w(\overleftarrow{e'e})$.

Definition 1.7. For every $e' \in \mathbf{P}(\Gamma)$ and every subset $J \subset \mathrm{Ch}(e')$ we let

$$F_{e'}^J = F_{e'}^J(\gamma) = \prod_{e \in (\mathrm{Ch}(e') \setminus J)} (X_{\mathrm{Par}(e')} - a_{e',e}) \in k[X_{\mathrm{Par}(e')}] \subset k[\Gamma].$$

The polynomial $F_{e'} := F_{e'}^{\emptyset}$ is called *the fatherhood polynomial* of e'.

The *ancestral polynomial* $A_e = A_e(\gamma)$ *of* $e \in \Gamma$ is the polynomial defined recursively by

$$A_{e_0} = 1 \quad \text{and} \quad A_e = F_{\mathrm{Par}(e)}^{\{e\}} A_{\mathrm{Par}(e)} \in k[X_{e_{-1}}, (X_{e'})_{e' \in \mathrm{Anc}(\mathrm{Par}(e))}] \subset k[\Gamma].$$

The *genealogical polynomial of* $e \in \mathbf{P}(\Gamma)$ *with respect to* $e' \in \mathrm{Anc}(e)$ is the polynomial

$$G_{e',e} = G_{e',e}(\gamma) = A_{e'}^{-1} A_e F_e \in k[X_{e_{-1}}, (X_{e''})_{e'' \in \mathrm{Anc}(e) \setminus \mathrm{Anc}(\mathrm{Par}(e'))}] \subset k[\Gamma].$$

The polynomial $G_e = G_{e_0,e}$ is simply referred to as *the genealogical polynomial of* e.

Remark 1.8. Up to changing the variables, $G_{e',e}(\gamma)$ coincides with the genealogical polynomial $G_e(\gamma')$ of e as an element of the maximal weighted subtree $\gamma(e') = ((\uparrow e')_\Gamma, w|_{\Gamma(e')})$ of γ rooted in e', considered as a fine k-weighted tree disregarding the inclusion $\gamma(e') \hookrightarrow \gamma$.

Definition 1.9. The *genealogical matrix* of a fine k-weighted tree $\gamma = (\Gamma, w)$ is the matrix $M(\gamma) \in \mathrm{Mat}_{d(\Gamma)-1,2}(k[\Gamma])$ with the rows $M_e = (G_e, X_e) \in \mathrm{Mat}_{1,2}(k[\Gamma])$, $e \in \mathbf{P}(\Gamma)$.

2. Danielewski surfaces defined by weighted trees

In [4], the author gives a method to construct a Danielewski surface $\pi\colon S^\gamma \to X$ over $X = \mathrm{Spec}(k[x])$ from the data consisting of a fine k-weighted tree γ. Here we review briefly this construction. Then we introduce a new procedure to associate to every such tree γ a second Danielewski surface $\pi\colon S_\gamma \to X$, which comes embedded in a relative affine space $\mathbb{A}^d_X = X \times \mathbb{A}^d_k$.

Notation 2.1. Throughout this section, we fix a field k of characteristic zero. We let $A = k[x]$, $X = \mathrm{Spec}(A) \simeq \mathbb{A}^1_k$, and we denote by $X_* \simeq \mathrm{Spec}(A_x)$ the open complement in X of the origin $x_0 \in \mathbb{A}^1_k$. We consider Danielewski surfaces over the fixed base X. We denote by $\mathrm{pr}_X\colon \mathbb{A}^1_X = \mathrm{Spec}(A[X_{e_{-1}}]) \to X$ the trivial line bundle over X. The additive group scheme with base X is denoted by $\mathbb{G}_{a,X} = \mathrm{Spec}(A[T])$.

Abstract Danielewski surface defined by a fine k-weighted tree. Given a fine k-weighted tree $\gamma = (\Gamma, w)$ of height $h = h(\Gamma)$ with leaves $e_{1,m_1}, \dots, e_{n,m_n}$, we construct a Danielewski surface $\pi\colon S^\gamma \to X$ as follows. Using the maximal weighted subchains

$$\gamma_{e_{i,m_i}} = ((\downarrow e_{i,m_i}), w) = \{e_0 = e_{i,0} < e_{i,1} < \dots < e_{i,m_i-1} < e_{i,m_i}\}_w, \quad i = 1, \dots, n,$$

of γ, we define a collection of polynomials

$$\sigma = \left\{\sigma_i = \textstyle\sum_{j=0}^{m_i-1} w(\overleftarrow{e_{i,j}e_{i,j+1}})x^j \in k[x]\right\}_{i=1,\dots,n}.$$

For every $i \neq j$, we let $g_{ij} = x^{-m_i}(\sigma_j - \sigma_i) \in A_x$. These *transition functions* g_{ij} satisfy the cocycle relation $g_{ik} = g_{ij} + x^{m_j - m_i} g_{jk}$ in A_x for every triple $i \neq j \neq k$.

2.2. We let $\pi\colon S^\gamma \to X$ be the X-scheme obtained by gluing n copies $S_i = \mathrm{Spec}(A[T_i])$ of $\mathbb{A}^1_X$ over X_* by means of the A_x-algebra isomorphisms

$$\tau_{ij}\colon A_x[T_i] \to A_x[T_j], \quad T_i \mapsto g_{ij} + x^{m_j - m_i} T_j, \quad i \neq j,\ i, j = 1, \dots, n.$$

Since γ is a fine k-weighted tree, it follows from 2.8 in [4] that S^γ is a Danielewski surface $\pi\colon S^\gamma \to X$. The irreducible components of $\pi^{-1}(x_0)$ are the curves $C_i = \pi^{-1}(x_0) \cap S_i \simeq \mathrm{Spec}(k[T_i])$, $i = 1, \dots, n$. It comes equipped with a canonical birational X-morphism $\psi\colon S^\gamma \to \mathbb{A}^1_X = \mathrm{Spec}(A[X_{e_{-1}}])$ corresponding to the section $s_{e_{-1}} \in B^\gamma = \Gamma(S^\gamma, \mathcal{O}_{S^\gamma})$ with restrictions $s_{e_{-1}}|_{S_i} = \sigma_i + x^{m_i} T_i \in A[T_i]$,

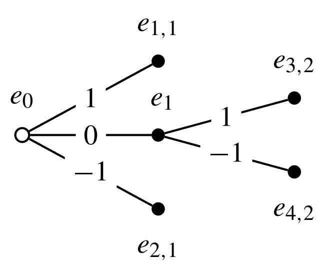
$i = 1, \dots, n$. By Theorem 3.2 in [4], every Danielewski surface $\pi\colon S \to X$ is X-isomorphic to an *abstract Danielewski surface* $\pi\colon S^\gamma \to X$ obtained by this procedure.

2.3. A Danielewski surface $\pi\colon S \to X$ admits nontrivial actions of the additive group scheme $\mathbb{G}_{a,X}$. Indeed, since by definition $S|_{X_*}$ is isomorphic to the trivial line bundle $\mathbb{A}^1_{X_*} = \operatorname{Spec}(A_x[X_{e_{-1}}])$ over X_*, there exists $r \geq 0$ such that the A-derivation $x^m \partial_{X_{e_{-1}}}$ extends to a locally nilpotent A-derivation ∂ of $\Gamma(S, \mathcal{O}_S)$, corresponding to a nontrivial $\mathbb{G}_{a,X}$-action on S. By Proposition 2.12 in [4], every nontrivial $\mathbb{G}_{a,X}$-action on a Danielewski surface S^γ is induced by the extension $\partial_{a,m}$ to B^γ of a locally nilpotent A-derivation $ax^m \partial_{X_{e_{-1}}}$ of $B^\gamma \otimes_A A_x \simeq A_x[X_{e_{-1}}]$, where $m \geq h(\Gamma)$ and $a \in A \setminus \{0\}$. We denote the corresponding $\mathbb{G}_{a,X}$-actions on $\mathbb{A}^1_X$ and S^γ by $\boldsymbol{t}_{a,m}$ and $\boldsymbol{t}^\gamma_{a,m}$ respectively. On the open subsets $S_i = \operatorname{Spec}(A[T_i])$, $\mathrm{t}^\gamma_{a,m}$ coincides with the *twisted translation* $\boldsymbol{t}_{a,m-m_i}$ defined by the group co-action homomorphism

$$A[T_i] \to A[T_i, T] \simeq A[T_i] \otimes_A A[T], \quad T_i \mapsto T_i + ax^{m-m_i}T, \quad i = 1, \dots, n.$$

The canonical morphism $\psi\colon S^\gamma \to \mathbb{A}^1_X$ is $\mathbb{G}_{a,X}$-equivariant when S^γ and $\mathbb{A}^1_X$ are equipped with the $\mathbb{G}_{a,X}$-actions $\boldsymbol{t}^\gamma_{a,m}$ and $\boldsymbol{t}_{a,m}$ respectively.

Example 2.4. The collection of polynomials σ corresponding to the following fine k-weighted tree $\gamma = (\Gamma, w)$ with leaves $e_{1,1}, e_{2,1}, e_{3,2}, e_{4,2}$



is $\sigma = \{1, -1, x, -x\}$. The associated transition functions $g = \{g_{ij}\}_{1 \leq i < j \leq 4}$ are

$$g_{12} = g_{34} = -2x^{-1}, \quad g_{13} = -g_{24} = x^{-1}(x-1),$$
$$g_{23} = -g_{14} = x^{-1}(x+1).$$

The gluing homomorphisms $\{\tau_{ij}\}_{1 \leq i < j \leq 4}$ are given by

$$\tau_{ij}\colon k[x, x^{-1}][T_i] \to k[x, x^{-1}][T_j],$$
$$T_i \mapsto \begin{cases} g_{ij} + T_j, & \text{if } (i,j) \in \{(1,2), (3,4)\}, \\ g_{ij} + xT_j, & \text{if } (i,j) \in \{(1,3), (1,4), (2,3), (2,4)\}. \end{cases}$$

The $\mathbb{G}_{a,X}$-action $\boldsymbol{t}^{\gamma}_{1,2}$ on $\pi : S^{\gamma} \to X$ is a non-free action which restricts on $S_i = \operatorname{Spec}(A[T_i])$ to the action

$$T_i \mapsto \begin{cases} T_i + xT, & \text{if } i = 1, 2, \\ T_i + T, & \text{if } i = 3, 4. \end{cases}$$

Letting $P(t) = t^2 - 1 \in k[t]$, we will see in Example 3.2 below that S^{γ} is X-isomorphic to the Bandman and Makar-Limanov surface [1] $S \subset \operatorname{Spec}(k[x][y, z, u])$ with equations

$$xz - yP(y) = 0, \quad yu - zP(z) = 0, \quad xu - P(y)P(z) = 0,$$

and that $\boldsymbol{t}^{\gamma}_{1,2}$ coincides with the action on S induced by the triangular derivation

$$\partial_{1,2} = x^2\partial_y + x(3y^2-1)\partial_z + (2P(y)(3y^2-1)z + 2xyP(z))\partial_u \in \operatorname{Der}_{k[x]}(k[x][y, z, u]).$$

Embedded Danielewski surface defined by a fine k-weighted tree. Given a fine k-weighted tree $\gamma = (\Gamma, w)$, we construct a Danielewski surface $\pi : S_{\gamma} \to X$ which comes embedded in a relative affine space $\mathbb{A}^{d(\Gamma)}_X$, where $d(\Gamma) = \operatorname{Card}(\mathbf{P}(\Gamma)) + 1$. These surfaces are canonically equipped with the restrictions of certain actions of the additive group $\mathbb{G}_{a,X}$ on the ambient space $\mathbb{A}^{d(\Gamma)}_X$, defined by explicit locally nilpotent derivations.

2.5. Given a fine k-weighted tree $\gamma = (\Gamma, w)$, we let $A[\Gamma] = A \otimes_k k[\Gamma] \simeq A[X_{e_{-1}}, (X_e)_{e\in\mathbf{P}(\Gamma)}]$ (see Definition 1.6). We let $\bar{M}(\gamma) \in \operatorname{Mat}_{d(\Gamma),2}(A[\Gamma])$ be the matrix with the rows $M_{e_{-1}} = (x, 1)$ and $M_e = (G_e(\gamma), X_e)$, $e \in \mathbf{P}(\Gamma)$, *i.e.* $\bar{M}(\gamma) = (M_{e_{-1}}, M(\gamma))$, where $M(\gamma) \in \operatorname{Mat}_{d(\Gamma)-1,2}(k[\Gamma])$ denotes the genealogical matrix of γ (Definition 1.9).

Definition 2.6. Given a fine k-weighted tree $\gamma = (\Gamma, w)$, we let $I_{\gamma} \subset A[\Gamma]$ be the ideal generated by the *simplified genealogical minors* of $\bar{M}(\gamma)$

$$\Delta_{e',e} = \Delta_{e',e}(\gamma) = A_{e'}^{-1} \det(M_{\operatorname{Par}(e')}, M_e) \in A[\Gamma], \quad (e, e') \in \mathbf{P}(\Gamma) \times (\downarrow e)_{\Gamma}. \quad (2.1)$$

We let $B_{\gamma} = A[\Gamma]/I_{\gamma}$, and we let $\pi : S_{\gamma} = \operatorname{Spec}(B_{\gamma}) \to X$ be the corresponding closed sub- X-scheme of the relative affine space $\mathbb{A}^{d(\Gamma)}_X = \operatorname{Spec}(A[\Gamma])$.

2.7. By construction, $\Delta_e := \Delta_{e_0,e} = xX_e - G_e \in A[(X_{e'})_{e'\in(\downarrow e)_{\Gamma}\cup\{e_{-1}\}}]$ for every $e \in \mathbf{P}(\Gamma)$, whereas $\Delta_{e',e} = (X_{\operatorname{Par}^2(e')} - a_{\operatorname{Par}(e'),e'})X_e - X_{\operatorname{Par}(e')}G_{e',e}$ for every pair $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_{\Gamma} \setminus \{e_0\})$. As a consequence, for every triple $e_0 < e'' \le e' \le e$ in $\mathbf{P}(\Gamma)$, the following relations hold in $A[\Gamma]$:

$$\left.\begin{aligned} A_{e'}\Delta_{e',e} &= X_{\operatorname{Par}(e')}\Delta_e - \Delta_{\operatorname{Par}(e')}X_e, \\ x\Delta_{e',e} &= (X_{\operatorname{Par}^2(e')} - a_{\operatorname{Par}(e'),e'})\Delta_e - \Delta_{\operatorname{Par}(e')}G_{e',e}, \\ (X_{\operatorname{Par}^2(e'')} - a_{\operatorname{Par}(e''),e''})\Delta_{e',e} &= (X_{\operatorname{Par}^2(e')} - a_{\operatorname{Par}(e'),e'})\Delta_{e'',e} - \Delta_{e'',e'}G_{e',e}. \end{aligned}\right\} \quad (2.2)$$

2.8. If $\gamma = (\Gamma, w)$ is the trivial tree with just one element e_0, then the first projection $\pi : S_\gamma = \mathrm{Spec}(k[x][X_{e_{-1}}]) \to X$ is a Danielewski surface. Similarly, if Γ has height 1, then $G_{e_0} \in k[X_{e_{-1}}]$ is a monic polynomial with simple roots $a_{e_0,e} = w(\overleftarrow{e_0 e}) \in k$, $e \in \mathrm{Ch}(e_0)$. Therefore,

$$\pi : S_\gamma = \mathrm{Spec}(A[\Gamma]/I_\gamma) = \mathrm{Spec}(k[x][X_{e_{-1}}, X_{e_0}]/xX_{e_0} - G_{e_0}(X_{e_{-1}})) \to X$$

is a Danielewski surface, and the irreducible components of $\pi^{-1}(x_0)$ are the curves $C_e \simeq \mathrm{Spec}(k[X_{e_0}])$ with defining ideals $I_{\gamma,e} = (I_\gamma, X_{e_{-1}} - a_{e,e_0}) \subset A[\Gamma]$, $e \in \mathrm{Ch}(e_0)$. More generally, we have the following result.

Theorem 2.9. *For every fine k-weighted tree $\gamma = (\Gamma, w)$ with leaves $e_{1,m_1}, \dots, e_{n,m_n}$, $\pi : S_\gamma \to X$ is a Danielewski surface. Furthermore, the fiber $\pi^{-1}(x_0)$ is the disjoint union of the curves $C_{e_{i,m_i}} \simeq \mathrm{Spec}(k[X_{e_{i,m_i-1}}])$ with defining ideals*

$$I_{\gamma,e_{i,m_i}} = (I_\gamma, x, (X_{e_{i,j-1}} - a_{e_{i,j},e_{i,j+1}})_{0 \le j \le m_i - 1}) \subset A[\Gamma], \quad i = 1, \dots, n.$$

The proof is divided as follows. In 2.10, Lemmas 2.11 and 2.12 below, we show that S_γ is an integral scheme. Then, in Lemma 2.13, we describe explicitly the irreducible components of $\pi^{-1}(x_0)$.

2.10. We first observe that S_γ restricts to the trivial line bundle $\mathbb{A}^1_{X_*} = \mathrm{Spec}(A_x[X_{e_{-1}}])$ over X_*. Indeed, the second relation of (2.2) guarantees that the ideal $I_\gamma A_x[\Gamma]$ of $A_x[\Gamma] \simeq A[\Gamma] \otimes_A A_x$ is generated by the polynomials $x^{-1}\Delta_e = X_e - x^{-1}G_e$, $e \in \mathbf{P}(\Gamma)$. Since G_e only involves the variables $X_{e'}$, where $e' \in \mathrm{Anc}(e)$, we recursively arrive at an A_x-algebra isomorphism $A_x[\Gamma]/I_\gamma A_x[\Gamma] \simeq A_x[X_{e_{-1}}]$. Thus S_γ is a Danielewski surface with base $(k[x], x)$ provided that x is not a zero divisor in B_γ and that B_γ/xB_γ is isomorphic to a nonempty direct product of polynomial rings in one variable over k. Indeed, the first condition guarantees that the canonical map $B_\gamma \to B_\gamma \otimes_A A_x \simeq A_x[X_{e_{-1}}]$ is injective. In turn, this implies that B_γ is a sub-domain of $A_x[X_{e_{-1}}]$. The second one means equivalently that the fiber $\pi^{-1}(x_0)$ decomposes as a nonempty disjoint union of affine lines $\mathbb{A}^1_k$.

To show that x is not a zero divisor in B_γ, it suffices to find a covering of S_γ by principal affine open subsets $Y_i = \mathrm{Spec}(B_i)$ such that x is not a zero divisor in B_i for every $i = 1, \dots, n$.

Lemma 2.11. *If $\gamma = (\Gamma, w)$ if a fine k-weighted tree with the leaves $e_1, \dots, e_n$, then S_γ is covered by the principal open subsets $Y_i = \mathrm{Spec}(A[\Gamma][T]/(I_\gamma, A_{e_i}T - 1))$, $i = 1, \dots, n$.*

Proof. For every $e \in \mathbf{P}(\Gamma)$ the polynomials $F_e^{\{e'\}} \in A[X_{\mathrm{Par}(e)}]$, $e' \in \mathrm{Ch}(e)$ generate the unit ideal of $A[X_{\mathrm{Par}(e)}]$ as γ is a fine k-weighted tree. Therefore, there exist polynomials $\Lambda_{e'} \in A[\Gamma]$, $e' \in \mathrm{Ch}(e)$, such that

$$A_e = A_e \sum_{e' \in \mathrm{Ch}(e)} \Lambda_{e'} F_e^{\{e'\}} = \sum_{e' \in \mathrm{Ch}(e)} \Lambda_{e'} A_{e'}.$$

It follows by induction that the image of $A_{e_0} = 1$ in B_γ belongs to the ideal generated by the images $a_i \in B_\gamma$ of the ancestral polynomials A_{e_i} of the leaves of Γ. This means equivalently that the open subsets $\mathrm{Spec}((B_\gamma)_{a_i}) \simeq \mathrm{Spec}(A[\Gamma][T]/(I_\gamma, A_{e_i}T - 1))$ cover S_γ. □

Lemma 2.12. *For every $i = 1, \dots, n$, Y_i is an integral scheme.*

Proof. Let us denote by $e_j = e_{i,j}$, $j = 0, \dots, m = m_i$, the elements of the maximal subchain $(\downarrow e_{i,m_i})_\Gamma$ of Γ associated with the leaf e_{i,m_i}. For every $i = 1, \dots, m-2$, the polynomial $A_{e_{i+1}}$divides A_{e_m}. Similarly, for every $e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m)$, the first common ancestor of e and e_m is an element e_i, $i \leq m-1$, such that $e' = \mathrm{Ch}(e_i) \cap (\downarrow e) \neq e_{i+1}$, and so $(X_{e_{i-1}} - a_{e_i,e'})$ divides A_{e_m}. Therefore, these polynomials become invertible in $A[\Gamma]_{A_{e_m}}$. We claim that the ideal $I_\gamma A[\Gamma]_{A_{e_m}}$ is generated by the polynomials

$$\delta_{e_i} = A_{e_{i+1}}^{-1} \Delta_{e_i} = -(X_{e_{i-1}} - a_{e_i,e_{i+1}}) + A_{e_{i+1}}^{-1} x X_{e_i}, \quad i = 1, \dots, m-2,$$

$$\begin{aligned} \delta_{e_i,e} &= (X_{e_{i-1}} - a_{e_i,e'})^{-1} \Delta_{e_i,e} \\ &= X_e - (X_{e_{i-1}} - a_{e_i,e'})^{-1} X_{e_i} G_{e_i,e}, \end{aligned} \qquad \begin{cases} e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m), \\ \mathrm{Anc}(e, e_m) = e_i, \\ e' = \mathrm{Ch}(e_i) \cap (\downarrow e)_\Gamma. \end{cases}$$

Indeed, the second relation of (2.2) guarantees that the polynomials Δ_e, where $e \in \mathbf{P}(\Gamma) \setminus (\downarrow e_m)_\Gamma$, can be expressed in $A[\Gamma]_{A_{e_m}}$ in terms of the δ_{e_i}'s and $\delta_{e_i,e}$'s. In turn, we deduce from the first and the third ones that all the polynomials $\Delta_{e',e}$, $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$ belong to the ideal of $A[\Gamma]_{A_{e_m}}$ generated by the δ_{e_i}'s and the $\delta_{e_i,e}$'s. Since the polynomials A_{e_i} and $G_{e_i,e}$ above only involve the variables corresponding to the elements in $(\downarrow e_{i-2})_\Gamma$ and $(\uparrow e')_\Gamma \cap (\downarrow \mathrm{Anc}(e))_\Gamma$ respectively, we conclude by induction that there exists a nonconstant polynomial $P \in A[X_{e_{m-1}}]$ such that $A[\Gamma]_{A_{e_m}}/I_\gamma A[\Gamma]_{A_{e_m}} \simeq A[X_{e_{m-1}}]_P$. Since A is a domain and P is nonconstant, it follows that $(B_\gamma)_{a_i} \simeq A[X_{e_{m-1}}]_P$ is a nonzero domain too. □

Summing up, we have established that for every fine k-weighted tree γ, $\pi\colon S_\gamma \to X$ is an integral affine scheme restricting to the trivial bundle $\mathbb{A}^1_{X_*}$ over X_*. The following result completes the proof of Theorem 2.9.

Lemma 2.13. *For every fine k-weighted tree $\gamma = (\Gamma, w)$ with leaves $e_{1,m_1}, \dots, e_{n,m_n}$, the fiber $\pi^{-1}(x_0)$ of $\pi\colon S_\gamma \to X$ is the disjoint union of the curves $C_{e_{i,m_i}} \simeq \mathrm{Spec}(k[X_{e_{i,m_i-1}}])$ with defining ideals*

$$I_{\gamma,e_{i,m_i}} = (I_\gamma, x, (X_{e_{i,j-1}} - a_{e_{i,j},e_{i,j+1}})_{0 \leq j \leq m_i - 1}) \subset A[\Gamma], \quad i = 1, \dots, n.$$

Proof. We proceed by induction on the height h of Γ. If $h = 0$ then $S_\gamma = \mathrm{Spec}(A[X_{e_{-1}}])$ and $\pi^{-1}(x_0) \simeq \mathrm{Spec}(k[X_{e_{-1}}])$. Otherwise, if $\mathrm{Ch}(e_0) \neq \emptyset$ then, since γ is a fine k-weighted tree, it follows that the polynomials $X_{e_{-1}} - a_{e_0,e}$, $e \in \mathrm{Ch}(e_0)$ are pairwise relatively prime. Therefore $\pi^{-1}(x_0) = \mathrm{Spec}(A[\Gamma]/(x, I_\gamma))$ decomposes as the disjoint union of curves $D_e = \mathrm{Spec}(A[\Gamma]/(x, X_{e_{-1}} - a_{e_0,e}, I_\gamma))$, $e \in \mathrm{Ch}(e_0)$. We let $\gamma(e) = (\Gamma(e), w|_{\Gamma(e)})$ be the maximal fine k-weighted subtree of γ rooted in e. Clearly, the ideal $(x, X_{e_{-1}} - a_{e_0,e}, I_\gamma)$ coincides with the ideal $I_e \subset A[\Gamma]$ generated by x, $X_{e_{-1}} - a_{e_0,e}$ and the polynomials

$$\begin{array}{ll} G_{e,e'}(\gamma), & e' \in \mathbf{P}(\Gamma(e)), \\ \Delta_{e'',e'}(\gamma), & (e', e'') \in \mathbf{P}(\Gamma(e)) \times (\mathrm{Anc}_{\Gamma(e)}(e')), \\ \delta_{e,e'} = (a_{e_0,e} - a_{e_0,e''})X_{e'} - X_{e_0} G_{e,e'}(\gamma), & \begin{cases} e' \in \mathbf{P}(\Gamma) \setminus (\{e_0\} \cup \mathbf{P}(\Gamma(e))), \\ e'' = \mathrm{Ch}(e_0) \cap (\downarrow e') \neq e. \end{cases} \end{array}$$

By definition, we have $A[\Gamma(e)] = A[X_{e_{-1}}, (X_{e'})_{e' \in \mathbf{P}(\Gamma)}] \simeq A[X_{e_0}, (X_{e'})_{e' \in \mathbf{P}(\Gamma)}]$ as $e_0 \notin \Gamma(e)$. This choice of coordinates yields the identities

$$\begin{aligned} G_{e'}(\gamma(e)) &= G_{e,e'}(\gamma), \quad e' \in \mathbf{P}(\Gamma(e)), \\ G_{e'',e'}(\gamma(e)) &= G_{e'',e'}(\gamma), \quad (e', e'') \in \mathbf{P}(\Gamma(e)) \times \mathrm{Anc}_{\Gamma(e)}(e'), \end{aligned}$$

and we conclude that $A[\Gamma]/(x, X_{e_{-1}} - a_{e_0,e}, I_\gamma) \simeq A[\Gamma]/I_e \simeq A[\Gamma(e)]/(x, I_{\gamma(e)})$. This means equivalently that $\pi^{-1}(x_0)$ is isomorphic to the disjoint union of the fibers $\pi_{\gamma(e)}^{-1}(x_0)$ of the corresponding surfaces $\pi_{\gamma(e)} \colon S_{\gamma(e)} \to X$, $e \in \mathrm{Ch}(e_0)$. Since the fine k-weighted tree $\gamma(e)$ has height $h-1$, it follows from our induction hypothesis that these fibers are nonempty and reduced, consisting of disjoint unions of affine lines $\mathbb{A}^1_k$. So the same holds for $\pi^{-1}(x_0)$. Finally, the precise description of the irreducible components of $\pi^{-1}(x_0)$ follows easily by induction again. □

Remark 2.14. A Danielewski surface $\pi \colon S_\gamma \to X = \mathbb{A}^1_k$ is a flat (or rather a smooth) X-scheme. In general, the scheme $\tilde{\pi} \colon \tilde{S}_\gamma \to X$ with defining ideal $\tilde{I}_\gamma$ generated only by the polynomials Δ_e, $e \in \mathbf{P}(\Gamma)$, is not flat over X. The above discussion together with the second relation of (2.2) imply that S_γ coincides with the flat limit over X of the trivial family of affine lines $\tilde{S}_\gamma|_{X_*} \simeq \mathbb{A}^1_{X_*}$ defined by the equations $\Delta_e = 0$, $e \in \mathbf{P}(\Gamma)$, in $\mathbb{A}^{d(\Gamma)}_{X_*} = \mathrm{Spec}(A_x[\Gamma])$. This explains why the polynomials $\Delta_{e',e}$, $(e, e') \in \mathbf{P}(\Gamma) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$, should be added to the obvious ones Δ_e, $e \in \mathbf{P}(\Gamma)$, to define the surface S_γ.

The following result shows that the *embedded Danielewski surface* $\pi \colon S_\gamma \to X$ defined by a fine k-weighted tree $\gamma = (\Gamma, w)$ admits nontrivial actions of the additive group $\mathbb{G}_{a,X}$, which come as the restrictions of certain $\mathbb{G}_{a,X}$-actions on the ambient space $\mathbb{A}^{d(\Gamma)}_X$.

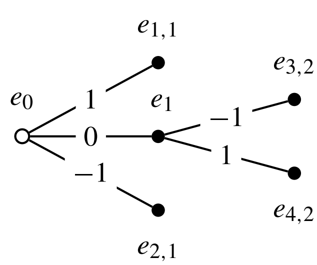
Proposition 2.15. *Let $\gamma = (\Gamma, w)$ be a fine k-weighted tree of height $h \geq 0$. Then, for every $m \geq h$ and every $a \in A \setminus \{0\}$, the derivation $\tilde{\partial}_{\gamma,a,m} \in \mathrm{Der}_A(A[\Gamma], A_x[\Gamma])$ defined recursively by*

$$\tilde{\partial}_{\gamma,a,m} = ax^m \partial_{X_{e_{-1}}} + x^{-1} \sum_{e \in \mathbf{P}(\Gamma)} \tilde{\partial}_{\gamma,a,m}(G_e(\gamma)) \partial_{X_e}$$

is a triangular derivation of $A[\Gamma]$ inducing a locally nilpotent A-derivation $\partial_{\gamma,a,m}$ of B_γ.

Proof. It suffices to prove the assertion for the derivation $\tilde{\partial} = \partial_{\gamma,1,h}$ as $\tilde{\partial}_{\gamma,a,m} = ax^{m-h}\tilde{\partial}$. For every $e \in \mathbf{P}(\Gamma)$ at level $i < h$, the polynomial G_e only involves the variables X_0 and $X_{e'}$, $e' \in \mathrm{Anc}(e)$. So we conclude recursively that $\tilde{\partial}(X_e) \in x^{h-i-1}A[X_{e_{-1}}, (X_{e'})_{e' \in \mathrm{Anc}(e)}]$. Thus $\tilde{\partial}$ restricts to a triangular A-derivation of $A[\Gamma]$. By construction, $\tilde{\partial}$ annihilates Δ_e for every $e \in \mathbf{P}(\Gamma)$. Moreover, $x\tilde{\partial}(\Delta_{e',e}) = \tilde{\partial}(x\Delta_{e',e}) \in I_\gamma$ for every pair $(e, e') \in (\mathbf{P}(\Gamma) \setminus \{e_0\}) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$ by virtue of (2.2). Thus $\tilde{\partial}(\Delta_{e',e}) \in I_\gamma$ as I_γ is a prime ideal which does not contain x. Hence $\tilde{\partial}(I_\gamma) \subset I_\gamma$ and so, $\tilde{\partial}$ induces a locally nilpotent A-derivation ∂ of B_γ. □

Example 2.16. We consider the following fine k-weighted tree $\tilde{\gamma} = (\Gamma, \tilde{w})$ with the leaves $e_{1,1}, e_{2,1}, e_{3,2}, e_{4,2}$.



We have $A[\Gamma] = k[x][X_{e_{-1}}, X_{e_0}, X_{e_1}]$ and

$${}^t\overline{M}(\tilde{\gamma}) = \begin{pmatrix} x & X_{e_{-1}}P(X_{e_{-1}}) & P(X_{e_{-1}})P(X_{e_0}) \\ 1 & X_{e_0} & X_{e_1} \end{pmatrix},$$

where $P(t) = t^2 - 1 \in k[t]$. Therefore $\pi : S_{\tilde{\gamma}} \to X$ is the surface with equations

$$xX_{e_0} - X_{e_{-1}}P(X_{e_{-1}}) = 0, \quad X_{e_{-1}}X_{e_1} - X_{e_0}P(X_{e_0}) = 0,$$
$$xX_{e_1} - P(X_{e_{-1}})P(X_{e_0}) = 0.$$

Letting $y = X_{e_{-1}}, z = X_{e_0}$ and $u = X_{e_1}$, the locally nilpotent derivation $\tilde{\partial}_{\tilde{\gamma},1,2} \in \mathrm{Der}_A(A[\Gamma])$ is simply the derivation $\partial_{1,2} \in \mathrm{Der}_{k[x]}(k[x][y, z, u])$ of Example 2.4.

3. Embeddings of Danielewski surfaces in affine spaces

In this section, we compare the two constructions of Danielewski surfaces by means of fine k-weighted trees. We describe a certain class of morphisms of Danielewski surfaces as the restrictions of suitable linear projections.

From abstract to embedded Danielewski surfaces. Here we prove the following result.

Theorem 3.1. *For every abstract Danielewski surface $\pi\colon S^{\gamma}\to X$ defined by a fine k-weighted tree $\gamma=(\Gamma,w)$, there exists another fine weight function $\tilde{w}\colon E(\Gamma)\to k$ on the tree Γ, and a closed embedding $\zeta\colon S^{\gamma}\hookrightarrow \mathbb{A}^{d(\Gamma)}_X$ inducing an isomorphism between S^{γ} and the embedded Danielewski surface $S_{\tilde{\gamma}}$ defined by the fine k-weighted tree $\tilde{\gamma}=(\Gamma,\tilde{w})$. Moreover, ζ is equivariant when we equip S^{γ} and $\mathbb{A}^{d(\Gamma)}_X$ with the $\mathbb{G}_{a,X}$-actions corresponding to the locally nilpotent A-derivations $\partial_{a,m}\in \mathrm{Der}_A(B^{\gamma})$ (see* 2.3) *and* $\tilde{\partial}_{\tilde{\gamma},a,m}\in \mathrm{Der}_A(A[\Gamma])$ (*Proposition* 2.15) *respectively.*

Example 3.2. We consider the abstract Danielewski surface $\pi\colon S^{\gamma}\to X$ defined by the fine k-weighted tree of Example 2.4. The canonical morphism $\psi\colon S^{\gamma}\to \mathbb{A}^1_X=\mathrm{Spec}(k[x][X_{e_{-1}}])$ is given by the section $s_{e_{-1}}\in B^{\gamma}$ whose restrictions on the canonical open subsets $S_i=\mathrm{Spec}(k[x][T_i])$ are given by

$$s_{e_{-1}}|_{S_i}=\begin{cases}(-1)^{i+1}+xT_i, & \text{if } i=1,2,\\ (-1)^{i+1}x+x^2T_i, & \text{if } i=3,4.\end{cases}$$

Letting $C_i=\pi^{-1}(x_0)\cap S_i$, $i=1,\dots,4$, be the irreducible components of $\pi^{-1}(x_0)$, we see that $s_{e_{-1}}$ restricts to a coordinate function on every fiber $\pi^{-1}(y)$, $y\in X_*$, and is locally constant on $\pi^{-1}(x_0)$ with the values 1, -1 and 0 on C_1, C_2 and $C_3\cup C_4$ respectively. Therefore, letting $P(t)=(t^2-1)\in k[t]$, the section $x^{-1}s_{e_{-1}}P(s_{e_{-1}})\in B^{\gamma}\otimes_{k[x]}k[x,x^{-1}]$ extends to a section $s_{e_0}\in B^{\gamma}$ whose restrictions on the S_i's are given by

$$s_{e_0}|_{S_i}=\begin{cases}2T_1+3xT_1^2+x^2T_1^3, & \text{if } i=1,\\ 2T_2-3xT_2^2+x^2T_2^3, & \text{if } i=2,\\ -1-xT_3+x^2\xi_3(x,T_3), & \text{if } i=3,\\ 1-xT_4+x^2\xi_4(x,T_4), & \text{if } i=4,\end{cases}$$

for certain polynomials $\xi_3(x,t),\xi_4(x,t)\in k[x,t]$. Thus s_{e_0} restricts to a coordinate function on C_1 and C_2, and is constant on C_3 and C_4 with the values -1 and 1 respectively. Again, $x^{-1}P(s_{e_0})\in B_{\gamma}\otimes_{k[x]}k[x,x^{-1}]$ extends to a regular function on $S_3\cup S_4\subset S^{\gamma}$ which restricts to a coordinate function on C_3 and C_4. Clearly, $x^{-1}P(s_{e_{-1}})P(s_{e_0})$ extends to a section $s_{e_1}\in B^{\gamma}$ with the same property as

$P(s_{e_{-1}})|_{C_i} = -1$, $i = 3, 4$. The A-algebra homomorphism $A[X_{e_{-1}}, X_{e_0}, X_{e_1}] \to B^\gamma$, $X_e \mapsto s_e$ defines a closed embedding $\zeta\colon S^\gamma \to \mathbb{A}^3_X$, inducing an X-isomorphism between S^γ and the embedded Danielewski surface $S_{\tilde{\gamma}}$ defined by the fine k-weighted tree $\tilde{\gamma} = (\Gamma, \tilde{w})$ of Example 2.16.

3.3. To prove Theorem 3.1, we proceed in a similar way as in the previous example. More precisely, given an abstract Danielewski surface S^γ defined by a fine k-weighted tree $\gamma = (\Gamma, w)$, we construct in 3.4 and Lemmas 3.5–3.7 below a fine weight function $\tilde{w}\colon E(\Gamma) \to k$ on Γ and a collection of sections $s_e \in B^\gamma$, $e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}$, which define a closed embedding $\zeta\colon S^\gamma \hookrightarrow \mathbb{A}^{d(\Gamma)}_X$ inducing an X-isomorphism $\phi\colon S^\gamma \overset{\sim}{\to} S_{\tilde{\gamma}}$ between S^γ and the embedded Danielewski surface defined by the tree $\tilde{\gamma} = (\Gamma, \tilde{w})$.

3.4. Given a fine k-weighted tree $\gamma = (\Gamma, w)$ with the leaves $e_{1,m_1}, \ldots, e_{n,m_n}$, we denote by $\tau_i\colon B^\gamma = \Gamma(S^\gamma, \mathcal{O}_{S^\gamma}) \to A[T_i]$ the localization homomorphisms corresponding to the canonical open covering of the abstract Danielewski surface S^γ by the open subsets $S_i = \operatorname{Spec}(A[T_i])$, $i = 1, \ldots, n$. The canonical X-morphism $\psi\colon S^\gamma \to \mathbb{A}^1_X = \operatorname{Spec}(A[X_{e_{-1}}])$ (2.2) corresponds to the section $s_{e_{-1}} \in B^\gamma$ such that

$$\tau_i(s_{e_{-1}}) = \sum_{j=0}^{m_i} w_{i,j} x^j \in A[T_i],$$

where

$$w_{i,j} = \begin{cases} w(\overleftarrow{e_{i,j}e_{i,j+1}}), & \text{if } 0 \leq j \leq m_i - 1, \\ T_i, & \text{if } j = m_i. \end{cases}$$

For every $e \in \Gamma$, we let

$$C_e = \bigsqcup_{\{e_{i,m_i} \in L((\uparrow e)_\Gamma)\}} (\pi^{-1}(x_0) \cap S_i) \simeq \operatorname{Spec}\Big(\prod_{\{e_{i,m_i} \in L((\uparrow e)_\Gamma)\}} \operatorname{Spec}(k[T_i])\Big).$$

If γ has height $h = 0$ then Γ is the trivial tree with one element $\{e_0\}$ and $\psi\colon S^\gamma \to \mathbb{A}^1_X$ is an isomorphism. Otherwise, if $h \geq 1$, then we have the following result.

Lemma 3.5. *If $h \geq 1$ then there exists a fine weight function $\tilde{w}\colon E(\Gamma) \to k$, $\overleftarrow{e'e} \mapsto \tilde{a}_{e',e}$ defining a fine k-weighted tree $\tilde{\gamma} = (\Gamma, \tilde{w})$, and a collection of sections $(s_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}} \in B^\gamma$ with the following properties.*

a) *For every $e_{i,j} \in \mathbf{P}(\Gamma)$, $s_{e_{i,j}} = x^{-1} G_{e_{i,j}}(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, s_{e_{i,1}}, \ldots, s_{e_{i,j-1}})$.*

b) *If $\operatorname{Ch}(e_{i,j}) = \{e_{i_1,j+1}, \ldots, e_{i_r,j+1}\}$, then $s_{e_{i,j-1}}$ is constant on $C_{e_{i_l,j+1}} \subset \pi^{-1}(x_0)$ with the value $\tilde{a}_{e_{i,j},e_{i_l,j+1}} \in k$, $l = 1, \ldots, r$.*

c) *For every leaf e_{i,m_i} of Γ, $s_{e_{i,m_i-1}}$ induces an coordinate function on $C_{e_{i,m_i}} \simeq \mathbb{A}^1_k$.*

Proof. We construct the weight function $\tilde{w}$ and the sections s_e by induction as follows. For every $m = 0, \dots, h$, we denote by Γ_m the subtree of Γ with the elements $e \in \Gamma$ at levels $l \leq m$. At step m, we suppose that the weight function $\tilde{w}_m \colon E(\Gamma_m) \to k$ is constructed on Γ_m, as well as the sections s_e for every $e \in \Gamma_{m-2}$, and we define the sections s_e, $e \in \Gamma_{m-1} \setminus \Gamma_{m-2}$. Then we extend $\tilde{w}_m$ to a weight function $\tilde{w}_{m+1} \colon E(\Gamma_{m+1}) \to k$.

Step 0. We let $s_{e_{-1}} \in B^{\gamma}$ be the section corresponding to the canonical morphism $\psi \colon S^{\gamma} \to \mathbb{A}^1_X$. By definition, $\tau_i(s_{e_{-1}}) = w_{i,0} + x\xi_i$ for a certain $\xi_i \in A[T_i]$ for every $i = 1, \dots, n$. Thus b) is satisfied provided that we define the weight function $\tilde{w}_1$ on $\Gamma_1 \setminus \{e_0\}$ by

$$\tilde{a}_{e_0, e_{i,1}} = \tilde{w}_1(\overleftarrow{e_0 e_{i,1}}) = s_{e_{-1}}|_{C_{e_{i,1}}} = w_{i,0} \in k$$

for every $e_{i,1} \in \mathrm{Ch}(e_0)$. Note that if $e_{j,1} = e_{i,1}$, then $w_{i,0} = w_{j,0}$ as $w_{i,0} \neq w_{j,0}$ if and only if e_0 is the first common ancestor of the leaves e_{i,m_i} and e_{j,m_j}. Thus $\tilde{\gamma}_1 = (\Gamma_1, \tilde{w}_1)$ is a fine k-weighted tree and we are done with Step 0.

Step 1. By construction, the rational section $x^{-1} G_{e_0}(\tilde{\gamma}_1)(s_{\gamma, e_{-1}}) \in B^{\gamma} \otimes_A A_x$ extends to a section s_{e_0} of B^{γ} satisfying a). Since γ is a fine k-weighted tree, we deduce from Taylor's Formula that for every $i = 1, \dots, n$, there exists a pair $(\alpha_{i,1} = F_{e_0}^{\{e_{i,1}\}}(w_{i,0}), \beta_{i,1}) \in k^* \times k$ depending only of the subchain $(\downarrow e_{i,1})_{\Gamma}$, and a polynomial $\xi_{i,1} \in A[T_i]$ such that

$$\tau_i(s_{e_0}) = \alpha_{i,1} w_{i,1} + \beta_{i,1} + x\xi_{i,1} \in A[T_i].$$

Thus, if $e_{i,1}$ is a leaf of Γ then $w_{i,1} = T_i$ and so c) is satisfied. Otherwise, if $e_{j,2}$ and $e_{j',2}$ are children of $e_{i,1}$ then $\alpha_{j,1} = \alpha_{j',1} = \alpha_{i,1}$ and $\beta_{j,1} = \beta_{j',1} = \beta_{i,1}$ as $e_{j,1} = e_{j',1} = e_{i,1}$, whereas $w_{j,1} \neq w_{j',1}$ as γ is a fine k-weighted tree. Thus $\tilde{\gamma}_2 = (\Gamma_2, \tilde{w}_2)$ is a fine k-weighted tree for the weight function $\tilde{w}_2 \colon E(\Gamma_2) \to k$ restricting to $\tilde{w}_1$ on $\Gamma_1 \subset \Gamma_2$ and such that

$$\tilde{a}_{e_{i,1}, e_{i,2}} = \tilde{w}_2(\overleftarrow{e_{i,1} e_{i,2}}) = s_{e_0}|_{C_{e_{i,2}}} = (\alpha_{i,0} w_{i,1} + \beta_{i,1}) \in k, \quad i = 1, \dots, n.$$

By construction, b) is also satisfied. This completes Step 1.

Step m, $m \geq 2$. By induction hypothesis, $\tilde{\gamma}_m = (\Gamma_m, \tilde{w}_m)$ is a fine k-weighted tree, and the sections $s_e \in B^{\gamma}$, $e \in \Gamma_{m-2}$, satisfying the hypothesis of Lemma 3.5 have been defined. So the formula

$$s_{e_{i,m-1}} = x^{-1} G_{e_{i,m-1}}(\tilde{\gamma}_m)(s_{e_{-1}}, s_{e_0}, s_{e_{i,1}}, \dots, s_{e_{i,m-2}})$$

makes sense and defines an element of $B^{\gamma} \otimes_A A_x$. Similarly as in Step 1, we deduce from Taylor's Formula that for every $j = 0, \dots, m-1$ there exists a pair $(\tilde{\alpha}_{i,j}, \tilde{\beta}_{ij}) \in k^* \times k$ depending only on the subchain $(\downarrow e_{i,j})_{\Gamma}$, and a polynomial $\tilde{\xi}_{i,j} \in A[T_i]$ such that

$$\tau_i(s_{e_{i,j-1}}) = a_{e_{i,j+1} e_{i,j}} + x(\tilde{\alpha}_{i,j} w_{i,j+1} + \tilde{\beta}_{i,j}) + x^2 \tilde{\xi}_{i,j} \in A[T_i].$$

By applying Taylor's Formula again, we conclude that there exists a pair $(\alpha_{i,m}, \beta_{i,m}) \in k^* \times k$ depending only on the subchain $(\downarrow e_{i,m})_\Gamma$ and a polynomial $\xi_{i,m} \in A[T_i]$ such that

$$\tau_i(s_{e_{i,m-1}}) = \alpha_{i,m} w_{i,m} + \beta_{i,m} + x\xi_{i,m} \in A[T_i].$$

Thus, if $e_{i,m-1} \in (\downarrow e_{j,m_j})$ then $e_{i,m-1} = e_{j,m-1}$ and so $\tau_j(s_{e_{i,m-1}}) \in A[T_j]$. Otherwise, for every index j such that $e_{i,m-1} \notin (\downarrow e_{j,m_j})_\Gamma$, the first common ancestor of $e_{i,m-1}$ and e_{j,m_j} is an element $e_{i,l} = e_{j,l}$ at level $l \leq \min(m-2, m_j - 1)$. Thus $(X_{e_{j,l-1}} - \tilde{a}_{e_{j,l},e_{j,l+1}})$ divides the genealogical polynomial $G_{e_{i,m-1}}(\tilde{\gamma}_m)$ of $e_{i,m-1}$. Since $\tau_j(s_{e_{j,l-1}} - \tilde{a}_{e_{j,l},e_{j,l+1}}) \in xA[T_j]$, we conclude that

$$x\tau_j(s_{e_{i,m-1}}) = G_{e_{i,m-1}}(\tilde{\gamma}_m)(\tau_j(s_{e_{-1}}), \tau_j(s_{e_{i,0}}), \tau_j(s_{e_{i,1}}), \ldots, \tau_j(s_{e_{i,m-2}})) \in xA[T_j].$$

Thus $\tau_j(s_{e_{i,m-1}}) \in A[T_j]$ for every $j = 1, \ldots n$, and hence, $s_{\gamma,e_{i,m-1}} \in B^\gamma$. If $e_{i,m}$ is a leaf of Γ then $w_{i,m} = w_{i,m_i} = T_i$ by definition. Thus $s_{e_{i,m-1}}$ satisfies a) and c). Finally, the same argument as in Step 1 shows that $\tilde{\gamma}_{m+1} = (\Gamma_{m+1}, \tilde{w}_{m+1})$ is a fine k-weighted tree for the weight function $\tilde{w}_{m+1} \colon E(\Gamma_{m+1}) \to k$ restricting to $\tilde{w}_m$ on $\Gamma_m \subset \Gamma_{m+1}$ and such that

$$\tilde{a}_{e_{i,m},e_{i,m+1}} = \tilde{w}_{m+1}(\overleftarrow{e_{i,m}e_{i,m+1}}) = s_{e_{i,m-1}}|_{C(e_{i,m+1})} = (\alpha_{i,m} w_{i,m} + \beta_{i,m}) \in k,$$

whenever $e_{i,m}$ is not a leaf of Γ. This completes Step m as b) is satisfied by construction.

After $h = h(\Gamma)$ steps, the above procedure stops, and we obtain a fine k-weighted tree $\tilde{\gamma} = \tilde{\gamma}_h = (\Gamma, \tilde{w}_h)$ and a collection of sections $(s_e)_{e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}} \in B^\gamma$ satisfying conditions a), b) and c). This completes the proof. □

The following lemma implies the first assertion of Theorem 3.1.

Lemma 3.6. *The X-morphism $\zeta \colon S^\gamma \to \mathbb{A}_X^{d(\Gamma)}$ induced by the A-algebra homomorphism $\zeta^* \colon A[\Gamma] \to B^\gamma$, $X_e \mapsto s_e$, $e \in \mathbf{P}(\Gamma) \cup \{e_{-1}\}$, is a closed embedding inducing an X-isomorphism $\phi \colon S^\gamma \xrightarrow{\sim} S_{\tilde{\gamma}}$.*

Proof. By construction, $s_{e_{-1}}$ corresponds to the canonical birational morphism $\psi \colon S^\gamma \to \mathbb{A}_X^1$, whence induces a X_*-isomorphism $S^\gamma|_{X_*} \xrightarrow{\sim} \mathbb{A}_{X_*}^1$. By b) of Lemma 3.5, for every pair e_{i,m_i}, e_{j,m_j} of leaves of Γ with first common ancestor $e \in \Gamma$, the section $s_{\mathrm{Par}(e)}$ takes distinct constant values on $C_{e_{i,m_i}}$ and $C_{e_{i,m_j}}$. Thus ζ distinguishes the irreducible components of the fiber $\pi^{-1}(x_0)$. Finally, c) of Lemma 3.5 implies that for every $i = 1, \ldots, n$, $s_{e_{i,m_i-1}}$ induces a coordinate function on $C_{e_{i,m_i}} \simeq \mathbb{A}_k^1$. This proves that $\zeta \colon S^\gamma \to \mathbb{A}_X^{d(\Gamma)}$ is an embedding. By construction, $\zeta^*(\Delta_e(\tilde{\gamma})) = 0$ in B^γ for every $e \in \mathbf{P}(\Gamma)$. Thus $x\zeta^*(\Delta_{e',e}(\tilde{\gamma})) = \zeta^*(x\Delta_{e',e}(\tilde{\gamma})) = 0$ for every $(e, e') \in (\mathbf{P}(\Gamma) \setminus \{e_0\}) \times ((\downarrow e)_\Gamma \setminus \{e_0\})$ by virtue of (2.2), and so, $\zeta^*(\Delta_{e',e}(\tilde{\gamma})) = 0$

as B^γ is an integral A-algebra. This proves that the image of ζ in contained in the embedded Danielewski surface $S_{\tilde{\gamma}}$. It is clear by construction that the induced X-morphism $\phi\colon S^\gamma \to S_{\tilde{\gamma}}$ restricts to a bijection between the sets of closed points of S^γ and $S_{\tilde{\gamma}}$ respectively. So the result follows from Zariski's Main Theorem as $S_{\tilde{\gamma}}$ is smooth over k, whence, in particular, normal. □

The following result completes the proof of Theorem 3.1.

Lemma 3.7. *For every nontrivial $\mathbb{G}_{a,X}$-action $\boldsymbol{t}_{\gamma,a,m}$ (2.3) on an abstract Danielewski surface $\pi\colon S^\gamma \to X$ defined by a fine k-weighted tree $\gamma = (\Gamma, w)$, the closed embedding $\zeta\colon S^\gamma \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$ in Lemma 3.6 is equivariant when we equip $\mathbb{A}_X^{d(\Gamma)}$ with the $\mathbb{G}_{a,X}$-action induced by the locally nilpotent A-derivation $\tilde{\partial}_{\tilde{\gamma},a,m} \in \mathrm{Der}_A(A[\Gamma])$ (Proposition 2.15).*

Proof. By definition (see 2.3), the twisted translation $\boldsymbol{t}_{\gamma,a,m}$ on S^γ is induced by the extension $\partial_{a,m}$ to B^γ of the locally nilpotent derivation $\delta_{a,m} = ax^m\partial_{X_{e_{-1}}}$ of $B^\gamma \otimes_A A_x \simeq A_x[X_{e_{-1}}]$, where $m \geq h(\Gamma)$ and $a \in A \setminus \{0\}$. By construction, for every $e \in \mathbf{P}(\Gamma)$, we have $s_e = x^{-1}G_e(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, \dots, s_{\mathrm{Par}(e)}) \in B^\gamma \subset A_x[X_{e_{-1}}]$ and so,

$$\partial_{a,m}(s_e) = x^{-1} \sum_{e' \in \mathrm{Anc}(e)\cup\{e_{-1}\}} \partial_{X_{e'}} G_e(\tilde{\gamma})(s_{e_{-1}}, s_{e_0}, \dots, s_{\mathrm{Par}(e)})\partial_{a,m}(s_{e'}) \in B^\gamma \otimes_A A_x.$$

In view of the definition of $\tilde{\partial}_{\tilde{\gamma},a,m} \in \mathrm{Der}_A(A[\Gamma])$ (see Proposition 2.15), this means precisely that the embedding $\zeta\colon S^\gamma \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$ is equivariant when we equip S^γ and $\mathbb{A}_X^{d(\Gamma)}$ with the actions corresponding to the locally nilpotent derivation $\partial_{a,m}$ and $\tilde{\partial}_{\tilde{\gamma},a,m}$. □

Corollary 3.8. *Every Danielewski surface $\pi\colon S \to X$ equipped with a nontrivial $\mathbb{G}_{a,X}$-action is equivariantly X-isomorphic to an embedded Danielewski surface S_γ defined by a fine k-weighted tree $\gamma = (\Gamma, w)$, equipped with the $\mathbb{G}_{a,X}$-action corresponding to a suitable locally nilpotent derivation $\partial_{\gamma,a,m} \in \mathrm{Der}_A(B_\gamma)$, where $m \geq h(\Gamma)$ and $a \in A \setminus \{0\}$.*

Proof. By Theorem 3.2 in [4], every Danielewski surface S is isomorphic to an abstract Danielewski surface S^γ defined by a fine k-weighted tree γ. Moreover, by Proposition 2.12 in *loc. cit.*, every nontrivial $\mathbb{G}_{a,X}$-action on S^γ coincides with a twisted translation $\boldsymbol{t}_{\gamma,a,m}$ for a suitable pair $(m \geq h(\Gamma), a \in A \setminus \{0\})$. So the result follows from Theorem 3.1. □

Corollary 3.9. *Every $\mathbb{G}_{a,X}$-action on an embedded Danielewski surface S_γ defined by a fine k-weighted tree $\gamma = (\Gamma, w)$ is induced by a locally nilpotent derivation $\partial_{\gamma,a,m} \in \mathrm{Der}_A(B_\gamma)$.*

Since the locally nilpotent derivations $\partial_{\gamma,a,m} \in \mathrm{Der}_A(B_\gamma)$ are induced by locally nilpotent derivations $\tilde{\partial}_{\gamma,a,m} \in \mathrm{Der}_A(A[\Gamma])$, we obtain the following result.

Corollary 3.10. *Every Danielewski surface $\pi : S \to X$ admits a closed embedding $\zeta : S \hookrightarrow \mathbb{A}^d_X$ into a relative affine space $\mathbb{A}^d_X$, where $d \geq 1$, such that every $\mathbb{G}_{a,X}$-action on S extends to an action on $\mathbb{A}^d_X$.*

In particular, if the Makar-Limanov invariant of S is nontrivial, then $\pi : S \to X$ is a unique $\mathbb{A}^1$-fibration on S up to automorphisms of X. Therefore, the general orbits of a $\mathbb{G}_{a,k}$-action on S coincide with the general fibers of π. This leads to the following result.

Corollary 3.11. *Every Danielewski surface S with a nontrivial Makar-Limanov invariant admits a closed embedding into an affine space $\mathbb{A}^d_k$ in such a way that every $\mathbb{G}_{a,k}$-action on S extends to an action on $\mathbb{A}^d_k$.*

Morphisms of Danielewski surfaces as linear projections. A *morphism of Danielewski surfaces* is a birational X-morphism $\beta : S' \to S$, restricting to an isomorphism over X_*. In other words, β is an affine modification [7] restricting to an isomorphism over the complement of the support of the principal divisor $\pi^{-1}(x_0) = \mathrm{div}(x) \subset S$. Thus, letting $S = \mathrm{Spec}(B)$, there exists an ideal $I \subset B$ containing a power x^m of x such that S' is isomorphic to the open subset $\mathrm{Spec}(B[It]/(1 - x^m t))$ of the spectrum of the Rees algebra $B[It]$. In turn, this implies that $S' \simeq \mathrm{Spec}(B[t_1, \ldots, t_r]/J)$ for a certain ideal J. In these coordinates, the morphism $\beta : S' \to S$ coincides with the restriction to S' of the projection $\mathrm{pr}_S : \mathbb{A}^{r+1}_S = \mathrm{Spec}(B[t_1, \ldots, t_r]) \to S$. Here we give a more precise description of this situation.

3.12. To every morphism $\tau : \gamma' = (\Gamma', w') \to \gamma = (\Gamma, w)$ of fine k-weighted tree (see Definition 1.4), we associate a morphism $\beta_\tau : S^{\gamma'} \to S^{\gamma}$ between the associated abstract Danielewski surfaces in the following manner. We let $\sigma' = \{\sigma'_i \in A\}_{i=1,\ldots,n'}$ and $\sigma = \{\sigma_j \in A\}_{j=1,\ldots,n}$ be the collection of polynomials associated with γ' and γ, and we let $g' = \{g'_{ij} \in A_x\}$ and $g = \{g_{ij} \in A_x\}$ be the corresponding transition functions. We denote by $S'_i = \mathrm{Spec}(A[T'_i])$, $i = 1, \ldots, n'$, and $S_j = \mathrm{Spec}(A[T_j])$, $j = 1, \ldots, n$, the open subsets of the canonical coverings of $S^{\gamma'}$ and S^{γ} respectively. By Remark 1.5, the image of a leaf e'_{i,m'_i} of Γ' by τ is a leaf $e_{j(i),m_{j(i)}}$ of Γ such that $m'_i \geq m_{j(i)}$ and $\tau(e'_{i,k}) = e_{j(i),\min(k,m_{j(i)})}$ for every $k = 0, \ldots, m'_i$. Since $w(\overleftarrow{\tau(e'_{i,k})\tau(e'_{i,k+1})}) = w'(\overleftarrow{e'_{i,k}e'_{i,k+1}})$ whenever $\tau(e'_{i,k}) \neq \tau(e'_{i,k+1})$, we conclude that there exists a collection $\sigma'' = \{\sigma''_i \in A\}_{i=1,\ldots,n'}$ such that $\sigma'_i = \sigma_{j(i)} + x^{m_{j(i)}} \sigma''_i \in A$ for every $i = 1, \ldots, n'$. Then for every $i = 1, \ldots, n'$, the A-algebra homomorphism

$$A[T_{j(i)}] \longrightarrow A[T'_i], \quad T_{j(i)} \mapsto \sigma''_i + x^{m'_i - m_{j(i)}} T'_i$$

defines a birational X-morphism $\beta_\tau^{(i)}\colon S_i' \to S_{j(i)}$ restricting to an isomorphism over X_*. Since the transition functions satisfy the relation $x^{m_{i'}-m_{j(i)}}g_{il}' = g_{j(i)j(l)} + x^{m_{j(l)}-m_{j(i)}}\sigma_l'' + \sigma_i''$ for every $i, l = 1, \dots, n'$, it follows that these local morphisms $\beta_\tau^{(i)}$ glue to a morphism of Danielewski surfaces $\beta_\tau\colon S^{\gamma'} \to S^\gamma$. By Proposition 3.8 and Corollary 3.9 in [4], for every morphism of Danielewski surfaces $\beta\colon S' \to S$, there exists X-isomorphisms $\phi'\colon S' \xrightarrow{\sim} S^{\gamma'}$ and $\phi\colon S \xrightarrow{\sim} S^\gamma$ for suitable fine k-weighted trees γ' and γ such that $\phi \circ \beta \circ (\phi')^{-1}$ is the morphism β_τ induced by a morphism of fine k-weighted tree $\tau\colon \gamma' \to \gamma$.

3.13. Every morphism of fine k-weighted tree $\tau\colon \gamma' \to \gamma$ factors through a surjective morphism $\tau'\colon \gamma' \to \tau(\gamma')$ followed by an injection $\tau(\gamma') \hookrightarrow \gamma$. As a consequence, every morphism of Danielewski surfaces factors through a *quasi-surjective morphism* $\beta'\colon S^{\gamma'} \to S^{\tau(\gamma')}$, *i.e.* a morphism of Danielewski surfaces such that $\beta'^{-1}(C) \neq \emptyset$ for every irreducible component C of the fiber $\pi_{\tau(\gamma')}^{-1}(x_0) \subset S^{\tau(\gamma')}$ followed by the open immersion of $S^{\tau(\gamma')}$ in S^γ as the complement of irreducible components of $\pi_\gamma^{-1}(x_0) \subset S^\gamma$ corresponding to the leaves of Γ which are not in the image of τ.

3.14. Given a fine k-weighted tree $\gamma = (\Gamma, w)$, we consider the tree $\tilde{\gamma} = (\Gamma, \tilde{w})$ constructed in Lemma 3.5. For every edge $\overleftarrow{e'e}$ of Γ, the weight $\tilde{w}(\overleftarrow{e'e}) \in k$ is uniquely determined by the weights w of the edges of the subtree of Γ with elements $(\downarrow e)_\Gamma \cup \bigcup_{e' \in (\downarrow e)_\Gamma} \mathrm{Ch}(e')$. Therefore, every *surjective* morphism of fine k-weighted trees $\tau\colon \gamma' = (\Gamma', w') \to \gamma$ gives rise to a surjective morphism of fine k-weighted trees $\tilde{\tau}\colon \tilde{\gamma}' = (\Gamma', \tilde{w}') \to \tilde{\gamma}$ which restricts to the same morphism as τ between the underlying trees Γ' and Γ of $\tilde{\gamma}'$ and $\tilde{\gamma}$ respectively[1]. Since the subset $\Gamma'' = \{e' \in \Gamma',\ \tau^{-1}(\tau(e')) = \{e'\}\} \subset \Gamma'$ is a subtree of Γ' isomorphic to Γ, we obtain that

$$A[\Gamma'] = A[\Gamma'']\otimes_A A[(X_{e'})_{e' \in \mathbf{P}(\Gamma') \cap (\Gamma' \setminus \mathbf{P}(\Gamma''))}] \simeq A[\Gamma]\otimes_A A[(X_{e'})_{e' \in \mathbf{P}(\Gamma') \cap (\Gamma' \setminus \mathbf{P}(\Gamma''))}].$$

Moreover, for every $e' \in \mathbf{P}(\Gamma'')$, the genealogical polynomial $G_{e'}(\tilde{\gamma'})$ of e' is an element of $A[\Gamma''] \subset A[\Gamma']$ which coincides with the genealogical polynomial $G_{\tau(e')}(\tilde{\gamma}) \in A[\Gamma]$ of $\tau(e')$ via the isomorphism above. In turn, this implies that the genealogical matrix (see Definition 1.9) $M(\tilde{\gamma})$ of $\tilde{\gamma}$ is obtained from $M(\tilde{\gamma}')$ by deleting the rows corresponding to the elements in $\mathbf{P}(\Gamma') \setminus \mathbf{P}(\Gamma'')$. By construction of the embedding of S^γ into $\mathbb{A}_X^{d(\Gamma)}$ as the Danielewski surface $S_{\tilde{\gamma}}$, we obtain the following result.

Theorem 3.15. *Let $\tau\colon \gamma' = (\Gamma', w') \to \gamma = (\Gamma, w)$ be a surjective morphism of fine k-weighted trees and let $\tilde{\tau}\colon \tilde{\gamma}' \to \tilde{\gamma}$ be the morphism obtained above. Let*

[1] Actually, the functor $\gamma \mapsto \tilde{\gamma}$, $\tau \mapsto \tilde{\tau}$ is an automorphism of the category $\mathcal{T}_{w,k}^s$ of fine k-weighted trees equipped with surjective morphisms.

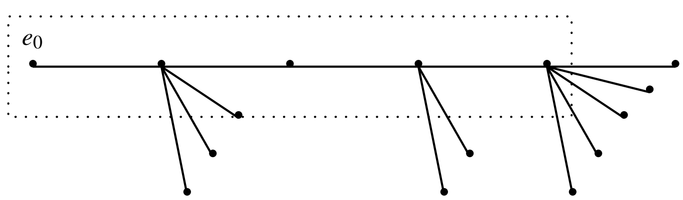
$\zeta'\colon S^{\gamma'} \hookrightarrow \mathbb{A}_X^{d(\Gamma')}$ and $\zeta\colon S^{\gamma} \hookrightarrow \mathbb{A}_X^{d(\Gamma)}$ are the embeddings from Lemma 3.6 *of $S^{\gamma'}$ and S^{γ} as the Danielewski surfaces $S_{\tilde{\gamma}'}$ and $S_{\tilde{\gamma}}$ respectively. Then $\zeta \circ \beta = p_{\Gamma'/\Gamma} \circ \zeta'$, where $p_{\Gamma'/\Gamma}\colon \mathbb{A}_X^{d(\Gamma')} \to \mathbb{A}_X^{d(\Gamma)}$ is the projection induced by the inclusion $A[\Gamma] \simeq A[\Gamma''] \subset A[\Gamma']$.*

4. Danielewski surfaces with a trivial Makar-Limanov invariant

The Makar-Limanov [6] invariant of an affine variety $V = \mathrm{Spec}(B)$ over a field k of characteristic zero is the sub-algebra $\mathrm{ML}(V) \subset B$ of regular functions on V which are invariant under *every* $\mathbb{G}_{a,k}$-action on V. A surface S has a trivial Makar-Limanov invariant $\mathrm{ML}(S) = k$ if and only if it admits two nontrivial $\mathbb{G}_{a,k}$-actions with distinct general orbits. In view of the correspondence between nontrivial $\mathbb{G}_{a,k}$-actions $\mathbb{G}_{a,k} \times S \to S$ on S and quotient $\mathbb{A}^1$-fibrations $\pi\colon S \to X = S//\mathbb{G}_{a,k}$, this means in turn that S has a trivial Makar-Limanov invariant if and only if it admits two $\mathbb{A}^1$-fibrations with distinct general fibers. In this section, we characterize among Danielewski surfaces the ones with a trivial Makar-Limanov invariant.

Danielewski surfaces defined by weighted combs

Definition 4.1. A nontrivial (*oriented*) *comb* of height $h \geq 1$ is a tree Γ such that for every $e \in \mathbf{P}(\Gamma)$ of degree $\deg_{\Gamma}(e) \geq 1$, all but possibly one of the children of e are leaves of Γ. This means equivalently that the subtree $C_{\Gamma} = \mathbf{P}(\Gamma) = \{e_0 < \cdots < e_{h-1}\}$ of Γ is a nonempty chain of length $h - 1$, called the *dorsal chain* of Γ.



A comb rooted in e_0.

4.2. By Theorem 5.4 in [4], a Danielewski surface S defined over an algebraically closed field $k = \bar{k}$ of characteristic zero has a trivial Makar-Limanov invariant if and only if it is isomorphic to an abstract Danielewski surface S^{γ} defined by a fine k-weighted comb. This result is based on a characterization of normal affine surfaces S with a trivial Makar-Limanov invariant in terms on the boundary divisors of certain minimal completions $\bar{S}$ of S (see [3]). Unfortunately, no such criterion exists for a normal affine surface defined over an arbitrary field k of characteristic zero. However, the following result shows that the combinatorial characterization of Danielewski surfaces with a trivial Makar-Limanov invariant remains valid in this more general setting.

Theorem 4.3. *A Danielewski surface $S \not\simeq \mathbb{A}^1_X$, defined over a field k of characteristic zero, has a trivial Makar-Limanov invariant if and only if it is isomorphic to an abstract Danielewski surface S^γ defined by a fine k-weighted comb. If this is the case, then there exist an integer $h \geq 1$ and a collection of monic polynomials $P_0, \dots, P_{h-1} \in k[t]$ with simple roots $a_{i,j} \in k^*$, $i = 0, \dots, h-1$, $j = 1, \dots, \deg_t(P_i)$, such that S is isomorphic to the surface $S_{P_0,\dots,P_{h-1}} \subset \mathrm{Spec}(k[x][y_{-1}, \dots, y_{h-2}][z])$ defined by the equations*

$$xz - y_{h-2}\prod_{l=0}^{h-1} P_l(y_{l-1}) = 0,$$

$$zy_{i-1} - y_i y_{h-2}\prod_{l=i+1}^{h-1} P_l(y_{l-1}) = 0, \quad xy_i - y_{i-1}\prod_{l=0}^{i} P_l(y_{l-1}) = 0, \quad 0 \leq i \leq h-2,$$

$$y_{i-1}y_j - y_i y_{j-1}\prod_{l=i+1}^{j} P_l(y_{l-1}) = 0, \quad 0 \leq i < j \leq h-2.$$

4.4. The proof is given in 4.5–4.7 below. We first observe that the condition is necessary. Indeed, suppose that the Makar-Limanov invariant of S is trivial. We let $\gamma = (\Gamma, w)$ be a fine k-weighted tree such that $S \simeq S^\gamma$, and we let $i : k \hookrightarrow \bar{k}$ be the injection of k in an algebraic closure $\bar{k}$. Then the Danielewski surface $S_{\bar{k}} = S \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k}) \to X_{\bar{k}} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ is $X_{\bar{k}}$-isomorphic to the abstract Danielewski surface $S^\gamma \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ defined by the tree γ considered a fine $\bar{k}$-weighted tree via the weight function $i \circ w : E(\Gamma) \to \bar{k}$. Since every nontrivial $\mathbb{G}_{a,k}$-action on S lifts to a nontrivial action of $\mathbb{G}_{a,\bar{k}} = \mathbb{G}_{a,k} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ on $S_{\bar{k}}$, we conclude that $S_{\bar{k}}$ has a trivial Makar-Limanov invariant too. Thus the tree γ is a comb by virtue of Theorem 5.4 in [4].

4.5. Conversely, the same argument shows that if S is isomorphic to an abstract Danielewski surface S^γ defined by a fine k-weighted comb γ, then $S_{\bar{k}}$ has a trivial Makar-Limanov invariant. Unfortunately, in general, there is no guarantee that a given $\mathbb{G}_{a,\bar{k}}$-action on $S_{\bar{k}}$ appears as the lifting of an action of $\mathbb{G}_{a,k}$ on S. Therefore, to show that the condition is sufficient, we must proceed in a different way. We will exploit the fact that S is isomorphic to an embedded surface S_γ defined by a fine k-weighted comb γ to construct two explicit $\mathbb{A}^1$-fibrations on S with distinct general fibers.

4.6. By construction, a Danielewski surface S is isomorphic to $\mathbb{A}^1_X$ if and only if it is isomorphic to an abstract surface S^γ defined by a fine k-weighted chain γ. In this case it is also isomorphic to the surface $S_{\{e_0\}}$ defined by the trivial tree with one element $\{e_0\}$. More generally, it follows from Theorem 3.10 in [4] that every Danielewski surface $S \not\simeq \mathbb{A}^1_X$ isomorphic to an abstract Danielewski surface S^γ defined by a fine k-weighted comb γ is also isomorphic to a surface S^{γ_0} defined by a fine k-weighted comb $\gamma_0 = (\Gamma, w_0)$ of height $h \geq 1$, with dorsal chain $C_\Gamma = \{e_0 < e_1 < \cdots < e_{h-1}\}$, satisfying the following properties:

a) The root e_0 of Γ as at least two children.

b) For every $i = 0, \dots, h-2$, $w_0(\overleftarrow{e_i e_{i+1}}) = 0 \in k$.

c) There exists $e_h \in \mathrm{Ch}(e_{h-1})$ such that $w_0(\overleftarrow{e_{h-1} e_h}) = 0 \in k$.

By definition, the restriction of the canonical morphism $\psi\colon S^{\gamma_0} \to \mathbb{A}^1_X$ to an open subset $S_i = \mathrm{Spec}(A[T_i])$ corresponding to a leaf e_{i,m_i} of Γ at level $m_i \geq 1$ is induced by the section $w_0(\overleftarrow{e_{m_i-1} e_{i,m_i}}) x^{m_i-1} + x^{m_i} T_i$. Thus, by applying the procedure used in the proof of Lemma 3.5 to this comb γ_0, we obtain a fine k-weighted comb $\tilde{\gamma}_0 = (\Gamma, \tilde{w}_0)$ with the same underlying comb Γ as γ_0 such that $\tilde{w}_0(\overleftarrow{e_i e_{i+1}}) = 0 \in k$ for every $i = 0, \dots, h-1$.

4.7. By construction of the tree $\tilde{\gamma}_0$, there exists monic polynomials $P_0, \dots, P_{h-1} \in k[t]$, of degrees $\deg(P_i) = \deg_\Gamma(e_i) - 1$, with simple roots $\tilde{a}_{e,e_i} \in k^*$, $e \in \mathrm{Ch}(e_i) \setminus \{e_{i+1}\}$ respectively, such that $F_{e_i}(\tilde{\gamma}_0) = X_{e_{i-1}} P_i(X_{e_{i-1}})$ for every $i = 0, \dots, h-1$. Letting $y_{-1} = X_{e_{-1}}$, $y_0 = X_{e_0}, \dots, y_{h-2} = X_{e_{h-2}}$, $z = X_{e_{h-1}}$, we conclude that the embedded Danielewski surface $S_{\tilde{\gamma}_0}$ is X-isomorphic to the surface $S_{P_0,\dots,P_{h-1}}$ of Theorem 4.3. This shows that every abstract Danielewski surface $S^\gamma \not\simeq \mathbb{A}^1_X$ defined by a fine k-weighted comb γ is X-isomorphic to a surface $S_{P_0,\dots,P_{h-1}} \subset \mathbb{A}^{h+1}_X$. Thus, to complete the proof of Theorem 4.3, it suffices to show that a surface $S = S_{P_0,\dots,P_{h-1}} = \mathrm{Spec}(B)$ has a trivial Makar-Limanov invariant. A similar argument as in 2.10 shows that $B \otimes_{k[z]} k[z, z^{-1}] \simeq k[z, z^{-1}][y_{h-2}]$. This means equivalently that the projection $\pi_2 = \mathrm{pr}_z\,|_S\colon S \to Z = \mathrm{Spec}(k[z])$ in an $\mathbb{A}^1$-fibration restricting to the trivial line bundle $\mathbb{A}^1_{Z_*} = \mathrm{Spec}(k[z, z^{-1}][y_{h-2}])$ over Z_*. Since the general fibers of the two projections $\pi_1 = \mathrm{pr}_x\,|_S\colon S \to X = \mathrm{Spec}(k[x])$ and $\pi_2\colon S \to Z$ do not coincide, we conclude that S has a trivial Makar-Limanov invariant. This completes the proof of Theorem 4.3.

Remark 4.8. The same argument as in the proof of Proposition 2.15 applied to the fibration π_2 shows that the locally nilpotent derivation $z^h \partial_{y_{h-2}}$ of $B \otimes_{k[z]} k[z, z^{-1}] \simeq k[z, z^{-1}][y_{h-2}]$ extends to a locally nilpotent derivation of B, induced by a triangular $k[z]$-derivation of $k[z][y_{h-2}, \dots, y_{-1}, x]$. This proves that every Danielewski surface S with a trivial Makar-Limanov invariant can be embedded in an affine space $\mathbb{A}^d_k$ in such a way that at least two algebraically independent $\mathbb{G}_{a,k}$-actions on S extend to $\mathbb{G}_{a,k}$-actions on $\mathbb{A}^d_k$.

Nonconjugated $\mathbb{G}_a$-actions on a Danielewski surface. By a result of Daigle [2], all the $\mathbb{G}_{a,k}$-actions on a Danielewski surface $S_{P,1} = \{xz - P(y)\}$ are conjugated under the action of the automorphism group $\mathrm{Aut}(S_{P,1})$ of $S_{P,1}$.

4.9. This means that for every pair of nontrivial locally nilpotent derivations ∂_1 and ∂_2 of $B = \Gamma(S_{P,1}, \mathcal{O}_{S_{P,1}})$, there exists a k-algebra automorphism ϕ of B such that $\phi(\mathrm{Ker}(\partial_1)) = \mathrm{Ker}(\partial_2)$. This implies in particular that the fibers of corresponding quotient $\mathbb{A}^1$-fibrations $\pi_1\colon S_{P,1} \to \mathbb{A}^1_k$ and $\pi_2\colon S_{P,1} \to \mathbb{A}^1_k$ have the same scheme-theoretic structures. By 4.7 above, a Danielewski surface $S = S_{P_0,\ldots,P_{h-1}} = \mathrm{Spec}(B)$ admits two $\mathbb{A}^1$-fibrations $\pi_1\colon S \to X = \mathrm{Spec}(k[x])$ and $\pi_2\colon S \to Z = \mathrm{Spec}(k[z])$. Moreover π_2 restricts to the trivial line bundle over $Z_* = \mathrm{Spec}(k[z, z^{-1}])$, and a similar argument as in Lemma 2.13 shows that the fiber $(\pi_2^{-1}(0))_{\mathrm{red}}$ decomposes as a disjoint union of curves isomorphic to the affine line $\mathbb{A}^1_k$. However, we have the following result.

Lemma 4.10. *If $h \geq 2$, then $\pi_2 : S = S_{P_0,\ldots,P_{h-1}} \to Z$ is not a Danielewski surface over Z.*

Proof. It suffices to show that the intersection of the fiber $\pi_2^{-1}(0)$ with the complement of the fiber $\pi_1^{-1}(0)$ is a nonreduced scheme. By (2.2), the defining ideal I_* of $S \setminus \pi_1^{-1}(0) \simeq \mathbb{A}^1_{X_*}$ in $k[x, x^{-1}][y_{-1}, \ldots, y_{h-2}][z]$ is generated by the polynomials $c_i = y_i - x^{-1} y_{i-1} \prod_{l=0}^{i} P_l(y_{l-1})$, $i = 0, \ldots, h-2$ and $d = z - x^{-1} y_{h-2} \prod_{l=0}^{h-1} P_l(y_{l-1})$. We conclude recursively that there exists a polynomial $R \in k[x, x^{-1}][y_{-1}]$ such that

$$d \equiv z - x^{-h} y_{-1} (P_0(y_{-1}))^h R(y_{-1})$$

modulo $c_0, \ldots, c_{h-2}$. Since the polynomial P_0 is nonconstant (see 4.6),

$$\begin{aligned}\left(S \setminus \pi_1^{-1}(0)\right) \cap \pi_2^{-1}(0) &\simeq \mathrm{Spec}\left(k[x, x^{-1}]\,[y_{-1}, \ldots, y_{h-2}, z]/(I_*, z)\right)\\ &\simeq \mathrm{Spec}\left(k[x, x^{-1}][y_{-1}]/(x^{-h} y_{-1} (P_0(y_{-1}))^h R(y_{-1}))\right)\end{aligned}$$

is clearly nonreduced whenever $h \geq 2$. This completes the proof. □

4.11. The above result implies that if $h \geq 2$, then the degenerate fibers of π_1 and π_2 have different scheme-theoretic structures. Therefore two $\mathbb{G}_{a,k}$-actions on $S_{P_0,\ldots,P_{h-1}}$ with associated quotient $\mathbb{A}^1$-fibrations $\pi_1\colon S \to X$ and $\pi_2\colon S \to Z$ respectively can not be conjugated in the sense of (4.9) above. This leads to the following result.

Theorem 4.12. *A Danielewski surface $S \not\simeq S_{P,1}$ with a trivial Makar-Limanov invariant admits two algebraically independent nonconjugated $\mathbb{G}_{a,k}$-actions.*

As a consequence of this description, we obtain the following characterization of ordinary Danielewski surfaces $S_{P,1}$.

Corollary 4.13. *Let $\pi : S \to X = \mathrm{Spec}(k[x])$, where k is an* arbitrary *field of characteristic zero, be a Danielewski surface with a trivial Makar-Limanov invariant. Then the following are equivalent.*

a) *S admits a free $\mathbb{G}_{a,X}$-action.*

b) *S is isomorphic to a surface $S_{P,1} = \{xz - P(y) = 0\}$ in $\mathbb{A}^3_k = \mathrm{Spec}(k[x, y, z])$, where P is a nonconstant polynomial with* deg *P simple roots.*

c) *The canonical sheaf ω_S is trivial.*

d) *All $\mathbb{G}_{a,k}$-actions on S are conjugated.*

Proof. The equivalence b)⇔d) follows from [2] and the above discussion. All the other equivalences can be obtained in the same way as in Corollary 5.7 in [4]. □

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