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Erratum to "A gap theorem for hypersurfaces with constant scalar curvature one"

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Proof of Lemma 4.1

In our paper [AdCS], the following lemma is stated.

Lemma 4.1. *Let* $M^n \to \mathbb{S}^{n+1}$ *be a closed orientable hypersurface with scalar curvature* $R = 1$ (*i.e.* $S_2 = 0$). *Then the index of the quadratic form*

$$I(f, f) = -\int_M \{f L_1 f + ((n-1)S_1 - 3S_3) f^2\}\, dM$$

is greater than one.

Here S_r, $r = 1, \dots, n$, is the r^{th}-symmetric function of the principal curvatures of the immersion and L_1 is the linearized operator corresponding to the equation of hypersurfaces of $\mathbb{S}^{n+1}$ with $S_2 \equiv 0$.

The proof of Lemma 4.1 presented in [AdCS] is incorrect. We had set $N = \sum_{i=1}^{n+2} n_i e_i$, where $\{e_1, \dots, e_{n+2}\}$ is an orthonormal basis of $\mathbb{R}^{n+2}$ and N is the unit normal vector to the immersion of $M^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$. By assuming the index of I is greater than one and noticing that $I(n_i, n_i) \leq 0$, we concluded that all n_i but one were zero. This is not true. Replace the proof of the lemma by the following.

Proof. To prove that $\operatorname{Ind}(M)$ is greater than one, we will follow Simons [S], where a similar proposition is proved for the minimal case in arbitrary codimension (Proposition 5.1.1 of [S]). The crucial point is that Simons argument in codimension one does not depend on minimality. For convenience of the reader we present here the details.

Let ξ denote the $(n+2)$-dimensional space of vector fields that are tangential projections to $\mathbb{S}^{n+1}$ of parallel vector fields in $\mathbb{R}^{n+2} \supset \mathbb{S}^{n+1}$. Clearly at each point $p \in M$, ξ spans the entire tangent space $T_p(\mathbb{S}^{n+1})$. Let ξ^N be the space of all normal vector fields of the form $\langle a, N\rangle N$, where a is a fixed vector of $\mathbb{R}^{n+2}$ and N is the unit normal of M.

We first show that restricted to ξ^N, the index form is negative definite. Indeed, setting $f_a = \langle a, N\rangle$, $g_a = \langle a, x\rangle$ and noticing that

$$L_1(f_a) = -(S_1S_2 - 3S_3)f_a - 2S_2g_a,$$

we obtain, since $S_2 = 0$, that $L_1(f_a) = 3S_3 f_a$.

Thus,

$$\begin{aligned} I(f_a, f_a) &= -\int_M (f_a L_1(f_a) + [(n-1)S_1 - 3S_3]f_a^2)\, dM \\ &= -\int_M (n-1)S_1 f_a^2\, dM < 0, \end{aligned}$$

since $S_1 > 0$. It follows that the index formula restricted to ξ^N is negative definite, hence $\mathrm{Ind}(M) \geq 1$.

To show that $\mathrm{Ind}(M) > 1$, we need a few lemmas taken from [S]. Let ∇, $\overline{\nabla}$ and $\overline{\overline{\nabla}}$ the covariant derivatives of M^n, $\mathbb{S}^{n+1}$ and $\mathbb{R}^{n+2}$, respectively.

Lemma 1.1. *Let $Z \in \xi$. Then, given $p \in \mathbb{S}^{n+1}$, there exists a λ such that, for any $x \in T_p(\mathbb{S}^{n+1})$,*

$$\overline{\nabla}_x Z = \lambda x. \tag{1}$$

Proof. Since $Z = W^T$, where W is a parallel field in $\mathbb{R}^{n+2}$, we have, by setting $W = W^T + W^N$,

$$\overline{\nabla}_x Z = (\overline{\overline{\nabla}}_x W^T)^T = -(\overline{\overline{\nabla}}_x W^N)^T = \lambda x,$$

because $\mathbb{S}^{n+1}$ is umbilic. □

Lemma 1.2. *Let $Z \in \xi$ and write $Z = Z^T + Z^N$, where Z^N is the projection into $N(M)$ and Z^T is the projection into $T(M)$. Then, for any $x \in T_p(M)$,*

$$\nabla_x^N Z^N = -B(x, Z^T). \tag{2}$$

Proof. By using (1),

$$\begin{aligned} \nabla_x^N Z^N &= (\overline{\nabla}_x Z^N)^N = (\overline{\nabla}_x Z - \overline{\nabla}_x Z^T)^N \\ &= (\lambda x - \overline{\nabla}_x Z^T)^N = -(\overline{\nabla}_x Z^T)^N = -B(x, Z^T). \end{aligned}$$ □

Lemma 1.3. *Assume that $dim\ \xi^N = 1$. Then given $z \in T_p(M)$, there exists $Z \in \xi$ such that $Z(p) = z$ and Z is everywhere tangent to M.*

Proof. Let η be the kernel of the homomorphism $\xi \to \xi^N$. If $Z \in \eta$, then $Z^T = Z$ at every point of M. Now, for some $p \in M$, let β_p be the kernel of the homomorphism $\xi \to N_p(M)$ defined by $Z \mapsto Z^N(p)$. Clearly $\eta \subset \beta_p$. On the other hand, $\dim \beta_p = n+2-1$ and the assumption that $\dim \xi^N = 1$ implies that $\dim \eta = n+2-1$. Thus $\eta = \beta_p$. Since $Z \to Z^T(p)$ maps β_p onto $T_p(M)$, it also maps η onto $T_p(M)$. This implies the claim of the lemma. □

We can now conclude our proof. By using Lemma 1.3 and (2), we obtain

$$B(x, z) = B(x, Z) = B(x, Z^T) = -\nabla_x^N Z^N = 0.$$

Thus $B \equiv 0$ and M is totally geodesic. Since $S_1 > 0$, this is a contradiction and shows that $\mathrm{Ind}(M) > 1$. □

References

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