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On the extraction of roots in exponential A-groups

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Abstract. An exponential A-group is a group which comes equipped with an A-action (A is a commutative ring with unity), satisfying certain axioms. In this paper, we investigate some aspects of root extraction in the category of exponential A-groups. Of particular interest is the extraction of roots in nilpotent R-powered groups. Among other results, we prove that if R is a PID and G is a nilpotent R-powered group for which root extraction is always possible, then the torsion R-subgroup of G lies in the center. Furthermore, if the torsion R-subgroup is finitely R-generated, then G is torsion-free.

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1. Introduction

Let G be a group, n a positive integer, and g an element of G. Then g is said to have an n^{th} *root* if there exists $h \in G$ such that $h^n = g$. An element of G need not have an n^{th} root. On the other hand, there may be elements of G with more than one n^{th} root.

A group G in which every element has an n^{th} root for every integer $n > 0$ is termed a *radicable* or *complete* (or *divisible* when G is abelian) group. Thus, for every $g \in G$ and every integer $n > 0$, there exists $h \in G$ such that $g = h^n$. One can interpret this definition in terms of mappings: G is radicable if and only if the map

$$\psi\colon G \to G \quad \text{defined by} \quad \psi(g) = g^n$$

is surjective for every $n > 0$.

If every element of G has at most one n^{th} root (that is, n^{th} roots are unique when they exist), then G is called an *R-group*. Thus, if $g, h \in G$ and $g^n = h^n$ for some integer $n > 0$, then $g = h$. Put another way, G is an R-group if and only if the mapping ψ defined above is injective.

In [1], Baumslag developed the theory of certain groups containing radicable groups, R-groups, and radicable R-groups as special cases. For each non-empty set of primes ω, he defined the following classes:

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- $\mathcal{E}_\omega$ denotes the class of groups in which p^{th} roots exist for all $p \in \omega$;
- $\mathcal{U}_\omega$ denotes the class of groups in which p^{th} roots are unique (whenever they exist) for all $p \in \omega$;
- $\mathcal{D}_\omega$ denotes the class $\mathcal{E}_\omega \cap \mathcal{U}_\omega$.

As before, these notions can be described in terms of maps. If G is a group and $\psi : G \to G$ is defined as $\psi(g) = g^p$ for some prime $p \in \omega$, then $G \in \mathcal{E}_\omega$ if and only if ψ is surjective, $G \in \mathcal{U}_\omega$ if and only if ψ is injective, and $G \in \mathcal{D}_\omega$ if and only if ψ is bijective. In case ω is the set of all primes, then the classes $\mathcal{E}_\omega$, $\mathcal{U}_\omega$, and $\mathcal{D}_\omega$ are denoted by $\mathcal{E}$, $\mathcal{U}$, and $\mathcal{D}$, respectively. Thus, in the terminology set forth by Baumslag, a $\mathcal{U}$-group is an R-group, an $\mathcal{E}$-group is a radicable group, and a $\mathcal{D}$-group is a radicable R-group.

Our research focuses on the study of the classes $\mathcal{E}_\omega$, $\mathcal{U}_\omega$ and $\mathcal{D}_\omega$ in the category of exponential A-groups, where A is an integral domain and ω is a non-empty set of primes in A. Of particular interest is the category of nilpotent R-powered groups, where R is a binomial ring. The results presented in this paper have been selected from a work in progress by the authors and deal mainly with nilpotent R-powered groups.

We recall the definition of an exponential A-group (see [12]).

Definition. An *exponential A-group* is a group G, equipped with an action by a commutative ring with unity A, such that for all $g \in G$ and for all $\alpha \in A$, the element $g^\alpha \in G$ is uniquely defined and the following axioms hold:

(1) $g^1 = g$, $g^\alpha g^\beta = g^{\alpha+\beta}$, and $(g^\alpha)^\beta = g^{\alpha\beta}$ for all $g \in G$ and $\alpha, \beta \in A$.
(2) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ for all $g, h \in G$ and $\alpha \in A$.
(3) If g and h are commuting elements of G, then $(gh)^\alpha = g^\alpha h^\alpha$ for all $\alpha \in A$.

Examples of exponential A-groups include A-modules, Lyndon's free $\mathbb{Z}[x]$-group [6] and Baumslag's $\mathbb{Q}$-completion of a free group [1]. Categorical notions such as A-subgroup, normal A-subgroup, and A-homomorphism are defined in the obvious way. The interested reader should consult the works of Majewicz ([8] and [10]), and Myasnikov and Remeslennikov [12] for more details.

In this paper, A will always be an integral domain and $\mathcal{M}_A$ will denote the class of exponential A-groups.

A rich collection of exponential A-groups consists of the nilpotent R-powered groups, where R is a binomial ring. Recall that a *binomial ring* R is an integral domain of characteristic zero with unity such that for any $r \in R$ and $k \in \mathbb{Z}^+$,

$$\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!} \in R.$$

The definition of a nilpotent R-powered group is due to Hall (see [3]).

Definition. Let G be a (locally) nilpotent group, and suppose that G is equipped with an action by a binomial ring R, such that for all $g \in G$ and for all $\alpha \in R$, the element $g^\alpha \in G$ is uniquely defined. Then G is termed a *nilpotent R-powered group* if the following axioms hold:

(1) $g^1 = g$, $g^\alpha g^\beta = g^{\alpha+\beta}$ and $(g^\alpha)^\beta = g^{\alpha\beta}$ for all $g \in G$ and $\alpha, \beta \in R$.
(2) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$ for all $g, h \in G$ and for all $\alpha \in R$.
(3) If $\{g_1, \ldots, g_n\} \subset G$ and $\alpha \in R$, then

$$g_1^\alpha \ldots g_n^\alpha = \tau_1(\bar{g})^\alpha \tau_2(\bar{g})^{\binom{\alpha}{2}} \ldots \tau_{k-1}(\bar{g})^{\binom{\alpha}{k-1}} \tau_k(\bar{g})^{\binom{\alpha}{k}},$$

where k is the class of the nilpotent group generated by $\{g_1, g_2, \ldots, g_n\}$ and $\bar{g} = (g_1, \ldots, g_n)$.

Axiom (3) is called the *Hall–Petresco axiom* and the $\tau_i(\bar{g})$'s are the *Hall–Petresco words*. By setting $\alpha = 1$, $\alpha = 2$, and so on, one can compute the Hall–Petresco words:

$$\tau_1(\bar{g}) = g_1 \ldots g_n, \quad \tau_2(\bar{g}) = (g_1 \ldots g_n)^{-2}(g_1^2 \ldots g_n^2), \text{ etc.}$$

By a theorem of Hall [3], each $\tau_i(\bar{g})$ is contained in $\gamma_i(G)$, the i^{th} term of the lower central series of G. This allows one to deduce that a nilpotent R-powered group is, indeed, an exponential R-group.

A well-known example of a nilpotent R-powered group is the Mal'cev completion of a torsion-free locally nilpotent group G, which is a torsion-free locally nilpotent *radicable* group containing G (see [7]). In the terminology set forth, this is just a nilpotent $\mathbb{Q}$-powered group. Other examples of nilpotent R-powered groups can be found in the works of Hall [3], Majewicz [9], and Warfield [14].

From this point on, R will always be a binomial ring and $\mathcal{N}_R$ will denote the class of nilpotent R-powered groups.

In Section 2 we provide some preliminary material on exponential A-groups and nilpotent R-powered groups.

Section 3 contains a selection of results on $\mathcal{U}_\omega$-groups and $\mathcal{E}_\omega$-groups in the class $\mathcal{N}_R$. We begin by proving

Theorem 3.1. *Suppose that $G \in \mathcal{N}_R$, where R contains $\mathbb{Q}$. If $\{g_1, \ldots, g_m\} \subset G$, then the product $g_1^\beta \ldots g_m^\beta$ has a β^{th} root for any $\beta \in R$.*

A well-known theorem of Mal'cev [7] states that if ω is a non-empty set of primes, then every ω-torsion-free nilpotent group is a $\mathcal{U}_\omega$-group. Our next theorem generalizes this result for nilpotent R-powered groups.

Theorem 3.2. *Let ω be a non-empty set of primes in R. A nilpotent R-powered group G is a $\mathcal{U}_\omega$-group if and only if it is ω-torsion-free.*

Next we prove a theorem similar to one due to Baumslag [1]. In [8], Majewicz introduced the notion of a π-primary component of a nilpotent R-powered group. If $G \in \mathcal{N}_R$ and $\pi \in R$ is prime, then the *π-primary component* of G is the set

$$G_\pi = \{g \in G \mid g^{\pi^k} = 1 \text{ for some } k \in \mathbb{Z}^+\}.$$

Theorem 3.3. *If $G \in \mathcal{N}_R$ is an $\mathcal{E}_\omega$-group for some non-empty set of primes ω and $\pi \in \omega$, then G_π is an R-subgroup of $Z(G)$, the center of G.*

A consequence of this theorem which generalizes a result of Černikov (see [2] or [5], p. 234) is

Corollary 3.4. *Let R be a PID. If $G \in \mathcal{N}_R$ is an $\mathcal{E}$-group, then $\tau(G)$ is an R-subgroup of $Z(G)$, where $\tau(G)$ is the torsion R-subgroup of G.*

Another theorem of Černikov that carries over to nilpotent R-powered groups is

Theorem 3.7. *If $G \in \mathcal{N}_R$ is an $\mathcal{E}$-group and $\tau(G)$ is finitely R-generated, then G is torsion-free.*

2. Preliminaries

In this section we provide some elementary results on exponential A-groups and nilpotent R-powered groups.

Notation: If H is an A-subgroup (a normal A-subgroup) of an exponential A-group G, then we write $H \leq_A G$ ($H \trianglelefteq_A G$).

We begin by stating a useful computational lemma. The proof follows from a result of Hall [3] which states that $\tau_i(\bar{g}) \in \gamma_i(G)$ for each $i \geq 1$.

Lemma 2.1. *Let $G \in \mathcal{N}_R$ and $\alpha \in R$. If g_1 and g_2 commute in G, then*

$$(g_1 g_2)^\alpha = g_1^\alpha g_2^\alpha.$$

We remind the reader of some well-known commutator identities.

Lemma 2.2. *Let x, y and z be elements of any group. Then*

$$[xy, z] = y^{-1}[x, z]y[y, z] \quad \textit{and} \quad [x, yz] = [x, z]z^{-1}[x, y]z.$$

Using the Hall–Petresco axiom, one can establish the following identity (see [14], p. 86):

Lemma 2.3. *Let $G \in \mathcal{N}_R$, and suppose that $[g,h] \in Z(G)$ for some $g,h \in G$. Then for any $\mu \in R$,*

$$[g^\mu, h] = [g, h^\mu] = [g,h]^\mu.$$

Using Lemmas 2.2 and 2.3, one can prove

Lemma 2.4. *If $G \in \mathcal{N}_R$ is non-abelian, then there exists an R-homomorphism from G into $Z(G)$ whose image is non-trivial.*

The factor group of an exponential A-group by a normal A-subgroup *need not* be an exponential A-group (see [10] or [12]).

Definition 2.1. Let $G \in \mathcal{M}_A$ and $N \trianglelefteq_A G$. We call N an *ideal* of G if

$$[g,h] \in N \implies h^{-\alpha} g^{-\alpha} (gh)^\alpha \in N \quad \text{for any } g,h \in G \text{ and } \alpha \in A.$$

Lemma 2.5. *If $G \in \mathcal{M}_A$ and N is an ideal of G, then the A-action on G induces an A-action on G/N,*

$$(gN)^\mu = g^\mu N \quad \text{for all } gN \in G/N \text{ and } \mu \in A,$$

which turns G/N into an exponential A-group.

If $G \in \mathcal{N}_R$, then an application of the Hall–Petresco axiom shows that every normal R-subgroup of G is an ideal. It can readily be verified that the isomorphism theorems hold for nilpotent R-powered groups.

Definition 2.2. Let $G \in \mathcal{M}_A$, and let $S = \{s_1, \dots, s_j\}$ be a subset of G. Then

$$H = \bigcap_{S \subset H_i \leq_A G} \{H_i\} = \mathrm{gp}_A(s_1, \dots, s_j)$$

is called the *A-subgroup of G, which is A-generated by $s_1, \dots, s_j$*. We term S a set of *A-generators* for H.

The next theorem can be found in [14], p. 87.

Theorem 2.6. *The upper and lower central subgroups of a nilpotent R-powered group G, denoted by $\zeta_i(G)$ and $\gamma_i(G)$ respectively, are R-subgroups of G.*

Recall that if $\{g_1, \dots, g_n\}$ is a set of elements in a group G, then

$$[g_1, \dots, g_n] = [[g_1, \dots, g_{n-1}], g_n]$$

is a *simple commutator of weight n* (see [13], p. 123).

Lemma 2.7. *If $G \in \mathcal{N}_R$, then*

$$\gamma_n(G) = \text{gp}([g_1, \ldots, g_n] \mid g_i \in G).$$

The next lemma is useful for proving nilpotency by induction on the class.

Lemma 2.8. *If $G \in \mathcal{N}_R$ is of class $c \geq 2$ and $g \in G$, then $H = \text{gp}_R(g, \gamma_2(G))$ is of class at most $c - 1$.*

It is well known that subgroups of finitely generated nilpotent groups are finitely generated. In the case of finitely R-generated nilpotent R-powered groups, this property is inherited by R-subgroups provided that R is a certain type of ring. The next theorem is mentioned in [4].

Theorem 2.9. *If R is a noetherian ring and G is a finitely R-generated nilpotent R-powered group, then every R-subgroup of G is finitely R-generated.*

The notion of a torsion element in an exponential A-group is defined in the obvious way.

Definition 2.3. If $G \in \mathcal{M}_A$, then an element $g \in G$ is called a *torsion element* if there exists a non-zero element $\alpha \in A$ for which $g^\alpha = 1$. The set of torsion elements of G is denoted by $\tau(G)$. We call G a *torsion A-group* if $\tau(G) = G$, and a *torsion-free A-group* if $\tau(G) = 1$.

In [14], p. 87, Warfield proves the next theorem which does not hold for exponential A-groups in general.

Theorem 2.10. *If $G \in \mathcal{N}_R$, then $\tau(G) \trianglelefteq_R G$ and $G/\tau(G)$ is torsion-free.*

From this point on, if $G \in \mathcal{N}_R$, then we refer to $\tau(G)$ as the *torsion R-subgroup* of G.

In the remainder of this paper, ω will always denotes a non-empty set of primes in A and ω' will denote the set of all primes in A which are not in ω.

Definition 2.4. An element $\alpha \in A$ is an *ω-member* if either $\alpha = 1$ or all prime divisors of α are in ω. If $G \in \mathcal{M}_A$, then an element $g \in G$ is an *ω-torsion element* if $g^\alpha = 1$ for some ω-member α. If every element of G is an ω-torsion element, we say that G is an *ω-torsion group*. If the only ω-torsion element of G is the identity, then G is *ω-torsion-free*.

In case $\omega = \{\pi\}$ for a prime $\pi \in A$, we use the terms π-torsion and π-torsion-free.

Theorem 2.11. *If $G \in \mathcal{N}_R$ and $Z(G)$ is ω-torsion-free, then each factor R-group $\zeta_{i+1}(G)/\zeta_i(G)$ is ω-torsion-free. Consequently, G is ω-torsion-free.*

The theorem follows from Lemma 2.3 and induction on i.

Corollary 2.12. *If $G \in \mathcal{N}_R$ is ω-torsion-free, then $G/Z(G)$ is ω-torsion-free.*

Next we provide the definition of a nilpotent R-powered group of finite type introduced by Majewicz in [9].

Definition 2.5. A nilpotent R-powered group is of *finite type* if it is a finitely R-generated torsion R-group.

By Theorem 2.9, if R is a noetherian ring and G is a finitely R-generated nilpotent R-powered group, then $\tau(G)$ is of finite type.

One can understand nilpotent R-powered groups of finite type by examining their π-primary components.

Definition 2.6. Let $G \in \mathcal{N}_R$ and let $\pi \in R$ be prime. The *π-primary component* of G is the set

$$G_\pi = \{g \in G \mid g^{\pi^k} = 1 \text{ for some } k \in \mathbb{Z}^+\}.$$

If $G = G_\pi$, then G is referred to as a *π-primary R-group*. A finitely R-generated π-primary R-group is said to be of *finite π-type*.

Nilpotent R-powered groups of finite π-type are the analogues of p-groups in the category of finite groups.

The following theorem is due to Majewicz and Zyman [11]:

Theorem 2.13. *Let R be a noetherian ring and π a prime in R. Consider the short exact sequence*

$$1 \to H \to G \to G/H \to 1$$

in the category of nilpotent R-powered groups. Then G is of finite type (finite π-type) if and only if H and G/H are both of finite type (finite π-type).

Consequently, every R-subgroup of a nilpotent R-powered group of finite π-type is again of finite π-type when R is noetherian.

The next result can be proven in a similar way to Theorem 3.25 in [14].

Theorem 2.14. *If $G \in \mathcal{N}_R$ and $\pi \in R$ is prime, then $G_\pi \trianglelefteq_R G$.*

The direct product of nilpotent R-powered groups whose classes are bounded can be turned into a nilpotent R-powered group in the obvious way.

In [9], Majewicz proved the following:

Theorem 2.15. *Suppose that R is a PID and G is a torsion R-group. If $\{\pi_i \mid i \in I\}$ is the set of all primes in R, then G is the direct product of the G_{π_i}.*

3. $\mathcal{U}_\omega$-groups and $\mathcal{E}_\omega$-groups in the class $\mathcal{N}_R$

The first theorem is proven in [9] for the ring $\mathbb{Q}[x]$. For completeness, we recreate the proof for any binomial ring containing $\mathbb{Q}$.

Theorem 3.1. *Suppose that R contains $\mathbb{Q}$, and let $G \in \mathcal{N}_R$ be of class c. Let β be any element of R. If $g_1, \dots, g_m \in G$, then there exists $h \in G$ such that $g_1^\beta \dots g_m^\beta = h^\beta$.*

Thus, every element of the form $g_1^\beta \dots g_m^\beta$ in $G \in \mathcal{N}_R$ has a β^{th} root whenever R contains $\mathbb{Q}$.

Proof. The proof is by induction on c. If $c = 1$, then $h = g_1 \dots g_m$ satisfies the theorem by Lemma 2.1.

Suppose that $c > 1$ and assume that the result holds for every nilpotent R-powered group of class less than c. Suppose that $\mathrm{gp}(g_1, \dots, g_m)$ is of class $k \leq c$. By the Hall–Petresco axiom,

$$\begin{aligned} g_1^\beta \dots g_m^\beta &= \tau_1(\bar{g})^\beta \tau_2(\bar{g})^{\binom{\beta}{2}} \dots \tau_{k-1}(\bar{g})^{\binom{\beta}{k-1}} \tau_k(\bar{g})^{\binom{\beta}{k}} \\ &= \tau_1(\bar{g})^\beta [\tau_2(\bar{g})^{j_2}]^\beta \dots [\tau_{k-1}(\bar{g})^{j_{k-1}}]^\beta [\tau_k(\bar{g})^{j_k}]^\beta, \end{aligned}$$

where

$$j_i = \frac{(\beta - 1)(\beta - 2)\dots(\beta - i + 1)}{i!} \quad \text{for } 2 \leq i \leq k.$$

By the comment preceding Lemma 2.1, $\tau_i(\bar{g}) \in \gamma_2(G)$ for each $i = 2, \dots, k$. Consequently, each $\tau_1(\bar{g})$, $\tau_2(\bar{g})^{j_2}$, ..., $\tau_k(\bar{g})^{j_k}$ is contained in $\mathrm{gp}_R(\tau_1(\bar{g}), \gamma_2(G))$ which, by Lemma 2.8, is an R-subgroup of G of class less than c. By induction,

$$\tau_1(\bar{g})^\beta [\tau_2(\bar{g})^{j_2}]^\beta \dots [\tau_{k-1}(\bar{g})^{j_{k-1}}]^\beta [\tau_k(\bar{g})^{j_k}]^\beta = h^\beta$$

for some $h \in \mathrm{gp}_R(\tau_1(\bar{g}), \gamma_2(G))$. Therefore, $g_1^\beta \dots g_m^\beta = h^\beta$. □

A fundamental result of Mal'cev in the theory of nilpotent groups is that every torsion-free nilpotent group admits unique root extraction whenever roots exist (see [7]). This result carries over to nilpotent R-powered groups.

Theorem 3.2. *A nilpotent R-powered group G is a $\mathcal{U}_\omega$-group if and only if it is ω-torsion-free.*

Proof. Suppose that G is an ω-torsion-free group of class c. We prove that G is a $\mathcal{U}_\omega$-group by induction on c. If $c = 1$, there is nothing to prove.

Let $c > 1$, and suppose $g, h \in G$ such that $g^\pi = h^\pi$ for some $\pi \in \omega$. By Corollary 2.12, $G/Z(G)$ is ω-torsion-free. Hence, by induction, there exists an element $z \in Z(G)$ such that $g = hz$. Lemma 2.1 yields

$$g^\pi = (hz)^\pi = h^\pi z^\pi = g^\pi z^\pi.$$

Consequently, $z^\pi = 1$. Since G is ω-torsion-free, $z = 1$; that is, $g = h$.

Conversely, if G is a $\mathcal{U}_\omega$-group and $g \in G$, then $g^\pi = 1 = 1^\pi$ for some $\pi \in \omega$ implies $g = 1$. Thus, G is ω-torsion-free. □

We remark that every exponential A-group which is a $\mathcal{U}_\omega$-group is ω-torsion-free, but not every ω-torsion-free exponential A-group is a $\mathcal{U}_\omega$-group.

In [2], Černikov proved that the torsion elements of a complete ZA-group G lie in $Z(G)$ (see [5], p. 234). A similar result holds for nilpotent R-powered groups in the class $\mathcal{E}$. We establish this by first proving a generalization of a theorem of Baumslag [1].

Theorem 3.3. *If $G \in \mathcal{N}_R$ is an $\mathcal{E}_\omega$-group of class c and $\pi \in \omega$, then $G_\pi \leq_R Z(G)$.*

Proof. The bulk of the proof rests on proving that $G/Z(G) \in \mathcal{U}_\omega$ or equivalently, by Theorem 3.2, that $G/Z(G)$ is ω-torsion-free. Let $g \in G$, $g \neq 1$, and let $\pi \in \omega$ such that $g^\pi \in Z(G)$. There exists an integer k, $0 \leq k < c$, such that $g \notin \zeta_k(G)$ and $g \in \zeta_{k+1}(G)$. We claim that $k = 0$; that is, $g \in Z(G)$.

Let h be any element of G. If $k = 0$, we are done. Assume $k = 1$, and suppose $h_0 \in G$ is a π^{th} root of h. Then $[g, h] \in Z(G)$ because $g \in \zeta_2(G)$. Thus, by Lemma 2.3,

$$[g, h] = [g, h_0^\pi] = [g^\pi, h_0] = 1.$$

Therefore, $g \in Z(G)$, a contradiction. Hence k cannot be 1.

Suppose that $k > 1$, and assume that the set

$$S = \{\tilde{g} \in \zeta_i(G) \mid \tilde{g}^{\pi^n} \in Z(G) \text{ for some } n \in \mathbb{Z}^+\}$$

is contained in $Z(G)$ for $1 < i \leq k$. Notice that $g^\pi \in Z(G)$ implies both $(g^{-1}Z(G))^\pi = Z(G)$ and $(h^{-1}ghZ(G))^\pi = Z(G)$ in $G/Z(G)$. Hence, by Theorem 2.14, $[g, h]^{\pi^m} Z(G) = Z(G)$ in $G/Z(G)$ for some integer $m \geq 0$; that is, $[g, h]^{\pi^m} \in Z(G)$. Since $g \in \zeta_{k+1}(G)$, $[g, h] \in \zeta_k(G)$ and so $[g, h] \in S$. Hence, $[g, h] \in Z(G)$. Since this holds for all $h \in G$, we have $g \in \zeta_2(G)$, which implies $g \in Z(G)$ as before. This contradicts the assumption that $g \notin \zeta_k(G)$ and $k > 1$. We conclude that $k \not> 1$, so we must have $k = 0$, as claimed.

We have established that $g^\pi \in Z(G)$ implies $g \in Z(G)$. To complete the proof, observe that if $g \in G_\pi$, then there exists an integer $t \geq 0$ such that $g^{\pi^t} = 1$. Therefore, $g^{\pi^t} \in Z(G)$ and, consequently, $g \in Z(G)$. □

Our analogue of Černikov's result is:

Corollary 3.4. *If R is a PID and $G \in \mathcal{N}_R$ is an $\mathcal{E}$-group, then $\tau(G) \leq_R Z(G)$.*

Proof. This follows from Theorems 2.15 and 3.3. □

Next we prove that if R is a PID and a nilpotent R-powered group in $\mathcal{E}$ has a finitely R-generated torsion R-subgroup, then the torsion R-subgroup must be trivial. This generalizes another result due to Černikov (see [5], p. 235). First we mention an easy generalization of a well-known fact about abelian groups.

Lemma 3.5. *If R is a PID and G is a non-trivial divisible abelian R-group, then G is not finitely R-generated.*

Another useful result is

Lemma 3.6. *If $G \in \mathcal{N}_R$ is an $\mathcal{E}$-group, then any R-homomorphic image of G is also an $\mathcal{E}$-group.*

Theorem 3.7. *Let R be a PID, and suppose $G \in \mathcal{N}_R \cap \mathcal{E}$. If $\tau(G)$ is finitely R-generated, then $\tau(G) = 1$.*

The proof of the theorem further shows that G has an ascending central R-series, all of whose factors are divisible abelian torsion-free R-groups.

Proof. Suppose that G is a divisible abelian R-group and $\tau(G)$ is finitely R-generated, and assume $\tau(G) \neq 1$. Let $g \in \tau(G)$, $g \neq 1$, satisfy $g^\alpha = 1$ for some $\alpha \in R$. If $\mu \in R$, then there exists $h \in G$ such that $g = h^\mu$. Since $h^{\mu\alpha} = (h^\mu)^\alpha = g^\alpha = 1$, it follows that $h \in \tau(G)$. Thus, $\tau(G)$ is divisible, contradicting Lemma 3.5.

Next let $G \in \mathcal{E}$ be a non-abelian nilpotent R-powered group and suppose that $\tau(G)$ is finitely R-generated. We claim that G has a strictly ascending central R-series

$$H_1 < H_2 < \cdots < H_i < \cdots$$

satisfying

(1) $H_i \cap \tau(G) = 1$;
(2) $\tau(G/H_i) = \tau(G)H_i/H_i$ and is finitely R-generated;
(3) H_{i+1}/H_i is a divisible torsion-free abelian R-group.

To begin, we show that H_1 exists and satisfies (1) and (2). By Lemmas 2.4 and 3.6, there exist non-trivial R-homomorphic images of G in $Z(G)$ which are $\mathcal{E}$-groups. Let $H_1 \in \mathcal{E}$ be one such R-subgroup of $Z(G)$. Then H_1 is abelian and $\tau(H_1) <_R G$ is finitely R-generated by Theorem 2.9. Hence,

$$H_1 \cap \tau(G) = \tau(H_1) = 1$$

and (1) holds. Next we prove that $\tau(G/H_1) = \tau(G)H_1/H_1$ and is finitely R-generated, establishing (2). Observe that if $gH_1 \in \tau(G/H_1)$, then there exists $\alpha \in R$ such that $(gH_1)^\alpha = H_1$; that is, $g^\alpha \in H_1$. Since $H_1 \in \mathcal{E}$, there exists $k \in H_1$ such that $g^\alpha = k^\alpha$. By Lemma 2.1, $(gk^{-1})^\alpha g^\alpha k^{-\alpha} = 1$ because $k \in Z(G)$. Thus,

$gk^{-1} \in \tau(G)$ and $gH_1 = (gk^{-1})H_1 \in \tau(G)H_1/H_1$. Now, by the R-isomorphism theorems,

$$\tau(G)H_1/H_1 \cong_R \tau(G)/(\tau(G) \cap H_1) = \tau(G).$$

Since $\tau(G)$ is finitely R-generated by hypothesis, so is $\tau(G/H_1)$.

Next we concoct an R-subgroup H_k, assuming that H_{k-1} has been constructed and satisfies (1)–(3). Notice that $G/H_{k-1} \in \mathcal{E}$ by Lemma 3.6 and $\tau(G/H_{k-1})$ is finitely R-generated by Theorem 2.9. By Lemmas 2.4 and 3.6, there exists a non-trivial normal R-subgroup H_k/H_{k-1} of $Z(G/H_{k-1})$ which is an $\mathcal{E}$-group. Using the same argument as before and the fact that $\tau(G/H_{k-1}) = \tau(G)H_{k-1}/H_{k-1}$, we have

$$(\tau(G)H_{k-1}/H_{k-1}) \cap (H_k/H_{k-1}) = H_{k-1}.$$

Thus,

$$(\tau(G)H_{k-1}) \cap H_k = H_{k-1}.$$

Since $H_{k-1} \cap \tau(G) = 1$, we also have $H_k \cap \tau(G) = 1$. Moreover,

$$\tau((G/H_{k-1})/(H_k/H_{k-1})) = (\tau(G/H_{k-1}) \cdot (H_k/H_{k-1}))/(H_k/H_{k-1}),$$

from which it follows that $\tau(G/H_k) = \tau(G)H_k/H_k$. Hence, H_k satisfies the required properties.

The ascending central series $\{H_k\}$ must reach G in a finite numbers of steps since G is nilpotent. Thus, there exists $n > 0$ such that $H_n = G$. Consequently, $G \cap \tau(G) = 1$; that is, $\tau(G) = 1$. □

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