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Tangent bundle embeddings of manifolds in Euclidean space

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Abstract. For a given n-dimensional manifold M^n we study the problem of finding the smallest integer $N(M^n)$ such that M^n admits a smooth embedding in the Euclidean space $\mathbb{R}^N$ without intersecting tangent spaces. We use the Poincaré–Hopf index theorem to prove that $N(\mathbb{S}^1) = 4$, and construct explicit examples to show that $N(\mathbb{S}^n) \leq 3n + 3$, where $\mathbb{S}^n$ denotes the n-sphere. Finally, for any closed manifold M^n, we show that $2n + 1 \leq N(M^n) \leq 4n + 1$.

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1. Introduction

Every C^1-immersion $f\colon M^n \to \mathbb{R}^N$, where M^n is an n-manifold and $\mathbb{R}^N$ is the Euclidean N-space, induces a mapping of the tangent bundle TM via the differential map $df : TM \to \mathbb{R}^N$. We say that f is a *tangent bundle embedding*, or a *T-embedding* for short, provided that df is one-to-one. In other words, a submanifold of Euclidean space is T-embedded provided that it has no pairs of intersecting tangent spaces. The aim of this note is to begin the study of and call attention to the following basic question:

Problem 1.1. *For a given manifold M^n, what is the smallest integer $N(M^n)$ such that M^n admits a T-embedding in $\mathbb{R}^N$?*

The above problem may be regarded as a generalization of the investigations conducted in the 1940's by H. Whitney [12], culminating in his celebrated theorem that every n-manifold may be embedded in $\mathbb{R}^{2n}$. The prime stimulus for this work, however, is due to the recent renewed interest in studying global geometry of tangent

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lines of closed curves and knots ([1], [2], [4], [11], [13]), see Note 1.8. Our first result shows that the circle $\mathbb{S}^1$ admits no T-embedding in $\mathbb{R}^3$.

Theorem 1.2. *Every closed C^1-immersed curve in $\mathbb{R}^3$ has uncountably many pairs of intersecting tangent lines.*

The proof of Theorem 1.2, which we present in Section 2, follows from the Poincaré–Hopf index theorem (Proposition 2.3) together with a perturbation argument for bitangent planes (Proposition 2.4). The remaining results of this paper, which are proved in Section 3, are concerned mainly with some explicit constructions of T-embeddings:

Theorem 1.3. *Every torus $\mathbb{T}^n$ admits a smooth T-embedding in $\mathbb{R}^{4n}$ given by*

$$\mathbb{C}^n \supset \mathbb{T}^n \ni (z_1, \ldots, z_n) \longmapsto (z_1, z_1^2, \ldots, z_n, z_n^2) \in \mathbb{C}^{2n}.$$

In particular, there exists a T-embedded closed curve in $\mathbb{R}^4$.

Where $\mathbb{C}^n$ denotes the complex n-space. The previous two theorems solve Problem 1.1 for the case $M^n = \mathbb{S}^1$:

Corollary 1.4. $N(\mathbb{S}^1) = 4$. □

Another class of T-embeddings may be constructed using cubic curves, and exploiting the fact (Proposition 3.2) that T-embeddings are preserved under cartesian product:

Theorem 1.5. *Let $f : M^n \to \mathbb{R}^N$ be any C^1-embedding, and f_i, $i = 1, \ldots, N$ be the components of f. Then*

$$M \ni p \longmapsto \big(f_1(p), f_1^2(p), f_1^3(p), \ldots, f_N(p), f_N^2(p), f_N^3(p)\big) \in \mathbb{R}^{3N}$$

is a T-embedding. In particular, $N(\mathbb{R}^n) \leq 3n$ and $N(\mathbb{S}^n) \leq 3n + 3$.

Since every planar curve has intersecting tangents, the above theorem immediately yields that $N(\mathbb{R}^1) = 3$. Further, note that the above result, via Whitney's $2n$-embedding theorem [12], implies that any n-manifold admits a T-embedding in $\mathbb{R}^{6n}$. Working a bit harder, via successive projections into subspaces of lower dimension, we obtain:

Corollary 1.6. $N(M^n) \leq 4n + 1$, *for any manifold M^n.*

Using Thom's transversality theorem, it can be shown that the above result is in fact a generic property [5]. That is, any immersion of M^n in $\mathbb{R}^{4n+1}$ can be turned into a T-embedding by an arbitrarily small perturbation. In Section 3, we also obtain a lower bound:

Theorem 1.7. *$N(M^n) \geq 2n + 1$, for any non-contractible manifold M^n.*

Note that since $\dim(TM^n) = 2n$, we trivially have $N(M^n) \geq 2n$ for all manifolds. The last theorem improves this lower bound for manifolds whose homotopy type is different from that of a point. In particular, when M^n is a compact manifold without boundary, $N(M^n) \geq 2n + 1$. However, the author does not know if the non-contractibility assumption in Theorem 1.7 is necessary. More generally, the author does not know if the estimates in the last two results can be improved. But it would be reasonable to conjecture that $N(M^n) \leq 4n$.

Note 1.8 (Terminology and some history). The class of mappings we study in this paper, the T-embeddings, are not to be confused with *skew* immersions, or S-immersions ([1], [2], [4], [11], [13], [8]), which are defined as immersions without any pairs of *p*arallel tangent lines. The first proof of the existence of an S-embedding of a circle, or skew loop, in $\mathbb{R}^3$ is due to B. Segre [8]. For an explicit formula for such a curve see [2]; there skew loops where used to solve Wente's shadow problem which is related to stability questions concerning surfaces of constant mean curvature [3]. Skew loops are also of interest due to their connection with quadric surfaces: the author and B. Solomon [4] showed that the absence of skew loops characterizes ellipsoids, and S. Tabachnikov [11] has ruled out the existence of skew loops on any quadric surface.

One may also introduce a notion of *totally skew* embedding, or TS-embedding [5], which is defined as an embedding which is both a T-embedding and an S-embedding; an example is the cubic curve $x \longmapsto (x, x^2, x^3)$, and another example is given by Proposition 3.1. Though in this paper we confine our attention primarily to T-embeddings, Problem 1.1 can be stated for S-embeddings and TS-embeddings as well. The case of TS-embeddings will be studied in [5], and is related to the existence of nonsingular bilinear maps, and the "generalized vector field problem". The present work has a different flavor which is in part due to the fact that, unlike T-embeddings, TS-embeddings are not preserved under cartesian product (see Note 3.3).

2. Proof of Theorem 1.2

The basic idea for the proof of Theorem 1.2 is as follows. Let $C \subset \mathbb{R}^3$ be (the image of) a closed C^1-immersed curve. For every unit vector $u \in \mathbb{S}^2$, let $\Pi(u, \lambda)$ be the

plane which is orthogonal to u and passes through λu; that is,

$$\Pi(u, \lambda) := \{\, x \in \mathbb{R}^3 : \langle x - \lambda u, u\rangle = 0 \,\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product. Let λ_u be the infimum of all λ such that $\Pi(u, \lambda)$ is disjoint from C. Then $\Pi_u := \Pi(u, \lambda_u)$ is tangent to C. Let L_u be the collection of all tangent lines of C which lie in Π_u. We claim that, for some $u \in \mathbb{S}^2$, L_u must contain a pair of intersecting lines. Suppose not. Then, for each $u \in \mathbb{S}^2$, let ℓ_u be a line which passes through u and is parallel to the elements of L_u. Since Π_u is orthogonal to u, this yields a tangent line field on $\mathbb{S}^2$. As we show in Lemma 2.1 below, $u \mapsto \ell_u$ is continuous. But it follows from the Poincaré–Hopf index theorem (Lemma 2.2), that there are no continuous line fields on the sphere. Hence we obtain a contradiction. So at least one of the planes Π_u must have contained a pair of intersecting tangent lines. A perturbation argument (Proposition 2.3) applied to Π_u then yields the existence of infinitely many pairs of intersecting tangent lines in nearby bitangent planes, and completes the proof.

To proceed more formally, let $c\colon \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^3$ be a C^1 unit speed curve, i.e., a C^1 mapping with $\|c'\| = 1$. For any $u \in \mathbb{S}^2$, let $t_u \in \mathbb{S}^1$ be a maximum point of $t \mapsto \langle c(t), u\rangle$, and set $\ell(u) := \{\pm c'(t_u)\}$.

Lemma 2.1. *Suppose that $c'(t_1) = \pm c'(t_2)$ whenever t_1 and t_2 are maximum points of $t \mapsto \langle c(t), u\rangle$. Then $\ell\colon \mathbb{S}^2 \to \mathbb{S}^2/\{\pm 1\} \simeq \mathbb{RP}^2$ is well-defined and continuous.*

Proof. Our hypothesis, together with the definition of ℓ, implies that

$$\ell(u) = \{\pm c'(t_u)\} \iff \langle c(t), u\rangle \le \langle c(t_u), u\rangle, \quad \text{for all } t \in \mathbb{S}^1. \tag{1}$$

Thus ℓ is well-defined. To prove the continuity of ℓ, let $u_i \in \mathbb{S}^2$ be a sequence, and t_{u_i} be a maximum point of $t \mapsto \langle c(t), u_i\rangle$. Then

$$\ell(u_i) = \{\pm c'(t_{u_i})\}. \tag{2}$$

Since $\mathbb{S}^1$ is compact, t_{u_i} has a limit point $\bar{t}$. Thus, since c is C^1, $c'(t_{u_i})$ has a limit point at $c'(\bar{t})$. So, (2) yields that

$$\{\pm c'(\bar{t})\} \text{ is a limit point of } \ell(u_i). \tag{3}$$

Next note that (1) and (2) imply

$$\langle c(t), u_i\rangle \le \langle c(t_{u_i}), u_i\rangle. \tag{4}$$

Now let $\bar{u}$ be a limit point of u_i. Then, since $\langle \cdot, \cdot \rangle$ is continuous, (4) implies

$$\langle c(t), \bar{u}\rangle \le \langle c(\bar{t}), \bar{u}\rangle.$$

Consequently, (1) yields

$$\ell(\bar{u}) = \{\pm c'(\bar{t})\}.$$

So, by (3), $\ell(\bar{u})$ is a limit point of $\ell(u_i)$; therefore, we conclude that ℓ is continuous. □

We also need to recall the following well-known fact:

Lemma 2.2. *There exists no continuous tangent line fields on* $\mathbb{S}^2$.

Proof. By a generalization of the Poincaré–Hopf theorem on vector fields [10], the sum of the index of the singularities of a line field on a 2-manifold M is equal to the Euler characteristic $\chi(M)$. In particular, if M has a line field which is defined everywhere, i.e., it has no singularities, then we must have $\chi(M) = 0$, which rules out $\mathbb{S}^2$. □

Combining Lemmas 2.1 and 2.2, we obtain:.

Proposition 2.3. *For every* C^1 *immersed curve* $c\colon \mathbb{S}^1 \to \mathbb{R}^3$, *there exists a plane* $\Pi \subset \mathbb{R}^3$ *such that*

(1) $c(\mathbb{S}^1)$ *lies entirely on one side of* Π

(2) c *has a pair of tangent lines in* Π *which intersect transversely*

In particular, there exist t_0, $s_0 \in \mathbb{S}^1$, *such that* $c(t_0)$, $c(s_0) \in \Pi$ *and* $c'(t_0) \times c'(s_0) \neq 0$.

Proof. After a reparametrization, we may assume that c has unit speed. For each $u \in \mathbb{S}^2$, let t_u be a maximum point of $t \mapsto \langle c(t), u\rangle$, and let Π_u be the plane given by

$$\Pi_u := \{x \in \mathbb{R}^3 : \langle x, u\rangle = \langle c(t_u), u\rangle\}.$$

Note that t_u is a maximum point of $t \mapsto \langle c(t), u\rangle$ if and only if $c(t_u) \in \Pi_u$.

Suppose now that c has no pairs of intersecting tangent lines in Π_u. Then at all the maximum points t_u of $t \mapsto \langle c(t), u\rangle$, the tangent vectors $c'(t_u)$ should be equal up to a sign. So, by Lemma 2.1, the mapping $\ell\colon \mathbb{S}^2 \to \mathbb{S}^2/\{\pm 1\}$, given by $\ell(u) := \{\pm c'(t_u)\}$, is well defined and continuous. But $\langle c'(t_u), u\rangle = 0$, so ℓ determines a tangent line field on $\mathbb{S}^2$, given by

$$L_u := \{\, u + \lambda v : v \in \ell(u), \text{ and } \lambda \in \mathbb{R}\,\}.$$

This contradicts Lemma 2.2 and thus completes the proof. □

Next we show that it is possible to perturb the plane Π in the previous proposition to produce infinitely many pairs of intersecting tangent lines. For every $t \in \mathbb{S}^1$ and $\epsilon \in \mathbb{R}$ we define

$$U_\epsilon(t) := \{\, x \in \mathbb{S}^1 : \operatorname{dist}_{\mathbb{S}^1}(x, t) < \epsilon\,\},$$

where $\mathrm{dist}_{\mathbb{S}^1}$ is the intrinsic distance in $\mathbb{S}^1$. The end points of the above interval are denoted by $\partial U_\epsilon(t)$.

Proposition 2.4. *Let $c\colon U_\eta(t_0) \cup U_\delta(s_0) \to \mathbb{R}^3$ be a C^1 immersion for some t_0, $s_0 \in \mathbb{S}^1$, $t_0 \neq s_0$, and $\eta, \delta > 0$. Suppose that there exists a plane $\Pi \subset \mathbb{R}^3$ such that*

(1) *$c(U_\eta(t_0) \cup U_\delta(s_0))$ lies on one side of Π;*

(2) *$c(t_0)$, $c(s_0) \in \Pi$;*

(3) *$c'(t_0) \times c'(s_0) \neq 0$;*

(4) *For every $0 < \alpha \leq \delta$, there is a $0 < \beta < \alpha$ such that $c(\partial U_\beta(s_0)) \cap \Pi = \emptyset$.*

Then there exists an $0 < \epsilon \leq \eta$ with the following property: For every $t \in U_\epsilon(t_0)$ there exists an $s \in U_\delta(s_0)$ such that the tangent lines of c at $c(t)$ and $c(s)$ intersect transversely.

Proof. Since $(t, s) \mapsto c'(t) \times c'(s)$ is continuous and $c'(t_0) \times c'(s_0) \neq 0$, we may choose ϵ small enough so that

$$c'(t) \times c'(s) \neq 0, \quad \text{for all } (t, s) \in U_\epsilon(t_0) \times U_\epsilon(s_0). \tag{5}$$

In particular, $U_\epsilon(t_0) \cap U_\epsilon(s_0) = \emptyset$. Further, item 4 above implies that we may assume

$$c(\partial U_\epsilon(s_0)) \cap \Pi = \emptyset, \tag{6}$$

where ∂U denotes the endpoints of U.

Now let ℓ_x denote the tangent line of c at $c(x)$. Using (6), and the assumption that $c'(t_0) \times c'(s_0) \neq 0$, we may rotate Π around ℓ_{t_0} to obtain a new tangent plane, say Π_{t_0}, such that

$$\Pi_{t_0} \text{ separates } c(s_0) \text{ from } c(\partial U_\epsilon(s_0)). \tag{7}$$

That is, $c(s_0)$ lies in one of the open half-spaces determined by Π_{t_0} while $c(\partial U_\epsilon(s_0))$ lies in the other open half-space.

Let n_0 denote a unit normal to Π_{t_0}, and $n\colon U_\epsilon(t_0) \to \mathbb{S}^2$ be any continuous unit normal vector field of c with $n(t_0) = n_0$. For every $t \in U_\epsilon(t_0)$, let Π_t be the plane which pass through $c(t)$ and is perpendicular to $n(t)$. Then, by continuity of $t \mapsto \Pi_t$, (7) implies that there exists $0 < \epsilon' \leq \epsilon$ such that

$$\Pi_t \text{ separates } c(s_0) \text{ from } c(\partial U_\epsilon(s_0)),$$

for all $t \in U_{\epsilon'}(t_0)$.

So, for each $t \in U_{\epsilon'}(t_0)$, we may rotate Π_t around ℓ_t until we obtain a plane, say Π'_t, which is tangent to $c(U_\epsilon(s_0))$. Then, in addition to ℓ_t, Π'_t will contain another tangent line ℓ_s, for some $s \in U_\epsilon(s_0)$. Thus (5) implies that ℓ_t and ℓ_s intersect transversely, which completes the proof. □

The above propositions now yield:

Proof of Theorem 1.2. For any C^1 immersed closed curve $c\colon \mathbb{S}^1 \to \mathbb{R}^3$, Proposition 2.3 ensures that all the conditions of Proposition 2.4 are satisfied except possibly Condition 4. If Condition 4 is also satisfied, then Proposition 2.4 yields that c has uncountably many intersecting tangent lines. On the other hand, if Condition 4 is not satisfied, then it follows that c maps an open neighborhood of $\mathbb{S}^1$ into Π. In particular, c is tangent to Π uncountably often. But this again results in uncountably many intersecting tangents, because Π contains a pair of nonparallel tangent lines. □

Note 2.5. It is comparatively easy to prove the existence of intersecting tangents for knotted curves $c\colon \mathbb{S}^1 \to \mathbb{R}^3$. To see this let d be the maximum distance of $c(\mathbb{S}^1)$ from the origin $o := (0, 0, 0)$, and $S \subset \mathbb{R}^3$ be a sphere of radius $r > d$ centered at o. Then, for each $t \in \mathbb{S}^1$, the ray

$$R_t := \{c(t) + \lambda c'(t) : \lambda \geq 0\}$$

has a point inside and a point outside of S. So R_t intersects S at a unique point, say $c(t) + \lambda_t c'(t)$. Since c is continuous, $t \mapsto \lambda_t$ is continuous as well. Thus

$$\bar{c} := c(t) + \lambda_t c'(t)$$

gives a closed curve in S. Suppose that c has no pairs of intersecting tangents. Then

$$R_t \cap R_s = \emptyset \tag{8}$$

for all $t \neq s$. Consequently $\bar{c}$ is one-to-one. So, by Jordan's curve theorem, $\bar{c}(\mathbb{S}^1)$ bounds an embedded disk, and is therefore unknotted. Now define $h\colon \mathbb{S}^1 \times [0, 1] \to \mathbb{R}^3$ by

$$h(t, s) := c(t) + s\lambda_t c'(t).$$

Clearly h is continuous. Further, (8) implies that h is one-to-one. Thus h gives an isotopy between $c(\mathbb{S}^1)$ and $\bar{c}(\mathbb{S}^1)$, which is a contradiction. So we conclude that c must have a pair of intersecting tangent lines

Note 2.6. The set of all tangent lines to a C^1 curve $c\colon \mathbb{S}^1 \to \mathbb{R}^3$ determines a surface parametrized by $c(t) + sc'(t)$, and called the *tangent developable* of c. Theorem 1.2 implies that the tangent developable of any closed C^1 curve in $\mathbb{R}^3$ has infinitely many double points. The multiple points and singularities of the tangent developable of "*generic*" curves have been studied by a number of authors. For a survey of results of this type and references see [7].

3. Proof of other results and examples

In the previous section we established the nonexistence of closed T-embedded curves in $\mathbb{R}^3$. The following result shows that such curves may be constructed in $\mathbb{R}^4$.

Proposition 3.1. *The mapping* $c\colon \mathbb{S}^1 \to \mathbb{R}^4$*, given by*

$$\mathbb{C} \supset \mathbb{S}^1 \ni z \overset{c}{\longmapsto} (z, z^2) \in \mathbb{C}^2,$$

is a TS-embedding.

Proof. We may parametrize c as

$$c(t) := (\cos t, \sin t, \cos 2t, \sin 2t).$$

First we verify the skewness. To see this let

$$T(t) := \frac{c'(t)}{\|c'(t)\|} = \frac{1}{\sqrt{3}}(-\sin t, \cos t, -2\sin 2t, 2\cos 2t).$$

It is enough to check that $T(t) \neq \pm T(s)$, unless $t \equiv s \pmod{2\pi}$. Suppose $T(t) = T(s)$. Then $\sin t = \sin s$, and $\cos t = \cos s$. This yields $t \equiv s \pmod{2\pi}$. Next suppose that $T(t) = -T(s)$. Then $\sin t = -\sin s$, and $\cos t = -\cos s$. This yields

$$t \equiv s + \pi \pmod{2\pi}. \tag{9}$$

But we also have $\sin 2t = -\sin 2s$, and $\cos 2t = -\cos 2s$. So

$$2t \equiv 2s + \pi \pmod{2\pi}. \tag{10}$$

Subtracting (9) from (10) we get $t \equiv s \pmod{2\pi}$. So c is skew.

To see that c is totally skew, note that

$$c(t) = \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & \cos 2t & -\sin 2t \\ 0 & 0 & \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

That is, c is the image of a one parameter subgroup of the special orthogonal group SO(4) acting on $c(0)$. In particular, c is invariant under such rotations. Thus to show that the tangent lines of c do not intersect, it is enough to check that the tangent line of c at $c(0)$ does not intersect any other tangent line. Let

$$\begin{aligned} \ell_t(s) &:= c(t) + sc'(t) \\ &= (\cos t - s\sin t,\ \sin t + s\cos t,\ \cos 2t - 2s\sin 2t,\ \sin 2t + 2s\cos 2t\,). \end{aligned}$$

be a parametrization for the tangent line of c at $c(t)$. Setting $\ell_t(s_1) = \ell_0(s_2)$ yields:

$$\left.\begin{aligned} \cos t - s_1 \sin t &= 1, \\ \sin t + s_1 \cos t &= s_2, \\ \cos 2t - 2s_1 \sin 2t &= 1, \\ \sin 2t + 2s_1 \cos 2t &= 2s_2. \end{aligned}\right\} \tag{11}$$

Eliminating s_1 from the first and third equations, we get

$$2 \sin 2t \cos t - \sin t \cos 2t = 2 \sin 2t - \sin t.$$

Using the identities $\sin 2t = 2 \sin t \cos t$ and $\cos 2t = 1 - 2 \sin^2 t$, we may rewrite the above equation as

$$2 \sin t \, (1 - \cos t)^2 = 0.$$

The solutions to this equation, modulo 2π, are $t = 0$ and $t = \pi$. But a quick examination of (11) reveals that only $t = 0$ satisfies all the equations. Thus the tangent line ℓ_0 is disjoint from all other tangent lines of c, and we conclude that c is totally skew. □

Proposition 3.2. *If $M_1 \subset \mathbb{R}^{n_1}$ and $M_2 \subset \mathbb{R}^{n_2}$ are T-embedded submanifolds, then so is $M_1 \times M_2 \subset \mathbb{R}^{n_1+n_2}$.*

Proof. Suppose, towards a contradiction, that $M_1 \times M_2$ has intersecting tangent lines. Then there are distinct points p, q in $M_1 \times M_2$ and vectors $v \in T_p(M_1 \times M_2)$ and $w \in T_q(M_1 \times M_2)$, such that

$$p + v = q + w.$$

Note that $p = (p_1, p_2)$ and $q = (q_1, q_2)$ where p_i, q_i are points in M_i, $i = 1, 2$. Further, $v = (v_1, v_2)$ and $w = (w_1, w_2)$ where $v_i \in T_{p_i} M_i$ and $w_i \in T_{q_i} M_i$. Thus the above equality implies that

$$p_1 + v_1 = q_1 + w_1 \quad \text{and} \quad p_2 + v_2 = q_2 + w_2.$$

Since, by assumption, $p \neq q$, we must have $p_1 \neq p_2$ or $q_1 \neq q_2$. Thus the above equalities imply that M_1 or M_2 must have a pair of intersecting tangent lines respectively. □

Note 3.3. In contrast to the previous proposition, S-embeddings are not closed under cartesian product. Indeed, the cartesian product of any two submanifolds will always have parallel tangent lines. To see this let $M_1 \subset \mathbb{R}^{n_1}$ and $M_2 \subset \mathbb{R}^{n_2}$ be a pair of submanifolds, and $p \in M_1$, $v \in T_p M_1$, and $q, r \in M_2$. Then $(v, 0) \in T_{(p,q)} M_1 \times M_2$, and $(v, 0) \in T_{(p,r)} M_1 \times M_2$. Thus $M_1 \times M_2$ has parallel tangent vectors at all pairs of points (p, r) and (p, q).

Combining the previous two propositions immediately yields Theorem 1.3. Further, the last proposition also yields:

Proof of Theorem 1.5. Any submanifold of a T-embedded submanifold is also T-embedded. Thus, by Proposition 3.2, it suffices to check that

$$\mathbb{R} \ni x \mapsto (x, x^2, x^3) \in \mathbb{R}^3$$

has no intersecting tangent lines. Suppose otherwise. Then there are $x, y \in \mathbb{R}$, $x \neq y$, such that

$$(x, x^2, x^3) + \lambda(1, 2x, 3x^2) = (y, y^2, y^3) + \mu(1, 2y, 3y^2),$$

for some $\lambda, \mu \in \mathbb{R}$. The above equality yields three equations:

$$\begin{aligned} x - y &= \mu - \lambda, \\ x^2 - y^2 &= 2(\mu y - \lambda x), \\ x^3 - y^3 &= 3(\mu y^2 - \lambda x^2). \end{aligned}$$

Solving for y in the first equation and substituting in the second yields that $\lambda = \pm\mu$. If $\lambda = \mu$, then the first equation yields $x = y$, which is a contradiction. If $\lambda = -\mu$, then dividing the third equation by the first yields that

$$x^2 + xy + y^2 = \frac{3}{2}(y^2 + x^2).$$

The above equation is equivalent to $(x - y)^2 = 0$, which yields $x = y$, another contradiction. □

From this we immediately obtain the following result.

Corollary 3.4. *If a manifold admits an embedding in $\mathbb{R}^m$, then it admits a T-embedding in $\mathbb{R}^{3m}$.* □

Further, we can use Theorem 1.5 to etablish Corollary 1.6.

Proof of Corollary 1.6. As is well-known, every compact manifold M^n admits a smooth embedding into $\mathbb{R}^m$, provided that m is sufficiently large [6, p. 23]. Thus, by the previous corollary, M^n admits a smooth embedding without intersecting tangents into $\mathbb{R}^N$, where $N = 3m$. If $N \leq 4n + 1$ we are done. So suppose that $N > 4n + 1$. Define $g\colon (TM \times TM - \Delta TM) \to \mathbb{S}^{N-1}$, by

$$g(x, y) := \frac{x - y}{\|x - y\|},$$

where Δ denotes the diagonal elements of a cartesian product. Since

$$\dim(TM \times TM - \Delta TM) = 4n < N - 1 = \dim(\mathbb{S}^{N-1}),$$

and g is a C^1 mapping, the image of g must have measure zero [6, p. 68]. In particular, g is not unto. Let $u \in \mathbb{S}^{N-1}$ be a vector in the complement of the image of g, H_u be the hyperplane through the origin and orthogonal to u, and $\pi_u : \mathbb{R}^N \to H_u$ be the orthogonal projection

$$\pi_u(x) := x - \langle x, u\rangle u.$$

Then π is an embedding on TM. But $\pi(TM) = T(\pi(M))$. Thus $\pi(M)$ has no intersecting tangent spaces, and we obtain a T-embedding of M in $H_u \simeq \mathbb{R}^{N-1}$. We may repeat this procedure until we reach the desired dimension, $4n + 1$, for the ambient space. □

Finally, we prove the last observation mentioned in the introduction. The proof below uses the notions of contractibility and retract of a topological space X. Recall that X is *contractible* if there exists a point $x_0 \in X$ and a continuous mapping $f : X \times [0, 1] \to X$ such that $f(x, 0) = x$, and $f(x, 1) = x_0$. A subset $Y \subset X$ is called a *retract* of X if there exists a continuous map $f : X \to Y$ such that $f(y) = y$ for all $y \in Y$.

Proof of Theorem 1.7. Suppose that there exists a non-contractible manifold M^n which is T-embedded in $\mathbb{R}^{2n}$. Since $\dim(TM) = 2n$, it follows from the theorem on the invariance of domain that TM is an open subset of $\mathbb{R}^{2n}$. We claim that TM is closed in $\mathbb{R}^{2n}$ as well. To see this let p_i be a sequence of points in TM converging to a point p in $\mathbb{R}^{2n}$. Let q_i be the corresponding sequence of points in M such that $p_i \in T_{q_i}M$. Since M is compact, q_i have a limit point q in M. Since the tangent space T_qM is a limit point of $T_{q_i}M$, it follows that $p \in T_qM \subset TM$. Thus TM is both open and closed in $\mathbb{R}^{2n}$, which yields that $TM = \mathbb{R}^{2n}$. In particular TM is contractible. But M is a retract of TM, and the retract of a contractible space is contractible. So M is contractible – a contradiction. □

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