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Fixed point sets of parabolic isometries of CAT(0)-spaces

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Abstract. We study the fixed point set in the ideal boundary of a parabolic isometry of a proper CAT(0)-space. We show that the radius of the fixed point set is at most $\pi/2$, and study its centers. As a consequence, we prove that the set of fixed points is contractible with respect to the Tits topology.

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Contents

1 Introduction 305
2 Preliminaries 309
3 Estimate of radii of fixed point sets 311
4 Dimension of Tits ideal boundaries 315
5 CAT(1)-spaces of small diameter 317
6 Example of a symmetric space 323

1. Introduction

CAT(0)-spaces are generalizations of Hadamard manifolds in Riemannian geometry to geodesic spaces. The classification of isometries of the hyperbolic plane into elliptic, hyperbolic, and parabolic applies to the CAT(0) setting. The flat torus theorem (cf. [B], [BH]), which is one of the important results concerning hyperbolic isometries, remains true for CAT(0)-spaces.

In the study of isometries of Hadamard manifolds and CAT(0)-spaces, hyperbolic isometries have been extensive. We remark that in a cocompact, proper, isometric

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group action, there are no parabolic isometries. However, if one does not include the cocompactness assumption, then the group could contain parabolics.

In this paper, we focus on parabolic isometries of CAT(0)-spaces. We generalize the results of Schroeder from Appendix 3 in [BGS] on the fixed point set of a parabolic isometry in the ideal boundary of an Hadamard manifold to the setting of proper CAT(0)-spaces. Our results are not straightforward generalizations since much less has been known concerning analysis on proper CAT(0)-spaces than on Hadamard manifolds. As a new ingredient in our argument, we study the geometry of complete improper CAT(1)-spaces. We then apply the results to the Tits ideal boundary of a CAT(0)-space. At the end, as an example of our theorems, we examine a symmetric space in detail.

1.1. Main theorems and examples. Let X be a complete CAT(0)-space and $X(\infty)$ the ideal boundary of X defined as the asymptotic classes of rays in X. We classify an isometry f of X as elliptic, hyperbolic (or axial), or parabolic. f is called *elliptic* if it has a fixed point in X, and *hyperbolic* if there exists an invariant geodesic line, called axis, γ in X such that f acts on γ by a non-trivial translation. If f is neither elliptic nor hyperbolic, then it is called *parabolic*. We recall that f is parabolic if and only if the displacement function $d_f(p) := d(p, f(p))$ of f does not attain its infimum in X. $X(\infty)$ is equipped with a natural topology called sphere topology. Any isometry of X also acts as a homeomorphism of $X(\infty)$ since the isometry takes geodesics to geodesics.

It is known that if X is proper (i.e., any closed bounded subset is compact), then any parabolic isometry of X has at least one fixed point in $X(\infty)$ (cf. [B], [BH]). This does not necessarily hold if X is improper. In fact, there is an example of a parabolic isometry f of a separable Hilbert space X of infinite dimension, which is an improper CAT(0)-space, such that f has no fixed point in $X(\infty)$ (and in X) (cf. [BH]). We denote by $X_f(\infty)$ the fixed point set of f in $X(\infty)$.

The ideal boundary $X(\infty)$ has a natural metric, called the *Tits metric*, denoted Td. The metric space $(X(\infty), Td)$, the *Tits ideal boundary*, is a complete CAT(1)-space. The topology defined by the Tits metric on $X(\infty)$ is stronger than the sphere topology.

For a metric space (Y, d), we define

$$\operatorname{rad} Y = \operatorname{rad}(Y, d) := \inf_{x \in Y} \sup_{y \in Y} d(x, y),$$

which is called the *radius* of Y. For $p \in X$, we denote by $\Sigma_p X$ the space of directions at p. As one of the main results of this paper, we state the following:

Theorem 1.1. *Let X be a proper* CAT(0)*-space such that $\Sigma_p X$ is compact for every $p \in X$, and let f be a parabolic isometry of X. Then we have* $\operatorname{rad}(X_f(\infty), Td) \le \pi/2$. *In particular, $(X_f(\infty), Td)$ is contractible.*

Schroeder has proved Theorem 1.1 for smooth Hadamard manifolds in Appendix 3 in [BGS]. We remark that $\pi/2$ is the optimal upper bound of $\operatorname{rad} X_f(\infty)$ even for Hadamard manifolds (cf. Example 1.4). We also have some examples with $0 < \operatorname{rad} X_f(\infty) < \pi/2$ (cf. Examples 1.5 and 1.6). Notice that if X is proper and geodesically complete, then $\Sigma_p X$ is compact for any $p \in X$.

Recall that X is *visible* if and only if $Td(x, y) = \infty$ for any distinct points $x, y \in X(\infty)$. By Theorem 1.1, we immediately obtain:

Corollary 1.2. *Under the same assumption as in Theorem* 1.1*, if X is visible, then $X_f(\infty)$ consists of a single point.*

Buyalo [Bu] has shown that if X is a complete, not necessarily proper, Gromov-hyperbolic CAT(0)-space, then the infimum of the displacement function of any parabolic isometry f of X is equal to zero, and $X_f(\infty)$ consists of a single point. Let X be a proper CAT(0)-space. If X is Gromov-hyperbolic, then X is visible. If X admits a cocompact group action, then the converse is true (cf. [BH]).

Next, we study the centers of $X_f(\infty)$. A *center* of a metric space (Y, d) is a point in Y where the function $Y \ni x \mapsto \sup_{y \in Y} d(x, y) \in [0, \infty]$ attains the infimum, $\operatorname{rad} Y$. We denote by $C(A)$ the set of all centers of A, and define $C^2(Y) := C(C(Y))$.

Theorem 1.3. *Let X be a proper* CAT(0)*-space of finite covering dimension such that $\Sigma_p X$ is compact for every $p \in X$. Let f be a parabolic isometry of X. Then $C^2(X_f(\infty))$ consists of a single point, which is fixed by any isometry of X leaving $X_f(\infty)$ invariant. In particular, the point is a fixed point of any isometry of X commuting f.*

Theorem 1.3 for Hadamard manifolds has been shown by Eberlein [E] following Schroeder's work in Appendix 3 in [BGS].

We give some examples.

Example 1.4. Let us denote the hyperbolic plane by $\mathbb{H}^2$. We consider the product Riemannian manifold $X := \mathbb{R} \times \underbrace{\mathbb{H}^2 \times \cdots \times \mathbb{H}^2}_{m \text{ times}}$, $m \geq 1$. For m parabolic isometries $h_1, h_2, \ldots, h_m$ of $\mathbb{H}^2$, we define the product map $f := (\mathrm{id}_{\mathbb{R}}, h_1, \ldots, h_m)$, where $\mathrm{id}_{\mathbb{R}}$ is the identity map on $\mathbb{R}$. f is a parabolic isometry of X. We denote by $\mathbb{S}^{m-1}(1)$ the standard unit $(m-1)$-sphere in the Euclidean m-space $\mathbb{E}^m$ and set

$$\triangle_1^{m-1} := \{(x_1, \ldots, x_m) \in \mathbb{S}^{m-1}(1) \subset \mathbb{E}^m \mid x_i \geq 0 \text{ for all } i\}, \tag{1.1}$$

which we call the *standard spherical* $(m-1)$*-simplex*. $X_f(\infty)$ is isometric to the spherical suspension over $\triangle_1^{m-1}$. We refer [BBI] for the definition of spherical suspension. We have $\operatorname{rad} X_f(\infty) = \pi/2$. $C(X_f(\infty))$ is isometric to $\triangle_1^{m-1}$ and $C^2(X_f(\infty))$ consists of the barycenter of $\triangle_1^{m-1}$.

The following example is discussed in Section 6.

Example 1.5. We consider $X := \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$, which is a five-dimensional, irreducible symmetric space of non-compact type and rank two. $\mathrm{SL}(3,\mathbb{R})$ is the identity component of the isometry group of X. The Tits ideal boundary $(X(\infty), Td)$ is a thick spherical building of dimension one. Weyl chambers of X are corresponding to edges of the building $(X(\infty), Td)$ and any edge has length $\pi/3$. By Theorem 6.1, for any parabolic isometry $f \in \mathrm{SL}(3,\mathbb{R})$ of X, $X_f(\infty)$ is one of the following:

(1) an edge,

(2) a closed interval of length π consisting of three edges,

(3) the union of an edge c and all edges incident to c.

In (3), $X_f(\infty)$ has uncountably many edges.

For the irreducible symmetric space $X := \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$, $n \geq 3$, let f be any isometry of X. Since for any Weyl chamber c at infinity, $fc \cap c$ is a (possibly empty) face of c and since $\operatorname{rad} c \geq \pi/6$ (cf. [BH]), we have either $\operatorname{rad} X_f(\infty) = 0$ or $\geq \pi/6$.

For any given $\theta \in (0, \pi/2)$, we have an example with $\operatorname{rad} X_f(\infty) = \theta$, where X is a manifold with boundary.

Example 1.6. Let h be a parabolic isometry of $\mathbb{H}^2$ and x its fixed point in $\mathbb{H}^2(\infty)$. Let γ be a ray in $\mathbb{H}^2$ tending to x, and b_γ the Busemann function associated with γ (see Section 2 for the definition of b_γ). Note that h leaves every horosphere $b_\gamma^{-1}(t)$ invariant. For an arbitrarily given $\theta \in (0, \pi/2)$, we consider the closed convex subset

$$X := \{(p, s) \in \mathbb{H}^2 \times \mathbb{R} \mid b_\gamma(p) \leq -t,\ |s| \leq t \sin\theta \text{ for some } t \geq 0\}$$

of $\mathbb{H}^2 \times \mathbb{R}$. X is a proper CAT(0)-space and $(X(\infty), Td)$ is isometric to a closed interval of length 2θ whose midpoint corresponds to x. The product map $(h, \mathrm{id}_{\mathbb{R}})$ leaves X invariant, and its restriction, f, on X is a parabolic isometry of X. Since $X_f(\infty)$ coincides with $X(\infty)$, we have $\operatorname{rad} X_f(\infty) = \theta$.

1.2. Key ideas of the proof of main theorems. We prove Theorem 1.1 in Section 3 by using the gradient curves for the displacement function, the existence of which is established by Jost and Mayer ([J], [M]). Our proof is based on Schroeder's original argument for Hadamard manifolds in Appendix 3 in [BGS]. Since a CAT(0)-space X is not differentiable in general, we need to investigate the directional derivatives of a Lipschitz continuous, convex function on X. It is non-trivial to prove a first variation formula for such a function (see Lemma 3.5).

For Theorem 1.3, the original proof in [BGS] does not seem to work for a CAT(0)-space. We take a new approach using the geometry of the Tits ideal boundary $(X(\infty), Td)$ as explained in the following.

For a topological space Y, $\dim_C Y$ is defined as the supremum of the covering dimensions of compact subsets of Y (cf. [K]). A key theorem needed for understanding the set of centers of $X_f(\infty)$ is the following.

Theorem 1.7. *Let Y be a complete* CAT(1)*-space of* $\dim_C Y < \infty$ *and diameter* $\operatorname{diam} Y \le \pi/2$. *Then there exists a constant $\delta > 0$, which depends only on* $\dim_C Y$, *such that* $\operatorname{rad} Y \le \pi/2 - \delta$. *In particular, $C(Y)$ consists of a single point.*

Schroeder has shown the same statement if Y is a closed convex subset of the unit sphere of dimension n in Appendix 3 in [BGS]. The basic strategy of the proof is following [BGS], however the proof is more delicate because Y is possibly noncompact. Namely, we cannot avoid a discussion of error estimates (i.e., the estimate of δ). We only need $\operatorname{rad} Y < \pi/2$ for Theorem 1.3.

It is necessary for Theorem 1.7 that $\dim_C Y$ is finite. In fact, the inductive limit, Y, of the standard spherical $(m-1)$-simplices $\triangle_1^{m-1}$, $m = 1, 2, \dots$, given in (1.1) is a complete CAT(1)-space such that $\dim_C Y = \infty$, $\operatorname{diam} Y = \pi/2$, and $\operatorname{rad} Y = \pi/2$.

For applying Theorem 1.7 to $Y := X_f(\infty)$, we need the next result.

Proposition 1.8. *For a proper* CAT(0)*-space X we have*

$$\dim_C(X(\infty), Td) \le \dim X - 1,$$

where $\dim X$ *denotes the covering dimension of X.*

Theorem C in [K] implies Proposition 1.8, provided that X has a cocompact group action. For the proof of the proposition, we use a result in [FSY] on the dimension of $X(\infty)$ with sphere topology. There is another way to obtain the proposition using Lemma 11.1 of [L]. We would like to thank A. Lytchak for bringing his work to our attention. We do not know whether $\dim_C$ in Proposition 1.8 can be replaced with the covering dimension.

Theorem 1.3 is proved as follows. By Theorem 1.1, $Y := C(X_f(\infty))$ has $\operatorname{diam} Y \le \pi/2$. Proposition 1.8 implies $\dim_C Y < \infty$. Therefore, applying Theorem 1.7, we obtain Theorem 1.3. The details are given in Section 3.5.

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2. Preliminaries

A *minimizing geodesic* is, by definition, a length-minimizing curve joining two points in a metric space. We assume that all minimizing geodesics have unit speed parameters. Denote by γ_{pq} a minimizing geodesic from a point p to a point q, and by $[p, q]$

its image. A *geodesic triangle* $\triangle(p,q,r)$ means a triple of minimizing geodesics γ_{pq}, γ_{qr}, and γ_{rp} for three points p, q, and r, called *vertices*.

For $\kappa \in \mathbb{R}$, let M_κ^2 be a complete, simply connected model surface of constant curvature κ. We set $D_\kappa := \operatorname{diam} M_\kappa^2$. Note that D_κ is equal to $\pi/\sqrt{\kappa}$ if $\kappa > 0$, and to ∞ if $\kappa \le 0$. We say that a metric space X is a $\mathrm{CAT}(\kappa)$-*space* if the following properties (1) and (2) are satisfied.

(1) Any two points $p, q \in X$ with $d(p,q) < D_\kappa$ can be joined by a minimizing geodesic in X.

(2) ($\mathrm{CAT}(\kappa)$-inequality) Let $\triangle(p,q,r)$ be any geodesic triangle in X with perimeter $< 2D_\kappa$ and $\triangle(\tilde{p},\tilde{q},\tilde{r})$ a comparison triangle of it in M_κ^2, i.e., having the same side lengths as $\triangle(\tilde{p},\tilde{q},\tilde{r})$. For any four points $x \in [p,q]$, $y \in [r,p]$, $\tilde{x} \in [\tilde{p},\tilde{q}]$, and $\tilde{y} \in [\tilde{r},\tilde{p}]$ such that $d(p,x) = d(\tilde{p},\tilde{x})$ and $d(p,y) = d(\tilde{p},\tilde{y})$, we have
$$d(x,y) \le d(\tilde{x},\tilde{y}),$$
where d denotes the distance function.

Let X be a $\mathrm{CAT}(\kappa)$-space. A minimizing geodesic γ_{pq} joining two points $p, q \in X$ with $d(p,q) < D_\kappa$ is unique. For $p \in X$ and $q_1, q_2 \in X \setminus \{p\}$, we denote by $\angle_p(\gamma_{pq_1}, \gamma_{pq_2})$ the angle at p between γ_{pq_1} and γ_{pq_2}. $\angle_p$ is a pseudo-distance function on the set of all minimizing geodesics emanating from p. The quotient metric space by the relation $\angle_p = 0$ is denoted by $\Sigma_p^* X$. Let $\Sigma_p X$ be the $\angle_p$-completion of $\Sigma_p^* X$, which is called the *space of directions at* p. We denote by $C_p X$ the Euclidean cone over $\Sigma_p X$, and call this the *tangent cone at* p. $\Sigma_p X$ is a complete $\mathrm{CAT}(1)$-space and $C_p X$ is a complete $\mathrm{CAT}(0)$-space. We denote by $\dot{\gamma}(0)$ the equivalence class in $\Sigma_p^* X$ of a minimizing geodesic γ from p.

Assume that X is a complete $\mathrm{CAT}(0)$-space. Two rays $\gamma, \sigma\colon [0,+\infty) \to X$ are said to be *asymptotic* if $d(\gamma(t), \sigma(t))$ is uniformly bounded for all $t \ge 0$. The *ideal boundary* $X(\infty)$ of X is defined as the set of all asymptotic equivalence classes of rays in X. $X(\infty)$ is equipped with the *sphere topology*, which is the restriction of the cone topology (cf. II.8 [BH]) on $X \sqcup X(\infty)$. We denote by $\gamma(\infty)$ the equivalence class in $X(\infty)$ of a ray γ in X. For any $p \in X$ and $x \in X(\infty)$ there exists a unique ray $\gamma_{px}\colon [0,\infty) \to X$ from p to $\gamma(\infty) = x$.

For $x, y \in X(\infty)$, we set $\angle(x,y) := \sup_{p \in X} \angle_p(x,y)$, the *angle distance* between x and y, where we write $\angle_p(x,y) := \angle_p(\gamma_{px}, \gamma_{py})$. Note that $\angle$ is a distance function on $X(\infty)$ and is lower semi-continuous with respect to the sphere topology. We remark that if X is proper, then $X(\infty)$ is compact with respect to the sphere topology. The *Tits distance* on $X(\infty)$, denoted by Td, is the interior distance induced from $\angle$. We have $\angle = \min\{Td, \pi\}$. The Tits ideal boundary $(X(\infty), Td)$ of X is a complete $\mathrm{CAT}(1)$-space, which is non-compact in general. The *Busemann function* $b_\gamma\colon X \to \mathbb{R}$ associated to a ray γ in X is defined as
$$b_\gamma(p) := \lim_{t\to\infty} \{d(p,\gamma(t)) - t\}.$$

This is a 1-Lipschitz continuous, convex function with $b_\gamma(\gamma(0)) = 0$.

A subset A of a metric space X is said to be *convex* in X if any $x, y \in A$ can be joined by a minimizing geodesic and the image of every such geodesic is contained in A. If this condition holds only for any $x, y \in A$ with $d(x, y) < r$, then A is said to be *r-convex in X*.

Let B be a closed subset of a metric space X. We define a function $d_B\colon X \to [0, \infty)$ by $d_B(p) := d(p, B)$, the *distance function* from B. For $p \in X \setminus B$, we denote by γ_{pB} a minimizing geodesic in X from p to B, i.e., to a point $q \in B$ with $d_B(p) = d(p, q)$.

Assume that B is a closed, convex subset of a complete CAT(0)-space. Then, for any $p \in X$ there exists a unique point $q \in B$ with $d_B(p) = d_B(p, q)$, in particular, $\gamma_{pq} = \gamma_{pB}$. We note that d_B is a 1-Lipschitz continuous, convex function.

3. Estimate of radii of fixed point sets

We prove Theorem 1.1.

3.1. Directional derivatives of convex functions. Let X be a complete CAT(0)-space and $F\colon X \to \mathbb{R}$ a locally Lipschitz continuous, convex function. We discuss the directional derivatives of F. For any geodesic γ in X, $F \circ \gamma$ has the left and right derivatives. Recall that the tangent cone C_pX is the quotient space $[0, +\infty) \times \Sigma_pX/\{0\} \times \Sigma_pX$. We identify the subspace $\{1\} \times \Sigma_pX$ of C_pX with Σ_pX. Denote any element $(t, v) \in C_pX$ by tv and define $|tv| := t$. Let $C_p^*X := [0, \infty) \times \Sigma_p^*X/\{0\} \times \Sigma_p^*X \subset C_pX$. The *directional derivative* $D_pF\colon C_p^*X \to \mathbb{R}$ of F at a point $p \in X$ is defined as

$$D_pF(tv) := \lim_{s \to 0+} \frac{F(\gamma_v(s)) - F(\gamma_v(0))}{s} t,$$

where γ_v is a minimizing geodesic from p with $v = \dot\gamma_v(0)$. The existence of the limit above is guaranteed by the convexity of F. $D_pF(tv)$ is independent of the choice of γ_v. D_pF extends to a unique Lipschitz continuous function on C_pX, which is convex (cf. Lemma 2.4 in [K]). Moreover, it is linear along each ray from the vertex o_p of C_pX.

Assume that Σ_pX is compact for every $p \in X$. We say that a point $p \in X$ is a *critical point* of F if $D_pF(u) \ge 0$ for every $u \in \Sigma_pX$. Note that, by the convexity of F, a point is critical for F if and only if it is a minimizer of F. For more general functions, such as c-convex functions (cf. [BBI]), this is not true and we still have some local properties stated below, e.g. Theorem 3.1 and Lemma 3.5. By the convexity of D_pF and the compactness of Σ_pX, for any non-critical point p of F, there exists a unique direction $u_p \in \Sigma_pX$ where $D_pF|_{\Sigma_pX}$ attains its minimum (< 0). We call u_p the *gradient direction* of $-F$ at p. Define the *gradient vector* $\mathrm{grad}_p(-F) \in C_pX$

of $-F$ at a point p by

$$\operatorname{grad}_p(-F) := |D_pF(u_p)|u_p \in C_pX$$

if p is non-critical, and by $\operatorname{grad}_p(-F) := o_p$ (the vertex) if p is critical. It follows that $|\operatorname{grad}_p(-F)| = -D_pF(u_p)$.

3.2. Jost–Mayer's gradient curves. The following theorem is a restricted version of a result in [J], [M].

Theorem 3.1 ([J], [M]). *Let X be a complete* $\mathrm{CAT}(0)$*-space such that Σ_pX is compact for every $p \in X$, and let $F\colon X \to \mathbb{R}$ be a convex function. Then, for every $p \in X$ there exists a Lipschitz continuous curve $c_p\colon [0,\infty) \to X$ from $p = c_p(0)$, called the* gradient curve *from p for $-F$, such that for any $t \ge 0$ we have*

$$\begin{aligned}(1)\qquad \lim_{s\to 0+}\frac{d(c_p(t+s),c_p(t))}{s} &= \lim_{s\to 0+}\frac{-F\circ c_p(t+s)+F\circ c_p(t)}{d(c_p(t+s),c_p(t))}\\ &= \limsup_{q\to c_p(t)}\frac{-F(q)+F(c_p(t))}{d(q,c_p(t))}\\ &= |\operatorname{grad}_{c_p(t)}(-F)|,\end{aligned}$$

$$(2)\qquad (F\circ c_p)'_+(t) = |\operatorname{grad}_{c_p(t)}(-F)|^2,$$

where $(F\circ c_p)'_+(t)$ is the right derivative of $F\circ c_p$ at t. Moreover, for any $r \ge 0$, the gradient curve $c_{c_p(t)}$ from $c_p(t)$ for $-F$ satisfies

$$c_{c_p(t)}(r) = c_p(t+r).$$

Under the same assumption as in Theorem 3.1, we have:

Lemma 3.2. *For the gradient curve c_p from p of $-F$, the right tangent vector $(\dot c_p)_+(0) \in C_pX$ exists and coincides with* $\operatorname{grad}_p(-F)$.

Proof. By taking a sequence $\{s_i\}$ with $s_i \to 0+$, we have a limit $v \in \Sigma_pX$ of the direction $\dot\gamma_{pc_p(s_i)}(0)$ as $i \to \infty$. By Theorem 3.1(1), $D_pF(v)$ must be equal to $D_pF(u_p) = -|\operatorname{grad}_p(-F)|$. We see that $v = u_p$ by the uniqueness of the gradient direction u_p. □

3.3. First variation formula. The following is well known.

Lemma 3.3. *Let X be a complete* $\mathrm{CAT}(0)$*-space.*

(1) *Let B be a closed, convex subset of X. Then for any $p \in X \setminus B$ and $v \in \Sigma_p X$ we have*

$$D_p d_B(v) = -\cos \angle_p(\dot{\gamma}_{pB}(0), v).$$

(2) *Let γ be a ray in X. Then for any $p \in X$ and $v \in \Sigma_p X$ we have*

$$D_p b_\gamma(v) = -\cos \angle_p(\dot{\gamma}_{p\gamma(\infty)}(0), v).$$

Proof. (1) follows from a standard argument (cf. Section 4.5 of [BBI]).

We prove (2). Set $B_t := b_\gamma^{-1}(-\infty, -t]$ for $t > 0$. B_t is convex in X. Let $p \in X$ be any point. If $t > 0$ is large enough for p, then $p \in X \setminus B_t$ and $d_{B_t}(p) = b_\gamma(p) + t$ (cf. Proposition II.8.22 in [BH]), which and (1) imply (2). □

Some variants of Lemma 3.3(1) are seen in Section 4.5 of [BBI]. Note that the CAT(0)-condition for X is not essential for Lemma 3.3.

To prove a first variation formula for convex functions, we need a lemma.

Lemma 3.4. *Let S be a sector in $\mathbb{E}^2$ bounded by two distinct rays from the origin o. Let $F\colon S \to \mathbb{R}$ be a function that is linear along each ray from o. If the directional derivative $D_u F\colon C_u S \to \mathbb{R}$ of F at a point $u \in S \setminus \{o\}$ exists, then $D_u F$ is linear on $C_u S$.*

Lemma 3.4 is shown by a standard argument. We omit the proof.

We prove the following first variation formula.

Lemma 3.5. *Let $F\colon X \to \mathbb{R}$ be a locally Lipschitz continuous, convex function on a complete* CAT(0)*-space X. Let $p \in X$ be a non-critical point of F such that $\Sigma_p X$ is compact. Then for any $v \in \Sigma_p X$ we have*

$$D_p F(v) \geq -|\operatorname{grad}_p(-F)| \cos \angle_p(u_p, v),$$

where $u_p \in \Sigma_p X$ is the gradient direction of $-F$ at p.

Proof. Let $v \in \Sigma_p X$ be a direction. If $\angle_p(u_p, v) = 0$, the lemma is obvious. In the case where $\angle_p(u_p, v) = \pi$, the minimizing geodesic $\gamma_{u_p v}$ in $C_p X$ from u_p to v passes through the vertex o_p, so that the convexity of $D_p F$ along $\gamma_{u_p v}$ implies the lemma.

We assume that $0 < \angle_p(u_p, v) < \pi$. Now consider the second derivative $D_{u_p} D_p F\colon C_{u_p} C_p X \to \mathbb{R}$. Let $S \subset C_p X$ be the 2-dimensional flat sector generated by $\gamma_{u_p v}$. S is convex in $C_p X$. We set $\xi := \dot{\gamma}_{u_p v}(0)$ and $\eta := \dot{\gamma}_{u_p o_p}(0)$, both which belong to $\Sigma_{u_p} S$. Note that $C_{u_p} S$ is a flat half plane in $C_{u_p} C_p X$. Take the direction $\zeta \in \Sigma_{u_p} S$ perpendicular to η. Setting $\theta := \angle_{u_p}(o_p, v)$, we see

$\xi = (\cos\theta)\eta + (\sin\theta)\zeta$. Since Lemma 3.4 implies the linearity of $D_{u_p}D_pF$, we have

$$D_{u_p}D_pF(\xi) = D_{u_p}D_pF(\eta)\cos\theta + D_{u_p}D_pF(\zeta)\sin\theta.$$

The linearity of D_pF along $\gamma_{u_po_p}$ shows that $D_{u_p}D_pF(\eta) = -D_pF(u_p) > 0$. Since u_p is the minimum point of D_pF on Σ_pX, we have $D_{u_p}D_pF(\zeta) \geq 0$. Thus, by noting $0 < \theta < \pi/2$,

$$D_{u_p}D_pF(\xi) \geq -D_pF(u_p)\cos\theta \ (> 0). \tag{3.1}$$

It follows that the distance between u_p and v in C_pX is equal to $2\cos\theta$, so that, by the convexity of D_pF along γ_{u_pv},

$$D_pF(v) \geq D_pF(u_p) + 2\, D_{u_p}D_pF(\xi)\cos\theta. \tag{3.2}$$

Combining (3.1) and (3.2) yields

$$D_pF(v) \geq -D_pF(u_p)\cos 2\theta = D_pF(u_p)\cos\angle_p(u_p, v),$$

which completes the proof of Lemma 3.5. □

Note that the equality in Lemma 3.5 does not necessarily hold. Lemma 3.5 remains true for a locally Lipschitz continuous, c-convex function F on a locally CAT(κ)-space X, $c, \kappa \in \mathbb{R}$.

3.4. Monotone points. Let X be a complete CAT(0)-space and $F\colon X \to \mathbb{R}$ a convex function. The following terminology was introduced by Eberlein in Section 4.1 of [E] for a Riemannian manifold. A point $x \in X(\infty)$ is said to be *F-monotone* if there exists a ray $\gamma\colon [0,\infty) \to X$ with $x = \gamma(\infty)$ such that $F \circ \gamma(t)$ is monotone non-increasing in $t \geq 0$. We denote by $X_F(\infty)$ the set of all F-monotone points in $X(\infty)$, called the *F-monotone set*. For an isometry f of X, we recall the displacement function $d_f(p) := d(p, f(p))$, which is a 1-Lipschitz continuous, convex function on X. For a ray γ in X, $\gamma(\infty)$ is d_f-monotone if and only if $f \circ \gamma$ is asymptotic to γ. This leads to $X_{d_f}(\infty) = X_f(\infty)$.

The following lemma is obtained by the same discussion as in Section 4.1 of [E]. We omit the proof.

Lemma 3.6. *Let $F\colon X \to \mathbb{R}$ be a convex function.*

(1) *For a point $x \in X(\infty)$, the following properties are equivalent to each other.*

(a) *x is F-monotone.*

(b) *For any ray γ with $x = \gamma(\infty)$, $F \circ \gamma(t)$ is monotone non-increasing in $t \geq 0$.*

(c) *There exists a sequence* $\{p_i\}$ *of points in* X *converging to* x *in the cone topology such that* $F(p_i)$ *is uniformly bounded from above.*

(2) $X_F(\infty)$ *is closed with respect to the sphere topology.*

(3) *If* X *is proper, then* $X_F(\infty)$ *is a closed,* π*-convex subset of* $(X(\infty), Td)$.

3.5. Proof of Theorem 1.1. We prove the theorem in the same way as in [E] by using Lemma 3.5. Let X be a proper CAT(0)-space such that $\Sigma_p X$ is compact for every $p \in X$, and f a parabolic isometry of X. Since the displacement function d_f has no minimal (or critical) point in X, we have the gradient direction u_p of $-d_f$ at any $p \in X$, which satisfies $D_p d_f(u_p) < 0$. We fix a point $p \in X$ and take the gradient curve c_p from p for $-d_f$. By Lemma 3.2, the right tangent vector $(\dot{c}_p)_+(t) \in C_p X$ satisfies $(\dot{c}_p)_+(t) = \mathrm{grad}_{c_p(t)}(-d_f)$ for any $t \geq 0$. It follows from Theorem 3.1(1) that $d_f \circ c_p(t)$ is strictly monotone decreasing in $t \geq 0$. There exists a sequence $t_i \to \infty$ such that $c_p(t_i)$ converges to some point $x \in X(\infty)$ in the cone topology. Lemma 3.6(1) implies $x \in X_f(\infty)$.

We take any $y \in X_f(\infty)$ and fix it. It suffices to prove that $Td(x, y) \leq \pi/2$. Let $v_t := \dot{\gamma}_{c_p(t)y}(0)$. Consider the Busemann function $b := b_{\gamma_{py}}$ associated with γ_{py}. Since y is d_f-monotone and by Theorem 3.1(1), Lemma 3.3(2), and Lemma 3.5, we have

$$(b \circ c_p)'_+(t) = -|\,\mathrm{grad}_{c_p(t)}(-d_f)| \cos \angle_{c_p(t)}(u_{c_p(t)}, v_t) \leq D_{c_p(t)} d_f(v_t) \leq 0$$

for any $t \geq 0$, and hence $b \circ c_p(t)$ is monotone non-increasing in t. By Lemma 3.6(1), x is b-monotone and, for any $q \in X$, $b \circ \gamma_{qx}(t)$ is monotone non-increasing in t. It follows from Lemma 3.3(2) that

$$-\cos \angle_q(x, y) = (b \circ \gamma_{qx})'_+(0) \leq 0,$$

which proves $Td(x, y) \leq \pi/2$.

Since $(X(\infty), Td)$ is CAT(1), $X_f(\infty)$ is contractible. This completes the proof of Theorem 1.1. □

Let X be as in Theorem 1.1. Then we have $\mathrm{rad}\, X_F(\infty) \leq \pi/2$ for any locally Lipschitz continuous, convex function F on X with no minimum in X.

4. Dimension of Tits ideal boundaries

We need the following to prove Proposition 1.8.

Proposition 4.1 ([FSY]). *Let* X *be a proper* CAT(0)*-space. Then, the covering dimension of* $X(\infty)$ *for the sphere topology satisfies*

$$\dim X(\infty) \leq \dim X - 1.$$

Proof of Proposition 1.8. By Proposition 4.1, it suffices to show that

$$\dim_C(X(\infty), Td) \leq \dim X(\infty).$$

We consider the identity map $\iota\colon (X(\infty), Td) \to X(\infty)$, which is continuous. Take any compact subset $K \subset (X(\infty), Td)$. Since $X(\infty)$ is Hausdorff, $\iota|_K\colon K \to \iota(K)$ is a homeomorphism. Thus, we have $\dim K = \dim \iota(K) \leq \dim X(\infty)$. This completes the proof. □

We denote by $\triangle^n = \triangle^n(a_0, a_1, \dots, a_n)$ a (closed) n-simplex with vertices $a_0, a_1, \dots, a_n$. Let $F_i \subset \partial\triangle^n$ be the $(n-1)$-simplex that is the opposite face to a_i, where $\partial\triangle^n$ is the boundary of $\triangle^n$. We say that a map ψ from $\triangle^n$ to a set *collapses* $\partial\triangle^n$ if

$$\psi(F_0) \cap \psi(F_1) \cap \cdots \cap \psi(F_n) \neq \emptyset.$$

The following is a consequence of Sperner's lemma (cf. 2.1 in [F]).

Lemma 4.2. *Let Y be a Hausdorff space of* $\dim Y \leq n-1$, $n \geq 1$. *Then any continuous map $\psi\colon \triangle^n \to Y$ collapses $\partial\triangle^n$.*

Proof. Suppose that there exists a continuous map $\psi\colon \triangle^n \to Y$ that does not collapse $\partial\triangle^n$. We set $U_i := Y \setminus \psi(F_i)$, $i = 0, 1, \dots, n$, which are open in the Hausdorff space Y. Since ψ does not collapse $\partial\triangle^n$, $\{U_i\}_{i=0}^n$ is an open covering of Y. By $\dim Y \leq n-1$, there exists a refinement $\{V_i\}$ of $\{U_i\}$ of order at most n. Since ψ is continuous and the order of $\{V_i\}$ is at most n, we can take a sufficiently refined triangulation of $\triangle^n$ such that for each simplex s of it, $\psi(s)$ intersects at most n members of $\{V_i\}$. Then we give a label by $i = 0, 1, \dots, n$ to each vertex of the refinement as follows. A label of a vertex a is i if $\psi(a) \in V_i$, which implies that this label is a Sperner label on $\triangle^n$. Namely, each original vertex a_i has the label i, and each vertex in the refinement contained in a j-dimensional simplex $\triangle^j = \triangle^j(a_{i_0}, a_{i_1}, \dots, a_{i_j})$ is labelled by one of $i_0, i_1, \dots, i_j$; e.g., a vertex contained in F_i does not have the label i. Therefore, by Sperner's lemma there exists at least one n-simplex s^n in the refined triangulation of $\triangle^n$ such that the vertices of s^n have the $n+1$ different labels, $0, 1, \dots, n$.

On the other hand, since $\psi(s^n)$ is contained in at most n different V_i's, the simplex s^n has at most n different labels. This is a contradiction. □

Lemma 4.2 plays a key role in the proof of Theorem 1.7 in Section 5. As another application of Lemma 4.2, we have:

Proposition 4.3. *Let Y be a* CAT(1)*-space of* $\dim_C Y \leq m$, $m \geq 1$. *Then, for any embedding ψ from an m-sphere $\mathbb{S}^m$ into Y we have* $\operatorname{rad} \psi(\mathbb{S}^m) \geq \pi$. *In particular, if $m = 1$, then Y is locally an $\mathbb{R}$-tree.*

Proof. Suppose that there exists an embedding $\psi : \mathbb{S}^m \to Y$ satisfying rad $\psi(\mathbb{S}^m) < \pi$. Since Y is CAT(1), $\psi(\mathbb{S}^m)$ is contractible in Y. Hence, for a closed $(m+1)$-disk D^{m+1} there is a continuous extension $\overline{\psi} \colon D^{m+1} \to Y$ of ψ. By identifying D^{m+1} with an $(m+1)$-simplex, $\overline{\psi}$ does not collapse ∂D^{m+1} and dim $\overline{\psi}(D^{m+1}) \leq \dim_C Y \leq m$. This contradicts Lemma 4.2. □

Remark 4.4. Let X be a proper CAT(0)-space of dim $X \leq n$. By Proposition 1.8, we can apply Proposition 4.3 to $Y = (X(\infty), Td)$ and $m = n - 1$.

5. CAT(1)-spaces of small diameter

In this section we shall prove Theorems 1.3 and 1.7.

5.1. Small triangles. Let Y be a CAT(1)-space. For $x, y, z \in Y$ we set $\angle_x(y, z) := \angle_x(\gamma_{xy}, \gamma_{xz})$. Denote the image of γ_{xy} by $[x, y]$. Let $\triangle = \triangle(a_0, a_1, a_2)$ be a geodesic triangle in Y with sides $[a_0, a_1]$, $[a_1, a_2]$, $[a_2, a_0]$, and $\widetilde{\triangle} = \triangle(\widetilde{a}_0, \widetilde{a}_1, \widetilde{a}_2)$ a comparison triangle in $\mathbb{S}^2(1)$ of $\triangle$ with the same side-lengths as $\triangle$. Recall that $\angle_{a_i}(a_j, a_k) \leq \angle_{\widetilde{a}_i}(\widetilde{a}_j, \widetilde{a}_k)$ for distinct $i, j, k = 0, 1, 2$. We say that $\triangle(a_0, a_1, a_2)$ is *small* if $d(a_i, a_j) \leq \pi/2$ for any $i, j = 0, 1, 2$. If $\triangle(a_0, a_1, a_2)$ is small, then we have $d(a_2, x) \leq \pi/2$ for any $x \in [a_0, a_1]$ by the CAT(1)-inequality. If $\triangle(a_0, a_1, a_2)$ is small and if $d(a_2, x) = \pi/2$ for some $x \in [a_0, a_1] \setminus \{a_0, a_1\}$, then the triangle is an isosceles triangle and bounds a convex spherical surface.

As usual, $O(\varepsilon)$ denotes Landau's symbol, i.e., some universal function such that $\limsup_{\varepsilon \to 0} |O(\varepsilon)|/\varepsilon$ is finite. We assume that $O(\varepsilon)$ is positive.

For the proof of Theorem 1.7, we first show:

Lemma 5.1. *Let $\varepsilon \in (0, 1)$ be a positive number. Let $\triangle = \triangle(a_0, a_1, a_2)$ and $\triangle' = \triangle(a'_0, a'_1, a'_2)$ be small geodesic triangles in Y and in $\mathbb{S}^2(1)$, respectively. Then the following holds:*

(1) *if $|d(a_i, a_j) - d(a'_i, a'_j)| \leq \varepsilon$ for any $i, j = 0, 1, 2$ and if $d(a_0, a_j) \geq \varepsilon^{1/2}$ for each $j = 1, 2$, then we have*

$$\angle_{a_0}(a_1, a_2) < \angle_{a'_0}(a'_1, a'_2) + O(\varepsilon^{1/2});$$

(2) *if $\angle_{a_0}(a_1, a_2) \geq \angle_{a'_0}(a'_1, a'_2) - \varepsilon$ and $|d(a_0, a_j) - d(a'_0, a'_j)| \leq \varepsilon$ for each $j = 1, 2$, then we have*

$$d(a_1, a_2) > d(a'_1, a'_2) - O(\varepsilon).$$

Proof. (1) Let $\widetilde{\triangle} = \triangle(\widetilde{a}_0, \widetilde{a}_1, \widetilde{a}_2)$ be a comparison triangle in $\mathbb{S}^2(1)$ of $\triangle$. Since Y is CAT(1), we have $\angle_{a_0}(a_1, a_2) \leq \angle_{\widetilde{a}_0}(\widetilde{a}_1, \widetilde{a}_2)$. By the assumption of $\triangle$ and $\triangle'$, we have the conclusion of (1).

We omit the proof of (2). □

We next prove the following lemma.

Lemma 5.2. *Let $\varepsilon \in (0,1)$, and let $\triangle = \triangle(a_0, a_1, a_2)$ be a small geodesic triangle in Y. Assume that there exists a point $y \in [a_0, a_1]$ such that $\min_{i=0,1} d(a_i, y) \geq \varepsilon^{1/2}$ and $d(a_2, y) \geq \pi/2 - \varepsilon$. Then we have*

(1) $|\angle_y(a_2, a_i) - \pi/2| < O(\varepsilon^{1/2})$, $i = 0, 1$,

(2) $d(a_2, x) > \pi/2 - O(\varepsilon^{1/2})$

for any $x \in [a_0, a_1]$.

Proof. (1) Let $\triangle_i' = \triangle(y', a_i', a_2')$, $i = 0, 1$, be two spherical triangles in $\mathbb{S}^2(1)$ such that $d(y', a_i') = d(y, a_i)$, $d(a_i', a_2') = d(a_i, a_2)$, and $d(a_2', y') = \pi/2$. Since each $\triangle_i'$ is small, we have $\angle_{y'}(a_2', a_i') \leq \pi/2$. By $d(a_2, y) \geq \pi/2 - \varepsilon$, we have $|d(a_2, y) - d(a_2', y')| \leq \varepsilon$. Applying Lemma 5.1(1) to $\triangle(y, a_i, a_2)$ and $\triangle_i'$ yields that $\angle_y(a_2, a_i) < \pi/2 + O(\varepsilon^{1/2})$. Therefore, by $\pi \leq \angle_y(a_2, a_0) + \angle_y(a_2, a_1)$ we have $\angle_y(a_2, a_i) > \pi/2 - O(\varepsilon^{1/2})$.

(2) For any given $x \in [a_0, a_1] \setminus \{y\}$, let us take a small spherical isosceles triangle $\triangle'' = \triangle(y'', x'', a_2'')$ such that $d(a_2'', x'') = d(a_2'', y'') = \pi/2$ and $d(x'', y'') = d(x, y)$. Since $\angle_{y''}(a_2'', x'') = \pi/2$ and by (1) we have $\angle_y(a_2, x) > \angle_{y''}(a_2'', x'') - O(\varepsilon^{1/2})$. Applying Lemma 5.1(2) to $\triangle(y, x, a_2)$ and $\triangle''$ shows (2). □

Lemma 5.3. *Let $\varepsilon \in (0,1)$ be a positive number. For two small geodesic triangles $\triangle = \triangle(a_0, a_1, a_2)$ in Y and $\triangle' = \triangle(a_0', a_1', a_2')$ in $\mathbb{S}^2(1)$, we assume that*

(1) $d(a_2, x) > \pi/2 - \varepsilon$ *for any* $x \in [a_0, a_1]$;

(2) $d(a_2', x') = \pi/2$ *for any* $x' \in [a_0', a_1']$;

(3) $|d(a_0, a_1) - d(a_0', a_1')| < \varepsilon$.

For any four points $x_i \in [a_2, a_i]$, $x_i' \in [a_2', a_i']$, $i = 0, 1$, such that

$$d(a_2, x_i)/d(a_2, a_i) = d(a_2', x_i')/d(a_2', a_i'),$$

we have

$$|d(x_0, x_1) - d(x_0', x_1')| < O(\varepsilon^{1/4}). \tag{5.1}$$

Proof. Take such four points x_0, x_1, x_0', and x_1'. We may assume that $d(a_0', a_1') \geq 4\varepsilon^{1/2}$. Note that $d(a_0, a_1) \geq 3\varepsilon^{1/2}$. Take $\overline{y}_i \in [a_0, a_1]$ and $\overline{y}_i' \in [a_0', a_1']$ with $d(a_i, \overline{y}_i) = d(a_i', \overline{y}_i') = \varepsilon^{1/2}$ for $i = 0, 1$. Let $y_i \in [a_2, \overline{y}_i]$ and $y_i' \in [a_2', \overline{y}_i']$ be the points determined by

$$\frac{d(a_2, y_i)}{d(a_2, \overline{y}_i)} = \frac{d(a_2, x_i)}{d(a_2, a_i)}, \quad \frac{d(a_2', y_i')}{d(a_2', \overline{y}_i')} = \frac{d(a_2', x_i')}{d(a_2', a_i')}$$

(cf. Figure 1). Let $\widetilde{\triangle} = \triangle(\widetilde{\overline{y}}_0, \widetilde{\overline{y}}_1, \widetilde{a}_2)$ be a spherical comparison triangle in $\mathbb{S}^2(1)$ of $\triangle = \triangle(\overline{y}_0, \overline{y}_1, a_2)$, and $\widetilde{y}_0, \widetilde{y}_1 \in \widetilde{\triangle}$ the corresponding points to y_0, y_1. Considering the two geodesic triangles $\widetilde{\triangle}$ and $\triangle(\overline{y}'_0, \overline{y}'_1, a'_2)$ in $\mathbb{S}^2(1)$, we have $|d(\widetilde{y}_0, \widetilde{\overline{y}}_1) - d(y'_0, \overline{y}'_1)|, |d(\widetilde{y}_0, \widetilde{y}_1) - d(y'_0, y'_1)| < O(\varepsilon)$,

$$d(y_0, \overline{y}_1) < d(y'_0, \overline{y}'_1) + O(\varepsilon), \tag{5.2}$$

and $d(y_0, y_1) < d(y'_0, y'_1) + O(\varepsilon)$. By $d(x_i, y_i), d(x'_i, y'_i) < O(\varepsilon^{1/2})$, we have $d(x_0, x_1) < d(x'_0, x'_1) + O(\varepsilon^{1/2})$. To obtain the opposite inequality, it suffices to prove

$$d(y_0, y_1) > d(y'_0, y'_1) - O(\varepsilon^{1/4}). \tag{5.3}$$

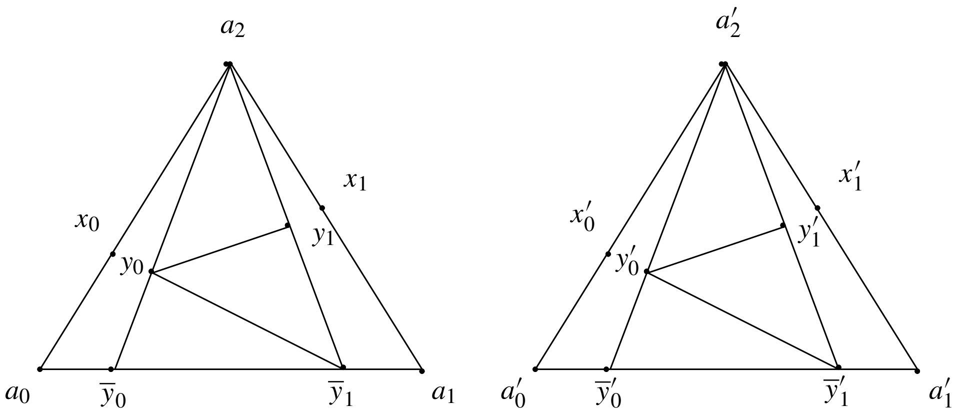

Figure 1. $\triangle = \triangle(a_0, a_1, a_2)$ and $\triangle' = \triangle(a'_0, a'_1, a'_2)$.

Applying Lemma 5.2(1) to $\triangle(a_0, \overline{y}_1, a_2)$ and $\triangle(\overline{y}_0, a_1, a_2)$ yields

$$\pi/2 - O(\varepsilon^{1/2}) < \angle_{\overline{y}_0}(a_2, \overline{y}_1),\ \angle_{\overline{y}_1}(a_2, \overline{y}_0) < \pi/2 + O(\varepsilon^{1/2}). \tag{5.4}$$

Consider $\triangle(\overline{y}_0, \overline{y}_1, y_0)$ and $\triangle(\overline{y}'_0, \overline{y}'_1, y'_0)$. By $\angle_{\overline{y}'_0}(a'_2, \overline{y}'_1) = \pi/2$ and (5.4), we have $\angle_{\overline{y}_0}(a_2, \overline{y}_1) > \angle_{\overline{y}'_0}(a'_2, \overline{y}'_1) - O(\varepsilon^{1/2})$. Hence Lemma 5.1(2) implies $d(y_0, \overline{y}_1) > d(y'_0, \overline{y}'_1) - O(\varepsilon^{1/2})$. This together with (5.2) implies

$$|d(y_0, \overline{y}_1) - d(y'_0, \overline{y}'_1)| < O(\varepsilon^{1/2}). \tag{5.5}$$

Therefore, by Lemma 5.1(1) we see that $\angle_{\overline{y}_1}(\overline{y}_0, y_0) < \angle_{\overline{y}'_1}(\overline{y}'_0, y'_0) + O(\varepsilon^{1/4})$. It follows from (5.4) and $\angle_{\overline{y}'_1}(a'_2, \overline{y}'_0) = \pi/2$ that

$$\begin{aligned} \angle_{\overline{y}_1}(y_0, y_1) &\geq \angle_{\overline{y}_1}(y_1, \overline{y}_0) - \angle_{\overline{y}_1}(\overline{y}_0, y_0) \\ &> \pi/2 - \angle_{\overline{y}'_1}(\overline{y}'_0, y'_0) - O(\varepsilon^{1/4}) = \angle_{\overline{y}'_1}(y'_0, y'_1) - O(\varepsilon^{1/4}). \end{aligned} \tag{5.6}$$

By (5.5), (5.6), and Lemma 5.1(2), we have (5.3). This completes the proof. □

5.2. Proof of Theorem 1.7. We need a lemma.

Lemma 5.4. *Let ε and l be positive numbers, and let $c\colon [0,l] \to Y$ be a 1-Lipschitz continuous curve from a point x_0 to a point x_1 in a metric space Y such that*

$$l < d(x_0, x_1) + \varepsilon. \tag{5.7}$$

Assume that there exists a minimizing geodesic $\gamma_{x_0x_1}$ joining x_0 to x_1. Then, for any $s \in [0,1]$, setting $x_s := \gamma_{x_0x_1}(s\, d(x_0,x_1))$ we have

$$d(x_s, c(sl)) < 2\varepsilon.$$

Note that the parameter of c is not necessarily proportional to the arc-length.

Proof. Since c is 1-Lipschitz continuous, it follows from (5.7) that

$$d(x_0, c(sl)) + d(c(sl), x_1) \le sl + (1-s)l < d(x_0,x_1) + \varepsilon,$$

and hence, by the triangle inequality,

$$0 \le sl - d(x_0, c(sl)) < \varepsilon. \tag{5.8}$$

By (5.7) and $d(x_s, x_1) \le (1-s)l$, we have

$$sl \ge d(x_0, x_s) = d(x_0, x_1) - d(x_s, x_1) > sl - \varepsilon.$$

Combining this and (5.8) yields

$$|d(x_0, x_s) - d(x_0, c(sl))| < 2\varepsilon.$$

By the triangle inequality, this completes the proof. □

Let Y be a CAT(1)-space with $\operatorname{diam} Y \le \pi/2$, and let $\rho\colon Y \to \mathbb{R}$ be the function defined by $\rho(x) := \sup_{y\in Y} d(x,y)$. By the definition, $\operatorname{rad} Y = \inf_{x\in Y} \rho(x) \le \pi/2$. We define the constant $\delta_m := \pi/2 - \operatorname{rad} \triangle_1^m$, where $\triangle_1^m$ is the standard spherical simplex defined in (1.1). δ_m is strictly monotone decreasing in $m = 1, 2, \dots$. Denote the barycenter of $\triangle_1^m$ by b_m'.

The *distortion* $\operatorname{dis}\varphi$ of a map $\varphi\colon A_1 \to A_2$ between metric spaces is defined by

$$\operatorname{dis}\varphi := \sup_{x,y\in A_1} |d(\varphi(x), \varphi(y)) - d(x,y)|.$$

We prove the following:

Lemma 5.5. *Let ε be a positive number with $\varepsilon \ll \delta_m$. Assume that there exists a 1-Lipschitz continuous map $\varphi_m\colon \triangle_1^m \to Y$ such that $\operatorname{dis}\varphi_m < \varepsilon$ and $\rho(b_m) > \pi/2 - \varepsilon$, where $b_m := \varphi_m(b_m')$. Then, there exists a 1-Lipschitz continuous map $\varphi_{m+1}\colon \triangle_1^{m+1} \to Y$ such that $\operatorname{dis}\varphi_{m+1} < O(\varepsilon^{1/8})$.*

Proof. Denote by $a'_0, \dots, a'_{m+1}$ the vertices of $\triangle_1^{m+1}$, and set $a_i := \varphi_m(a'_i)$. Let $\triangle_1^m \subset \partial\triangle_1^{m+1}$ be the face opposite to a'_{m+1}. There exists a point $a_{m+1} \in Y$ with $d(a_{m+1}, b_m) > \pi/2 - \varepsilon$. We construct a map $\varphi_{m+1} \colon \triangle_1^{m+1} \to Y$ as follows. For any given $x' \in \triangle_1^{m+1}$, the segment $[a'_{m+1}, x']$ extends to a segment $[a'_{m+1}, \bar{x}']$ with $\bar{x}' \in \triangle_1^m$. Set $\bar{x} := \varphi_m(\bar{x}')$. There is a unique point $x \in [a_{m+1}, \bar{x}]$ such that

$$\frac{d(a_{m+1}, x)}{d(a_{m+1}, \bar{x})} = \frac{d(a'_{m+1}, x')}{d(a'_{m+1}, \bar{x}')}.$$

We then define $\varphi_{m+1}(x') := x$. It follows that $\varphi_{m+1}(a'_{m+1}) = a_{m+1}$ and $\varphi_{m+1}|_{\triangle_1^m} = \varphi_m$. Note that φ_m and φ_{m+1} are not necessarily injective.

Let us prove that for any $z \in \varphi_m(\triangle_1^m)$,

$$d(a_{m+1}, z) > \pi/2 - O(\varepsilon^{1/2}). \tag{5.9}$$

Take a point $z' \in \triangle_1^m$ with $\varphi_m(z') = z$. The segment $[b'_m, z']$ extends to a segment $[z'_0, z'_1]$ with $z'_0, z'_1 \in \partial\triangle_1^m$. Since δ_m coincides with the radius of the inscribed sphere of $\triangle_1^m$ centered at b'_m, we have $d(b'_m, z'_i) \geq \delta_m$ for each $i = 0, 1$. Set $z_i := \varphi_m(z'_i)$. Consider the 1-Lipschitz continuous curve $c := \varphi_m \circ \gamma_{z'_0 z'_1}$ joining z_0 and z_1. Note that c passes through z and b_m. Choose a number $s \in [0, 1]$ with $c(s\, d(z'_0, z'_1)) = b_m$ and let $b := \gamma_{z_0 z_1}(s\, d(z_0, z_1))$ (cf. Figure 2). Since $\operatorname{dis} \varphi_m < \varepsilon$,

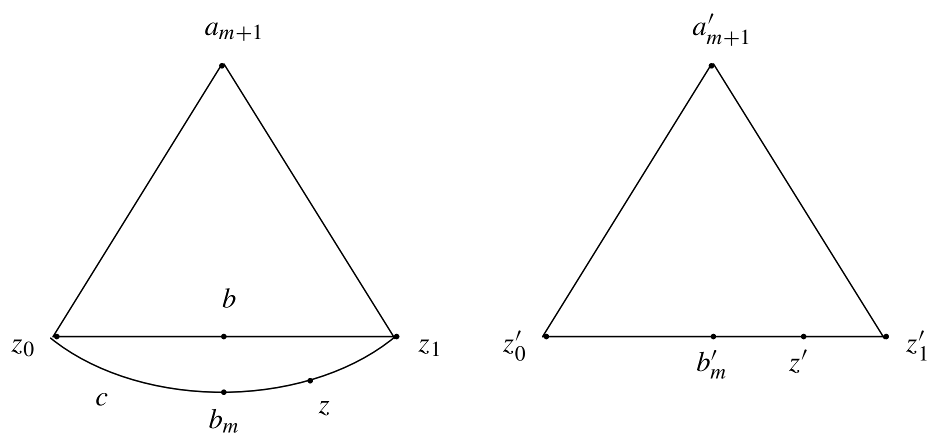

Figure 2. $\triangle(z_0, z_1, a_{m+1})$ and $\triangle(z'_0, z'_1, a'_{m+1})$.

we see that $d(z'_0, z'_1) < d(z_0, z_1) + \varepsilon$. Lemma 5.4 implies that $d(b, b_m) < 2\varepsilon$ and so $d(a_{m+1}, b) > \pi/2 - 3\varepsilon$ by the assumption for a_{m+1}. By $\varepsilon \ll \delta_m$ we have

$$d(b, z_i) > d(b'_m, z'_i) - 3\varepsilon \geq \delta_m - 3\varepsilon \geq \varepsilon^{1/2}$$

for each $i = 0, 1$. Applying Lemma 5.2(2) to $\triangle(z_0, z_1, a_{m+1})$ yields that $d(a_{m+1}, y) > \pi/2 - O(\varepsilon^{1/2})$ for any $y \in [z_0, z_1]$. Therefore, by Lemma 5.4 we obtain (5.9).

For any given two points $x'_0, x'_1 \in \triangle_1^{m+1}$, either segment $[a'_{m+1}, x'_i]$ extends to a segment $[a'_{m+1}, \bar{x}'_i]$ with $\bar{x}'_i \in \triangle_1^m$. Let $x_i := \varphi_{m+1}(x'_i)$ and $\bar{x}_i := \varphi_{m+1}(\bar{x}'_i)$. Since φ_m is 1-Lipschitz continuous, we have $d(\bar{x}_0, \bar{x}_1) \le d(\bar{x}'_0, \bar{x}'_1)$. Comparing $\triangle(\bar{x}_0, \bar{x}_1, a_{m+1})$ and $\triangle(\bar{x}'_0, \bar{x}'_1, a'_{m+1})$, the CAT(1)-inequality leads to $d(x_0, x_1) \le d(x'_0, x'_1)$. Thus, φ_{m+1} is 1-Lipschitz continuous. It remains to prove that

$$d(x_0, x_1) > d(x'_0, x'_1) - O(\varepsilon^{1/8}). \tag{5.10}$$

By Lemma 5.4, for any point $w \in [\bar{x}_0, \bar{x}_1]$ there exists a point $z \in \varphi_m([\bar{x}'_0, \bar{x}'_1])$ with $d(z, w) < 2\varepsilon$. This and (5.9) imply

$$d(a_{m+1}, w) > \pi/2 - O(\varepsilon^{1/2}). \tag{5.11}$$

Consider the small geodesic triangles $\triangle(\bar{x}_0, \bar{x}_1, a_{m+1})$ in Y and $\triangle(\bar{x}'_0, \bar{x}'_1, a'_{m+1})$ in a unit 2-sphere in $\triangle_1^{m+1}$. By $\operatorname{dis}\varphi_m < \varepsilon$, (5.11), and applying Lemma 5.3 to their triangles, we obtain (5.10). This completes the proof of Lemma 5.5. □

Let $n := \dim_C Y + 1 < \infty$ and let ε be a positive number with $\varepsilon \ll \delta_n$. To prove Theorem 1.7, we suppose that $\operatorname{rad} Y > \pi/2 - \varepsilon$. Note that $\rho(y) > \pi/2 - \varepsilon$ holds for any $y \in Y$. Take a point $a_0 \in Y$. There exists a point $a_1 \in Y$ with $d(a_1, a_0)\pi/2 - \varepsilon$. Let $\varphi_1 \colon \triangle_1^1 \to [a_0, a_1]$ be the linear bijective map. Since $\pi/2 - \varepsilon < d(a_0, a_1) \le \pi/2$, this is a 1-Lipschitz continuous map $\triangle_1^1 \to Y$ with $\operatorname{dis}\varphi_1 < \varepsilon$. By Lemma 5.5, we inductively have 1-Lipschitz continuous maps $\varphi_m \colon \triangle_1^m \to Y$, $m = 1, 2, \dots, n$, such that $\operatorname{dis}\varphi_m < O(\varepsilon^{1/8^m})$. Since $\dim \varphi_n(\triangle_1^n) \le \dim_C Y = n - 1$, Lemma 4.2 implies that φ_n collapses $\partial\triangle_1^n$. Hence, there exist $n+1$ points $y'_i \in F_i$, $i = 0, 1, \dots, n$, that are all mapped by φ_n to a common point of Y, where $F_i \subset \triangle_1^n$ is the opposite face to a'_i. We set

$$\alpha_n := \inf\{ \max_{i,j} d(x'_i, x'_j) \mid x'_i \in F_i,\ x'_i \neq x'_j \text{ for any } i \neq j \} > 0.$$

Then for some $i_0 \neq j_0$ we have $\alpha_n \le d(y'_{i_0}, y'_{j_0}) \le \operatorname{dis}\varphi_n < O(\varepsilon^{1/8^n})$, which gives a lower estimate of ε. Therefore we obtain $\operatorname{rad} Y < \pi/2 - \delta$ for some positive constant δ depending only on n.

Since Y is complete, $C(Y)$ consists of a single point (cf. Proposition 3.1 in [LS2]). This completes the proof of Theorem 1.7. □

Remark 5.6. For $A \subset Y$, we denote by $C_Y(A)$ the set of all points where the function $Y \ni x \mapsto \sup_{y \in A} d(x, y) \in [0, \infty]$ attains the infimum. For an arbitrary subset A of a CAT(1)-space Y with $\operatorname{diam} A \le \pi/2$, we have $\operatorname{diam} A = \operatorname{diam} B$

for the closure B of the convex hull of A (cf. Lemma 4.1 in [LS1]). By applying Theorem 1.7 to B, we obtain the following generalization. Let Y be a complete CAT(1)-space of $\dim_C Y < \infty$, and $A \subset Y$ a subset of $\operatorname{diam} A \leq \pi/2$. Then $\inf_{x \in Y} \sup_{y \in A} d(x, y) < \pi/2 - \delta$, and $C_Y(A)$ consists of a single point.

5.3. Proof of Theorem 1.3. Let f be a parabolic isometry of a proper CAT(0)-space X and let $B := X_f(\infty)$. It follows from Theorem 1.1 and Lemma 3.6(3) that B is a closed, π-convex subset of $(X(\infty), Td)$ with $\operatorname{rad} B \leq \pi/2$. Hence, B itself is a complete CAT(1)-space.

First, we verify that $C(B)$ is non-empty. Let $\rho\colon B \to \mathbb{R}$ be the function defined by $\rho(x) := \sup_{y \in B} d(x, y)$. There exists a sequence $\{x_i\}$ in B with $\rho(x_i) \to \operatorname{rad} B$ as $i \to \infty$. Since $X(\infty)$ is compact with respect to the sphere topology, some subsequence of $\{x_i\}$ converges to a point x. We have $x \in B$ because B is closed. By the lower semi-continuity of Tits distances, we have $\rho(x) \leq \operatorname{rad} B$. Thus, $\rho(x) = \operatorname{rad} B$ and $x \in C(B)$.

By the convexity of B, $C(B)$ is a closed, convex subset of $(X(\infty), Td)$ with the property that $\operatorname{diam} C(B) \leq \operatorname{rad} B \leq \pi/2$. By setting $Y := C(B)$, it is a complete CAT(1)-space of $\operatorname{diam} Y \leq \pi/2$. By Proposition 1.8, $\dim_C(X(\infty), Td)$ is finite. Therefore, by Theorem 1.7 we have $\operatorname{rad} Y < \pi/2$, and $C(Y)$ consists of a single point. Moreover, the second half follows from the uniqueness of the point and its property. This completes the proof of Theorem 1.3. □

Theorem 1.7 and the proof of Theorem 1.3 imply the following:

Proposition 5.7. *Let Y be a compact* CAT(1)*-space of* $\dim Y < \infty$ *and* $\operatorname{rad} Y \leq \pi/2$. *Then $C^2(Y)$ consists of a single point.*

Remark 5.8. Let X be a complete CAT(0)-space and G a subgroup of the isometry group of X. Set $X_G(\infty) := \bigcap\{X_g(\infty) \mid g \in G\}$. We say that G is *admissible* if $X_G(\infty) \neq \emptyset$ and $\operatorname{rad} X_G(\infty) \leq \pi/2$. It follows from Theorem 1.3 that if G is an abelian group containing a parabolic element, then G is admissible, provided X is as in Theorem 1.3. This is an extension of Proposition 4.4.2 of [E]. Similarly, we can obtain some extensions of Propositions 4.4.3, 4.4.4, and Corollary 4.4.5 of [E] for CAT(0)-spaces. Proposition 4.4.6 of [E] can be also extended by using the flat torus theorem for CAT(0)-spaces (cf. Theorem II.7.1 of [BH]).

6. Example of a symmetric space

In this section we discuss the symmetric space $\mathrm{SL}(3, \mathbb{R})/\mathrm{SO}(3, \mathbb{R})$ in detail as an example for Theorems 1.1 and 1.3. A good reference for standard facts we use here is II.10 in [BH]. We would like to thank M. Bestvina for suggesting this example, and also K. Wortman for useful discussions and informations.

6.1. Manifolds $P(n,\mathbb{R})$ and $P(n,\mathbb{R})_1$. Let $P(n,\mathbb{R})$ denote the space consisting of all positive definite, symmetric $(n\times n)$-matrices with real coefficients. Naturally, $P(n,\mathbb{R})$ is a differentiable manifold of dimension $n(n+1)/2$. The tangent space $T_pP(n,\mathbb{R})$ at a point p is naturally isomorphic (via translation) to the space of all symmetric $(n\times n)$- matrices, $S(n,\mathbb{R})$. The inner product $(u,v)_p = \operatorname{tr}(p^{-1}up^{-1}v)$ on $T_pP(n,\mathbb{R}) \simeq S(n,\mathbb{R})$ defines a Riemannian metric on $P(n,\mathbb{R})$, where $\operatorname{tr} u$ is the trace of a matrix u. $P(n,\mathbb{R})$ is a simply connected, complete, non-positively curved Riemannian manifold, so that it is a proper CAT(0)-space.

Let $P(n,\mathbb{R})_1 \subset P(n,\mathbb{R})$ be the subset of matrices with determinant 1. $P(n,\mathbb{R})_1$ is a totally geodesic submanifold, whose tangent space at p is the subspace in $S(n,\mathbb{R})$ of matrices with trace 0. $P(n,\mathbb{R})$ is a simply connected, complete, non-positively curved Riemannian manifold of dimension $n(n+1)/2-1$, so that it is a proper CAT(0)-space as well.

$\mathrm{SL}(n,\mathbb{R})$ acts on $P(n,\mathbb{R})$ by isometries according to the rule

$$f(p) := fp\,{}^tf, \quad p\in P(n,\mathbb{R}),\ f\in \mathrm{SL}(n,\mathbb{R}),$$

where ${}^tf$ is the transpose of f. The right hand side of the definition is by the multiplication of matrices. We may write $f\cdot p$ instead of $f(p)$. $P(n,\mathbb{R})_1$ is invariant by this action, and the action is transitive on this submanifold. Let e be the identity matrix. The stabilizer of e is $\mathrm{SO}(n)$, so that $P(n,\mathbb{R})_1$ is identified as $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

6.2. Geometry of $P(3,\mathbb{R})_1$ and Tits boundary. We collect some standard facts on $P(3,\mathbb{R})_1$ from II.10 in [BH]. Most of them are true for all $P(n,\mathbb{R})_1, n\geq 3$ with appropriate change. Put $X := P(3,\mathbb{R})_1$. X is a 5-dimensional, irreducible symmetric space of non-compact type of rank 2, which is a proper CAT(0)-space.

Let us denote the Tits boundary $(X(\infty), Td)$ by $X(\infty)$ for simplicity. $X(\infty)$ is a "thick spherical building" of dimension 1 such that each apartment is isometric to $\mathbb{S}^1(1)$ and each Weyl chamber at infinity is an edge of length $\pi/3$. Moreover, $\operatorname{diam} X(\infty)=\pi$. Since $X(\infty)$ is a spherical building, any two Weyl chambers at infinity are contained in at least one apartment.

The action of $\mathrm{SL}(3,\mathbb{R})$ induced on $X(\infty)$ is by simplicial isometries. It is transitive on pairs (A,c), where A is an apartment, and $c\subset A$ is a Weyl chamber at infinity. A Weyl chamber is a fundamental domain for the action. (cf. II.10.71, 75, 76, 77 in [BH]). Therefore there are two orbits in the vertices of $X(\infty)$ by the group action, so that $X(\infty)$ is a bi-partite graph. It follows that any loop in $X(\infty)$ consists of an even number of edges.

The isometry group of X, $I(X)$, has two connected components, and the one which contains the identity map, $I_0(X)$, is $\mathrm{SL}(3,\mathbb{R})$. Let σ be the involution of X at e, which is an orientation reversing isometry. It is given by $\sigma(f) = {}^tf^{-1}$. $I(X) = I_0(X)\cup \sigma I_0(X)$.

Let f be an isometry of X. $\mathrm{Min}(f)$ denotes the set of all points in X at which the displacement function d_f of f attains its infimum $|f| := \inf_{p\in X} d_f(p)$, which is the translation length. If f is elliptic, then $\mathrm{Min}(f)$ coincides with the fixed point set $\mathrm{Fix}(f)$ of f in X. If f is hyperbolic, then the axes of f are parallel to each other, and the union of their images is $\mathrm{Min}(f)$. If f is parabolic, then $\mathrm{Min}(f) = \emptyset$. f is said to be *semi-simple* if f is elliptic or hyperbolic.

In this section we calculate those geometric characters of $f \in \mathrm{SL}(3,\mathbb{R})$.

6.3. Real Jordan forms. It is known that $f \in \mathrm{SL}(3,\mathbb{R})$ is semi-simple as an isometry of X if and only if it is semi-simple as a matrix, i.e., diagonalizable in $\mathrm{GL}(3,\mathbb{C})$. (cf. II.10.61 in [BH]).

Calculation of $X_f(\infty)$ and $\mathrm{Min}(f)$ of $f \in \mathrm{SL}(3,\mathbb{R})$ is mostly by linear algebra. Each $f \in \mathrm{SL}(3,\mathbb{R})$ is conjugate to g in $\mathrm{SL}(3,\mathbb{R})$ such that g is one (and only one) of the following list. g is a real Jordan form of f. The symbol $\mathrm{diag}(a,b,c)$ is for the (3×3)-diagonal matrix with entries a, b, c.

Since f and g are conjugate in $I(X)$, f is elliptic, hyperbolic, or parabolic if and only if so is g, respectively. If $h \in I(X)$ is a conjugating element, i.e., $hfh^{-1} = g$, then $X_f(\infty) = h \cdot X_g(\infty)$, $\mathrm{Min}(f) = h \cdot \mathrm{Min}(g)$, and $|f| = |g|$. We discuss g instead of f.

List of real Jordan forms in SL(3, $\mathbb{R}$).

(1) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(2) $\begin{pmatrix} 1/a^2 & 0 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$, where $0, 1 \neq a \in \mathbb{R}$.

(3) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(4) $\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1/(a^2+b^2) \end{pmatrix}$ where $a, b \in \mathbb{R}$ with $a^2 + b^2 \neq 0$ and $b \neq 0$.

This one is conjugate to $\mathrm{diag}(a+ib, a-ib, 1/(a^2+b^2))$ by an element in $\mathrm{GL}(3,\mathbb{C})$.

(5) $\mathrm{diag}(a,b,c)$ such that $a, b, c \in \mathbb{R}$, $abc = 1$, $a \neq b \neq c \neq a$.

(6) $\mathrm{diag}(a,a,1/a^2)$, $a \in \mathbb{R}$, $a \neq 0, 1$.

(7) e.

6.4. Flat and Weyl chambers. Consider the following linear subspace in T_eX.

$$\mathfrak{a}_0 := \{u \mid u = \operatorname{diag}(u_1, u_2, u_3),\ \operatorname{tr} u = 0\} \subset T_eX.$$

Let

$$F_0 := \{ \exp(u) \mid u \in \mathfrak{a}_0 \} \subset X.$$

F_0 is a flat plane in X and $A_0 := F_0(\infty)$ is an apartment in $X(\infty)$.

For $x \in A_0$, let γ_{ex} be the geodesic in F_0 from e to x. γ_{ex} is $\exp(tu(x)), t \geq 0$ for some $u(x) \in \mathfrak{a}_0$. The tangent vector at e, $u(x)$, is uniquely determined by x up to scaling by a positive number, so that let us denote the one of unit length by $u(x)$, which we write as $\operatorname{diag}(u_1(x), u_2(x), u_3(x))$.

A_0 is a 6-gon as a building with the following Weyl chambers (see Figure 3). $\{u_i(x) \geq u_j(x) \geq u_k(x)\}$ means the set $\{x \in A_0 \mid u_i(x) \geq u_j(x) \geq u_k(x)\}$.

$$\begin{aligned}
c_1 &:= \{ u_1(x) \geq u_2(x) \geq u_3(x) \}, & c_2 &:= \{ u_2(x) \geq u_1(x) \geq u_3(x) \},\\
c_3 &:= \{ u_2(x) \geq u_3(x) \geq u_1(x) \}, & c_4 &:= \{ u_3(x) \geq u_2(x) \geq u_1(x) \},\\
c_5 &:= \{ u_3(x) \geq u_1(x) \geq u_2(x) \}, & c_6 &:= \{ u_1(x) \geq u_3(x) \geq u_2(x) \}.
\end{aligned}$$

$A_0 = c_1 \cup \cdots \cup c_6$. Define $v_i := c_{i-1} \cap c_i$, $i = 1, 2, \ldots, 6$, where $c_0 = c_6$. They are the vertices of $X(\infty)$ in A_0. We may write $c_i = [v_i, v_{i+1}]$, $1 \leq i \leq 6$, where $v_7 = v_1$.

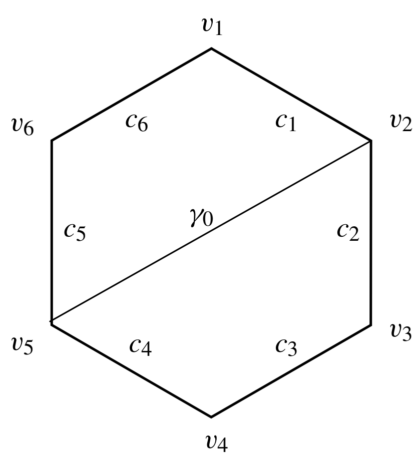

Figure 3. 6-gon.

A bi-infinite geodesic, or simply line, in X is always contained in some flat plane because X is a symmetric space. If a line is contained in a unique flat, then it is called regular (cf. 10.46 [BH]), and otherwise it is called singular. There are three singular bi-infinite geodesics (without orientation) on F_0, which are $\gamma_{ev_1} \cup \gamma_{ev_4}$, $\gamma_{ev_2} \cup \gamma_{ev_5}$, $\gamma_{ev_3} \cup \gamma_{ev_6}$.

Set $w_2 := u(v_2) = (1/\sqrt{6})\,\mathrm{diag}(1, 1, -2)$. This is a unit vector at e tangent to F_0, pointing the vertex v_2 at infinity. Define a line in F_0 by

$$\gamma_0(t) := \exp(t w_2), \quad t \in \mathbb{R}.$$

γ_0 is a line through e such that $\gamma_0(\infty) = v_2, \gamma_0(-\infty) = v_5$. As a set, $\gamma_0 = \{\mathrm{diag}(s, s, 1/s^2) \mid 0 < s \in \mathbb{R}\} = \gamma_{ev_2} \cup \gamma_{ev_5}$. γ_0 is a singular geodesic.

For a line γ in X, let $P(\gamma)$ denote the union of all lines in X parallel to γ. This is a convex subset in X, so that let $P(\gamma)(\infty) \subset X(\infty)$ denote the set of points at infinity of $P(\gamma)$.

Denote by $\mathcal{F}_0$ the set of all flat planes in X containing γ_0. Then, $P(\gamma_0) = \bigcup\{F \mid F \in \mathcal{F}_0\}$. $P(\gamma_0)$ is a totally geodesic, 3-dimensional submanifold, which is naturally isometric to $P(2, \mathbb{R})_1 \times \mathbb{R}$ (cf. Proposition II.10.67 in [BH]). $P(2, \mathbb{R})_1$ is isometric to $\mathbb{H}^2$ up to a scaling factor. We note that $P(\gamma_0)(\infty) = \bigcup\{F(\infty) \mid F \in \mathcal{F}_0\}$.

6.5. Theorem

Theorem 6.1. *Suppose $g \in \mathrm{SL}(3, \mathbb{R})$ is one in the list of Subsection 6.3. Then we have the following in the order of the list:*

(1) *g is parabolic and $X_g(\infty)$ is the union of all edges incident to c_2. $X_g(\infty)$ is not compact in $(X(\infty), Td)$, with uncountably many edges; $|g| = 0$.*

(2) *g is parabolic and $X_g(\infty) = c_1 \cup c_2 \cup c_3$; $|g| = 2\sqrt{6}\log|a|$.*

(3) *g is parabolic and $X_g(\infty) = c_1$; $|g| = 0$.*

(4) *g is semi-simple, and*

  (a) $X_g(\infty) = \{v_2, v_5\}$.

  (b) *If $a^2 + b^2 = 1$, then g is elliptic, and* $\mathrm{Fix}(g) = \gamma_0$.

  (c) *If $a^2 + b^2 \neq 1$, then g is hyperbolic and $|g| = \sqrt{6}\log(a^2 + b^2)$;* $\mathrm{Min}(g) = \gamma_0$.

(5) *g is hyperbolic, and $|g| = 2\sqrt{(\log|a|)^2 + (\log|b|)^2 + (\log|c|)^2}$.*

  (a) $X_g(\infty) = A_0$.

  (b) $\mathrm{Min}(g) = F_0$.

(6) *g is hyperbolic, and $|g| = 2\sqrt{6}\log|a|$.*

  (a) $X_g(\infty) = P(\gamma_0)(\infty)$.

  (b) $\mathrm{Min}(g) = P(\gamma_0)$.

(7) *g is the identity map, i.e. elliptic, with* $\mathrm{Fix}(g) = X$ *and $X_g(\infty) = X(\infty)$.*

6.6. Stabilizers. The analysis of the stabilizing subgroup in $\mathrm{SL}(3,\mathbb{R})$ of a point $v \in X(\infty)$ is important for the proof of the theorem. We quote Proposition II.10.64 in [BH] in the following form.

Lemma 6.2. *Let $g = (g_{ij}) \in \mathrm{SL}(3,\mathbb{R})$, and $x \in A_0$. Then $g(x) = x$ if and only if $g_{ij}e^{-t(u_i(x)-u_j(x))}$ converges as $t \to \infty$ for all i, j.*

This implies the following.

Proposition 6.3. *Let G_i be the subgroups of* $\mathrm{SL}(3,\mathbb{R})$ *stabilizing v_i. Then,*

$$G_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}, \quad G_2 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}, \quad G_3 = \left\{ \begin{pmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{pmatrix} \right\},$$

$$G_4 = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \right\}, \quad G_5 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \right\}, \quad G_6 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \right\},$$

where $* \in \mathbb{R}$.

Let H_1 be the following subgroup, parameterized by $t \in \mathbb{R}$, which fixes edges c_1, c_2, c_3, pointwise.

$$H_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

H_1 fixes $v_1 \in X(\infty)$. H_1 acts transitively on the set of all edges incident to v_1 other than c_1. To see it, consider the following subgroup in $\mathrm{SL}(3,\mathbb{R})$ containing H_1.

$$J_1 = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \;\middle|\; * \in \mathbb{R} \right\}.$$

For a given edge $c \neq c_1$, incident to v_1, we will find $h \in H_1$ with $h(c) = c_6$. Take an apartment, A, containing c and c_3. Then it automatically contains c_1, c_2 as well. Recall that $\mathrm{SL}(3,\mathbb{R})$ acts transitively on the set of pairs of an apartment, A', in $X(\infty)$ and a Weyl chamber, c', in A', (A', c'). Take $j \in \mathrm{SL}(3,\mathbb{R})$ which maps (A, c) to (A_0, c_6). Clear that $j \in J_1$ since it fixes c_1, c_2, c_3. Let $j = \left(\begin{smallmatrix} p & 0 & 0 \\ 0 & q & s \\ 0 & 0 & r \end{smallmatrix}\right)$, $pqr = 1$. Take $k = \left(\begin{smallmatrix} 1/p & 0 & 0 \\ 0 & 1/q & 0 \\ 0 & 0 & 1/r \end{smallmatrix}\right) \in \mathrm{SL}(3,\mathbb{R})$. Then $kj = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & s/q \\ 0 & 0 & 1 \end{smallmatrix}\right) = h \in H_1$. We have $k(c_i) = c_i$ for all i, so it follows from $j(c) = c_6$ that $h(c) = kj(c) = k(c_6) = c_6$.

Also, H_1 acts transitively on the set of all edges incident at v_4 other than c_3. Each of the two sets is parameterized by t by the action.

Off-diagonal entries except for the $(2,3)$-entries of matrices in H_1 are 0. Since there are 6 off-diagonal entries in (3×3)-matrices, we consider 5 other similar subgroups, H_2, H_3, H_4, H_5, H_6, which we define later.

6.7. Proof. We discuss each case in the order and prove Theorem 6.1. Case 7 is trivial.

Case 1. g is parabolic since it is not diagonalizable as a matrix. By Proposition 6.3, $X_g(\infty) \cap A_0 = c_1 \cup c_2 \cup c_3$. To see that any edge, $c \neq c_2, c_3$, incident to v_3 is fixed by g, take a (unique) element $h \in H_6$ such that $h(c) = c_3$, where

$$H_6 = \left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Since $g(c_3) = c_3$, $h^{-1}gh(c) = c$. Then it follows from $hg = gh$ that $g(c) = c$, pointwise.

To see any edge $c \neq c_2$, incident to v_2 is fixed by g, take a (unique) element $h \in H_2$ such that $h(c) = c_1$, where

$$H_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

As before, we have $hg = gh$, therefore $g(c) = c$, pointwise. We know $X_g(\infty)$ has no more edges because it is connected and its diameter is at most π.

The edges in $X_g(\infty)$ other than c_1, c_2, c_3 are those which are parametrized by H_2 and the others which are parametrized by H_6, so that uncountable. It is not compact because the mid points of the edges are at least $\pi/3$ apart from each other.

Let us prove $|g| = 0$ by computation. To deal with Case 3 at one time, suppose $g = \left(\begin{smallmatrix} 1 & k & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. It suffices for us to show that there exists a geodesic, $\gamma(t)$, such that $\lim_{t\to\infty} d(g(\gamma(t)), \gamma(t)) = 0$. We use the notations from the subsection 6.4. For $x \in A_0$, set $\gamma(t) = \gamma_{ex}(t) = \exp(tu)$, where u is the diagonal matrix $u(x) = \operatorname{diag}(u_1, u_2, u_3)$. For simplicity, we write the result of the action by a group element g on a point p as $g \cdot p$, instead of $g(p)$, in this discussion. Then,

$$\begin{aligned} d(g \cdot \exp(tu), \exp(tu)) &= d\left(\exp\left(-\frac{t}{2}u\right) \cdot g \cdot \exp(tu), \exp\left(-\frac{t}{2}u\right) \cdot \exp(tu)\right) \\ &= d\left(\exp\left(-\frac{t}{2}u\right) g \exp\left(\frac{t}{2}u\right) \cdot e, e\right), \end{aligned}$$

because $\exp\left(\frac{t}{2}u\right)\cdot e = \exp(tu)$. By the computation of matrix multiplications, $\exp\left(-\frac{t}{2}u\right) g \exp\left(\frac{t}{2}u\right)$ is

$$\begin{pmatrix} 1 & k\exp(t(u_2-u_1)/2) & 0 \\ 0 & 1 & \exp(t(u_3-u_2)/2) \\ 0 & 0 & 1 \end{pmatrix}.$$

For g in Case 1, we have $k=0$, so that if $u_2 > u_3$, then as $t \to \infty$, this matrix tends to e, which means that $d(g\cdot\gamma(t), \gamma(t)) \to 0$. We got $|g| = 0$. We remark that $u_2 > u_3$ is satisfied for $x \in A_0$ if and only if $x \in (c_1 \cup c_2 \cup c_3)\setminus(v_1 \cup v_4)$.

For g in Case 3, we have $k=1$. So, if $u_1 > u_2 > u_3$, then $d(g\cdot\gamma(t), \gamma(t)) \to 0$, which shows that $|g| = 0$. The condition $u_1 > u_2 > u_3$ holds for $x \in A_0$ if and only if $x \in c_1\setminus(v_1 \cup v_2)$.

Case 2. As in Case 1, g is parabolic and $X_g(\infty)\cap A_0 = c_1\cup c_2\cup c_3$. To see there are not more edges than those in $X_g(\infty)$, suppose there was an edge, $c \neq c_2, c_3$, incident to v_3 with $g(c) = c$. Take, as before, $h \in H_6$ such that $h(c) = c_3$.

$$h = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, $hgh^{-1}(c_3) = c_3$. Since $c \neq c_3$, we have $t \neq 0$, which is important to get a contradiction in this case. By computation

$$hgh^{-1} = \begin{pmatrix} 1/a^2 & 0 & t(a-1/a^2) \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}.$$

Since $t(a-1/a^2) \neq 0$, hgh^{-1} is not in G_4, so that does not fix $v_4 \in c_3$, a contradiction.

To see there is no edge $c \neq c_1, c_2$ at v_2 with $g(c) = c$, use H_2, as before. If there was, take $h \in H_2$ with $h(c) = c_1$ such that

$$h = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t \neq 0.$$

Then $hgh^{-1}(c_1) = c_1$, pointwise. By computation,

$$hgh^{-1} = \begin{pmatrix} 1/a^2 & 0 & 0 \\ t(1/a^2 - a) & a & 1 \\ 0 & 0 & a \end{pmatrix}$$

such that $t(1/a^2 - a) \neq 0$, therefore $hgh^{-1} \notin G_1$ does not fix v_1, which gives a contradiction since it is supposed to fix $c_1 = [v_1, v_2]$.

We get the claim because $X_g(\infty)$ is connected and with diameter at most π.

We postpone the computation of $|g|$ until the proof of Case 6. It is not a coincidence that $|g|$ is the same number in Cases 2 and 6.

Case 3. As in Case 1, g is parabolic and $X_g(\infty) \cap A_0 = c_1$. To see this is all, suppose there was an edge, $c \neq c_1$, incident to v_1 with $g(c) = c$. Take $h \in H_1$ such that $h(c) = c_6$. It follows that $hgh^{-1}(c_6) = c_6$. Let

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

By computation

$$hgh^{-1} = \begin{pmatrix} 1 & 1 & -t \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

therefore $hgh^{-1} \notin G_6$ does not fix $v_6 \in c_6$, a contradiction.

To see g does not fix any edge incident to v_2 other than c_1, use H_5 and do the same argument, where

$$H_5 = \left\{ \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

We get the claim since $X_g(\infty)$ is connected.

Since we already showed that $|g| = 0$ in the proof of Case 1, we finish Case 3.

Case 4. g is semi-simple because it is diagonalizable in $\mathrm{GL}(3, \mathbb{C})$.

(a) By Proposition 6.3, $X_g(\infty) \cap A_0 = \{v_2, v_5\}$. To see this is all, we first show that there is no edge incident to v_2 in $X_g(\infty)$. Suppose there was one, c. We know that $c \neq c_1, c_2$. Take $h \in H_2$ such that $h(c) = c_1$. If $h = \left(\begin{smallmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, then

$$hgh^{-1} = \begin{pmatrix} a - tb & b & 0 \\ -b(1+t^2) & a + tb & 0 \\ 0 & 0 & 1/(a^2+b^2) \end{pmatrix},$$

which does not fix v_1 because $-b(1+t^2) \neq 0$. But $hgh^{-1}(c_1) = hg(c) = h(c) = c_1$, so that it fixes $v_1 \in c_1$, a contradiction. Similarly there is no edge in $X_g(\infty)$ incident to v_5.

To finish, suppose there was a vertex, v, in $X_g(\infty) \backslash A_0$. Then $Td(v, v_2) = \pi$, because if it was less than π, then the unique geodesic from v to v_2 would have to be in $X_g(\infty)$, which is impossible since there is no edge incident to v_2 fixed by g. By the same reason, $Td(v, v_5) = \pi$. Consider a loop made of three geodesics: one from

v to v_2, one from v_2 to v_5 and one from v_5 to v. This loop consists of 9 edges, which is impossible because $X(\infty)$ is a bi-partite graph.

(b) Recall that we have $\gamma_0 = \{\mathrm{diag}(s, s, 1/s^2) \mid 0 < s \in \mathbb{R}\}$. Under the condition $a^2 + b^2 = 1$, by computation, $gp\,{}^t g = p$, $p \in P(3, \mathbb{R})_1$ if and only if $p = \mathrm{diag}(s, s, 1/s^2)$, $0 < s \in \mathbb{R}$. We get the claim.

(c). By computation, $g(\gamma_0) = \gamma_0$. Since $e \in \gamma_0$ and $e \neq g(e)$, g is hyperbolic, and γ_0 is an axis. The translation length, $|g|$, is $d(e, g(e)) = d(e, ge^t g) = d(e, g^t g)$, where

$$g^t g = \mathrm{diag}(a^2 + b^2, a^2 + b^2, 1/(a^2 + b^2)^2) = \exp[\log(a^2 + b^2)\,\mathrm{diag}(1, 1, -2)].$$

Since the norm $|\,\mathrm{diag}(1, 1, -2)|_e = \sqrt{6}$, $|g| = \sqrt{6}\log(a^2 + b^2)$.

We know that $\gamma_0 \subset \mathrm{Min}(g) \subset P(\gamma_0)$ because an axis of g is parallel to γ_0. Since g leaves γ_0 invariant, g leaves $P(\gamma_0)$ invariant as well. We remark that the action of g is by a shift and a rotation about γ_0. We define the following subgroup in $\mathrm{SL}(3, \mathbb{R})$, which is in fact in $\mathrm{SO}(3)$.

$$R = \{h = (h_{ij}) \in \mathrm{SO}(3) \mid (h_{ij})_{1 \le i,j \le 2} \in \mathrm{SO}(2),\ h_{33} = 1, \\ h_{13} = h_{31} = h_{23} = h_{32} = 0\}.$$

If $h \in R$, h fixes e, v_2, v_5, so that h fixes all points on γ_0. Therefore h leaves $P(\gamma_0)$ invariant, and acts on $\mathcal{F}_0$.

Claim. The action of R on $\mathcal{F}_0$ is transitive.

To see it, let $F \in \mathcal{F}_0$ be a flat. Then there is an element $w \in T_e X$ such that w and w_2 commute as matrices and the image by exp of the subspace spanned by w, w_2 in $T_e X$ is F. The two commuting symmetric matrices w, w_2 are simultaneously diagonalizable by an element, h, in $\mathrm{SO}(3)$. Moreover since w_2 is diagonal, one may assume that h commutes with w_2. By computation, this implies that h is in R. We claim that h maps F to F_0. Indeed, let γ be the geodesic through e defined by $\gamma = \exp(sw)$, $s \in \mathbb{R}$. It is in F. Since $h(\gamma_0) = \gamma_0$, it suffices to show $h(\gamma) \subset F_0$. Since $h \in \mathrm{SO}(3)$,

$$h(\gamma) = h\exp(sw)^t h = h\exp(sw)h^{-1} = \exp(shwh^{-1}),$$

which is in F_0 because hwh^{-1} is diagonal. We got the claim.

Suppose there was an axis of g, α, which is not γ_0. Take the plane $F \in \mathcal{F}_0$ which contains α. Such F exists since α is parallel to γ_0. Take $h \in R$ with $h(F) = F_0$. Since h commutes with g, $h(\alpha)$ is an axis of g as well. It implies that F_0 is invariant by g, so that $F_0(\infty) \subset X_g(\infty)$, which is impossible. We got $\mathrm{Min}(g) = \gamma_0$. We finished Case 4.

We are left with those g which are diagonal. Suppose $e \neq g = \mathrm{diag}(a, b, c) \in \mathrm{SL}(3, \mathbb{R})$. g is a semi-simple isometry and the flat F_0 is g-invariant, so that $A_0 \subset X_g(\infty)$.

Set

$$N = \sqrt{(\log|a|)^2 + (\log|b|)^2 + (\log|c|)^2}$$

and define a unit length element $u_g \in T_e F_0$ by

$$u_g := \frac{1}{N}\operatorname{diag}(\log|a|, \log|b|, \log|c|).$$

Let γ_g be the bi-infinite geodesic in F_0 through e defined by

$$\gamma_g(t) := \exp(tu_g), \quad t \in \mathbb{R}.$$

This is of unit speed. Computation shows that γ_g is g-invariant, therefore it is an axis. $|g| = d(e, g(e)) = d(e, g^t g)$. $g^t g = \operatorname{diag}(a^2, b^2, c^2) = \exp(2Nu_g) = \gamma_g(2N)$. Since γ_g has unit speed, $|g| = 2N$.

There are two cases: γ_g is regular (Case 5) or singular (Case 6). We already know that g is hyperbolic and calculated $|g|$.

Case 5.

(a) Since F_0 is invariant by g, $A_0 = F_0(\infty) \subset X_g(\infty)$. Let $v \in X_g(\infty)$. Then there is a flat F with $\gamma_g \subset F$ and $v \in F(\infty)$. Indeed, if γ is a bi-infinite geodesic through e with $\gamma(\infty) = v$, then since $g(v) = v$, γ and γ_g is on some flat.

Since γ_g is a regular geodesic, it is contained in only one flat, so that $F = F_0$. We get $v \in F_0(\infty) = A_0$.

(b) Since F_0 is g-invariant, $F_0 \subset \operatorname{Min}(g)$. $\operatorname{Min}(g)$ consists of axes of g. Let γ be an axis different from γ_g. Then there is a flat strip between them, so that there is indeed a flat, F, containing both of them because it is in a symmetric space. Since γ_g is regular, we have $F = F_0$, so that γ is in F_0.

Case 6.

(a) As in Case 5, $A_0 \subset X_g(\infty)$. Since g commutes with any element in R, $X_g(\infty)$ is R-invariant, so that $R \cdot A_0 \subset X_g(\infty)$. $R \cdot A_0 = P(\gamma_0)(\infty)$ implies that $P(\gamma_0)(\infty) \subset X_g(\infty)$. To see the other inclusion, let $v \in X_g(\infty)$. Then there is a flat, F, such that $\gamma_0 \subset F$ and $v \in F(\infty)$ (cf. (a) in Case 5). By definition, $F \in \mathcal{F}_0$, so that $F \subset P(\gamma_0)$. We get $v \in P(\gamma_0)(\infty)$.

(b) In this case, $\gamma_g = \gamma_0$. F_0 is g-invariant, so that $F_0 \subset \operatorname{Min}(g)$. Since g commutes with any element in R, $\operatorname{Min}(g)$ is R-invariant, so that $R \cdot F_0 \subset \operatorname{Min}(g)$. Because $R \cdot F_0 = P(\gamma_0)$, $P(\gamma_0) \subset \operatorname{Min}(g)$. On the other hand, since $P(\gamma_0)$ is the union of all geodesics parallel to γ_0, $\operatorname{Min}(g) \subset P(\gamma_0)$, therefore $\operatorname{Min}(g) = P(\gamma_0)$. Case 6 is done.

To finish the proof, we show $|g| = 2\sqrt{6}\log|a|$ for g in Case 2. It is easy to see that g is conjugate in $\mathrm{SL}(3,\mathbb{R})$ to the matrix, $h = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1/a^2 \end{pmatrix}$, so that it suffices to show $|h| = 2\sqrt{6}\log|a|$.

Although γ_0 is not h-invariant, h fixes $\gamma_0(\infty) = v_2, \gamma_0(-\infty) = v_5$ because $h \in G_2 \cap G_5$, so that h leaves not only the subset $P(\gamma_0)$, but also its product structure $P(\gamma_0) = \mathbb{H}^2 \times \mathbb{R}$ invariant.

The restriction of h to $P = P(\gamma_0)$, $h|_P$, is also parabolic. Since P is convex in X and h-invariant, $|h|_P| = |h|$, so that we compute $|h|_P|$. $h|_P$ acts on $P = \mathbb{H}^2 \times \mathbb{R}$ by a product of isometries: a parabolic isometry on $\mathbb{H}^2$, denoted by $h|_{\mathbb{H}^2}$, and a translation on $\mathbb{R}$, denoted by $h|_{\mathbb{R}}$. Since $|h|_{\mathbb{H}^2}| = 0$, we have $|h|_P| = |h|_{\mathbb{R}}|$.

Consider the following matrix in $\mathrm{SL}(3, \mathbb{R})$, $k = \begin{pmatrix} 1 & -1/a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This is also a parabolic isometry, which leaves $P(\gamma_0)$ invariant such that it acts on it as a product of isometries of $\mathbb{H}^2$ and $\mathbb{R}$. The action of k on $\mathbb{R}$ is trivial since the $(3,3)$-entry of k is 1. This is because one can show by computation that the geodesic from $e \in X$ to $k(e)$ is perpendicular to γ_0 at e, so that $k(e) \in \mathbb{H}^2$ in $\mathbb{H}^2 \times \mathbb{R}$. Or, one may use the fact that $P(\gamma_0)$ is the union of matrices of the following form in $P(3, \mathbb{R})_1$; $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$, where the set of top-left (2×2)-matrices corresponds to $\mathbb{H}^2$ and the $(3,3)$-entries, which are positive numbers, (by taking log) correspond to $\mathbb{R}$ in the product decomposition $P(\gamma_0) = \mathbb{H}^2 \times \mathbb{R}$. By the definition of the action, k acts trivially on the second factor. Therefore, $|hk|_P| = |h|_P|$. By the same reason as h, $|hk|_P| = |hk|$. By computation, $hk = \mathrm{diag}(a, a, 1/a^2)$, which is hyperbolic. We have just computed that $|\mathrm{diag}(a, a, 1/a^2)| = 2\sqrt{6} \log |a|$. To summarize,

$$|g| = |h| = |h|_P| = |hk|_P| = |hk| = 2\sqrt{6} \log |a|.$$

We finished the proof. $\square$

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