Bubbling location for *F*-harmonic maps and inhomogeneous Landau–Lifshitz equations

Yuxiang Li and Youde Wang*

Abstract. Let f be a positive smooth function on a closed Riemann surface (M, g). The f-energy of a map u from M to a Riemannian manifold (N, h) is defined as

$$E_f(u) = \int_M f|\nabla u|^2 \, dV_g.$$

In this paper, we will study the blow-up properties of Palais–Smale sequences for E_f. We will show that, if a Palais–Smale sequence is not compact, then it must blow up at some critical points of f. As a consequence, if an inhomogeneous Landau–Lifshitz system, i.e. a solution of

$$u_t = u \times \tau_f(u) + \tau_f(u), \quad u \colon M \to S^2,$$

blows up at time ∞, then the blow-up points must be the critical points of f.

1. Introduction

Let (M, g) and (N, h) be two Riemannian manifolds. A C^1-smooth map u from M into N is called a harmonic map if and only if u is a critical point of the energy functional $E(v)$, which is defined in local coordinates by

$$E(v) \equiv \int_M \text{Trace}_g(v^*h) \, dV_g,$$

where

$$\text{Trace}_g(v^*h) = g^{ij} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} h_{\alpha\beta}(u).$$

It is well known that the energy functional is conformally invariant when $\dim(M) = 2$.

In this paper we would like to study a class of C^1-smooth maps from a Riemann surface into a compact Riemannian manifold which are defined as the critical points of the inhomogeneous energy functional written as

$$E_f(v) \equiv \int_M \mathrm{Trace}_g(v^*h)f \, dV_g,$$

where f is a smooth real function. In [L] and [E-L] (see page 48, (10.20)), such maps are called f-harmonic from M into N. Obviously, they are just harmonic maps if $f \equiv 1$. Moreover, when $m = \dim(M) \neq 2$, an f-harmonic map is nothing but a harmonic map from $\left(M, f^{\frac{2}{n-2}}g\right)$ to (N, h). In local coordinates, the f-harmonic map satisfies the following Euler–Lagrange equation

$$f\tau(u) + \nabla f \cdot \nabla u = 0.$$

Here $\tau(u)$ is the tension field of u which can be written as

$$\tau^\alpha(u) = \Delta_g u^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j}.$$

To see the physical motivation for the f-harmonic maps, we consider a smooth domain Ω in the Euclidean space $\mathbb{R}^m$. An inhomogeneous Heisenberg spin system is given by

$$\partial_t u = f(u \wedge \Delta u) + \nabla f \cdot (u \wedge \nabla u),$$

where f is a real-valued function defined on Ω, $u(x, t) \in S^2$, $\wedge$ denotes the cross products in $\mathbb{R}^3$ and Δ is the Laplace operator on $\mathbb{R}^m$. Physically, the function f is called the coupling function, and is the continuum limit of the coupling constants between the neighboring spins. It is easy to see that if u is a smooth stationary solution of the above equation, then u is just an f-harmonic map from Ω into S^2. Indeed, in this case the tension field of u can be written as $\Delta u + |\nabla u|^2 u$, therefore, the right hand side of the above equation can be expressed by $u \wedge (f\tau(u) + \nabla f \cdot \nabla u)$, and u satisfies the following equation

$$f\tau(u) + \nabla f \cdot \nabla u = 0.$$

The above inhomogeneous Heisenberg spin system is also called inhomogeneous Landau–Lifshitz system. Landau and Lifshitz also suggested considering the following dispersive system

$$\partial_t u = u \wedge (f\tau(u) + \nabla f \cdot \nabla u) - u \wedge (u \wedge (f\tau(u) + \nabla f \cdot \nabla u)),$$

with an initial value condition

$$u(0) = u_0.$$

For the well-known equation, Tang [T] proved that it admits a global weak solution which is smooth except for finitely many points, if the domain manifold M is 2-dimensional closed , f is a smooth positive function and the initial value map belongs to $W^{1,2}(M, S^2)$ (see also [St] and [G-H]). The bubbles which the weak solution blows are called the magnetic bubbles ([Sh]). A natural question arises: *Where do the bubbling points of the Landau–Lifshitz equation locate?* In this paper, we intend to answer this problem partially.

Throughout this paper, we will always assume that f is smooth and positive. In order to answer the above question, mathematically we need to consider the convergence and bubbling of the sequence of f-harmonic maps with coupling function f. Precisely, we obtain the following results.

Theorem 1. *Let D be the unit disc in* $\mathbb{R}^2$. *If* $u\colon D \setminus \{0\} \to N$ *is a* $W^{2,2}_{\text{loc}}$-*map with finite energy and satisfies the following equation*

$$\tau(u) = \alpha \nabla u + g,$$

where $\alpha \in C^0(D)$ *and* $g \in L^p(D, TN)$ *for some* $p > 2$, *then u may be extended to a map* $\tilde{u} \in W^{2,p}(D, N)$.

Theorem 2. *Let* (M, g) *be a closed Riemann surface and N a compact submanifold of* $\mathbb{R}^K$. *Let* f *be a smooth positive function on M. Assume that* $u_k \in W^{2,2}(M, N)$ *is a sequence which satisfies*

$$f\tau(u_k) + \nabla f \cdot \nabla u_k = \alpha_k$$

and

$$\int_M |\nabla u_k|^2 f \, dV_g \leq C,$$

where α_k *lies in* $L^2(u_k^{-1}(TN))$ *and satisfies*

$$||\alpha_k||_{L^2} \to 0 \quad as \ k \to +\infty.$$

If p is a blow-up point of the sequence, i.e.,

$$\lim_{r \to 0} \liminf_{k \to +\infty} \int_{B_r(p)} |\nabla u_k|^2 f \, dV_g > 0,$$

then p must be a critical point of f.

Applying the above theorem to the inhomogeneous Landau–Lifshitz equation we can partially answer the above question. Concretely, we come to the following result.

Theorem 3. *Let (M, g) be a closed Riemann surface, and let S^2 be the unit sphere with standard metric. Suppose that the coupling function f is smooth and positive on M and $u \in L^2((0, \infty); W^{2,2}(M, S^2))$ is the unique weak solution for the initial value problem of the inhomogeneous Landau–Lifshitz equation with initial map $u_0 \in W^{1,2}(M, S^2)$. If $u(t) \equiv u(\cdot, t)$ blows up at time infinity, then the blow-up points must be the critical points of the coupling function f.*

2. Removable singularity

It is well known that the removable singularity theorem of Sacks and Uhlenbeck says that a harmonic map from $D \setminus \{0\} \to N$ with finite energy can be extended to 0 smoothly. The main aim of this section is to generalize Sacks–Uhlenbeck's theorem to the present case, i.e., to prove Theorem 1. The method adopted here is essentially due to Sacks and Uhlenbeck. One still sees that the Hopf differential is the key in the proof. However, in our case the Hopf differential is no longer holomorphic, thus the proof will be a little more delicate than theirs.

Let us first recall the ε-regularity discovered by Sacks and Uhlenbeck.

Lemma 2.1. *Suppose that $u \in W^{2,2}(D, N)$ satisfies*

$$\tau(u) = g \in L^2(D, TN).$$

Then there exits $\varepsilon > 0$ such that if $\int_D |\nabla u|^2 \leq \varepsilon$ we have

$$||u - \bar{u}||_{W^{2,2}(D_{\frac{1}{2}})} \leq C(||\nabla u||_{L^2(D)} + ||g||_{L^2(D)}).$$

Here $\bar{u}$ is the mean value of u over the unit disc and $D_{\frac{1}{2}}$ is a disc with radius $\frac{1}{2}$ and centered at the origin.

Proof. Cf. [S-U], or [D], or [D-T]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Using the standard elliptic estimate, we have

Corollary 2.2. *Suppose that $u \in W^{2,2}(D, N)$ satisfies*

$$\tau(u) = \alpha \nabla u + g, \qquad\qquad\qquad\qquad (2.1)$$

where $\alpha(x) \in C^0(D)$ and $g \in L^p(D, TN)$ for some $p > 2$. Then there exists $\varepsilon > 0$ such that whenever $\int_D |\nabla u|^2 \leq \varepsilon$ we have

$$|\nabla u|(0) \leq C(||\alpha||_{C^0(D)}, p)(||\nabla u||_{L^2(D)} + ||g||_{L^p(D)}).$$

In this section, we always assume u to be a map from $D \setminus \{0\}$ to N which belongs to $W^{2,2}_{loc}(D \setminus \{0\}, N)$ and satisfies the equation (2.1). In order to prove Theorem 1, we need to prove the following lemmas. First, we have

Lemma 2.3. *There exists $\varepsilon > 0$ such that if $\int_D |\nabla u|^2 dx < \varepsilon$, then there holds true*

$$|x||\nabla u|(x) \le C(||\nabla u||_{L^2(D_{2|x|})} + |x|^{2-\frac{2}{p}} \times ||g||_{L^p(D_{2|x|})}) \quad \text{for all } x \in D_{\frac{1}{2}},$$

where C is a positive constant which depends only on ε.

Proof. Fix an $x_0 \in D_{\frac{1}{2}}$, we define $\tilde{u} = u(x|x_0| + x_0)$. Then we have

$$\tau(\tilde{u}) = |x_0|^2 g + |x_0|\nabla \tilde{u}.$$

Notice that $|\nabla \tilde{u}|(0) = |\nabla u||x_0|$, hence we get this lemma from Corollary 2.2.   □

Now, let

$$\Psi = \langle u_x, u_x \rangle - \langle u_y, u_y \rangle - 2i \langle u_x, u_y \rangle = 4 \left\langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial z} \right\rangle,$$

where $z = x + iy$. It is easy to see that

$$
\begin{aligned}
\partial_{\bar{z}} \Psi &= 8 \left\langle \Delta u, \frac{\partial u}{\partial z} \right\rangle = 8 \left\langle A(u)(du, du) + \alpha \nabla u + g, \frac{\partial u}{\partial z} \right\rangle \\
&= 8 \left\langle \alpha \nabla u + g, \frac{\partial u}{\partial z} \right\rangle.
\end{aligned}
\tag{2.2}
$$

We need to prove a Stokes type equality for the 1-form $z\Psi$.

Lemma 2.4. *There holds true that*

$$\int_{|z|=r} z\Psi \, dz = \int_{D_r} z\partial_{\bar{z}} \Psi \, d\bar{z} \wedge dz.$$

Proof. As

$$d(z\Psi dz) = \partial_{\bar{z}}(z\Psi dz) = z\partial_{\bar{z}}\Psi d\bar{z} \wedge dz,$$

by applying the Stokes formula, for any $r_0 < r$ we have

$$\int_{|z|=r \setminus |z|=r_0} z\Psi \, dz = \int_{D_r \setminus D_{r_0}} z\partial_{\bar{z}}\Psi \, d\bar{z} \wedge dz.$$

By (2.2),

$$\int_{D_{r_0}} |z\partial_{\bar{z}}\Psi d\bar{z} \wedge dz| \le Cr_0 \int_{D_{r_0}} (|\alpha \nabla u|^2 + |g|^2) \, dx \to 0$$

as $r_0 \to 0$. Therefore, to complete the proof of the lemma, we only need to prove

$$\int_{|z|=r_0} z \Psi \, dz = \sqrt{-1} \int_0^{2\pi} z^2 \Psi \, d\theta|_{|z|=r_0} \to 0.$$

However this last equality follows from Lemma 2.3. □

Lemma 2.5. *There holds*

$$\int_{D_r} \langle u_r, u_r \rangle - \int_{D_r} \langle u_\theta, u_\theta \rangle = O(r).$$

Proof. By a direct computation, we have

$$\text{Re}(z^2 \Psi) = -|u_\theta|^2 + |z|^2 |u_r|^2.$$

Then

$$\left| \text{Re} \int_0^{2\pi} z^2 \Psi \, d\theta \right| = \left| \text{Im} \int_{|z|=r} z \Psi \, dz \right| = \left| \text{Im} \int_{D_r} z \partial_{\bar{z}} \Psi \, dz \wedge d\bar{z} \right|$$

$$\leq \int_{D_r} |z \partial_{\bar{z}} \Psi \, dz \wedge d\bar{z}|$$

$$\leq r \int_{D_r} |\alpha \nabla u + g| |\nabla u| \, dx$$

$$\leq C r \int_{D_r} (|g|^2 + |\alpha \nabla u|^2) \, dx,$$

i.e.,

$$\int_0^{2\pi} |u_r(r, \theta)|^2 r^2 \, d\theta - \int_0^{2\pi} |u_\theta(r, \theta)|^2 \, d\theta = O(r).$$

Therefore

$$\int_{D_r} (\langle u_r, u_r \rangle - \langle u_\theta, u_\theta \rangle) = \int_0^r \int_0^{2\pi} \left(|u_r|^2 - \frac{1}{r^2} |u_\theta|^2 \right) r \, d\theta \, dr$$

$$= \int_0^r \frac{1}{r} O(r) \, dr = \int_0^r O(1) \, dr$$

$$= O(r). \qquad \square$$

Proof of Theorem 1. As in [S-U] we approximate u by the function q which is harmonic on every domain,

$$D_m(r_0) = \{z : 2^{-m-1} r_0 < |z| < 2^{-m} r_0\},$$

and equals

$$\frac{1}{2\pi} \int_0^{2\pi} u(2^{-m}r_0, \theta) \, d\theta$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} u(2^{-m-1}r_0, \theta) \, d\theta,$$

respectively, on the boundaries $\{z : |z| = 2^{-m}r_0\}$ and $\{z : |z| = 2^{-m-1}r_0\}$. Then q is piecewise linear in $\log r$ and depends only on the radial coordinate. Now, for $2^{-m-1}r_0 \le r \le 2^{-m}r_0$,

$$
\begin{aligned}
|q(r) - u(r, \theta)| &\le |q(r) - q(2^{-m}r_0)| + |q(2^{-m}r_0) - u(r, \theta)| \\
&\le |q(2^{-m-1}r_0) - q(2^{-m}r_0)| \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} |u(2^{-m}r_0, \theta') - u(r, \theta)| \, d\theta' \\
&\le C \sup_{2^{-m-1}r_0 \le |r| \le 2^{-m}r_0} r|\nabla u| \\
&\le C(\|\nabla u\|_{L^2(D_{2r})} + r^{2-\frac{2}{p}} \|g\|_{L^p(D_{2r})}).
\end{aligned}
$$

Now, we estimate the difference between q and u:

$$
\int_{D_{r_0}} |\nabla(u - q)|^2 = \sum_{m=0}^\infty r \int_0^{2\pi} (u(r, \theta) - q(r))(u_r(r, \theta) - q'(r)) \, d\theta \Big|_{2^{-m-1}r_0}^{2^{-m}r_0} \\
- \int_{D_r} (q - u)\Delta(q - u) \, dx.
$$

(2.3)

Since $q'(r) = \text{constant} \times \frac{1}{r}$ on $D_m(r_0)$,

$$\int_0^{2\pi} (u(r, \theta) - q(r))q'(r) \, d\theta = 0, \quad \text{for all } r = 2^{-m}r_0.$$

Hence,

$$
\begin{aligned}
\sum_{m=0}^\infty r \int_0^{2\pi} &(u(r, \theta) - q(r))(u_r(r, \theta) - q'(r)) \, d\theta \Big|_{2^{-m-1}r_0}^{2^{-m}r_0} \\
&= r_0 \int_0^{2\pi} (u(r_0, \theta) - q(r_0))u_r(r_0, \theta) \, d\theta \\
&\quad - \lim_{m\to+\infty} 2^{-m}r_0 \int_0^{2\pi} (u(2^{-m}r_0, \theta) - q(2^{-m}r_0))u_r(2^{-m}r_0, \theta)) \, d\theta.
\end{aligned}
$$

By Lemma 2.3, we have

$$2^{-m} r_0 \int_0^{2\pi} (u(2^{-m} r_0, \theta) - q(2^{-m} r_0)) u_r(2^{-m} r_0, \theta) \, d\theta$$

$$\leq ||u(2^{-m} r_0, \theta) - q(2^{-m} r_0)||_{L^\infty} \sup_{r=2^{-m} r_0} r |\nabla u(r, \theta)| \to 0$$

as $m \to +\infty$.

Moreover, we have

$$r_0 \int_0^{2\pi} (u(r_0, \theta) - q(r_0)) u_r(r_0, \theta) \, d\theta$$

$$\leq r_0 \left(\int_0^{2\pi} (u(r_0, \theta) - q(r_0))^2 \, d\theta \int_0^{2\pi} |u_r(r_0, \theta)|^2 d\theta \right)^{\frac{1}{2}}$$

$$\leq \left(\int_0^{2\pi} |u_\theta(r_0, \theta)|^2 d\theta \right)^{\frac{1}{2}} \left(\int_0^{2\pi} r_0^2 |u_r(r_0, \theta)|^2 d\theta \right)^{\frac{1}{2}} \tag{2.4}$$

$$\leq \frac{1}{2} \int_0^{2\pi} (|u_\theta(r_0, \theta)|^2 + |u_r(r_0, \theta)|^2 r_0^2) \, d\theta = \frac{r_0}{2} \int_0^{2\pi} |\nabla u(r_0, \theta)|^2 r_0 \, d\theta$$

and

$$\int_{D_{r_0}} |(q-u)(\Delta(q-u))| = \int_{D_{r_0}} |q-u| \times |A(u)(du, du) - \alpha \nabla u - g| \, dx$$

$$\leq ||q-u||_{L^\infty(D_{r_0})} (||A||_{L^\infty} \int_{D_{r_0}} |\nabla u|^2 dx + \sqrt{\pi} r_0 ||\alpha \nabla u + g||_{L^2(D_{r_0})}). \tag{2.5}$$

Obviously, for any $1 > \delta > 0$, we can always pick up r_0 which is small enough such that

$$\int_{D_{r_0}} |(q-u)(\Delta(q-u))| \leq \delta \left(\int_{D_{r_0}} |\nabla u|^2 dx + r_0 \right).$$

Applying Lemma 2.5, we get

$$\int_{D_{r_0}} |\nabla(u-q)|^2 \, dx \geq \int_{D_{r_0}} \langle u_\theta, u_\theta \rangle \, dx$$

$$= \frac{1}{2} \int_{D_{r_0}} (\langle u_\theta, u_\theta \rangle + \langle u_r, u_r \rangle) \, dx$$

$$+ \frac{1}{2} \int_{D_{r_0}} (\langle u_\theta, u_\theta \rangle - \langle u_r, u_r \rangle) \, dx \tag{2.6}$$

$$= \frac{1}{2} \int_{D_{r_0}} |\nabla u|^2 \, dx + O(r_0).$$

Then, from (2.3), (2.4), (2.5) and (2.6) we can derive that

$$\lambda \int_{D_{r_0}} |\nabla u|^2 \le r_0 \int_0^{2\pi} |\nabla u(r_0, \theta)|^2 r_0 \, d\theta + Cr_0,$$

where λ is a positive constant which is smaller than 1.

Set

$$f(r) = \int_{D_r} |\nabla u|^2 \, dx.$$

Then we have

$$\lambda f(r) < rf'(r) + Cr,$$

and hence

$$\left(\frac{f}{r^\lambda} \right)' \ge -Cr^{-\lambda}.$$

By integrating the above differential inequality over the interval $[r, 1]$ we obtain

$$f(r) \le Cr^\lambda \int_r^1 s^{-\lambda} ds + f(1) r^\lambda \le Cr^\lambda.$$

By applying Lemma 2.3, it follows from the above inequality that

$$|\nabla u|(x) \le |x|^{\lambda - 1}.$$

Thus, we can complete the proof of the theorem by standard elliptic estimate theory.

$\square$

3. A variational formula

For the inhomogeneous functional $E_f(\cdot)$ defined on $W^{1,2}(M, N)$, we can easily see that the first variational formula at point $u \in W^{2,2}(M, N)$ can be written as

$$dE_f(\xi) = \int_M \langle f\tau(u) + \nabla u \nabla f, \xi \rangle \, dV_g,$$

for any $\xi \in T_u W^{1,2}(M, N)$. Here, we need to derive another formula for $E_f(\cdot)$ with respect to the variation of the domain manifold. The following calculation is essentially due to Price ([P]).

Take a 1-parameter family of transformations $\{\phi_s\}$ of M which is generated by the vector field X. We have

$$
\begin{aligned}
E_f(u \circ \phi_s) &= \frac{1}{2} \int_M |\nabla(u \circ \phi_s)|^2 f(x) \, dV_g \\
&= \frac{1}{2} \int_M \sum_\alpha |d(u \circ \phi_s)(e_\alpha)|^2 f(x) \, dV_g(x) \\
&= \frac{1}{2} \int_M \sum_\alpha |du(\phi_{s*}(e_\alpha))|^2 (\phi_s(x)) f(x) \, dV_g(x) \\
&= \frac{1}{2} \int_M \sum_\alpha |du(\phi_{s*}(e_\alpha))|^2 (x) f(\phi_{-s}) \mathrm{Jac}(\phi_s^{-1}) \, dV_g,
\end{aligned}
$$

where $\{e_\alpha\}$ is a local orthonormal basis of TM. Noting

$$
\frac{d}{ds} \mathrm{Jac}(\phi_s^{-1}) \, dV_g \big|_{s=0} = -\mathrm{div}(X) \, dV_g, \qquad \frac{d}{ds} f(\phi_{-s}) = -df(X),
$$

we have

$$
\begin{aligned}
\frac{d}{ds} E_f(u \circ \phi_s)\big|_{s=0} = &-\frac{1}{2} \int_M |\nabla u|^2 f \mathrm{div}(X) \, dV_g - \frac{1}{2} \int_M df(X) |\nabla u|^2 \, dV_g \\
&+ \sum_\alpha \int_M \langle du(\nabla_{e_\alpha} X), du(e_\alpha)\rangle f \, dV_g.
\end{aligned}
$$

So, we have proved the formula

$$
\begin{aligned}
dE_f(u)(u_*(X)) = &-\frac{1}{2} \int_M |\nabla u|^2 f \mathrm{div}(X) dV_g - \frac{1}{2} \int_M df(X) |\nabla u|^2 dV_g \\
&+ \sum_\alpha \int_M \langle du(\nabla_{e_\alpha} X), du(e_\alpha)\rangle f \, dV_g.
\end{aligned}
$$

4. The proof of the theorems

The task of this section is to prove Theorems 2 and 3. In fact, what concerns Theorem 2 this just means the blow-up analysis for a so-called Palais–Smale sequence of $E_f(u)$. We will focus on what occurs if the sequence is not compact in the Sobolev space $W^{1,2}(M, N)$.

Proof of Theorem 2. By the assumptions stated in Theorem 2, $\{u_k\} \subset W^{2,2}(M, N)$ is a Palais–Smale sequence of maps from M into N. Then it satisfies

$$
f\tau(u_k) + \nabla f \nabla u_k = \alpha_k, \tag{4.1}
$$

$$E_f(u_k) \le C, \tag{4.2}$$

where $\alpha_k \in u_k^{-1}(TN)$ fulfills

$$||\alpha_k||_{L^2} \to 0. \tag{4.3}$$

First we note that in local complex coordinates (4.1) can be written as

$$f\tau_0(u_k) + \nabla_0 f \cdot \nabla_0 u_k = |\beta|\alpha_k,$$

where τ_0 and ∇_0 are the operators defined on $\mathbb{R}^2$ with standard Euclidean metric, since τ is a conformally invariant operator. Without loss of generality, we may assume that $g = dx^2 + dy^2$ in a complex coordinate system.

Set

$$\mathcal{S} \equiv \left\{ x : \lim_{r \to 0} \liminf_{k \to +\infty} \int_{D_r(x)} |\nabla u_k|^2 \, dV_g > 0 \right\}.$$

Usually, we say that x is a bubbling point for the sequence $\{u_k\}$ if and only if $x \in \mathcal{S}$. It is easy to see that $\mathcal{S}$ contains only finitely many points. By the Lemma 2.1, for any $x_0 \in \mathcal{S}$, we have

$$\liminf_{k \to +\infty} \int_{D_r(x)} |\nabla u_k|^2 \, dV_g > \varepsilon \quad \text{for any } r > 0.$$

By the weak compactness of $W^{1,2}(M, N)$ we know that there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, and $u \in W^{1,2}(M, N)$ with

$$E_f(u) < +\infty,$$

such that $\{u_k\}$ converges weakly to u in $W^{1,2}(M, N)$, which is an f-harmonic map. Moreover, Theorem 1, Lemma 2.1, Corollary 2.2 and elliptic estimate theory tell us that $u \in C^\infty(M, N)$ and

$$u_k \to u$$

in $W^{1,q}(\Omega, N)$ for any $\Omega \subset\subset M \setminus \mathcal{S}$ and $q > 1$.

Thus, to prove Theorem 2 we only need to show that

$$\mathcal{S} \subset \{\text{the critical points of } f\}.$$

Now, pick up a point $p \in S$. As we have pointed out, we may assume $g = dx^2 + dy^2$ in a complex coordinate chart $\mathcal{N}$ around p. Without loss of generality, we may assume that $p = (0, 0)$, $Q = [-1, 1] \times [-1, 1] \subset \mathcal{N}$ and $Q \cap S = \{p\}$. If p is not a critical point of f, then, without loss of generality, we may suppose that

$$df(0) = \lambda dx,$$

where λ is a positive constant. Thus, in a neighborhood of p, $df(x) = \lambda dx + O(r)$ where $r^2 = x^2 + y^2$.

We need to choose two functions to cut off the vector field $\frac{\partial}{\partial x}$ in x and y directions, respectively. First, we take a cut off function $\sigma \in C^\infty(\mathbb{R})$ which is 1 on $[-\delta, \delta]$, and 0 on $[-2\delta, 2\delta]^c$, where

$$\delta = \frac{\lambda \varepsilon}{16||\nabla u||^2_{C^0(Q)}||f||_{C^0}}.$$

Then we define the second function as follows:

$$\eta(t) = \begin{cases} 1 & \text{if } |t| \leq b', \\ (b-t)/(b-b') & \text{if } b' \leq t \leq b, \\ (b+t)/(b-b') & \text{if } -b \leq t \leq -b', \\ 0 & \text{if } t > b \text{ or } t < -b. \end{cases}$$

Here b and b' are chosen to satisfy $0 < a < b' < b < 2a < 1$, where a is a constant such that

$$\int_{[-2a,2a]\times[-1,1]} |\nabla u|^2 < \frac{\lambda \varepsilon}{8||\sigma'||_{C^0}||f||_{C^0}}. \tag{4.4}$$

Set

$$X = \eta(x)\sigma(y)\frac{\partial}{\partial x}.$$

By a direct computation we have

$$\text{div}(X) = \eta'(x)\sigma(y),$$

and

$$\sum_\alpha \langle du_k(\nabla_{e_\alpha} X), du_k(e_\alpha) \rangle = \eta'(x)\sigma(y)\left|\frac{\partial u_k}{\partial x}\right|^2 + \eta(x)\sigma'(y)\left\langle \frac{\partial u_k}{\partial x}, \frac{\partial u_k}{\partial y} \right\rangle.$$

By the formula derived in Section 3, we have

$$-\int_Q \eta'(x)\sigma(y)\left(\left|\frac{\partial u_k}{\partial x}\right|^2 - \left|\frac{\partial u_k}{\partial y}\right|^2\right) f\, dxdy - 2\int_Q \eta(x)\sigma'(y)\left\langle \frac{\partial u_k}{\partial x}, \frac{\partial u_k}{\partial y} \right\rangle f\, dxdy$$

$$= \int_Q (\lambda + O(r))\eta(x)\sigma(y)|\nabla u_k|^2\, dxdy + 2\int_Q \langle \alpha_k, u_{k*}(X) \rangle\, dxdy.$$

Note that

$$\text{supp}(\eta'(x)\sigma(y)) \cup \text{supp}(\eta(x)\sigma'(y)) \subset Q \setminus (-a, a) \times (-\delta, \delta),$$

we can replace Q in the left side of the above equality with $Q \setminus (-a, a) \times (-\delta, \delta)$.

For arbitrarily fixed a and δ, u_k is bounded in $W^{2,2}(Q \setminus (-\frac{a}{2}, \frac{a}{2}) \times (-\frac{\delta}{2}, \frac{\delta}{2}))$. So, by taking a subsequence, we have $\nabla u_k \to \nabla u$ in $L^2(Q \setminus (-a, a) \times (\delta, \delta))$. Therefore

$$
\begin{aligned}
- \int_Q \eta'(x)\sigma(y)&\left(\left|\frac{\partial u_k}{\partial x}\right|^2 - \left|\frac{\partial u_k}{\partial y}\right|^2 \right) f \, dxdy \\
&\longrightarrow - \int_Q \eta'(x)\sigma(y)\left(\left|\frac{\partial u}{\partial x}\right|^2 - \left|\frac{\partial u}{\partial y}\right|^2 \right) f \, dxdy,
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
- 2 \int_Q \eta(x)\sigma'(y)&\langle \frac{\partial u_k}{\partial x}, \frac{\partial u_k}{\partial y}\rangle f \, dxdy \\
&\longrightarrow -2 \int_{[-2a,2a]\times[-1,1]} \eta(x)\sigma'(y)\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\rangle f \, dxdy \\
&\le 2\|\sigma'\|_{C^0}\|f\|_{C^0} \int_{[-2a,2a]\times[-1,1]} |\nabla u|^2 \, dxdy \\
&< \frac{\lambda\varepsilon}{4},
\end{aligned}
\tag{4.6}
$$

where we have used (4.4) in the last inequality. Moreover, once δ and a are chosen, then

$$
\int_{[-a,a]\times[-\delta,\delta]} |\nabla u_k|^2 \, dxdy \ge \varepsilon
$$

when k is sufficiently large. Hence

$$
\int_Q (\lambda + O(r))\eta(x)\sigma(y)|\nabla u_k|^2 dxdy + 2 \int_Q \langle \alpha_k, u_{k*}(X)\rangle \, dxdy > \frac{1}{2}\lambda\varepsilon. \tag{4.7}
$$

In view of (4.5), (4.6) and (4.7), we have

$$
\int_Q \eta'(x)\sigma(y)\left(-\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 \right) f \, dxdy > \frac{1}{4}\lambda\varepsilon.
$$

Letting $b' \to b$, we get

$$
\int_{|x|=b} \sigma(y)\left(-\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 \right) f \, dy > \frac{1}{4}\lambda\varepsilon.
$$

Recall that $\mathrm{supp}\,\sigma \subset [-2\delta, 2\delta]$. Then

$$
\delta > \frac{\lambda\varepsilon}{16\|\nabla u\|_{C^0}^2\|f\|_{C^0}},
$$

which contradicts the definition of δ. This means that λ must be zero. Therefore p is a critical point of f. This completes the proof of Theorem 2. $\qquad\square$

Now we return back to our problem on the location of the bubbling points of the weak solutions to the inhomogeneous Landau–Lifshitz equations. Consider the following initial value problem:

$$\partial_t u = u \wedge (f(x)\Delta u + \nabla f \cdot \nabla u) - u \wedge (u \wedge (f(x)\Delta u + \nabla f \cdot \nabla u)),$$
$$u(0) = u_0(x) \in W^{1,2}(M, S^2).$$

Noting $|u|^2 \equiv 1$ and the identity

$$u \wedge (u \wedge (f(x)\Delta u + \nabla f \cdot \nabla u)) = (u \cdot (f(x)\Delta u + \nabla f \cdot \nabla u))u - (u \cdot u)(f(x)\Delta u + \nabla f \cdot \nabla u),$$

we can see easily that the above equations may be rewritten as

$$\left.\begin{aligned}
\partial_t u &= f(x)\tau(u) + \nabla f \cdot \nabla u + u \wedge (f(x)\tau(u) + \nabla f \cdot \nabla u), \\
u(0) &= u_0(x) \in W^{1,2}(M, S^2).
\end{aligned}\right\} \qquad (4.8)$$

Here $\tau(u) = \Delta u + |\nabla u|^2 u$ is the tension field of the map $u : M \to S^2$.

Tang has ever employed Struwe's method to study the existence and uniqueness of the above equation. We outline the argument in [T] as follows.

1. There exists $T > 0$ such that (4.8) is solvable in $M \times [0, T)$.

2. $u(t)$ blows up at finitely many points.

3. $u(t)$ converges to a $u(T) \in W^{1,2}(M, N)$ weakly, and on any sub-domain which does not contain a bubbling point, $u(t)$ strongly converges to $u(T)$ locally.

Then we construct a new flow which stems from u_T. Then, by the same argument as in [St], we know that there exists $T_1 > 0$ such that the new flow exists on the interval $[T, T_1)$ and blows up at T_1. At each bubbling point $u(t)$ blows one or more bubbles, i.e. one or more non-constant harmonic maps. It is well known that $u(t)$ must lose energy at every bubbling point. Hence, we always have a $\tilde{T}$ such that

$$\partial_t u = f(x)\tau(u) + \nabla f \cdot \nabla u + u \wedge (f(x)\tau(u) + \nabla f \cdot \nabla u),$$
$$u(0) = u(\tilde{T}) \in W^{1,2}(M, S^2)$$

is solvable on $[0, \infty)$. The results in [T] can be summarized in the following lemma.

Lemma 4.1. *Let (M, g) be a closed Riemann surface and f be a smooth positive function on M. For any $u_0 \in W^{1,2}(M, S^2)$ there exists a distribution solution $u : M \times \mathbb{R}^+ \to S^2$ of the above equation which is smooth on $M \times \mathbb{R}^+$ away from at most finitely many points (x_k, t_k), $1 \le k \le K_0$, $0 < t_k \le \infty$, which satisfies the energy inequality $E_f(u(s)) \le E_f(u(t))$ for all $0 \le s \le t$, and which assumes its initial data continuously in $W^{1,2}(M, S^2)$. The solution is unique in this class.*

It is easy to see that the following identity holds for the solution to (4.8) and any $0 < t_1 < t_2 \leq \infty$:

$$E_f(u(t_1)) - E_f(u(t_2)) = -\int_{t_1}^{t_2} ||\partial_t u||_{L^2}^2 dt.$$

Hence, it follows that

$$\int_0^{+\infty} ||\partial_t u||_{L^2}^2 < +\infty.$$

This implies that there exists a sequence $\partial_t u(x, t_i)$ such that

$$||\partial_t u(x, t_i)||_{L^2} \to 0.$$

So, $\{u(t_i)\}$ is a Palais–Smale sequence of $E_f(u)$. Therefore, if $u(t)$ does blow up at infinity, then by applying Theorem 2 we obtain the conclusion of Theorem 3.

As another example, we may also consider the gradient flow of the function E_f, i.e. a solution of

$$u_t = f(x)\tau(u) + \nabla f \cdot \nabla u,$$
$$u(0) = u_0(x) \in W^{1,2}(M, N).$$

If $u(t)$ does blow up at infinity, we also have results similar to Theorem 3.

References

[D]   W. Ding, An Introduction to harmonic maps. Manuscript.

[D-T]   W. Ding and G. Tian, Energy identity for a class of approximate harmonic maps from surfaces. *Comm. Anal. Geom.* **3** (1996), 543–554. Zbl 0855.58016 MR 1371209

[D-W]   W. Ding and Y. Wang, Local Schrödinger flow into Kähler manifolds. *Sci. China Ser. A* **44** (2001), 1446–1464. Zbl 1019.53032 MR 1877231

[E-L]   J. Eells and L. Lemaire, *Two reports on harmonic maps*. World Scientific Publishing Co., Inc., River Edge, NJ, 1995. Zbl 0836.58012 MR 1363513

[G-H]   B. Guo, and M. Hong, The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps. *Calc. Var. Partial Diff. Equations* **1** (1993), 311–334. Zbl 0798.35139 MR 1261548

[L]   A. Lichnerowicz, Applications harmoniques et variétés Kählériennes. In *1968/1969 Symposia Mathematica*, Vol. III (INDAM, Rome, 1968/69), Academic Press, London 1970, 341–402. Zbl 0193.50101 MR 0262993

[P]   P. Price, A monotonicity formula for Yang-Mills fields. *Manuscripta Math.* **43** (1983), 131–166. Zbl 0521.58024 MR 0707042

[S-U]   J. Sacks and K. Uhlenbeck, The existence of minimal immersions of 2-spheres. *Ann. of Math.* (2) **113** (1981), 1–24. Zbl 0462.58014 MR 0604040

[Sh]   J. Shatah, The stability of localized solutions of Landau-Lifshitz equations. *Comm. Pure Appl. Math* **55** (2002), 1136–1159. Zbl 1028.35059 MR 1908745

[St]   M. Struwe, On the evolution of harmonic maps of Riemannian surfaces. *Comment. Math. Helv.* **60** (1985) 558–581. Zbl 0595.58013 MR 0826871

[T]   H. Tang, Inhomogeneous Landau-Lifshitz systems from a Riemann surface into S^2. *Chinese J. Comtemp. Math.* **22** (2001), 215–230. Zbl 1007.53064 MR 1861308

Yuxiang Li, International Centre for Theoretical Physics, Mathematics Section, Strada Costiera 11, 34014 Trieste, Italy
E-mail: liy@ictp.it

Youde Wang, Institute of Mathematics, Academy of Mathematics and System Sciences, CAS, Beijing 100080, China
E-mail: wyd@math.ac.cn