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Cohomogeneity one hypersurfaces of Euclidean Spaces

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Abstract. We study isometric immersions $f: M^n \longrightarrow \mathbb{R}^{n+1}$ into Euclidean space of dimension $n+1$ of a complete Riemannian manifold of dimension n on which a compact connected group of intrinsic isometries acts with principal orbits of codimension one. We give a complete classification if either $n \geq 3$ and M^n is compact or if $n \geq 5$ and the connected components of the flat part of M^n are bounded. We also provide several sufficient conditions for f to be a hypersurface of revolution.

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1. Introduction

An interesting problem in submanifold theory is to study isometric immersions $f: M^n \longrightarrow \mathbb{R}^N$ into Euclidean space of a connected complete Riemannian manifold of dimension n acted on by a closed connected subgroup of its isometry group $\mathrm{Iso}(M^n)$. This study was initiated by Kobayashi [8], who proved that if $N = n + 1$ and M^n is compact and *homogeneous*, i.e., $\mathrm{Iso}(M^n)$ acts transitively on M^n, then $f(M^n)$ must be a round sphere.

In this paper we consider isometric immersions $f: M^n \longrightarrow \mathbb{R}^{n+1}$ of a complete Riemannian manifold M^n on which a *compact, connected* subgroup G of $\mathrm{Iso}(M^n)$ acts with maximal dimensional orbits of codimension one. We call f a *hypersurface of G-cohomogeneity one*. Observe that the group G may not be realizable as a group of extrinsic isometries of the ambient space. For instance, consider the cohomogeneity one action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ and isometrically immerse $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ as a cylinder over a plane curve. However, such examples can only arise if f is not rigid. Recall that f is *rigid* if any other isometric immersion $\tilde{f}: M^n \to \mathbb{R}^{n+1}$ differs from f by an isometry of $\mathbb{R}^{n+1}$.

Examples of cohomogeneity one hypersurfaces may be obtained as follows. Start with a cohomogeneity two compact subgroup $G \subset \mathrm{SO}(n+1)$, so that the orbit space $\mathbb{R}^{n+1}/G$ is a two dimensional manifold, possibly with boundary. Now consider a curve that is either contained in the interior of $\mathbb{R}^{n+1}/G$ or meets its boundary orthogonally. Then the inverse image of such a curve by the canonical projection onto the orbit space is a cohomogeneity one hypersurface. We shall call these examples the *standard examples*. Among them, the simplest ones are the *hypersurfaces of revolution*, which are invariant by the action of $\mathrm{SO}_l(n+1)$, the subgroup of $\mathrm{SO}(n+1)$ that fixes a straight line l.

Our main result states that, under natural global assumptions, the standard examples comprise all cohomogeneity one hypersurfaces.

Theorem 1.1. *Let $f: M^n \longrightarrow \mathbb{R}^{n+1}$ be a complete hypersurface of G-cohomogeneity one. Assume either that $n \geq 3$ and M^n is compact or that $n \geq 5$ and the connected components of the flat part of M^n are bounded. Then f is either rigid or a hypersurface of revolution. In particular, f is a standard example.*

We also provide several sufficient conditions for a hypersurface of G-cohomogeneity one as in Theorem 1.1 to be a hypersurface of revolution.

Theorem 1.2. *Under the assumptions of Theorem* 1.1*, any of the following conditions implies that f is a hypersurface of revolution:*

(i) *there exists a principal orbit with positive curvature;*

(ii) *there exists a principal orbit that is totally geodesic in M^n;*

(iii) *the principal orbits are umbilical in M^n;*

(iv) *$n \neq 4$ and there exists a principal orbit that is homeomorphic to a sphere.*

Moreover, in this case G is isomorphic to one of the closed subgroups of $\mathrm{SO}(n)$ *that act transitively on S^{n-1}.*

Theorem 1.2 generalizes and gives new (and shorter) proofs of various known results. Namely, it was proved under condition (iii) in [12] in the compact case for $n \geq 4$ and later in [9] in the general case (even for $n = 3, 4$). It was also proved in [4] (resp., [2]) in the compact case for $n \geq 5$ (resp., $n \geq 4$) under the assumption that *all* orbits have positive (resp., constant) sectional curvature. We also point out that closed subgroups of $\mathrm{SO}(n)$ that act transitively on the sphere are completely classified (cf. [7], p. 392).

2. The proofs

Given an isometric immersion $f: M^n \longrightarrow \mathbb{R}^{n+1}$, let A_{ξ_p} denote the shape operator of f at $p \in M^n$ with respect to a normal vector $\xi_p \in T_p^{\perp} M^n$, that is, the symmetric endomorphism of $T_p M^n$ given by $A_{\xi_p} X = -\tilde{\nabla}_X \xi$ for any $X \in T_p M^n$, where ξ is a smooth local normal vector field extending ξ_p and $\tilde{\nabla}$ stands for the derivative of $\mathbb{R}^{n+1}$. Recall that the *relative nullity subspace* of f at $p \in M^n$ is the kernel of A_{ξ_p}. It is well-known that on any open subset of M^n where the relative nullity subspaces of f have constant positive dimension, they define a smooth distribution whose leaves (called the *leaves of relative nullity*) are mapped by f onto open subsets of affine subspaces of $\mathbb{R}^{n+1}$.

Our approach to the study of hypersurfaces of cohomogeneity one is based on the following variant due to Ferus of a rigidity theorem of Sacksteder [14].

Theorem 2.1. *Let $f, \widetilde{f}\colon M^n \longrightarrow \mathbb{R}^{n+1}$ be isometric immersions of a complete Riemannian manifold of dimension $n \geq 3$. If there exists no complete leaf of relative nullity of dimension $n-1$ or $n-2$ (in particular if M^n is compact), then the shape operators of f and $\tilde{f}$ satisfy $A(p) = \pm\widetilde{A}(p)$ for every $p \in M^n$. As a consequence, if the subset of totally geodesic points of f does not disconnect M^n then f is rigid.*

The relation between the shape operators of f and $\tilde{f}$ in the statement means, more precisely, that $\tilde{A}_{\psi(\xi_p)} = \pm A_{\xi_p}$ for any $p \in M^n$ and for any $\xi_p \in T_p^{\perp} M_f^n$, where $\psi\colon T^{\perp}M_f^n \to T^{\perp}M_{\tilde{f}}^n$ is one of the two vector bundle isometries between the normal bundles of f and $\tilde{f}$.

By means of Theorem 2.1 we now derive the following result for hypersurfaces of G-cohomogeneity one, which is the main tool for the proofs of Theorems 1.1 and 1.2. We refer to [1] and the references therein for the basic facts on cohomogeneity one manifolds that are used in the sequel.

Proposition 2.2. *Let $f\colon M^n \longrightarrow \mathbb{R}^{n+1}$ be a complete hypersurface of G-cohomogeneity one. If either f is rigid or there exists no complete leaf of relative nullity of f of dimension $n-1$ or $n-2$ (in particular if M^n is compact), then*

(i) *B, the set of totally geodesic points of f, is G-invariant.*

(ii) *There exists a Lie group homomorphism $\Psi\colon G \longrightarrow \mathrm{SO}(n+1)$ such that $f \circ g = \Psi(g) \circ f$ for every $g \in G$, that is, f is G-equivariant.*

(iii) *If Σ is a principal orbit of G, then $f(\Sigma)$ is a principal orbit of the action of $\widetilde{G} = \Psi(G)$ on $\mathbb{R}^{n+1}$. In particular, $f(\Sigma)$ is an isoparametric hypersurface of a sphere.*

(iv) *If $f(\Sigma)$ is a round sphere for some principal orbit Σ of G, then f is a hypersurface of revolution and Ψ is a monomorphism. In particular, G is isomorphic to one of the closed subgroups of $\mathrm{SO}(n)$ that act transitively on S^{n-1}.*

Proof. Given $g \in G$, let A^g denote the shape operator of $f \circ g$. If f is rigid then $A^g = A$ for every $g \in G$. We claim that this is also the case if there exists no complete leaf of relative nullity of f of dimension $n-1$ or $n-2$. In fact, on one hand we have

$$g_*(p) \circ A^g(p) = A(g(p)) \circ g_*(p) \text{ for each } p \in M. \tag{1}$$

This implies that for each fixed $p \in M^n$ the map $\phi_p\colon G \to \mathrm{End}(T_pM^n)$ given by

$$\phi_p(g) = A^g(p) = (g_*(p))^{-1} \circ A(g(p)) \circ g_*(p)$$

is continuous. On the other hand, it follows from Theorem 2.1 that for each $p \in M^n$ either $A^g(p) = A(p)$ or $A^g(p) = -A(p)$. We obtain that ϕ_p is a continuous map taking values in $\{A(p), -A(p)\}$. Since G is connected and $\phi_p(\mathrm{I}) = A(p)$, our claim follows.

In particular, the set B^g of totally geodesic points of $f \circ g$ coincides with B for every $g \in G$. In view of (1), this is equivalent to saying that B is G-invariant. Moreover, by the Fundamental Theorem of Hypersurfaces, for each $g \in G$ there exists $\tilde{g} \in \mathrm{Iso}(\mathbb{R}^{n+1})$ such that $f \circ g = \tilde{g} \circ f$. It now follows from standard arguments (cf. [12]) that $\Psi\colon G \longrightarrow \mathrm{Iso}\,\mathbb{R}^{n+1}$, $\Psi(g) = \tilde{g}$, is a Lie-group homomorphism whose image lies in (a conjugacy class of) $\mathrm{SO}(n+1)$, because it is compact (and hence has a fixed point) and connected. Assertion (iii) now follows from (ii).

Finally, if $f(\Sigma)$ is a round sphere for some principal orbit Σ of G then, since G is connected, it must fix the line ℓ orthogonal to the linear span of $f(\Sigma)$. Hence f is a hypersurface of revolution with ℓ as axis. Moreover, the restriction of f to Σ must be injective. Since $f \circ g = \Psi(g) \circ f$ for any $g \in G$, if $\Psi(g) = I \in \mathrm{SO}(n+1)$ for some $g \in G$ we obtain that $g(y) = y$ for all $y \in \Sigma$. Now, since Σ is a principal orbit, this implies that, for every $y \in \Sigma$, g_* acts trivially on the normal space at y to the inclusion of Σ into M^n. As a consequence, if $\gamma\colon \mathbb{R} \to M^n$ is a normal geodesic through $y \in \Sigma$, i.e., a complete geodesic that crosses Σ (and hence any other G-orbit) orthogonally, then g fixes any point of $\gamma(\mathbb{R})$. Since every point of M^n lies in a normal geodesic through a point of Σ, we obtain that $g = I \in G$, and the last assertion in (iv) follows. □

Our next result classifies complete hypersurfaces of G-cohomogeneity one with dimension $n \geq 5$ that carry a complete leaf of relative nullity of dimension $n-2$.

Proposition 2.3. *Let $f\colon M^n \longrightarrow \mathbb{R}^{n+1}$, $n \geq 5$, be a complete hypersurface of G-cohomogeneity one. If there exists a complete leaf of relative nullity of dimension $n-2$ then $M^n = S^2 \times \mathbb{R}^{n-2}$ and f splits as $f = i \times \mathrm{id}$, where $i\colon S^2 \to \mathbb{R}^3$ is an umbilical inclusion and $\mathrm{id}\colon \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is the identity map. In particular, f is rigid.*

Proof. Since M^n carries a complete leaf of relative nullity $\mathcal{F}$, it can not be compact. Thus the orbit space $\Omega = M^n/G$ is homeomorphic to either $\mathbb{R}$ or $[0, \infty)$. Moreover, if $\pi\colon M^n \to \Omega$ denotes the canonical projection and $\gamma\colon \mathbb{R} \to M^n$ is a normal geodesic parameterized by arc-length, then $\pi \circ \gamma$ maps $\mathbb{R}$ homeomorphically onto Ω in the first case, and it is a covering map of $\mathbb{R} \setminus \{0\}$ onto the subset Ω^0 of internal points of Ω in the latter. Set $I = \gamma^{-1}(G(\mathcal{F}))$. Since $G(\mathcal{F})$ is a closed unbounded connected subset, using that $G(\mathcal{F}) = G(\gamma(I))$ it follows easily that if $I \neq \mathbb{R}$ then $I = [a, \infty)$ for some $a \in \mathbb{R}$ in the first case and $I = (-\infty, -b] \cup [a, \infty)$ for some $a, b > 0$ in the latter. Now observe that the type number of f (i.e., the rank of its shape operator) is everywhere equal to 2 on $G(\mathcal{F})$. This is because the relative nullity subspace coincides with the nullity of the curvature tensor at a point where the type number is at least 2, whence the subset where the type number is 2 is invariant under isometries. Let $(t_0 - \varepsilon, t_0 + \varepsilon) \subset I$ be such that $\Phi\colon (t_0 - \varepsilon, t_0 + \varepsilon) \times \Sigma_p \to \pi^{-1}((t_0 - \varepsilon, t_0 + \varepsilon))$ given by $\Phi(t, g(p)) = g(\gamma(t))$, $p = \gamma(t_0)$, is a G-equivariant diffeomorphism. We call $\Gamma = \pi^{-1}((t_0 - \varepsilon, t_0 + \varepsilon))$ a *tube* around Σ_p. We have a well-defined vector field ξ on Γ given by $\xi(y) = g_*(\gamma(t))\gamma'(t)$ for $y = g(\gamma(t))$, $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, and $\xi(y)$ is orthogonal to $\Sigma_{\gamma(t)}$ at y.

Now let η be a local unit normal vector field to f on Γ and A^f_η the shape operator of f with respect to η. Given a principal orbit $\Sigma_q = G(q) \subset \Gamma$ of G, the vector fields $\bar{\xi} = f_*(\xi|_{\Sigma_q})$ and $\bar{\eta} = \eta|_{\Sigma_q}$ determine an orthonormal normal frame of the restriction $f|_{\Sigma_q}\colon \Sigma_q \to \mathbb{R}^{n+1}$ of f to Σ_q. Denote by $A_{\bar{\eta}}$ and $A_{\bar{\xi}}$ the corresponding shape operators. Notice that $A_{\bar{\xi}} = A^i_{\xi}$, where $i\colon \Sigma_q \to M^n$ is the inclusion of Σ_q into M^n. Thus $A_{\bar{\xi}} \circ g_* = g_* \circ A_{\bar{\xi}}$ for any $g \in G$, hence the eigenvalues of $A_{\bar{\xi}}$ are constant. On the other hand, $A_{\bar{\eta}} = \Pi \circ A_\eta$, where Π is the orthogonal projection of TM^n onto $T\Sigma_q$. In particular, rank $A_{\bar{\eta}} \leq$ rank A_η, so we have rank $A_{\bar{\eta}} \leq 2$ on Σ_q. We have two cases to consider:

(i) rank $A_{\bar{\eta}} \leq 1$ on each principal orbit contained in Γ;

(ii) rank $A_{\bar{\eta}} = 2$ on some principal orbit contained in Γ.

First we show that (i) can not occur. Assume otherwise. Then, it follows from Theorem 2 of [4] that the principal orbits in Γ are either isometric to Euclidean spheres or isometrically covered by Riemannian products $\mathbb{R} \times S^{n-2}(a)$ (in what follows we suppose $a = 1$). In the former case, for each principal orbit $\Sigma_q \subset \Gamma$ it follows from the Gauss equation of the restriction $f|_{\Sigma_q}\colon \Sigma_q \to \mathbb{R}^{n+1}$ that $A_{\bar{\xi}}$ must be a multiple of the identity tensor, that is, the principal orbits in Γ are umbilical in M^n. This is in contradiction with Lemma 2.8 of [9], taking into account that $n \geq 5$ and that f has type number 2 on Γ.

Suppose now that the principal orbits are covered by $\mathbb{R} \times S^{n-2}$. In this case, for any fixed principal orbit $\Sigma_q \subset \Gamma$ there must exist an open subset $U_0 \subset \Sigma_q$ where rank $A_{\bar{\eta}} = 1$. In fact, otherwise $A_{\bar{\eta}}$ vanishes identically, hence the first normal spaces of $f|_{\Sigma_q}$ (i.e., the subspaces of the normal spaces spanned by the image of the second fundamental form) have dimension one everywhere (notice that $A_{\bar{\xi}}$ can not vanish anywhere, otherwise it would be identically zero and $f|_{\Sigma_q}$ would be totally geodesic, which is impossible). Then either $f(\Sigma_q)$ is contained in an affine hyperplane $\mathscr{H}$ of $\mathbb{R}^{n+1}$ or the first normal spaces of $f|_{\Sigma_q}$ are nonparallel along an open subset of Σ_q. Both possibilities lead to contradictions: the latter forces Σ_q to be flat (cf. [6], Theorem 1); in the former, since the shape operator of the isometric immersion $f\colon \Sigma_q \to \mathscr{H}$ is $A_{\bar{\xi}}$, which has constant eigenvalues, it follows that $f(\Sigma_q)$ is a round sphere, which is again impossible. We obtain that there exists an open subset $U \subset \Gamma$ where rank $A_{\bar{\eta}} = 1$ and $U \cap \Sigma_q = U_0$. Since the images of $A_{\bar{\eta}}$ and A_η are related by $\mathrm{Im}(A_{\bar{\eta}}) = \Pi(\mathrm{Im}(A_\eta))$, and on U the dimensions of $\mathrm{Im}(A_{\bar{\eta}})$ and $\mathrm{Im}(A_\eta)$ are 1 and 2, respectively, we must have $\xi \in \mathrm{Im}(A_\eta)$ everywhere on U. Therefore, at any point $x \in U$ we have that $\ker A_\eta(x) \subset T_x\Sigma_x$, and hence $\ker A_\eta(x) = \ker A_{\bar{\eta}}(x)$. It follows that the leaves of the distribution on U_0 given by $\ker A_{\bar{\eta}}$ are totally geodesic in Σ_q and $\mathbb{R}^{n+1}$. In particular, they are flat hypersurfaces of Σ_q. This is in contradiction with the fact that Σ_q is locally isometric to $\mathbb{R} \times S^{n-2}$. In fact, for any $x \in U_0$ let W be an $(n-2)$-dimensional subspace of $T_x(\Sigma_q)$ where the sectional curvatures of Σ_q are equal to 1 and let F_x be the totally geodesic flat hypersurface through x. Then $S = W \cap T_x(F_x)$ has dimension at least 2, since $n \geq 5$. At each bidimensional subspace of S, the sectional curvature of Σ_q is 1, because $S \subset W$ and, on the other hand, such a curvature must be zero, for $S \subset T_x(F_x)$. Therefore (i) is not possible, and we are left with (ii).

If rank $A_{\bar{\eta}} = 2$ along a principal orbit $\Sigma_q \subset \Gamma$, then rank $A_{\bar{\eta}} = 2$ on a possibly smaller tube around Σ_q contained in Γ, which we still denote by Γ. By Theorem 3 in [4], each principal orbit Σ_x contained in Γ is isometric to a Riemannian product $S^2(a) \times S^{n-3}(b)$ of spheres and $f|_{\Sigma_x}\colon \Sigma_x \to \mathbb{R}^{n+1}$ splits as a product $f|_{\Sigma_x} = i_1 \times i_2\colon S^2(a) \times S^{n-3}(b) \to \mathbb{R}^3 \times \mathbb{R}^{n-2} = \mathbb{R}^{n+1}$, where $i_1\colon S^2(a) \to \mathbb{R}^3$ and $i_2\colon S^{n-3}(b) \to \mathbb{R}^{n-2}$ are umbilical inclusions. Moreover, $\{\bar{\eta}, \bar{\xi}\}$ is precisely the orthonormal normal frame of $f|_{\Sigma_x}$ determined by the unit normal vector fields to the inclusions i_1 and i_2, respectively. In particular, $\bar{\xi}$ and $\bar{\eta}$ are parallel with respect to the normal connection of $f|_{\Sigma_x}$. Hence, $A_{\bar{\eta}}$ coincides with the restriction of A_η to $T\Sigma_x$, which in turn implies that ξ is an eigenvector of A_η along Γ. Now, since rank $A_\eta =$ rank $A_{\bar{\eta}} = 2$ on Γ, it follows that $\xi \in \ker A_\eta$. Therefore, the segments of normal geodesics in Γ are contained in the leaves of $\ker A_\eta$. Since these are assumed to be complete, we obtain that f has type number 2 on the whole M^n and that A_η is everywhere of the form $A_\eta = \mathrm{diag}(\varphi, \varphi, 0, \dots, 0)$, where φ is nonzero and constant along each principal orbit and the φ-eigenspaces of A_η (or $A_{\bar{\eta}}$) coincide with $\ker A_{\bar{\xi}} = \ker A^i_{\bar{\xi}}$. Now, let X be a vector field such that $A_\eta(X) = \varphi X$. By the Codazzi equation

$$\nabla_X(A_\eta(\xi)) - A_\eta(\nabla_X \xi) = \nabla_\xi(A_\eta(X)) - A_\eta(\nabla_\xi X)$$

we get

$$-A_\eta(\nabla_X \xi) = \xi(\varphi)X + \varphi \nabla_\xi X - A_\eta(\nabla_\xi X) = \xi(\varphi)X,$$

where the last equality follows from $\nabla_\xi X \in (\ker A_\eta)^\perp = \ker(A_\eta - \varphi I)$, using that $\ker A_\eta$ is totally geodesic. Since $\nabla_X \xi = -A^i_\xi(X) = -A_{\bar{\xi}}(X) = 0$, it follows that $\xi(\varphi) = 0$. Therefore φ is a constant, which we may suppose to be 1. Standard arguments now show that M^n splits as $M^n = S^2 \times \mathbb{R}^{n-2}$ (cf. [13]). By the main lemma in [10], f also splits as stated. □

Proof of Theorem 1.1. Suppose that f is not rigid. If M^n is compact and $n \geq 3$, it follows from Theorem 2.1 that B, the set of totally geodesic points of f, disconnects M^n. In order to get the same conclusion in the non-compact case, we must show that there does not exist a complete leaf of relative nullity of f of dimension $\ell = n - 1$ or $\ell = n - 2$. For $\ell = n - 1$ this follows from our assumption on the flat part of M^n. Proposition 2.3 takes care of the case $\ell = n - 2$.

Since B disconnects M^n, it must contain a regular point p. Then the (principal) orbit Σ through p is contained in B, because B is G-invariant by Proposition 2.2 (i). It follows from Lemma 3.14 of [5] that $f(\Sigma)$ is contained in a hyperplane $\mathcal{H}$ which is tangent to f along Σ, for Σ is connected. But $f(\Sigma)$ is an isoparametric hypersurface of a sphere by Proposition 2.2 (iii), and hence $f(\Sigma)$ must be a round hypersphere of $\mathcal{H}$. Proposition 2.2 (iv) now completes the proof. □

Remark 2.4. In case M^n is complete non-compact of dimension $n \geq 5$, the arguments in the beginning of the proof of Proposition 2.3 show, more precisely, that the conclusion of Theorem 1.1 fails only when every point of $G(\gamma(I))$ is flat, where $\gamma\colon \mathbb{R} \to M^n$ is a normal geodesic parameterized by arc-length and I is either $[a, \infty)$ for some $a \in \mathbb{R}$ or $(-\infty, -b] \cup [a, \infty)$ for some $a, b > 0$, according to the orbit space being homeomorphic to $\mathbb{R}$ or $[0, \infty)$, respectively. Notice that in the latter case M^n is flat outside a compact subset. We also point out that, since G is assumed to be compact, our assumption on the flat part of M^n is equivalent to M^n being *unflat at infinity* in the sense of [9].

Proof of Theorem 1.2. We already know from Theorem 1.1 that f is either rigid or a hypersurface of revolution. Thus, by Proposition 2.2 (iv) it suffices to prove that any of the conditions in the statement implies that $f(\Sigma)$ is a round sphere for some principal orbit Σ of G.

Assume first that Σ is a positively curved principal orbit. By Proposition 2.2 (iii), f immerses Σ as a positively curved isoparametric hypersurface of some hypersphere of $\mathbb{R}^{n+1}$. It follows easily from the Cartan identities for isoparametric hypersurfaces of the sphere (cf. [4], Corollary 2) that $f(\Sigma)$ is a round sphere.

As for condition (ii), if Σ is a totally geodesic principal orbit, then it is immersed by f as an isoparametric hypersurface of a sphere S^n whose first normal spaces in $\mathbb{R}^{n+1}$ are one-dimensional. This can only happen if it is umbilical in S^n, and hence again a round hypersphere of S^n.

Now assume that (iii) holds. First notice that the position vector of f can not be tangent to f along $f(\Sigma)$ for every principal orbit Σ of G, otherwise f would be a cone over an isoparametric hypersurface of the sphere, in contradiction with the completeness of M^n. Now, if Σ is a principal orbit along which the position vector is nowhere tangent to f, then the normal bundle of the restriction $f|_\Sigma\colon\ \Sigma \to \mathbb{R}^{n+1}$ is spanned by the position vector and by $f_*\xi$, where ξ is a unit normal vector field to the inclusion of Σ into M^n. Since the shape operators of $f|_\Sigma$ with respect to both vector fields are multiples of the identity tensor, it follows that $f|_\Sigma$ is umbilical, and we obtain again that $f(\Sigma)$ is a round sphere.

Finally, under condition (iv) the conclusion is a consequence of the following result. □

Proposition 2.5. *Let $P^n \subset S^{n+1}$, $n \geq 4$, be an isoparametric hypersurface. If the universal covering of P^n is* (*homeomorphic to*) *S^n, then P^n is isometric to a Euclidean sphere.*

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_g$ be the distinct (and constant) principal curvatures of P^n. Let m_1 be the common multiplicity of the λ_k, when k is odd, and let m_2 be the common multiplicity of the λ_k, when k is even. Denote by $\beta_0, \beta_1, \beta_2, \dots, \beta_n$ the $\mathbb{Z}_2$-Betti numbers of P^n. Then we have (cf. [11]):

(i) [F. Münzner] $g \in \{1, 2, 3, 4, 6\}$;

(ii) $2n = g(m_1 + m_2)$;

(iii) [E. Cartan] If $g = 3$, then $m_1 = m_2 \in \{1, 2, 4, 8\}$;

(iv) [U. Abresh] If $g = 6$, then $m_1 = m_2 \in \{1, 2\}$;

(v) [F. Münzner] $\sum_{i=0}^{n} \beta_i = 2g$.

Suppose first that n, the dimension of P^n, is odd. Then (ii) and (iv) imply that $g \in \{1, 2, 3\}$. Since $n \geq 4$, it follows from (iii) that $g \in \{1, 2\}$. If $g = 2$, then P^n is a Riemannian product of spheres and thus it cannot be covered by a sphere. Hence we must have $g = 1$ and this implies that P^n is a Euclidean sphere.

Let now n be even, say $n = 2q$. Then the Euler characteristics of S^{2q} and P^n are related by $\chi(S^{2q}) = m\chi(P^n)$, where m is the number of sheets of the covering. Thus, either $m = 1$ or $m = 2$, since $\chi(S^{2q}) = 2$. Suppose $m = 2$. Then $\chi(P^n) = \sum_{i=0}^{n}(-1)^i \beta_i = 1$, which implies, using Poincaré duality, that the Betti number β_q is odd. On the other hand, we get from (v) that β_q must be even. This contradiction tells us that $m = 1$ and, again using (v), we obtain that $g = 1$. Therefore P^n is a Euclidean sphere. □

Remark 2.6. Proposition 2.5 is no longer true for $n = 3$, as shown by Cartan isoparametric hypersurfaces of S^4 with three distinct principal curvatures [3], which are diffeomorphic to S^3/Q, where Q stands for the quaternion 8-group.

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