Comment. Math. Helv. 81 (2006), 643–652 **Commentarii Mathematici Helvetici**

Semi-continuity of the first ℓ^2-Betti number on the space of finitely generated groups

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Abstract. To each finitely generated group is associated a sequence $(\beta_0, \beta_1, \beta_2, \dots)$ of non negative real numbers, its ℓ^2-Betti numbers. On the other hand, the set of finitely generated groups *desingularizes* into a usual topological space, the space MG of finitely generated *marked* groups. It is proved in this note that if a sequence $\Gamma_n \in \mathrm{MG}$ of marked groups converges to a group Γ, then $\overline{\lim}\, \beta_1(\Gamma_n) \leq \beta_1(\Gamma)$.

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1. Introduction

Two countable groups Γ and Λ are said to be *measure equivalent* if they admit commuting and essentially free actions on some (infinite) measured space, which both preserve the measure and have fundamental domain of finite volume. Two lattices in a same Lie group are measure equivalent as they act by left and right multiplication on this Lie group. All infinite amenable groups are measure equivalent by a deep theorem of Ornstein–Weiss. On the other hand Kazhdan's Property T groups are never measure equivalent to non property T groups, in particular to infinite amenable groups.

The ℓ^2*-Betti numbers* of a countable group Γ are non negative real numbers $\beta_0, \beta_1, \beta_2, \dots$ coming from ℓ^2-homology as Γ-dimensions. Yet being non trivial, they enjoy remarkable stability features due to their invariance under rough kind of homotopies. The theory originated with the work of Atiyah [2] on the index theory on non compact (periodic) manifolds, and then ramified into several directions. It came to geometric group theory after the homotopy invariance results of Dodziuk [8] and the work of Cheeger–Gromov [6]. Among the important recent results about ℓ^2-Betti numbers in group theory is Gaboriau's proportionality theorem asserting

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their invariance under measure equivalence, up to a multiplicative constant [9] (where the measure equivalence can be considered as a homotopy).

It is a theorem of Cheeger–Gromov that all these numbers are zero for any infinite amenable group. For the (1-dimensional) free group on n letters, the β_i are also zero for $i \geq 2$, but $\beta_1 = n - 1$ as it is easy to see (we will forget here about β_0 which is always zero except in the trivial case of finite groups). This leads to the somewhat popular $\beta_1(\mathrm{SL}_2(\mathbb{Z})) = 1/12$ thanks to their multiplicative behavior under finite index extension. For property T groups one has $\beta_1 = 0$ and more generally, for non amenable groups, $\beta_1 = 0$ if and only if $H^1(\Gamma, \lambda_\Gamma) = 0$, where $H^1(\Gamma, \lambda_\Gamma)$ is the first cohomology group of Γ with coefficients in the left regular representation [3].

Two short and interesting (geometric) surveys on ℓ^2-Betti numbers can be found in [11], [15]. The reference treatise for the detailed (and more algebraic) story is [14].

A *marked group* is a countable group Γ together with a given generating set $S \subset \Gamma$. The marking S turns Γ into a connected (labeled) graph by putting an edge between two elements γ and $\gamma' \in \Gamma$ if $\gamma^{-1}\gamma' \in S$. This graph is called the *Cayley graph* of the marked group Γ_S. For instance this pictures the free group on n letters as a tree of valence $2n$.

Let Γ_S and $\Gamma'_{S'}$ be marked groups. We put $d(\Gamma_S, \Gamma'_{S'}) = e^{-n}$ where n is the largest integer such that the balls of radius n in the Cayley graphs of Γ_S and $\Gamma'_{S'}$ are simplicially isomorphic (with respect to a isomorphism respecting the labelings). This function is an ultrametric distance that gives a separated topology on the set MG of marked groups. Gromov showed that Baire's property mixed with hyperbolicity in MG lead to highly surprising results in group theory, concerning particularly the (generic) existence of finitely generated groups with unexpected properties.

Our aim here is to prove the following result.

Theorem 1. *Let* $(\Gamma_{n S_n})_n$ *be a sequence of marked groups converging in* MG *to a marked group* Γ_S. *Then*

$$\overline{\lim}\ \beta_1(\Gamma_n) \leq \beta_1(\Gamma).$$

Obviously the inequality can be strict. For instance the (residually finite) free group on 2 letters has $\beta_1 = 1$ and can be approximated in MG by finite groups with $\beta_1 = 0$. It is actually a result of Gromov that property T groups are dense in the adherence of (some) hyperbolic groups in MG, so that each such a group can be approximated by groups with $\beta_1 = 0$.

Let us observe that it was not known before Gaboriau's proportionality theorem that there are uncountably many classes of measure equivalence in MG. These classes are distinguished by their sequences of ℓ^2-Betti numbers.

The space of marked groups has been studied in details by Champetier in [4]. Recently it has found new applications by putting into a topological framework the study of the so-called 'limit groups' of Sela: the limit groups are those which appear as limits of free groups. We refer to [5], for instance (see also [7]), where a detailed presentation of the space MG is also given. Note that our result implies that a limit of free groups in MG has its first ℓ^2-Betti number greater or equal to 1 (when non abelian). Here too, the inequality can be strict as one can realize the free group on k letters as a limit of free groups on 2 letters. Another semi-continuity result on MG, the semi-continuity of the Følner isoperimetric constant, has been recently proved in [1].

This work is taken from the beginning of my Ph.D. thesis. I would like to thank my advisor Damien Gaboriau.

2. Preliminary

Let Γ be a countable group.

2.1. The Murray–von Neumann dimension. Consider the Hilbert space

$$\ell^2(\Gamma)$$

of square-integrable functions on Γ, where Γ acts by left translations (the left regular representation of Γ). It is a remarkable result of Murray and von Neumann that if Γ has infinite conjugacy classes, then the *Γ-invariant Hilbert subspaces* of $\ell^2(\Gamma)$ are classified up to equivariant isometry by a real number in $[0, 1]$, *their dimension*. More precisely for a closed and Γ-invariant subspace H of $\ell^2(\Gamma)$ the dimension is given by the explicit formula

$$\dim_\Gamma H = \langle P_H \delta_e \mid \delta_e \rangle$$

where P_H is the orthogonal projection onto H and δ_e is the characteristic function of the identity in Γ, *and* the function $H \mapsto \dim_\Gamma H$, called the Γ-dimension function, is the *unique* one (up to normalization) satisfying the usual properties of a dimension function. The map $\mathrm{Tr}_\Gamma(P_H) = \langle P_H \delta_e \mid \delta_e \rangle$ on Γ-equivariant projections in $B(\ell^2(\Gamma))$ extends to the unique normalized trace on the whole (von Neumann) algebra of Γ-equivariant operators.

The same is true for subspaces of any multiple $\bigoplus_0^N \ell^2(\Gamma)$ of the regular representation, where the dimension

$$\dim_\Gamma H = \sum_{i=0}^{N} \langle P_H \delta_e^i \mid \delta_e^i \rangle$$

now ranges in $[0, N]$ with $N \in \{1, 2, 3, \dots, \infty\}$. For non infinite conjugacy classes groups one also take this formula as the definition of the dimension.

We refer to [13] for more details.

2.2. The first ℓ^2-Betti number. Let Y be a locally finite oriented 2-dimensional cellular complex. For $i = 0, 1, 2$ denote by $\mathbb{C}[Y^i]$ the algebras of functions with finite support on the i-cells of Y and complete them to the ℓ^2 cochains spaces $\ell^2(Y^i)$ for the standard $\|c\|^2 = \sum_{\sigma\ i-\text{cells}} |c(\sigma)|^2$ hilbertian norm. The natural boundary operators

$$\partial_i\colon \mathbb{C}[Y^i] \to \mathbb{C}[Y^{i-1}]$$

coming from the 'attaching cells maps' extend to bounded operators $\partial_i^{(2)}\colon \ell^2(Y^i) \to \ell^2(Y^{i-1})$ *if* Y is uniformly locally finite (which means that the number of cells incident to any point is uniformly bounded). We define the *first reduced ℓ^2-homology space* of Y as the quotient space $\overline{H}_1^{(2)}(Y) = \ker \partial_1^{(2)}/\overline{\operatorname{Im} \partial_2^{(2)}}$. It is naturally isometric to the orthogonal complement $\mathcal{H}_1^{(2)}(Y)$ of $\overline{\operatorname{Im} \partial_2^{(2)}}$ in $\ker \partial_1^{(2)}$. The elements of $\mathcal{H}_1^{(2)}(Y)$ are called the ℓ^2 harmonic 1-cochains on Y.

Let Γ be a finitely presented group. Attaching the relations of a finite presentation to the Cayley graph leads to a *Cayley complex* of Γ. It is a simply connected cellular complex Y of dimension 2 on which Γ acts freely, so that the quotient $X = Y/\Gamma$ is a finite polyhedron of fundamental group Γ. The action of Γ on Y being free, $\ell^2(Y^i)$ is equivalent to a multiple $\bigoplus_1^{\alpha_i} \ell^2(\Gamma)$ of the regular representation, where α_i is the number of orbits of the action of Γ on the i-cells. The subspace $\mathcal{H}_1^{(2)}(Y)$ of $\ell^2(Y^1)$ is Γ-invariant and equivariantly isometric to $\overline{H}_1^2(Y)$. Moreover it does actually not depend on the choice of the presentation, up to equivariant isometry (and nor on the choice of any simply connected Y with free cocompact action of Γ for that matter [8]). The first ℓ^2-Betti number of Γ is then defined by $\beta_1(\Gamma) = \dim_\Gamma \overline{H}_1^2(Y)$ where Y is associated to any finite presentation.

This definition has been extended to arbitrary countable groups by Cheeger–Gromov [6]. This is done by using the following results. Let Y be an oriented 2-dimensional cellular complex on which the countable group Γ acts freely. Given a Γ-invariant and cocompact exhaustion $Y_1 \subset Y_2 \subset \cdots \subset Y$ of Y let

$$J_{m,n}\colon \overline{H}_1^{(2)}(Y_m) \to \overline{H}_1^{(2)}(Y_n)$$

be the morphism induced in homology by the inclusion maps $Y_m \subset Y_n$ for $m \leq n$. Then the number

$$\sup_{m\geq 0} \inf_{n\geq m} \dim_\Gamma \overline{\operatorname{Im} J_{m,n}}$$

is actually independent of the chosen cocompact and Γ-invariant exhaustion $(Y_n)_n$ of Y and is called the first ℓ^2-Betti number of Y. Let us denote it by $\beta_1(Y, \Gamma)$. The point of the definition is that this number is still invariant under equivariant homotopies of Y (and coincide with the previous $\beta_1(Y, \Gamma) = \dim_\Gamma \overline{H}_1^2(Y)$ in the cocompact case). So one defines $\beta_1(\Gamma)$ as $\beta_1(Y, \Gamma)$ for any simply connected Y on which Γ acts freely (for example the Cayley complex coming from a presentation of Γ). Note that this generalization was crucial in the equivalence relation case as shown in [9].

The other ℓ^2-Betti numbers are defined exactly in the same way and will not be used in this article.

2.3. The space of finitely generated groups. We refer here to the papers of Champetier [4] and Champetier–Guirardel [5]. Let us briefly summarize the context. A *marked group* is a finitely generated group and a finite ordered system of generators $S \subset \Gamma$. Let us consider the set MG' of finitely generated marked groups. If Γ_S and $\Gamma'_{S'}$ are two marked groups one puts $d(\Gamma_S, \Gamma'_{S'}) = e^{-r}$ where r is the largest integer for which one can find a simplicial isomorphism respecting the edge labelings between the balls of radius r in the Cayley graphs of Γ_S and $\Gamma'_{S'}$. Thus such groups have exactly the same relations of length $\leq r$. The map d defines an ultrametric distance on the quotient space $\mathrm{MG} = \mathrm{MG}'/\{d = 0\}$ of marked groups up to relabeling the edges (that is, changing the generating set while keeping the same relations), which is naturally called *the space of marked groups*. This gives a "singular topology" on the set of finitely generated groups up to isomorphism.

3. The rate of relations of a finitely generated group

Let Γ be a finitely generated group. In order to prove the semi-continuity of β_1 on MG it will be more convenient to deal with the "quantity of relations" associated to finite generating sets S of Γ rather than the dimension of the ℓ^2-harmonic cochains of a Cayley complex. The easy link between the two is as follow.

Let Γ_S be a marked group and Y be the associated Cayley graph (from Γ_S to Y one just forgets the labeling, keeping the orientation). The family of 1-dimensional cycles in Y is stable under the action of Γ and leads to the Γ-invariant vector subspace

$$Z_1(Y) = \ker \partial_1$$

of $\mathbb{C}[Y^1]$ (complex functions with finite support on Y^1). The closure of $Z_1(S)$ in $\ell^2(Y^1)$ for the norm topology is a Γ-invariant subspace and has a dimension

$$\tau(\Gamma_S) = \dim_\Gamma \overline{Z_1(Y)}.$$

We call this dimension the *rate of relations* of the marked group Γ_S. The *rate of relations of* Γ is then defined to be the infimum $\tau(\Gamma) = \inf_S \tau(\Gamma_S)$ over the finite generating sets S of Γ.

Example. The rate of relations in a finite group Γ is $\tau(\Gamma_S) = \#S - 1 + 1/\#\Gamma$. Indeed the space $\ell^2(Y^1)$ and $\ell^2(Y^0)$ have finite (complex) dimension $\#\Gamma \cdot \#S$ and $\#\Gamma$

respectively, and the dimension of $Z_1(\Gamma_S)$ is computed as follow. One easily sees that the image of ∂_1 coincide with the kernel of the linear map $\ell^2(Y^0) \to \mathbb{C}$ defined by $c \mapsto \sum_{\gamma\in\Gamma} c(\gamma)$, which has codimension 1 in $\ell^2(Y^0)$. In particular,

$$b_1(Y) = \dim_{\mathbb{C}} Z_1(Y) = \#\Gamma \cdot \#S - \#\Gamma + 1.$$

As $\dim_\Gamma = \frac{1}{\#\Gamma} \cdot \dim_{\mathbb{C}}$, the result follows immediately. Recall that the classical Betti number $b_1(Y)$ is the number of cycles in Y, whereas $b_0(Y) = 1$ is the number of its connected components.

Proposition 2. *Let Γ_S be an infinite marked group. Then*

$$\beta_1(\Gamma) = \#S - 1 - \tau(\Gamma_S).$$

Proof. Assume first that there exists a finite presentation $P = \langle S \mid R\rangle$ of Γ, and let Y be the 2-dimensional associated Cayley complex. It is by definition simply connected and cocompact, so that $\beta_1(\Gamma) = \dim_\Gamma \ker \partial_1^{(2)}/\overline{\operatorname{Im} \partial_2^{(2)}}$ for the associated closure of the boundary operators ∂_i. The simple connectedness expresses homologically by the relation $\ker \partial_1 = \operatorname{Im} \partial_2$. So $\tau(\Gamma_S) = \dim_\Gamma \overline{\ker \partial_1} = \dim_\Gamma \overline{\operatorname{Im} \partial_2}$, and one easily verifies that $\overline{\operatorname{Im} \partial_2} = \overline{\operatorname{Im} \partial_2^{(2)}}$ from the fact that $\partial_2^{(2)}$ is bounded. Therefore, as $\dim_\Gamma \ell^2(Y^0) = 1$, $\dim_\Gamma \ell^2(Y^1) = \#S$, and $\partial_1^{(2)}$ has dense image, the rank formula for $\partial_1^{(2)}$ (see [14]) with respect to the Γ-dimension gives

$$\beta_1(\Gamma) = \#S - 1 - \tau(\Gamma_S).$$

We now proceed with the approximation case of arbitrary (finitely generated) Γ_S. Let $P = \langle S \mid R\rangle$ be a presentation of Γ and Y be the associated Cayley complex. Denote by R_n the set of the n first relations in R, and Y_n be the Γ-invariant subcomplex of Y associated to R_n. Then Y_n is a cocompact exhaustion of Y. Let

$$J_{m,n} \colon \overline{H}_1^{(2)}(Y_m) \to \overline{H}_1^{(2)}(Y_n)$$

be the morphism induced in homology by the inclusion maps $Y_m \subset Y_n$ $(m \leq n)$. Note that the 1-skeleton of the Y_m is fixed so that $J_{m,n}$ is simply the orthogonal projection from the ℓ^2-harmonic chains $\mathcal{H}_1^{(2)}(Y_m) \subset \ell^2(Y^1)$ on Y_m to the subspace $\mathcal{H}_1^{(2)}(Y_n) \subset \mathcal{H}_1^{(2)}(Y_m)$ of ℓ^2-harmonic chains on Y_n. Due to the simple connectedness of Y the first ℓ^2-Betti number of Γ can be computed using the formula

$$\beta_1(\Gamma) = \sup_{m\geq 0} \inf_{n\geq m} \dim_\Gamma \overline{\operatorname{Im} J_{m,n}}.$$

On the other hand $\partial_{2,Y_n} \colon \mathbb{C}[Y_n^2] \to \mathbb{C}[Y_n^1]$ extends to a bounded operator $\partial_{2,Y_n}^{(2)}$ on $\ell^2(Y_n^2)$, so that $\overline{\partial_{2,Y_n}(\mathbb{C}[Y_n^2])} = \overline{\partial_{2,Y_n}^{(2)}(\ell^2(Y_n^2))}$. We have

$$\overline{\partial_{2,Y_1}^{(2)}(\ell^2(Y_1^2))} \subset \cdots \subset \overline{\partial_{2,Y_n}^{(2)}(\ell^2(Y_n^2))} \subset \cdots \subset \overline{Z_1(Y)},$$

and the union of these subspaces is dense in $\overline{Z_1(Y)}$ (indeed let $\sigma \in \overline{Z_1(Y)}$ be a cycle which is orthogonal to $\overline{\partial_{2,Y_n}^{(2)}(\ell^2(Y_n^2))}$ for all n and consider any finite cycle $c \in Z_1(Y)$. Then $c \in \partial_{2,Y_n}(\mathbb{C}[Y_n^2])$ for some large n, so that $\langle c \mid \sigma \rangle = 0$ and $\sigma = 0$). Thus

$$\dim_\Gamma \overline{\partial_{2,Y_n}^{(2)}(\ell^2(Y_n^2))} \xrightarrow[n\to\infty]{} \dim_\Gamma \overline{Z_1(Y)}$$

and

$$\inf_{n\geq m} \dim_\Gamma \overline{\operatorname{Im} J_{m,n}} = \dim_\Gamma \ker \partial_1^{(2)} - \dim_\Gamma \overline{Z_1(Y)}$$

(which is thus independent of m). As $\dim_\Gamma \ker \partial_1^{(2)} = \#S - 1$ we get

$$\beta_1(\Gamma) = \#S - 1 - \tau(\Gamma_S),$$

hence the proposition. □

In particular the infimum defining $\tau(\Gamma)$ is attained for any finitely generated Γ and we have $\tau(\Gamma_S) \in \tau(\Gamma) + \mathbb{N}$. Observe that $\tau(\Gamma) = 0$ if and only if Γ is a free group. In general we have $\tau(\Gamma) \leq g(\Gamma) - 1$ and $\tau(\Gamma) \leq r(\Gamma)$ for any infinite finitely generated Γ, where $g(\Gamma)$ is the minimal number of generator of Γ and $r(\Gamma)$ the minimal number of relations of a presentation. Let us recall here a still open (particular case of a) question due to Atiyah [2]: is it true that the rate of relations of a torsion free finitely presented group is an integer? (See [10].)

4. The semi-continuity of β_1

We now prove Theorem 1.

Lemma 3. *For any $\varepsilon > 0$ the condition $\tau(\Gamma'_{S'}) \geq \tau(\Gamma_S) - \varepsilon$ defines a neighborhood of Γ_S in* MG.

Proof. The finite groups being isolated in MG we may assume that Γ is infinite. Let $P = \langle S \mid R \rangle$ be a presentation of Γ associated to the finite generating set S and Y be the corresponding 2-dimensional Cayley complex.

Denote by $c_i \in Z_1(Y)$ the family of characteristic functions of the fundamental cycles in Y, that is the cycles given by the relations (i.e. the elements of R and their conjugates). As P is a presentation we get that $\{c_i\}$ generates $Z_1(Y)$ as a vector space. Alternatively we can take for c_i a (countable) generating set of the space of cycles with rational coefficients in $Z^1(Y)$. Assume moreover that $(c_i)_i$ is well ordered and let $(e_k)_k$ be the ordered family obtained from $(c_i)_i$ by the Gram–Schmidt orthonormalization procedure for the scalar product of $\ell^2(Y^1)$. Then $(e_k)_k$ is an orthonormal basis of $\overline{Z_1(Y)}$. Observe that by construction the support of e_n is then included into the union of the supports of c_i for $i \leq n$.

Let $P\colon \ell^2(Y^1) \to \ell^2(Y^1)$ be the orthogonal (equivariant) projection on $\overline{Z_1(Y)}$. By definition

$$\tau(\Gamma_S) = \dim_\Gamma \overline{Z_1(Y)} = \mathrm{Tr}(P) = \sum_{s\in S} \langle P(\delta_s) \mid \delta_s\rangle$$

where δ_s is the characteristic function of the edge $s \in Y^1$ starting at the origin (i.e. the identity of $\Gamma = Y^0$). The family $(e_k)_k$ being an orthonormal basis of $\overline{Z_1(Y)}$ we get

$$\dim_\Gamma \overline{Z_1(Y)} = \sum_{s\in S} \sum_1^\infty |\langle e_k \mid \delta_s\rangle|^2.$$

Let N be an integer such that

$$\dim_\Gamma \overline{Z_1(Y)} - \sum_{s\in S} \sum_1^N |\langle e_k \mid \delta_s\rangle|^2 \leq \varepsilon$$

and let us fix a real number r sufficiently large for the supports of the cycles c_i corresponding to e_k with $k \leq N$ to be included in the ball of radius r and center the identity element in Y. The set of marked groups such that the ball of radius r coincide with that of the (labeled) complex Y define a neighborhood V of Γ_S in MG.

Let $\Gamma'_{S'}$ be a point in V and denote by Y' the associated Cayley complex. Then there exists an isometry φ between the ball of radius r in Y^1 et Y'^1 which fix the identity and permutes the labelings $S \leftrightarrow_\varphi S'$. Observe that φ induces an isometry φ_* between the finite dimensional Hilbert spaces $\ell^2(B^1_Y(r), \langle\cdot \mid \cdot\rangle)$ and $\ell^2(B^1_{Y'}(r), \langle\cdot \mid \cdot\rangle')$ where $B_Y(r)$ is the ball of radius r in Y centered at the identity in $\Gamma = Y^0$ and B^1_Y is the 1-skeleton. As for the case of Γ let's number the fundamental cycles $(c'_i)_i$ of Y'. We may of course assume that $\varphi_*(c_i) = c'_i$ for the cycle c_i leading to e_k for $k \leq N$. Then the Gram–Schmidt orthonormalization procedure for the scalar product of $\ell^2(Y')$ gives a basis $(e'_k)_k$ of $\overline{Z_1(Y')}$ for which we have

$$\begin{aligned}\dim_{\Gamma'} \overline{Z_1(Y')} &= \sum_{s'\in S'} \sum_1^\infty |\langle e'_k | \delta'_s\rangle'|^2 \geq \sum_{s'\in S'} \sum_1^N |\langle e'_k | \delta'_s\rangle'|^2\\ &= \sum_{s\in S} \sum_1^N |\langle \varphi_*(e_k) | \varphi_*(\delta_s)\rangle'|^2 \geq \dim_\Gamma \overline{Z_1(Y)} - \varepsilon,\end{aligned}$$

so that V satisfies the required assumptions. □

Theorem 4. *Let* $\Gamma_{n_{S_n}}$ *be a sequence in* MG *converging to* Γ_S. *Then,*

$$\underline{\lim}\ \tau(\Gamma_{n_{S_n}}) \geq \tau(\Gamma_S) \quad \text{and} \quad \overline{\lim}\ \beta_1(\Gamma_n) \leq \beta_1(\Gamma).$$

The set $\{\Gamma_S \in \mathrm{MG} \mid \beta_1(\Gamma) < \varepsilon\}$ *is open in* MG *for any* $\varepsilon > 0$.

This follows immediately from the lemma and the proposition.

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