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When does the associated graded Lie algebra of an arrangement group decompose?

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Abstract. Let $\mathcal{A}$ be a complex hyperplane arrangement, with fundamental group G and holonomy Lie algebra $\mathfrak{H}$. Suppose $\mathfrak{H}_3$ is a free abelian group of minimum possible rank, given the values the Möbius function $\mu\colon \mathcal{L}_2 \to \mathbb{Z}$ takes on the rank 2 flats of $\mathcal{A}$. Then the associated graded Lie algebra of G decomposes (in degrees ≥ 2) as a direct product of free Lie algebras. In particular, the ranks of the lower central series quotients of the group are given by $\phi_r(G) = \sum_{X\in\mathcal{L}_2} \phi_r(F_{\mu(X)})$, for $r \geq 2$. We illustrate this new Lower Central Series formula with several families of examples.

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1. Introduction

1.1. The purpose of this paper is to give an answer to the question posed in the title. Let $\mathcal{A}$ be an arrangement of finitely many hyperplanes through the origin of $\mathbb{C}^\ell$, and denote by $G(\mathcal{A}) = \pi_1\big(\mathbb{C}^\ell \setminus \bigcup_{H\in\mathcal{A}} H\big)$ the fundamental group of its complement. In Section 2, we single out a class of arrangements, closely related to certain arrangements studied in [2], [15]. Roughly speaking, $\mathcal{A}$ is *decomposable* if a certain quadratic, graded Lie algebra $\mathfrak{H}(\mathcal{A})$, naturally defined in terms of the codimension 2 flats of $\mathcal{A}$, has minimal possible dimension in degree 3, over any ground field.

Our main result (Theorem 2.4) implies the following: If $\mathcal{A}$ is decomposable, then the associated graded Lie algebra of $G(\mathcal{A})$ decomposes as a direct product of free Lie algebras (in degrees $r \geq 2$):

$$\mathrm{gr}_{\geq 2}(G(\mathcal{A})) \cong \prod_{X\in\mathcal{L}_2(\mathcal{A})} \mathrm{gr}_{\geq 2}(F_{\mu(X)}). \tag{1.1}$$

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Here:

- $\mathcal{L}(\mathcal{A}) = \{X = \bigcap_{H\in\mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\}$ is the intersection lattice, $\mathcal{L}_2(\mathcal{A})$ is the set of codimension 2 flats, and $\mu\colon \mathcal{L}(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function.
- $\{\Gamma_r G\}_{r\geq 1}$ is the lower central series, given by $\Gamma_1 G = G$ and $\Gamma_{r+1} G = (\Gamma_r G, G)$.
- $\operatorname{gr}(G) = \bigoplus_{r\geq 1} \Gamma_r G/\Gamma_{r+1} G$, with Lie bracket induced by the group commutator.
- F_n is the free group of rank n, and $\operatorname{gr}(F_n) = \boldsymbol{L}_n$ is the free Lie algebra on n generators.

Moreover, as we show in Proposition 3.3, the decomposability property of $\mathcal{A}$ is inherited by all sub-arrangements of $\mathcal{A}$.

1.2. The associated graded Lie algebra $\operatorname{gr}(G(\mathcal{A}))$ is not *a priori* determined by the intersection lattice, and, as such, it is not easy to handle. We turn instead to a more manageable, combinatorial approximation: The *holonomy Lie algebra* of the arrangement, $\mathfrak{H}(\mathcal{A})$, defined as the quotient of $\boldsymbol{L}(\mathcal{A})$, the free Lie algebra on variables $\{x_H \mid H \in \mathcal{A}\}$, modulo the ideal $J(\mathcal{A})$ generated by relations corresponding to rank 2 flats:

$$\mathfrak{H}(\mathcal{A}) = \boldsymbol{L}(\mathcal{A})/\text{ideal}\left\{\left[x_H, \textstyle\sum_{H'\in\mathcal{A}\colon H'\supset X} x_{H'}\right] \mid X \in \mathcal{L}_2(\mathcal{A}) \text{ and } X \subset H\right\}. \tag{1.2}$$

As shown by Kohno [6] (based on foundational work by Sullivan [18] and Morgan [10]), the associated graded Lie algebra $\operatorname{gr}(G(\mathcal{A}))$ and the holonomy Lie algebra $\mathfrak{H}(\mathcal{A})$ are rationally isomorphic:

$$\operatorname{gr}(G(\mathcal{A})) \otimes \mathbb{Q} \cong \mathfrak{H}(\mathcal{A}) \otimes \mathbb{Q}. \tag{1.3}$$

At the integral level, there is a surjective Lie algebra map, $\Psi_{\mathcal{A}}\colon \mathfrak{H}(\mathcal{A}) \twoheadrightarrow \operatorname{gr}(G(\mathcal{A}))$, such that $\Psi_{\mathcal{A}} \otimes \mathbb{Q}$ is an isomorphism, see [9]. In general, there exist arrangements for which $\Psi_{\mathcal{A}}$ is not injective. Nevertheless, for the class of decomposable arrangements we consider here, $\Psi_{\mathcal{A}}$ gives an isomorphism $\operatorname{gr}(G(\mathcal{A})) \cong \mathfrak{H}(\mathcal{A})$, see Theorem 2.4(2).

1.3. The lower central series ranks of a finitely-generated group G are defined as $\phi_r(G) = \operatorname{rank} \operatorname{gr}_r(G)$. For a free group, the LCS ranks are given by Witt's formula: $\prod_{r=1}^{\infty}(1-t^r)^{\phi_r(F_n)} = 1-nt$. For an arrangement group, the LCS ranks are determined by the intersection lattice, via (1.3) and (1.2). Clearly, $\phi_1(G(\mathcal{A})) = |\mathcal{A}|$. Hence, to determine the LCS ranks of $G(\mathcal{A})$, we only need to compute the graded ranks of the derived holonomy algebra, $\mathfrak{H}'(\mathcal{A}) = \bigoplus_{r\geq 2} \mathfrak{H}_r(\mathcal{A})$.

From work of Falk [3], we know that $\dim_{\mathbb{Q}} \mathfrak{H}_r(\mathcal{A}) \otimes \mathbb{Q} \geq \sum_{X\in\mathcal{L}_2(\mathcal{A})} \phi_r(F_{\mu(X)})$, for all $r \geq 2$, with equality holding for $r = 2$. Guided by these facts, we say that $\mathcal{A}$ is decomposable if the lower bound is attained in degree 3, for every field $\Bbbk$:

$$\dim_{\Bbbk} \mathfrak{H}_3(\mathcal{A}) \otimes \Bbbk = \sum_{X\in\mathcal{L}_2(\mathcal{A})} \phi_3(F_{\mu(X)}). \tag{1.4}$$

Up to now, an explicit formula for the LCS ranks of an arrangement group has only been known in the case when the intersection lattice is supersolvable [4], or, more generally, hypersolvable [5]. The isomorphism (1.1) leads to a new LCS formula, for the combinatorially defined class of decomposable arrangements:

$$\prod_{r=1}^{\infty}(1-t^r)^{\phi_r(G(\mathcal{A}))} = (1-t)^{|\mathcal{A}|} \prod_{X\in\mathcal{L}_2(\mathcal{A})} \frac{1-\mu(X)t}{(1-t)^{\mu(X)}}\,. \tag{1.5}$$

This LCS formula verifies the more general "resonance LCS formula", conjectured in [17], in what is arguably the simplest, yet most basic case.

As a byproduct of our main theorem, we compute in Section 6 the integral *Chen Lie algebra* of a decomposable arrangement, and we also obtain the Chen analog of decomposition (1.1), thus improving upon results from [2].

1.4. Formula (1.5) is equivalent to $\phi_r(G(\mathcal{A})) = \sum_{X\in\mathcal{L}_2(\mathcal{A})}\phi_r(F_{\mu(X)})$, for all $r \geq 2$. In other words, the (higher) LCS ranks behave as if $G(\mathcal{A})$ were to decompose as a direct product of free groups, of ranks dictated by the Möbius function. This happens, for instance, for the class of (hypersolvable, decomposable) arrangements considered in [1], where the arrangement group is always a product of free groups. In general, though, the group of a decomposable arrangement does not decompose in this manner.

Figure 1. The X_2 and X_3 matroids.

For example, consider the X_2 and X_3 arrangements, whose matroids are depicted in Figure 1. It is readily checked that both arrangements are decomposable (compare with [2], [15]), but not hypersolvable (see Remark 7.3).

For the X_2 arrangement, we find that $\phi_r(G(\mathcal{A})) = \phi_r((F_2)^{\times 5})$, for all $r \geq 2$, yet $\phi_1(G(\mathcal{A})) < \phi_1((F_2)^{\times 5})$; thus, $G(\mathcal{A}) \ncong (F_2)^{\times 5}$.

For the X_3 arrangement, we find that $\phi_r(G(\mathcal{A})) = \phi_r((F_2)^{\times 3})$, for all $r \geq 1$. Even so, $G(\mathcal{A}) \ncong (F_2)^{\times 3}$. Indeed, it can be checked that $G(\mathcal{A}) \cong G \times \mathbb{Z}$, where G is the celebrated Stallings group,[1] equal to the kernel of the projection $(F_2)^{\times 3} \to \mathbb{Z}$, which sends each standard generator to 1. As shown in [16], the group $H_3(G)$ is

[1]As a consequence, we can compute the LCS ranks of the Stallings group: $\phi_1(G) = 5$, and $\phi_r(G) = \phi_r((F_2)^{\times 3})$, for $r \geq 2$. In [13], we give an LCS formula that applies to *any* Bestvina–Brady group associated to a connected flag complex.

not finitely generated. It follows that $G(\mathcal{A})$ does not admit a finite $K(G(\mathcal{A}),1)$; in particular, $G(\mathcal{A})$ cannot be isomorphic to *any* finite direct product of free groups of finite rank.

In view of these examples, and of the infinite families of decomposable, non-hypersolvable graphic arrangements from Section 7, we see that the LCS formula (1.5) is a genuinely new formula, with a range of applicability which overlaps only marginally with that of the classical LCS formula.

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2. Decomposable arrangements

In this section, we delineate the class of decomposable arrangements, and state in detail our main result.

2.1. For an arrangement $\mathcal{A}$, denote by $\mathbb{Z}^{\mathcal{A}}$ the free abelian group on $\mathcal{A}$, with basis $\{x_H \mid H \in \mathcal{A}\}$. For a sub-arrangement $\mathcal{B} \subset \mathcal{A}$, let $\pi_{\mathcal{B}}\colon \mathbb{Z}^{\mathcal{A}} \to \mathbb{Z}^{\mathcal{B}}$ be the canonical projection map, defined by $\pi_{\mathcal{B}}(x_H) = x_H$, if $H \in \mathcal{B}$, and $\pi_{\mathcal{B}}(x_H) = 0$, if $H \notin \mathcal{B}$, and let $\boldsymbol{L}(\pi_{\mathcal{B}})\colon \boldsymbol{L}(\mathcal{A}) \to \boldsymbol{L}(\mathcal{B})$ be its extension to free Lie algebras. Clearly, $\boldsymbol{L}(\pi_{\mathcal{B}})(J(\mathcal{A})) \subset J(\mathcal{B})$, and so we get a Lie algebra epimorphism,

$$\mathfrak{H}(\pi_{\mathcal{B}})\colon \mathfrak{H}(\mathcal{A}) \twoheadrightarrow \mathfrak{H}(\mathcal{B}). \tag{2.1}$$

For a flat $X \in \mathcal{L}_2(\mathcal{A})$, let $\mathcal{A}_X = \{H \in \mathcal{A} \mid H \supset X\}$ be the *localization* of $\mathcal{A}$ at X. This is a pencil of $|\mathcal{A}_X| = \mu(X)+1$ hyperplanes. The group $G(\mathcal{A}_X)$ is isomorphic to $F_{\mu(X)} \times \mathbb{Z}$; thus, $\operatorname{gr}(G(\mathcal{A}_X)) \cong \boldsymbol{L}_{\mu(X)} \times \boldsymbol{L}_1$. From the defining relations (1.2), we also have $\mathfrak{H}(\mathcal{A}_X) \cong \boldsymbol{L}_{\mu(X)} \times \boldsymbol{L}_1$, and so $\mathfrak{H}(\mathcal{A}_X) \cong \operatorname{gr}(G(\mathcal{A}_X))$.

Set $\pi_X = \pi_{\mathcal{A}_X}$. The maps $\mathfrak{H}(\pi_X)\colon \mathfrak{H}(\mathcal{A}) \to \mathfrak{H}(\mathcal{A}_X)$ assemble into a Lie algebra map from $\mathfrak{H}(\mathcal{A})$ to the direct product of the holonomy Lie algebras of its localized sub-arrangements:

$$\pi = (\mathfrak{H}(\pi_X))_X : \mathfrak{H}(\mathcal{A}) \longrightarrow \prod_{X \in \mathcal{L}_2(\mathcal{A})} \mathfrak{H}(\mathcal{A}_X). \tag{2.2}$$

The starting point of our investigation is the following result, to be proved in §3.2.

Proposition 2.1. *The restriction of π to derived subalgebras,*

$$\pi'\colon \mathfrak{H}'(\mathcal{A}) \to \prod_{X \in \mathcal{L}_2(\mathcal{A})} \mathfrak{H}'(\mathcal{A}_X),$$

is surjective.

By comparing ranks of the source and target of $\pi_r\colon \mathfrak{H}_r(\mathcal{A}) \to \prod_X \mathfrak{H}_r(\mathcal{A}_X)$ for $r \geq 2$, we recover a lower bound for the LCS ranks of an arrangement group, first obtained by M. Falk [3], by other methods.

Corollary 2.2 ([3]). *For all $r \geq 2$,*

$$\phi_r(G(\mathcal{A})) \geq \sum_{X \in \mathcal{L}_2(\mathcal{A})} \phi_r(F_{\mu(X)}). \tag{2.3}$$

2.2. Our main goal here is to understand when the natural map π' from Proposition 2.1 is, in fact, an isomorphism; in particular, when the inequalities (2.3) become equalities.

It is easy to see that π_2 is always an isomorphism. On the other hand, the maps π_r ($r \geq 3$) may not be isomorphisms, as illustrated by the braid arrangements $\mathcal{B}_\ell$ in $\mathbb{C}^\ell$ ($\ell \geq 4$). In this case, the LCS formula of Kohno [7] and Falk–Randell [4], when applied to the pure braid group $P_\ell = G(\mathcal{B}_\ell)$, shows that inequality (2.3) is strict in degree $r = 3$.

This prompts the following definition.

Definition 2.3. Let $r \geq 2$ be an integer, and let $\Bbbk$ be a field. We say that $\mathfrak{H}_r(\mathcal{A})$ is $\Bbbk$-*decomposable* if

$$\dim_{\Bbbk} \mathfrak{H}_r(\mathcal{A}) \otimes \Bbbk = \sum_{X \in \mathcal{L}_2(\mathcal{A})} \phi_r(F_{\mu(X)}). \tag{2.4}$$

An arrangement $\mathcal{A}$ is *decomposable* if $\mathfrak{H}_3(\mathcal{A})$ is $\Bbbk$-decomposable, for every field $\Bbbk$.

By Proposition 2.1, $\mathfrak{H}_r(\mathcal{A})$ is $\Bbbk$-decomposable if and only if $\pi_r \otimes \Bbbk$ is an isomorphism, whereas $\mathcal{A}$ is decomposable precisely when π_3 is an isomorphism.

2.3. Two other decomposability conditions were considered in [2] and [15]. Let us briefly compare those conditions to ours.

The condition from [2] entails the decomposability of the I-adic completion of the Alexander invariant of $G(\mathcal{A})$ as the direct sum of the I-adic completions of the Alexander invariants of $G(\mathcal{A}_X)$, taken over $X \in \mathcal{L}_2(\mathcal{A})$. It can be shown that this condition on Alexander invariants is equivalent, over $\mathbb{Q}$, to the decomposability of $\mathfrak{H}_3(\mathcal{A})$, in the sense of Definition 2.3.

The condition from [15] entails the minimality of the linear strand of the free resolution of the Orlik–Solomon algebra of $\mathcal{A}$ as a module over the corresponding exterior algebra. As stated in [15, Definition 2.10], the MLS condition is equivalent to the $\Bbbk$-decomposability of $\mathfrak{H}_3(\mathcal{A})$, for $\Bbbk$ a field of characteristic 0. Actually, the only place where the hypothesis $\operatorname{char} \Bbbk = 0$ is needed in that context is to insure that $\dim_{\Bbbk} \mathrm{gr}_*(G(\mathcal{A})) \otimes \Bbbk = \dim_{\Bbbk} \mathfrak{H}_*(\mathcal{A}) \otimes \Bbbk$. All the other homological algebra arguments work as well over a field of positive characteristic. Consequently, Theorem 5.6 from [15] gives the following: If $\mathfrak{H}_3(\mathcal{A})$ is $\Bbbk$-decomposable, then $\mathfrak{H}_4(\mathcal{A})$ is $\Bbbk$-decomposable. In particular, if $\mathcal{A}$ is decomposable (i.e., π_3 is an isomorphism), then π_4 is an isomorphism.

2.4. Our main result is Theorem 2.4 below, which improves upon the aforementioned result from [15], in several ways. For one, it pushes the range where π_r is an isomorphism from $r = 4$ to infinity. For another, it assembles the graded pieces π_r ($r \geq 2$) into a Lie algebra isomorphism between the derived holonomy Lie algebra of $\mathcal{A}$ and a product of derived free Lie algebras. Finally, it gives a new LCS-type formula for the group of a decomposable arrangement, thus verifying Conjecture 5.7 from [15].

Theorem 2.4. *Let $\mathcal{A}$ be a decomposable arrangement. Then:*

(1) $\mathrm{gr}(G(\mathcal{A})) \cong \mathfrak{H}(\mathcal{A})$*, as graded Lie algebras.*

(2) $\mathfrak{H}(\mathcal{A})$ *is torsion-free, as a graded abelian group.*

(3) $\pi' \colon \mathfrak{H}'(\mathcal{A}) \longrightarrow \prod_{X\in\mathcal{L}_2(\mathcal{A})} \mathfrak{H}'(\mathcal{A}_X)$ *is an isomorphism of graded Lie algebras.*

(4) *The LCS ranks $\phi_r = \phi_r(G(\mathcal{A}))$ are given by the following combinatorial formula:*
$$\prod_{r=1}^{\infty}(1-t^r)^{\phi_r} = (1-t)^{b_1-b_2} \prod_{X\in\mathcal{L}_2(\mathcal{A})} (1-\mu(X)t), \tag{2.5}$$
where $b_1 = |\mathcal{A}|$ and $b_2 = \sum_{X\in\mathcal{L}_2(\mathcal{A})} \mu(X)$.

Here is an immediate corollary, already mentioned in the Introduction.

Corollary 2.5. *If $\mathcal{A}$ is decomposable, then the associated graded Lie algebra of $G(\mathcal{A})$ decomposes as a direct product of free Lie algebras* (*in degrees $r \geq 2$*):
$$\mathrm{gr}_{\geq 2}(G(\mathcal{A})) \cong \prod_{X\in\mathcal{L}_2(\mathcal{A})} \mathrm{gr}_{\geq 2}(F_{\mu(X)}). \tag{2.6}$$

Over the rationals, we can be even more precise: $\mathfrak{H}_3(\mathcal{A})$ is decomposable over $\mathbb{Q}$, i.e., $\phi_3(G(\mathcal{A})) = \sum_{X\in\mathcal{L}_2(\mathcal{A})} \phi_3(F_{\mu(X)})$, if and only if the derived subalgebra of the rational associated graded Lie algebra of $G(\mathcal{A})$ decomposes as a direct product of derived free Lie algebras over $\mathbb{Q}$:
$$(\mathrm{gr}(G(\mathcal{A})) \otimes \mathbb{Q})' \cong \prod_{X\in\mathcal{L}_2(\mathcal{A})} \boldsymbol{L}'_{\mu(X)} \otimes \mathbb{Q}. \tag{2.7}$$

This follows from Proposition 4.1 below and formula (1.3).

Now set $\phi_r^{\Bbbk}(\mathcal{A}) := \dim_{\Bbbk} \mathfrak{H}_r(\mathcal{A}) \otimes \Bbbk$, for $\Bbbk$ a field and $r \geq 1$. The isomorphism (1.3) implies $\phi_r^{\mathbb{Q}}(\mathcal{A}) = \phi_r(G(\mathcal{A}))$, for all r. As a consequence of Theorem 2.4, we obtain the following characterization of decomposability, in terms of LCS-type formulas in arbitrary characteristic.

Corollary 2.6. *The arrangement $\mathcal{A}$ is decomposable if and only if, for every field $\Bbbk$,*

$$\prod_{r=1}^{\infty}(1-t^r)^{\phi_r^{\Bbbk}(\mathcal{A})} = (1-t)^{b_1-b_2} \prod_{X \in \mathcal{L}_2(\mathcal{A})} (1-\mu(X)t).$$

Finally, suppose $\mathcal{A}$ is hypersolvable, with exponents $d_1 = 1, d_2, \dots, d_\ell$. The Poincaré polynomial of the quadratic Orlik–Solomon algebra associated to $\mathcal{A}$ is then given by $\bar{P}_{\mathcal{A}}(t) = \prod_{i=1}^{\ell}(1+d_i t)$; see [5, Proposition 3.2]. Putting together the decomposable LCS formula (2.5) and the hypersolvable LCS formula from [5, Theorem C], we obtain the following relationship between the exponents d_i and the level-2 Möbius function $\mu\colon \mathcal{L}_2(\mathcal{A}) \to \mathbb{Z}$ of a decomposable, hypersolvable arrangement $\mathcal{A}$. (We will exploit this relationship in the last section, within the framework of graphic arrangements.)

Corollary 2.7. *If $\mathcal{A}$ is both hypersolvable and decomposable, then*

$$\prod_{i=1}^{\ell}(1+d_i t) = (1+t)^{|\mathcal{A}|} \prod_{X \in \mathcal{L}_2(\mathcal{A})} \frac{1+\mu(X)t}{(1+t)^{\mu(X)}}.$$

3. The ι map

In this section, we define the natural candidate for the inverse map to $\pi'\colon \mathfrak{H}'(\mathcal{A}) \to \prod_{X \in \mathcal{L}_2(\mathcal{A})} \mathfrak{H}'(\mathcal{A}_X)$, and discuss some of its properties.

3.1. Let $\mathcal{B}$ be a sub-arrangement of $\mathcal{A}$. Let $\iota_{\mathcal{B}}\colon \mathbb{Z}^{\mathcal{B}} \to \mathbb{Z}^{\mathcal{A}}$ be the canonical inclusion, defined by $\iota_{\mathcal{B}}(x_H) = x_H$, and let $\boldsymbol{L}(\iota_{\mathcal{B}})\colon \boldsymbol{L}(\mathcal{B}) \to \boldsymbol{L}(\mathcal{A})$ be its extension to free Lie algebras. In general, the map $\boldsymbol{L}(\iota_{\mathcal{B}})$ need not preserve the defining ideals of the holonomy Lie algebras of $\mathcal{B}$ and $\mathcal{A}$.

However, suppose $\mathcal{B}$ is *closed* in $\mathcal{A}$, i.e., the only linear combinations of defining forms for the hyperplanes in $\mathcal{B}$ which are defining forms for hyperplanes in $\mathcal{A}$ are (up to constants) the defining forms for the hyperplanes in $\mathcal{B}$. Then $\mathcal{L}_2(\mathcal{B}) = \{X \in \mathcal{L}_2(\mathcal{A}) \mid \mathcal{A}_X \subset \mathcal{B}\}$. Thus, $\boldsymbol{L}(\iota_{\mathcal{B}})(J(\mathcal{B})) \subset J(\mathcal{A})$, and so we get a map of graded Lie algebras,

$$\mathfrak{H}(\iota_{\mathcal{B}})\colon \mathfrak{H}(\mathcal{B}) \to \mathfrak{H}(\mathcal{A}). \tag{3.1}$$

For a flat $X \in \mathcal{L}_2(\mathcal{A})$, note that $\mathcal{A}_X$ is closed in $\mathcal{A}$. Set $\iota_X = \iota_{\mathcal{A}_X}$.

Lemma 3.1. *Let $X, Y \in \mathcal{L}_2(\mathcal{A})$, and $\mathcal{B} \subset \mathcal{A}$. Then:*

(1) $\mathfrak{H}(\pi_X) \circ \mathfrak{H}(\iota_X) = \mathrm{id}$.

(2) $\mathfrak{H}'(\pi_{\mathcal{B}}) \circ \mathfrak{H}'(\iota_X) = 0$, *if* $|\mathcal{B} \cap \mathcal{A}_X| \le 1$.

(3) $\mathfrak{H}'(\pi_X) \circ \mathfrak{H}'(\iota_Y) = 0$, *if* $X \neq Y$.

Proof. (1) Clearly, $\pi_X \circ \iota_X$ is the identity map on $\mathbb{Z}^{\mathcal{A}_X}$.

(2) For each $r \ge 2$, the group $\mathfrak{H}_r(\mathcal{A}_X)$ is generated by elements of the form $x = [x_{H_1}, [x_{H_2}, \dots [x_{H_{r-1}}, x_{H_r}] \dots]]$, where $H_1, \dots, H_r$ are hyperplanes in $\mathcal{A}_X$. Now, since $|\mathcal{B} \cap \mathcal{A}_X| \le 1$, one of those hyperplanes, say H_i, must not belong to $\mathcal{B}$; otherwise, $H_1 = H_2 = \cdots = H_r$, and so $x = 0$. Hence, by definition, $\pi_{\mathcal{B}}(x_{H_i}) = 0$, and so

$$\mathfrak{H}(\pi_{\mathcal{B}}) \circ \mathfrak{H}(\iota_X)(x) = [\pi_{\mathcal{B}}(x_{H_1}), [\pi_{\mathcal{B}}(x_{H_2}), \dots [\pi_{\mathcal{B}}(x_{H_{r-1}}), \pi_{\mathcal{B}}(x_{H_r})] \dots]] = 0.$$

Thus, $\mathfrak{H}(\pi_{\mathcal{B}}) \circ \mathfrak{H}(\iota_X) = 0$ in degrees ≥ 2.

(3) If $X \neq Y$, then $|\mathcal{A}_X \cap \mathcal{A}_Y| \le 1$. Hence (2) applies. □

3.2. Proof of Proposition 2.1 The maps $\mathfrak{H}(\iota_X)$ define a homomorphism of graded abelian groups,

$$\iota\colon \prod_{X \in \mathcal{L}_2(\mathcal{A})} \mathfrak{H}(\mathcal{A}_X) \longrightarrow \mathfrak{H}(\mathcal{A}). \tag{3.2}$$

Let $\iota'\colon \prod_X \mathfrak{H}'(\mathcal{A}_X) \to \mathfrak{H}'(\mathcal{A})$ be the restriction of ι to derived subalgebras. The orthogonality relations from Lemma 3.1 imply that $\pi' \circ \iota' = \mathrm{id}$. Thus, π' is surjective, and so Proposition 2.1 is proved. □

3.3. The following lemma (the proof of which is an exercise in linear algebra) will be used repeatedly later on.

Lemma 3.2. *Let U and $\{V_X\}_{X \in \mathcal{X}}$ be finite-dimensional vector spaces over a field $\Bbbk$, with $|\mathcal{X}|$ finite. Set $V = \bigoplus_X V_X$. Suppose we have linear maps $\pi_X\colon U \to V_X$ and $\iota_X\colon V_X \to U$ such that $\pi_X \circ \iota_Y = \delta_{X,Y}$. Set $\pi = (\pi_X)_X\colon U \to V$ and $\iota = \sum_X \iota_X\colon V \to U$. Then, the following conditions are equivalent:*

(1) *π is an isomorphism.*

(2) *ι is surjective.*

(3) $\sum_X \iota_X \circ \pi_X = \mathrm{id}_U$.

(4) $\dim_{\Bbbk} U = \sum_X \dim_{\Bbbk} V_X$.

3.4. By Lemmas 3.1 and 3.2, $\mathfrak{H}_3(\mathcal{A})$ is $\Bbbk$-decomposable if and only if the map

$$\iota_3 \otimes \Bbbk\colon \bigoplus_{X\in\mathcal{L}_2(\mathcal{A})} \mathfrak{H}_3(\mathcal{A}_X)\otimes\Bbbk \longrightarrow \mathfrak{H}_3(\mathcal{A})\otimes\Bbbk \tag{3.3}$$

is surjective. We use this criterion to show that decomposability is hereditary.

Proposition 3.3. *If $\mathcal{B}$ is a sub-arrangement of $\mathcal{A}$, and if $\mathfrak{H}_3(\mathcal{A})$ is $\Bbbk$-decomposable, then $\mathfrak{H}_3(\mathcal{B})$ is also $\Bbbk$-decomposable.*

Proof. Note that $\mathcal{L}_2(\mathcal{B}) = \{X \in \mathcal{L}_2(\mathcal{A}) \mid |\mathcal{A}_X \cap \mathcal{B}| \geq 2\}$. Furthermore, if $X \in \mathcal{L}_2(\mathcal{B})$, then $\mathcal{B}_X = \mathcal{A}_X \cap \mathcal{B}$. Consider the following diagram:

$$\begin{tikzcd} \bigoplus_{X\in\mathcal{L}_2(\mathcal{A})} \mathfrak{H}'(\mathcal{A}_X) \arrow[r, "\iota'^{\mathcal{A}}"] \arrow[d, "\rho"] & \mathfrak{H}'(\mathcal{A}) \arrow[d, "\mathfrak{H}'(\pi^{\mathcal{A}}_{\mathcal{B}})"] \\ \bigoplus_{X\in\mathcal{L}_2(\mathcal{B})} \mathfrak{H}'(\mathcal{B}_X) \arrow[r, "\iota'^{\mathcal{B}}"] & \mathfrak{H}'(\mathcal{B}) \end{tikzcd} \tag{3.4}$$

where ρ restricts to $\mathfrak{H}'\big(\pi^{\mathcal{A}_X}_{\mathcal{B}_X}\big)\colon \mathfrak{H}'(\mathcal{A}_X) \to \mathfrak{H}'(\mathcal{B}_X)$ if $X \in \mathcal{L}_2(\mathcal{B})$, and $\rho = 0$ otherwise. Diagram (3.4) commutes. Indeed, if $X \in \mathcal{L}_2(\mathcal{B})$, this is clear. If $X \notin \mathcal{L}_2(\mathcal{B})$, then $|\mathcal{A}_X \cap \mathcal{B}| \leq 1$, and so, by Lemma 3.1 (2), $\mathfrak{H}'\big(\pi^{\mathcal{A}}_{\mathcal{B}}\big) \circ \mathfrak{H}'\big(\iota^{\mathcal{A}}_{\mathcal{A}_X}\big) = 0$.

Now, if $\mathfrak{H}_3(\mathcal{A})$ is $\Bbbk$-decomposable, then $\iota_3^{\mathcal{A}} \otimes \Bbbk$ is surjective. From the commutativity of diagram (3.4), we infer that $\iota_3^{\mathcal{B}} \otimes \Bbbk$ is also surjective, and we are done. □

4. Surjectivity of ι'

In general, the map $\iota'\colon \prod_X \mathfrak{H}'(\mathcal{A}_X) \to \mathfrak{H}'(\mathcal{A})$ is not surjective. On the other hand, if $\mathcal{A}$ is decomposable, ι' is surjective. This we show in the next proposition, which is the key to our main result.

Proposition 4.1. *Suppose $\mathfrak{H}_3(\mathcal{A})$ is $\Bbbk$-decomposable. Then*

$$\iota_r \otimes \Bbbk\colon \prod_X \mathfrak{H}_r(\mathcal{A}_X)\otimes\Bbbk \longrightarrow \mathfrak{H}_r(\mathcal{A})\otimes\Bbbk$$

is surjective, for all $r \geq 2$.

Proof. For simplicity, we will suppress the field $\Bbbk$ from the notation. We will need to establish various commutation relations in $\mathfrak{H}(\mathcal{A})$, between elements in $\mathfrak{H}(\iota_X)(\mathfrak{H}(\mathcal{A}_X))$

and $\mathfrak{H}(\iota_Y)(\mathfrak{H}(\mathcal{A}_Y))$, where X and Y are distinct flats in $\mathcal{L}_2(\mathcal{A})$. Again for simplicity, we will suppress the inclusion ι from the notation, and work in $\mathfrak{H}(\mathcal{A})$.

Since $X \neq Y$, there are two possibilities: either $\mathcal{A}_X \cap \mathcal{A}_Y = \emptyset$, or $\mathcal{A}_X \cap \mathcal{A}_Y$ consists of a single hyperplane. Pick $H' \in \mathcal{A}_X$ and $H'' \in \mathcal{A}_Y$, so that, if $\mathcal{A}_X^* = \mathcal{A}_X \setminus \{H'\}$ and $\mathcal{A}_Y^* = \mathcal{A}_Y \setminus \{H''\}$ are the corresponding deletions, then

$$\mathcal{A}_X^* \cap \mathcal{A}_Y = \mathcal{A}_X \cap \mathcal{A}_Y^* = \emptyset. \tag{4.1}$$

Let us note the following fact, whose proof is immediate, and which will be used repeatedly in the sequel. For any flat $Z \in \mathcal{L}_2(\mathcal{A})$, and for any hyperplane $H \in \mathcal{A}_Z$,

$$\mathfrak{H}'(\mathcal{A}_Z) = \mathrm{Lie}^{>1}(\mathcal{A}_Z^*), \tag{4.2}$$

where $\mathcal{A}_Z^* = \mathcal{A}_Z \setminus \{H\}$, and where $\mathrm{Lie}^r(\mathcal{A}_Z^*)$ denotes the degree r piece of the Lie subalgebra generated by $\{x_K \mid K \in \mathcal{A}_Z^*\}$ inside $\mathfrak{H}(\mathcal{A}_Z)$.

Here is the first commutation property, for which the decomposability assumption on $\mathfrak{H}_3(\mathcal{A})$ is needed in a crucial way.

Claim I. *If $H_i \in \mathcal{A}_X^*$ and $c \in \mathfrak{H}_2(\mathcal{A}_Y)$, then $[x_{H_i}, c] = 0$.*

Proof. By (4.2), it is enough to verify the claim for $c \in \mathrm{Lie}^2(\mathcal{A}_Y^*)$. Apply $\mathfrak{H}(\pi_Z)$, for some $Z \in \mathcal{L}_2(\mathcal{A})$. If $Z \neq Y$, we get $[\mathfrak{H}(\pi_Z)(x_{H_i}), \mathfrak{H}(\pi_Z)(c)] = [\mathfrak{H}(\pi_Z)(x_{H_i}), 0] = 0$. If $Z = Y$, we get $[\mathfrak{H}(\pi_Y)(x_{H_i}), c] = [0, c] = 0$, since $\mathcal{A}_X^* \cap \mathcal{A}_Y = \emptyset$. Thus, $\pi_3([x_{H_i}, c]) = 0$, and so $[x_{H_i}, c] = 0$, since, by assumption, π_3 is an isomorphism. □

Using Claim I, we obtain the next commutation property.

Claim II. *If $b \in \mathfrak{H}_2(\mathcal{A}_X)$ and $c \in \mathfrak{H}_s(\mathcal{A}_Y)$ $(s \geq 2)$, then $[b, c] = 0$.*

Proof. As before, we may assume that $b \in \mathrm{Lie}^2(\mathcal{A}_X^*)$ and $c \in \mathrm{Lie}^s(\mathcal{A}_Y^*)$. The proof is by induction on s. For $s = 2$, Claim II follows from Claim I, via the Jacobi identity. For the induction step, take an element $c \in \mathrm{Lie}^{s+1}(\mathcal{A}_Y^*)$, and write it as $c = [x_{H_j}, c']$, with $H_j \in \mathcal{A}_Y^*$ and $c' \in \mathrm{Lie}^s(\mathcal{A}_Y^*)$. By the Jacobi identity,

$$[b, c] = [[b, x_{H_j}], c'] + [x_{H_j}, [b, c']].$$

Note that $[b, x_{H_j}] = 0$ by Claim I, and $[b, c'] = 0$ by induction. Thus, $[b, c] = 0$. □

Finally, using both Claims I and II, we prove the following key commutation property.

Claim III. *If $H_i \in \mathcal{A}_X^*$ and $c \in \mathfrak{H}_s(\mathcal{A}_Y)$ $(s \geq 2)$, then $[x_{H_i}, c] \in \mathfrak{H}_{s+1}(\mathcal{A}_Y)$.*

Proof. The proof is by induction on s. The case $s = 2$ follows from Claim I. For the induction step, take an element $c = [x_{H_j}, c'] \in \mathfrak{H}_{s+1}(\mathcal{A}_Y)$, with $H_j \in \mathcal{A}_Y$ and $c' \in \mathfrak{H}_s(\mathcal{A}_Y)$. By the Jacobi identity,

$$[x_{H_i}, c] = [[x_{H_i}, x_{H_j}], c'] + [x_{H_j}, [x_{H_i}, c']].$$

By induction, $[x_{H_i}, c'] \in \mathfrak{H}_{s+1}(\mathcal{A}_Y)$, and so $[x_{H_j}, [x_{H_i}, c']] \in \mathfrak{H}_{s+2}(\mathcal{A}_Y)$.

On the other hand, since $H_i \neq H_j$, there exists a flat $Z \in \mathcal{L}_2(\mathcal{A})$ such that $\{H_i, H_j\} \subset \mathcal{A}_Z$, and so $[x_{H_i}, x_{H_j}] \in \mathfrak{H}_2(\mathcal{A}_Z)$. If $Z = Y$, then $[[x_{H_i}, x_{H_j}], c'] \in [\mathfrak{H}_2(\mathcal{A}_Y), \mathfrak{H}_s(\mathcal{A}_Y)] \subset \mathfrak{H}_{s+2}(\mathcal{A}_Y)$. If $Z \neq Y$, then $[[x_{H_i}, x_{H_j}], c'] = 0$, by Claim II. Either way, we conclude that $[x_{H_i}, c] \in \mathfrak{H}_{s+2}(\mathcal{A}_Y)$. $\square$

Having established the above claims, we are now ready to prove Proposition 4.1, by induction on r. For $r = 2$, the map ι_2 is surjective, since $\pi_2 \circ \iota_2 = \mathrm{id}$, and π_2 is an isomorphism (for arbitrary $\mathcal{A}$). For the induction step, it is plainly enough to show that

$$[x_H, c] \in \mathfrak{H}_{r+1}(\mathcal{A}_Y), \tag{4.3}$$

for any $H \in \mathcal{A}$ and $c \in \mathfrak{H}_r(\mathcal{A}_Y)$, where $Y \in \mathcal{L}_2(\mathcal{A})$ and $r \geq 2$.

If $H \in \mathcal{A}_Y$, this is clear. Assuming $H \notin \mathcal{A}_Y$, pick any $H'' \in \mathcal{A}_Y$, and set $X = H \cap H'' \in \mathcal{L}_2(\mathcal{A})$. Note that $X \neq Y$, and $\mathcal{A}_X^* = \mathcal{A}_X \setminus \{H''\}$. Hence, $H \in \mathcal{A}_X^*$, and (4.3) now follows from Claim III. The proof of Proposition 4.1 is thus complete. $\square$

5. Proof of Theorem 2.4

We are now in the position to prove our main result.

Let $\mathcal{A}$ be an arbitrary arrangement. Recall we defined in §2.1 a homomorphism of graded Lie algebras, $\pi\colon \mathfrak{H}(\mathcal{A}) \to \prod_{X \in \mathcal{L}_2(\mathcal{A})} \mathfrak{H}(\mathcal{A}_X)$. Recall also we defined in §3.2 a homomorphism of graded abelian groups, $\iota\colon \prod_{X \in \mathcal{L}_2(\mathcal{A})} \mathfrak{H}(\mathcal{A}_X) \to \mathfrak{H}(\mathcal{A})$, with the property that $\pi' \circ \iota' = \mathrm{id}$, which showed that π' is an epimorphism.

Now suppose $\mathcal{A}$ is decomposable. By Proposition 4.1, each map ι_r ($r \geq 2$) is surjective. By Lemmas 3.1 and 3.2, each map π_r ($r \geq 2$) is an isomorphism. Hence, π' is an isomorphism of Lie algebras, with inverse ι'. This proves Part (3) of Theorem 2.4.

Part (2) follows at once from (3), and the fact that each Lie algebra $\mathfrak{H}(\mathcal{A}_X) \cong \boldsymbol{L}_{\mu(X)} \times \boldsymbol{L}_1$ is torsion-free.

Part (1) follows from (2), together with (1.3) and [9, Proposition 5.1].

Part (4) follows from (3), together with (1.3) and the discussion from [15, §1.5]. $\square$

6. Decomposable Chen Lie algebras

Another, much coarser approximation to the associated graded Lie algebra of a group is its Chen Lie algebra. We now study the effect of the decomposability condition on the Chen Lie algebra of an arrangement group.

Given a finitely-generated group G, let G/G'' be the quotient by its second derived subgroup. We call the associated graded Lie algebra $\operatorname{gr}(G/G'')$, the *Chen Lie algebra* of G. Set $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$. Plainly, $\theta_k(G) = \phi_k(G)$ for $k \leq 3$, and $\theta_k(G) \leq \phi_k(G)$ for $k > 3$.

Now suppose $G(\mathcal{A})$ is an arrangement group. Then, as shown in [12, Theorem 11.1], there is an isomorphism of graded Lie algebras,

$$\operatorname{gr}(G(\mathcal{A})/G''(\mathcal{A})) \otimes \mathbb{Q} \cong (\mathfrak{H}(\mathcal{A})/\mathfrak{H}''(\mathcal{A})) \otimes \mathbb{Q}. \tag{6.1}$$

Let $B(\mathcal{A}) = \mathfrak{H}'(\mathcal{A})/\mathfrak{H}''(\mathcal{A})$ be the *infinitesimal Alexander invariant* of $\mathcal{A}$. Taking graded ranks on both sides of (6.1), we find

$$\theta_k(G(\mathcal{A})) = \operatorname{rank} B_k(\mathcal{A}) \quad \text{for all } k \geq 2. \tag{6.2}$$

Recall once more the surjective map of graded Lie algebras from Proposition 2.1, $\pi' \colon \mathfrak{H}'(\mathcal{A}) \twoheadrightarrow \prod_{X\in\mathcal{L}_2(\mathcal{A})} \mathfrak{H}'(\mathcal{A}_X)$. By abelianization, we obtain an epimorphism of graded abelian groups,

$$B(\pi)\colon B(\mathcal{A}) \longrightarrow \bigoplus_{X\in\mathcal{L}_2(\mathcal{A})} B(\mathcal{A}_X)\,. \tag{6.3}$$

By comparing graded ranks of the source and target of $B(\pi)$, we recover a lower bound for the Chen ranks of an arrangement group, first obtained in [2] by other methods.

Corollary 6.1 ([2]). *For all $r \geq 2$,*

$$\theta_r(G(\mathcal{A})) \geq \sum_{X\in\mathcal{L}_2(\mathcal{A})} \theta_r(F_{\mu(X)}), \tag{6.4}$$

where $\theta_r(F_n) = (r-1)\binom{n+r-2}{r}$.

Note that $B_2(\pi) = \pi_2$ is an isomorphism, and thus equality holds in (6.4) for $r = 2$. For $r \geq 3$, though, the inequality can well be strict; see again [2].

As another application of our methods, we provide a complete description of the Chen Lie algebra of a decomposable arrangement.

Theorem 6.2. *If $\mathcal{A}$ is decomposable, then:*

(1) $\operatorname{gr}(G(\mathcal{A})/G''(\mathcal{A})) = \mathfrak{H}(\mathcal{A})/\mathfrak{H}''(\mathcal{A})$*, as graded Lie algebras over* $\mathbb{Z}$*.*

(2) $\operatorname{gr}(G(\mathcal{A})/G''(\mathcal{A}))$ *is torsion-free, as a graded abelian group.*

(3) *The Chen ranks of $G(\mathcal{A})$, for $r \geq 2$, are given by*

$$\theta_r(G(\mathcal{A})) = \sum_{X \in \mathcal{L}_2(\mathcal{A})} \theta_r(F_{\mu(X)}). \tag{6.5}$$

Proof. For any flat $X \in \mathcal{L}_2(\mathcal{A})$, we have $B(\mathcal{A}_X) = \boldsymbol{L}'_{\mu(X)}/\boldsymbol{L}''_{\mu(X)}$, which is known to be torsion-free. Now, since $\mathcal{A}$ is decomposable, Theorem 2.4(3) implies that $B(\pi)$ is an isomorphism, and consequently $B(\mathcal{A})$ is torsion-free, as well. Hence, $\mathfrak{H}(\mathcal{A})/\mathfrak{H}''(\mathcal{A})$ is also torsion-free. Parts (1) and (2) now follow from Theorem B in [12]. Part (3) follows from the fact that $B(\pi)$ is an isomorphism, and (6.2). □

Formula (6.5) was derived by other methods in [2], under the decomposability condition from that paper.

7. Decomposable graphic arrangements

To a (simple) graph G, with vertex set $\mathsf{V} = \{1, \dots, \ell\}$ and edge set E, there corresponds a *graphic arrangement* in $\mathbb{C}^\ell$, denoted by $\mathcal{A}_{\mathsf{G}}$. The hyperplane corresponding to an edge $e = (i, j)$ is $H_e = \{z_i - z_j = 0\}$. For example, if $\mathsf{G} = K_\ell$, the complete graph on ℓ vertices, then $\mathcal{A}_{K_\ell} = \mathcal{B}_\ell$, the braid arrangement in $\mathbb{C}^\ell$.

For each flat $X \in \mathcal{L}_2(\mathcal{A}_{\mathsf{G}})$, there are either 2 or 3 hyperplanes containing X. Under the identification $\mathcal{A}_{\mathsf{G}} = \mathsf{E}$, a flat of size 3 of corresponds to a triangle in the graph, while a flat of size 2 corresponds to a pair of edges which is not included in any element of the triangle-set T. Thus, the holonomy Lie algebra of $\mathcal{A}_{\mathsf{G}}$ can be identified with the quotient of the free Lie algebra on variables $e \in \mathsf{E}$ by the corresponding ideal of quadratic relations:

$$\mathfrak{H}(\mathsf{G}) = \boldsymbol{L}(\mathsf{E})\big/\text{ideal}\left\{ \begin{array}{ll} [e_1, e_2 + e_3], & \text{if } \{e_1, e_2, e_3\} \in \mathsf{T} \\ [e_1, e_2], & \text{if } \{e_1, e_2, e\} \notin \mathsf{T} \text{ for all } e \in \mathsf{E} \end{array} \right\}. \tag{7.1}$$

As shown in [15], the $\mathbb{Q}$-decomposability condition for a graphic arrangement can be read off the graph itself, as the absence of complete quadrangles in G. We present a strengthened form of this result, which nicely illustrates our methods.

Proposition 7.1. *For a graphic arrangement $\mathcal{A}_{\mathsf{G}}$, the following conditions are equivalent:*

(1) $\mathcal{A}_{\mathsf{G}}$ *is decomposable.*

(2) $\mathfrak{H}_3(\mathsf{G})$ *is decomposable over some field* $\Bbbk$.

(3) G *contains no complete subgraphs on* 4 *vertices.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious.

To show (2) $\Rightarrow$ (3), suppose K_4 is a subgraph of G. Then, the braid arrangement $\mathcal{B} = \mathcal{A}_{K_4}$ is a sub-arrangement of $\mathcal{A}_{\mathsf{G}}$. But $\mathcal{B}$ is not $\Bbbk$-decomposable, for any field $\Bbbk$. Indeed, $\sum_{X \in \mathcal{L}_2(\mathcal{B})} \phi_3(F_{\mu(X)}) = 8$, whereas $\dim_{\Bbbk} \mathfrak{H}_3(\mathcal{B}) \otimes \Bbbk = 10$ (see [7] for the case $\Bbbk = \mathbb{Q}$, and [5] for the general case). This contradicts Proposition 3.3.

To show (3) $\Rightarrow$ (1), we must check that $\mathfrak{H}_3(\mathsf{G})$ is spanned by $\{\iota_\tau(\mathfrak{H}_3(\tau)) \mid \tau \in \mathsf{T}\}$, where $\iota_\tau \colon \mathfrak{H}(\tau) \to \mathfrak{H}(\mathsf{G})$ is the natural inclusion. As an abelian group, $\mathfrak{H}_3(\mathsf{G})$ is generated by elements of the form $x = [e_1, [e_2, e_3]]$. Note that the edges e_2, e_3 must belong to a common triangle, say, τ, for, otherwise, $[e_2, e_3] = 0$ in $\mathfrak{H}(\mathsf{G})$. If $e_1 \in \tau$, then clearly $x \in \iota_\tau(\mathfrak{H}_3(\tau))$. If $e_1 \notin \tau$, we will show that $x = 0$, and that will finish the proof.

First, we claim that there are two edges, e and e', in τ such that

$$[e_1, e] = [e_1, e'] = 0. \tag{7.2}$$

To verify the claim, denote by G_0 the subgraph supported on the vertices of τ and e_1. Since $e_1 \notin \tau$, there are two possibilities:

(a) G_0 has 5 vertices. Then $[e_1, e] = 0$, for any edge $e \in \tau$.

(b) G_0 has 4 vertices. Since by assumption $\mathsf{G}_0 \neq K_4$, again there are two possibilities:

(b$_1$) G_0 has 4 edges. Then $[e_1, e] = 0$, for any edge $e \in \tau$.

(b$_2$) G_0 has 5 edges. Then G_0 is the union of two triangles, with an edge in common. If e, e' are the other two edges in τ, then $[e_1, e] = [e_1, e'] = 0$.

Thus, (7.2) holds in all cases.

Now, applying (4.2) to $\tau^* = \{e, e'\}$, we see that $[e_2, e_3]$ is a multiple of $[e, e']$ in $\mathfrak{H}_2(\tau)$. Hence, by (7.2) and the Jacobi identity, $x = 0$. □

Let $\kappa_s = \kappa_s(\mathsf{G})$ be the number of K_{s+1} subgraphs of G; for example, $\kappa_0 = |\mathsf{V}|$, $\kappa_1 = |\mathsf{E}|$, $\kappa_2 = |\mathsf{T}|$. If $\kappa_3 = 0$, then, by Proposition 7.1 and Theorem 2.4, we have

$$\prod_{r=1}^{\infty} (1 - t^r)^{\phi_r} = (1 - t)^{\kappa_1 - 2\kappa_2} (1 - 2t)^{\kappa_2}. \tag{7.3}$$

We now provide concrete examples where this LCS formula gives new information. For that, we need graphs which are not chordal (i.e., supersolvable), or, more generally, hypersolvable (in the sense of [11]), since otherwise, previously known formulas apply.

Proposition 7.2. *Let* G *be a graph with* $\kappa_1 \leq 2\kappa_2$ *and* $\kappa_3 = 0$*. Then* $\mathcal{A}_{\mathsf{G}}$ *is decomposable, but not hypersolvable.*

Proof. Since $\kappa_3 = 0$, the arrangement $\mathcal{A}_{\mathsf{G}}$ is decomposable. If $\mathcal{A}_{\mathsf{G}}$ were hypersolvable, then, by the LCS formula from [5], $\prod_{r=1}^{\infty}(1-t^r)^{\phi_r} = (1-t)P(t)$, for some polynomial P. In view of (7.3), this can only happen when $\kappa_1 - 2\kappa_2 > 0$. $\square$

Remark 7.3. Let $\mathcal{A}$ be an arrangement (not necessarily graphic) for which the Möbius function takes only the values 1 and 2 on $\mathcal{L}_2(\mathcal{A})$. Set

$$\kappa_1 = |\mathcal{A}| \quad \text{and} \quad \kappa_2 = |\{X \in \mathcal{L}_2(\mathcal{A}) \mid \mu(X) = 2\}|\,.$$

The same argument as in Proposition 7.2 shows the following: If $\mathcal{A}$ is decomposable and hypersolvable, then $\kappa_1 - 2\kappa_2 > 0$.

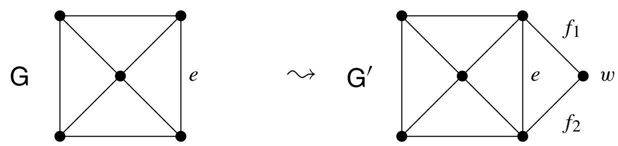

Figure 2. Coning an edge.

Now suppose G is a graph with $\kappa_1 - 2\kappa_2 \leq 0$ and $\kappa_3 = 0$. One can create a new graph, G', with the same properties, as follows. Choose an edge e of G, pick a new vertex w, and join it by edges f_1 and f_2 to the endpoints of e, as in Figure 2. Clearly, $\mathsf{V}' = \mathsf{V} \cup \{w\}$, $\mathsf{E}' = \mathsf{E} \cup \{f_1, f_2\}$, $\mathsf{T}' = \mathsf{T} \cup \{e, f_1, f_2\}$, and there are no complete quadrangles introduced. Thus, $\kappa_1' - 2\kappa_2' = (\kappa_1 + 2) - 2(\kappa_2 + 1) \leq 0$, and $\kappa_3' = 0$. Moreover, it is easy to check that $\mathcal{A}_{\mathsf{G}}$ is solvable in $\mathcal{A}_{\mathsf{G}'}$, in the sense of [5].

This permits us to create infinite families of graphs satisfying the hypothesis of Proposition 7.2. For instance, start with the graph $\mathsf{G}^0 = \mathsf{G}$ from the above figure (see also [15, Example 6.14]), and define inductively a sequence of graphs $\{\mathsf{G}^i\}$ by $\mathsf{G}^i = (\mathsf{G}^{i-1})'$. Since G satisfies $\kappa_1 - 2\kappa_2 = \kappa_3 = 0$, all the graphic arrangements $\mathcal{A}_{\mathsf{G}^i}$ are decomposable, but not hypersolvable. By (7.3), the LCS ranks of the corresponding arrangement groups are given by $\prod_{r=1}^{\infty}(1-t^r)^{\phi_r} = (1-2t)^{i+4}$.

Remark 7.4. To the best of our knowledge, the decomposable arrangements discussed in this paper provide the first non-hypersolvable examples where the LCS ranks ϕ_r are computed for *all* values of r. Note that the two graphic arrangements in Examples 3.7 and 5.4 from [14] are hypersolvable. Indeed, the two underlying graphs can be obtained from hypersolvable graphs, by iterating the above construction: the first one, starting from a 4-cycle, and the second one, starting from the K_4 graph. As such, both arrangements are hypersolvable, cf. [11, §6]. The first one has rank 4 and exponents $\{1, 1, 1, 1, 2\}$; the second one has rank 6 and exponents $\{1, 2, 2, 2, 2, 3\}$, and thus is actually supersolvable (by [5, Theorem D]), despite a claim to the contrary in [14].

Note added in proof. Using the holonomy Lie algebra approach, P. Lima-Filho and H. Schenck have recently announced in [8] a proof of the LCS formula for graphic arrangements, as conjectured in [15].

References

[1] A. D. R. Choudary, A. Dimca, S. Papadima, Some analogs of Zariski's theorem on nodal line arrangements. *Algebr. Geom. Topol.* **5** (2005), 691–711. Zbl 1081.32018 MR 2153112

[2] D. Cohen, A. Suciu, Alexander invariants of complex hyperplane arrangements. *Trans. Amer. Math. Soc.* **351** (1999), 4043–4067. Zbl 0945.20024 MR 1475679

[3] M. Falk, The cohomology and fundamental group of a hyperplane complement. In *Singularities* (Iowa City, IA, 1986), Contemp. Math. 90, Amer. Math. Soc, Providence, RI, 1989, 55–72. Zbl 0697.55013 MR 1000594

[4] M. Falk, R. Randell, The lower central series of a fiber-type arrangement. *Invent. Math.* **82** (1985), 77–88. Zbl 0574.55010 MR 0808110

[5] M. Jambu, S. Papadima, A generalization of fiber-type arrangements and a new deformation method. *Topology* **37** (1998), 1135–1164. Zbl 0988.52031 MR 1632975

[6] T. Kohno, On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces. *Nagoya Math. J.* **92** (1983), 21–37. Zbl 0503.57001 MR 0726138

[7] —, Série de Poincaré-Koszul associée aux groupes de tresses pures. *Invent. Math.* **82** (1985), 57–75. Zbl 0574.55009 MR 0808109

[8] P. Lima-Filho, H. Schenck, Holonomy Lie algebras and the LCS formula for graphic arrangements. Preprint, 2005.

[9] M. Markl, S. Papadima, Homotopy Lie algebras and fundamental groups via deformation theory. *Ann. Inst. Fourier* **42** (1992), 905–935. Zbl 0760.55010 MR 1196099

[10] J. Morgan, The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.* **48** (1978), 137–204. Zbl 0401.14003 MR 0516917

[11] S. Papadima, A. Suciu, Higher homotopy groups of complements of hyperplane arrangements. *Adv. Math.* **165** (2002), 71–100. Zbl 1019.52016 MR 1880322

[12] —, Chen Lie algebras. *Internat. Math. Research Notices* **2004** (2004), 1057–1086. Zbl 1076.17007 MR 2037049

[13] —, Algebraic invariants for Bestvina-Brady groups. Preprint, 2006; arXiv:math.GR/0603240.

[14] I. Peeva, Hyperplane arrangements and linear strands in resolutions. *Trans. Amer. Math. Soc.* **355** (2002), 609–618. Zbl 01821253 MR 1932716

[15] H. Schenck, A. Suciu, Lower central series and free resolutions of hyperplane arrangements. *Trans. Amer. Math. Soc.* **354** (2002), 3409–3433. Zbl 1057.52015 MR 1911506

[16] J. Stallings, A finitely presented group whose 3-dimensional integral homology is not finitely generated. *Amer. J. Math.* **85** (1963), 541–543. Zbl 0122.27301 MR 0158917

[17] A. Suciu, Fundamental groups of line arrangements: Enumerative aspects. In *Advances in algebraic geometry motivated by physics*, Contemp. Math. 276, Amer. Math. Soc, Providence, RI, 2001, 43–79. Zbl 0998.14012 MR 1837109

[18] D. Sullivan, Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 269–331. Zbl 0374.57002 MR 0646078

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