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Groups, Geometry, and Dynamics

Nil graded self-similar algebras

Victor M. Petrogradsky[1], Ivan P. Shestakov[2] and Efim Zelmanov[3]

Abstract. In [19], [24] we introduced a family of self-similar nil Lie algebras $\boldsymbol{L}$ over fields of prime characteristic $p > 0$ whose properties resemble those of Grigorchuk and Gupta–Sidki groups. The Lie algebra $\boldsymbol{L}$ is generated by two derivations

$$v_1 = \partial_1 + t_0^{p-1}(\partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots))))),$$
$$v_2 = \partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots))))$$

of the truncated polynomial ring $K[t_i, i \in \mathbb{N} \mid t_i^p = 0, i \in \mathbb{N}]$ in countably many variables. The associative algebra $\boldsymbol{A}$ generated by v_1, v_2 is equipped with a natural $\mathbb{Z} \oplus \mathbb{Z}$-gradation. In this paper we show that for p, which is not representable as $p = m^2 + m + 1$, $m \in \mathbb{Z}$, the algebra $\boldsymbol{A}$ is graded nil and can be represented as a sum of two locally nilpotent subalgebras. L. Bartholdi [3] and Ya. S. Krylyuk [15] proved that for $p = m^2 + m + 1$ the algebra A is not graded nil. However, we show that the second family of self-similar Lie algebras introduced in [24] and their associative hulls are always $\mathbb{Z}^p$-graded, graded nil, and are sums of two locally nilpotent subalgebras.

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1. Definitions and constructions

Let L be a Lie algebra over a field K of characteristic $p > 0$ and let $\operatorname{ad} x : L \to L$, $\operatorname{ad} x(y) = [x, y]$ for $x, y \in L$, be the adjoint map. Recall that L is called a *restricted Lie algebra* or *Lie p-algebra* [12], [26], [1] if L additionally affords a unary operation $x \mapsto x^{[p]}$, $x \in L$, satisfying

i) $(\lambda x)^{[p]} = \lambda^p x^{[p]}$ for all $\lambda \in K$, $x \in L$;
ii) $\operatorname{ad}(x^{[p]}) = (\operatorname{ad} x)^p$ for all $x \in L$;

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iii) for all $x, y \in L$ one has

$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x,y), \tag{1}$$

where $i s_i(x,y)$ is the coefficient of Z^{i-1} in the polynomial $(\mathrm{ad}(Zx+y))^{p-1}(x)$ in $L[Z]$, with Z is an indeterminate. Also, $s_i(x,y)$ is a Lie polynomial in x, y of degrees i and $p-i$, respectively.

Suppose that L is a restricted Lie algebra and $X \subset L$. Then by $\mathrm{Lie}_p(X)$ we denote the restricted subalgebra generated by X. Let $H \subset L$ be a Lie subalgebra, i.e., H is a vector subspace which is closed under the Lie bracket. Then by H_p we denote the restricted subalgebra generated by H. In what follows by an associative enveloping algebra of a Lie algebra we mean the associative algebra without 1.

We recall the notion of growth. Let A be an associative (or Lie) algebra generated by a finite set X. Denote by $A^{(X,n)}$ the subspace of A spanned by all monomials in X of length not exceeding n. If A is a restricted Lie algebra, then we define [16] $A^{(X,n)} = \langle [x_{i_1}, \dots, x_{i_s}]^{p^k} \mid x_{i_j} \in X,\ sp^k \le n \rangle_K$. In either situation, one defines the *growth function*:

$$\gamma_A(n) = \gamma_A(X,n) = \dim_K A^{(X,n)}, \quad n \in \mathbb{N}.$$

The growth function clearly depends on the choice of the generating set X. Furthermore, it is easy to see that the exponential growth is the highest possible growth for Lie and associative algebras. The growth function $\gamma_A(n)$ is compared with the polynomial functions n^k, $k \in \mathbb{R}^+$, by computing the *upper and lower Gelfand–Kirillov dimensions* [14], namely

$$\mathrm{GKdim}\, A = \overline{\lim_{n\to\infty}} \frac{\ln \gamma_A(n)}{\ln n},$$
$$\underline{\mathrm{GKdim}}\, A = \underline{\lim_{n\to\infty}} \frac{\ln \gamma_A(n)}{\ln n}.$$

This setting assumes that all elements of X have the same weight equal to 1. We shall mainly use a somewhat different growth function. Namely, we consider the weight function $\mathrm{wt}\, v$, $v \in A$, and the growth with respect to it: $\tilde{\gamma}_A(n) = \dim_K \langle y \mid y \in A,\ \mathrm{wt}\, y \le n \rangle$, $n \in \mathbb{N}$, where the elements of the generating set X have different weights. Standard arguments [14] prove that this growth function yields the same Gelfand–Kirillov dimensions.

Now suppose that $\mathrm{char}\, K = p > 0$. Denote $I = \{0,1,2,\dots\}$ and $\mathbb{N}_p = \{0,1,\dots,p-1\}$. Consider the truncated polynomial algebra

$$R = K[t_i, i \in I \mid t_i^p = 0, i \in I].$$

Let $\mathbb{N}_p^I = \{\alpha : I \to \mathbb{N}_p\}$ be the set of functions with finitely many nonzero values. For $\alpha \in \mathbb{N}_p^I$ denote $|\alpha| = \sum_{i\in I} \alpha_i$ and $\boldsymbol{t}^\alpha = \prod_{i\in I} t_i^{\alpha_i} \in R$. The set $\{\boldsymbol{t}^\alpha \mid \alpha \in \mathbb{N}_p^I\}$

is clearly a basis of R. Consider the ideal R^+ spanned by all elements $\boldsymbol{t}^\alpha$, $\alpha \in \mathbb{N}_p^I$, $|\alpha| > 0$. Let $\partial_i = \frac{\partial}{\partial t_i}$, $i \in I$, denote the partial derivatives of R.

We introduce the so-called Lie algebra of *special derivations* of R [22], [23], [20]:

$$\boldsymbol{W}(R) = \Big\{ \sum_{\alpha \in \mathbb{N}_p^I} \boldsymbol{t}^\alpha \sum_{j=1}^{m(\alpha)} \lambda_{\alpha,i_j} \frac{\partial}{\partial t_{i_j}} \mid \lambda_{\alpha,i_j} \in K,\ i_j \in I \Big\}.$$

It is essential that the sum at each $\boldsymbol{t}^\alpha$, $\alpha \in \mathbb{N}_p^I$, is finite.

Lemma 1.1 ([21]). *For arbitrary complex numbers $a_i \in \mathbb{C}$, $i \in \mathbb{N}$, there exist gradations on the algebras R, $\boldsymbol{W}(R)$ such that* $\operatorname{wt}(t_i) = -a_i$, $\operatorname{wt}(\partial_i) = a_i$.

Denote by $\tau\colon R \to R$ the shift endomorphism $\tau(t_i) = t_{i+1}$, $i \in I$. Extending it by $\tau(\partial_i) = \partial_{i+1}$, $i \in I$, we get the shift endomorphism $\tau\colon \boldsymbol{W}(R) \to \boldsymbol{W}(R)$.

2. First example

We define the following two derivations of R:

$$v_1 = \partial_1 + t_0^{p-1}(\partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots))))),$$
$$v_2 = \partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots)))).$$

These operators are special derivations $v_1, v_2 \in \boldsymbol{W}(R)$. Observe that we can write these derivations recursively:

$$v_1 = \partial_1 + t_0^{p-1}\tau(v_1), \quad v_2 = \tau(v_1).$$

Let $\boldsymbol{L} = \operatorname{Lie}_p(v_1, v_2) \subset \boldsymbol{W}(R) \subset \operatorname{Der} R$ be the restricted subalgebra generated by $\{v_1, v_2\}$. This algebra was introduced in [24]. In the case of characteristic $p = 2$, it coincides with the *Fibonacci restricted Lie algebra* introduced in [19]. Similarly, define

$$v_i = \tau^{i-1}(v_1) = \partial_i + t_{i-1}^{p-1}(\partial_{i+1} + t_i^{p-1}(\partial_{i+2} + t_{i+1}^{p-1}(\partial_{i+3} + \cdots))), \tag{2}$$

$i = 1, 2, \ldots$. We also can write

$$v_i = \partial_i + t_{i-1}^{p-1} v_{i+1}, \quad i = 1, 2, \ldots. \tag{3}$$

Lemma 2.1. *Let $\boldsymbol{L} = \operatorname{Lie}_p(v_1, v_2) \subset \operatorname{Der} R$ be the restricted Lie algebra generated by $\{v_1, v_2\}$. Then the following relations holds:*

(1) $[v_i, v_{i+1}] = -t_i^{p-2} v_{i+2}$ *for* $i = 1, 2, \ldots;$

(2) $[v_i, v_{i+2}] = -t_{i-1}^{p-1} t_{i+1}^{p-2} v_{i+3}$ *for* $i = 1, 2, \ldots;$

(3) *in general, for all* $1 \le i < j$ *we have*

$$[v_i, v_j] = -(t_{i-1}t_i \dots t_{j-3})^{p-1} t_{j-1}^{p-2} v_{j+1};$$

(4) *for all* $n \ge 1$, $j \ge 0$ *we have the action*

$$v_n(t_j) = \begin{cases} (t_{n-1}t_n \cdots t_{j-2})^{p-1}, & n < j, \\ 1, & n = j, \\ 0, & n > j; \end{cases}$$

(5) *for all* $k, n \ge 1$,

$$[\partial_n, v_k] = \begin{cases} -(t_{k-1}t_k \dots t_{n-1})^{p-1} t_n^{p-2} v_{n+2}, & k < n+1, \\ -t_n^{p-2} v_{n+2}, & k = n+1, \\ 0, & k > n+1; \end{cases}$$

(6) $v_i^p = -t_{i-1}^{p-1} v_{i+2}$ *for all* $i \ge 1$.

Proof. The claims (1)–(5) are proved in [24]. The last claim for $p = 2$ is checked in [19], we assume that $p \ge 3$. We have $v_i^p = (\partial_i + t_{i-1}^{p-1} v_{i+1})^p$. By formula (1) we obtain the sum of commutators of length p. We apply the previous claim $[\partial_i, t_{i-1}^{p-1} v_{i+1}] = -t_{i-1}^{p-1} t_i^{p-2} v_{i+2}$. In further commutators we cannot use $t_{i-1}^{p-1} v_{i+1}$ anymore because of the total power of t_{i-1}. Thus, only one term in (1) is nontrivial, namely $s_{p-1}(x, y) = (\operatorname{ad} x)^{p-1}(y)$. We get

$$\begin{aligned} v_i^p &= (\partial_i + t_{i-1}^{p-1} v_{i+1})^p \\ &= (\operatorname{ad} \partial_i)^{p-1}(t_{i-1}^{p-1} v_{i+1}) \\ &= (\operatorname{ad} \partial_i)^{p-2}(-t_{i-1}^{p-1} t_i^{p-2} v_{i+2}) \\ &= -t_{i-1}^{p-1} v_{i+2}. \end{aligned}$$

□

Lemma 2.2. *Let* H *be the* K*-linear span of all elements* $t_0^{\alpha_0} t_1^{\alpha_1} \dots t_{n-2}^{\alpha_{n-2}} v_n$*, where* $0 \le \alpha_i \le p-1$, $\alpha_{n-2} \le p-2$, $n \ge 1$. *Then* H *is a restricted subalgebra of* Der R *and* $\mathbf{L} \subset H$.

Proof. Let us prove that H is a Lie subalgebra. We apply Lemma 2.1 to check that the product of two monomials of type above is expressed via such monomials again. Let $n < m$. Then

$$\begin{aligned} &[t_0^{\alpha_0} \dots t_{n-2}^{\alpha_{n-2}} v_n, t_0^{\beta_0} \dots t_{m-2}^{\beta_{m-2}} v_m] \\ &\quad = -t_0^{\alpha_0} \dots t_{n-2}^{\alpha_{n-2}} (t_{n-1} \dots t_{m-3})^{p-1} t_0^{\beta_0} \dots t_{m-2}^{\beta_{m-2}} t_{m-1}^{p-2} v_{m+1} \\ &\qquad + t_0^{\alpha_0} \dots t_{n-2}^{\alpha_{n-2}} \sum_{\beta_j \ne 0} \Big(\prod_{i=0, i \ne j}^{m-2} t_i^{\beta_i} \Big) \beta_j t_j^{\beta_j - 1} v_n(t_j) v_m. \end{aligned} \tag{4}$$

The first term is of type $\dots t_{m-2}^{\beta_{m-2}} t_{m-1}^{p-2} v_{m+1}$, as required.

By claim (4), v_m acts on all $t_i^{\alpha_i}$s trivially because $m > n > n-2 \geq i$, and no respective terms appear. It remains to consider the second term above. Similarly, $v_n(t_j)$ is nonzero only for $n \leq j$, namely

$$v_n(t_j) = (t_{n-1} t_n \dots t_{j-2})^{p-1}, \quad n \leq j.$$

In this case, $n \leq j \leq m-2$ and the new t_i's above have indices such that $n-1 < \dots < j-2 \leq m-4$. We again obtain monomials of the required type. Hence, $H \subset \operatorname{Der} R$ is a Lie subalgebra. The last claim of Lemma 2.1 implies that the subalgebra $H \subset \operatorname{Der} R$ is restricted. □

Let H_n denote the K-linear span of all elements $t_0^{\alpha_0} \dots t_{m-2}^{\alpha_{m-2}} v_m$, where $0 \leq \alpha_i \leq p-1$, $\alpha_{m-2} \leq p-2$, $m \geq n$.

Corollary 2.3. (1) $H_n \triangleleft H$, $n \geq 1$; $H = H_1 \supset H_2 \supset \cdots$.

(2) *Let $\boldsymbol{L}_n = \boldsymbol{L} \cap H_n$ for $n \geq 1$. Then the factor algebras H_n/H_{n+2} and $\boldsymbol{L}_n/\boldsymbol{L}_{n+2}$ are abelian with the trivial p-mapping for all $n \geq 1$.*

Proof. The fact that H_n are ideals and H_n/H_{n+2} are abelian follows from eq. (4) and other arguments of Lemma 2.2. In order to check that the p-mapping on H_n/H_{n+2} is trivial, we use claim (6) of Lemma 2.1 and eq. (1). □

A Lie algebra L is said to be *just-infinite* if it is infinite-dimensional and any proper factor algebra L/J is finite dimensional.

Lemma 2.4. *The algebra $\boldsymbol{L}$ is not just-infinite.*

Proof. In the case $p = 2$ all elements $\{v_n \mid n \geq 1\}$ belong to $\boldsymbol{L}$; see [19]. In the case of arbitrary characteristic the situation is more complicated, nevertheless, we have $v_{2n} \in \boldsymbol{L}$ for all $n \geq 1$; see [24]. Now let J be the restricted ideal of $\boldsymbol{L}$ generated by the elements

$$[v_1, v_{2n}] = -(t_0 t_1 \dots t_{2n-3})^{p-1} t_{2n-1}^{p-2} v_{2n+1}, \quad n \geq 2.$$

Observe that they all contain the common factor t_0^{p-1}, which has no chances to disappear by any further commutation. So, all elements of J have the factor t_0^{p-1}. Hence, the ideal J is abelian, infinite-dimensional, and has the trivial p-mapping. Since the elements $\{v_{2n} \mid n \geq 1\}$ are linearly independent modulo J, we conclude that $\dim L/J = \infty$. □

In our constructions we are motivated by analogies with constructions of self-similar groups and algebras [9] [8], [2]. In particular, the following property is analogous to the *periodicity* of the Grigorchuk and Gupta–Sidki groups [7], [10].

Theorem 2.5 ([19], [24]). *Let $\boldsymbol{L} = \operatorname{Lie}_p(v_1, v_2) \subset \operatorname{Der} R$ be the restricted subalgebra generated by $\{v_1, v_2\}$. Then $\boldsymbol{L}$ has a nil p-mapping.*

3. The first example: the gradation

In this section we introduce a $\mathbb{Z}\oplus\mathbb{Z}$-gradation on our algebras. Suppose that all elements v_i are homogeneous, $\operatorname{wt} v_i = -\operatorname{wt} t_i = a_i \in \mathbb{R}$, where $i = 1, 2 \dots$, such that all terms in (3) are homogeneous. To achieve this, we assume that

$$a_i = \operatorname{wt} v_i = \operatorname{wt} \partial_i = (p-1)\operatorname{wt} t_{i-1} + \operatorname{wt} v_{i+1} = -(p-1)a_{i-1} + a_{i+1}.$$

Hence, we get the recurrence relation

$$a_{i+1} = a_i + (p-1)a_{i-1}, \quad i \in \mathbb{N}. \tag{5}$$

This equation has the characteristic polynomial $\phi(t) = t^2 - t - (p-1)$ with two different roots

$$\lambda = \frac{1+\sqrt{4p-3}}{2}, \quad \lambda_1 = \frac{1-\sqrt{4p-3}}{2}.$$

It is well known that all solutions of the recurrence relation (5) are linear combinations of the two sequences $a_i = \lambda^i$, $i \in \mathbb{N}$, and $a_i = \lambda_1^i$, $i \in \mathbb{N}$.

We distinguish two cases.

Irrational: λ, λ_1 are irrational, e.g., for primes $p = 2, 5, 11, 17, 19, \dots$.

Rational: λ, $\bar{\lambda}$ are rational (moreover, in this case λ, λ_1 are integers), e.g., for primes $p = 3, 7, 13, 31, 43, \dots$. Note that in this case $\lambda \in \mathbb{Z}$ and $p = \lambda^2 - \lambda + 1$.

Remark. To the best of our knowledge the question if there are infinitely many such primes is open. A more general question asks whether there are infinitely many primes of the form $an^2 + bn + c$, where a, b, c are relatively prime integers, a positive, $a + b$ and c are not both even, and $b^2 - 4ac$ is not a perfect square; see [11], p. 19.

The existence of two linearly independent weight functions yields a $\mathbb{Z} \oplus \mathbb{Z}$-gradation.

Theorem 3.1. *Let* $\boldsymbol{L} = \operatorname{Lie}_p(v_1, v_2) \subset \operatorname{Der} R$ *be the restricted subalgebra generated by* $\{v_1, v_2\}$. *We introduce weight and superweight functions as follows:*

$$\operatorname{wt} v_n = -\operatorname{wt} t_n = \lambda^n, \qquad n = 1, 2, \dots, \ \lambda = \frac{1+\sqrt{4p-3}}{2},$$
$$\operatorname{swt} v_n = -\operatorname{swt} t_n = \lambda_1^{n-2}, \quad n = 1, 2, \dots, \ \lambda_1 = \frac{1-\sqrt{4p-3}}{2}.$$

Then:

(1) *Both functions are additive on products of homogeneous elements of* $\boldsymbol{L}$.

(2) *We have the* $\mathbb{Z}\oplus\mathbb{Z}$*-gradation* $\boldsymbol{L} = \bigoplus_{a,b\ge 0} \boldsymbol{L}_{a,b}$, *where* $\boldsymbol{L}_{a,b}$ *is spanned by products with a factors* v_1 *and b factors* v_2.

(3) *Let* $v \in \boldsymbol{L}_{a,b}$*, where* $a, b \geq 0$*. Then*

$$\operatorname{wt} v = \lambda a + \lambda^2 b, \quad \operatorname{swt} v = -\frac{\lambda}{p-1} a + b.$$

Proof. Let us introduce one more function that takes values in $\mathbb{R}^2$:

$$\mathrm{Wt}(v_i) = -\mathrm{Wt}(t_i) = (\mathrm{wt}(v_i), \mathrm{swt}(v_i)), \quad i \in \mathbb{N}.$$

Consider a monomial $v \in \boldsymbol{L}$ that is a product of a elements v_1 and b elements v_2. Then both weight functions are well defined on v. Moreover, $\mathrm{Wt}(*)$ is additive on products of monomials in v_i and t_j. Therefore, we get

$$\mathrm{Wt}(v) = a\,\mathrm{Wt}(v_1) + b\,\mathrm{Wt}(v_2).$$

Consider another pair of integers $(a', b') \neq (a, b)$ and a monomial $v' \in \boldsymbol{L}$ that contains a', b' letters v_1, v_2, respectively. By construction,

$$\mathrm{Wt}(v_1) = (\lambda, \lambda_1^{-1}) = (\lambda, -\lambda/(p-1)),$$
$$\mathrm{Wt}(v_2) = (\lambda^2, 1) = (\lambda + p - 1, 1).$$

Since these two vectors are linearly independent over $\mathbb{R}$, we get $\mathrm{Wt}(v') = a'\,\mathrm{Wt}(v_1) + b'\,\mathrm{Wt}(v_2) \neq \mathrm{Wt}(v)$ and the claimed $\mathbb{Z} \oplus \mathbb{Z}$-gradation $\boldsymbol{L} = \bigoplus_{a,b \geq 0} \boldsymbol{L}_{a,b}$.

Let $v \in \boldsymbol{L}_{a,b}$, where $a, b \geq 0$. Then

$$\operatorname{wt} v = a \operatorname{wt} v_1 + b \operatorname{wt} v_2 = a\lambda + b\lambda^2,$$
$$\operatorname{swt} v = a \operatorname{swt} v_1 + b \operatorname{swt} v_2 = a\lambda_1^{-1} + b = -\frac{\lambda}{p-1} a + b. \qquad \square$$

Let us introduce a new coordinate system on the plane. For a point $A = (x, y) \in \mathbb{R}^2$ we define its new coordinates as

$$\begin{aligned} \xi &= \mathrm{wt}(x, y) = \lambda x + \lambda^2 y = \lambda(x + \lambda y), \\ \eta &= \mathrm{swt}(x, y) = -\frac{\lambda}{p-1} x + y = \lambda_1^{-1}(x + \lambda_1 y) \end{aligned} \quad (x, y) \in \mathbb{R}^2. \tag{6}$$

We will refer to these coordinates as the *weight* and the *superweight* of the point (x, y) respectively. Gradations by superweights yield *triangular decompositions*.

Corollary 3.2. *Consider the restricted Lie algebra* $\boldsymbol{L} = \mathrm{Lie}_p(v_1, v_2)$*, the associative algebra* $\boldsymbol{A} = \mathrm{Alg}(v_1, v_2)$ *generated by* v_1, v_2*, the universal enveloping algebra* $U = U(\boldsymbol{L})$*, and the universal restricted enveloping algebra* $\boldsymbol{u} = u(\boldsymbol{L})$*. Then:*

(1) *All these algebras have decompositions into direct sums of three subalgebras,*

$$\boldsymbol{L} = \boldsymbol{L}_+ \oplus \boldsymbol{L}_0 \oplus \boldsymbol{L}_-, \quad \boldsymbol{A} = \boldsymbol{A}_+ \oplus \boldsymbol{A}_0 \oplus \boldsymbol{A}_-,$$
$$U = U_+ \oplus U_0 \oplus U_-, \quad \boldsymbol{u} = \boldsymbol{u}_+ \oplus \boldsymbol{u}_0 \oplus \boldsymbol{u}_-,$$

where $\boldsymbol{L}_+$, $\boldsymbol{L}_0$, *and* $\boldsymbol{L}_-$ *are spanned by homogeneous elements* $v \in \boldsymbol{L}$ *such that* $\operatorname{swt} v > 0$, $\operatorname{swt} v = 0$, *and* $\operatorname{swt} v < 0$, *respectively. The decompositions of other algebras are defined similarly.*

(2) *In the irrational case we have* $\boldsymbol{L}_0 = \{0\}$, $\boldsymbol{A}_0 = \{0\}$, $U_0 = \{0\}$, $\boldsymbol{u}_0 = \{0\}$.

Proof. Suppose that λ is irrational. Consider $0 \neq v \in \boldsymbol{L}_{a,b}$, where $(a,b) \in \mathbb{Z}^2$. Suppose that $\operatorname{swt}(v) = -a\lambda/(p-1) + b = 0$. If $b \neq 0$ then $\lambda \in \mathbb{Q}$, a contradiction. □

Lemma 3.3. *In the irrational case for an arbitrary lattice point* $(a,b) \in \mathbb{Z}^2 \subset \mathbb{R}^2$ *we have*

$$|\operatorname{wt}(a,b) \cdot \operatorname{swt}(a,b)| \geq \frac{\lambda^2}{p-1}, \quad (a,b) \in \mathbb{Z}^2.$$

Proof. Note that the polynomial $\psi(t) = t^2 + t - (p-1)$ has the discriminant $D = 4p - 3$ and no rational roots. For arbitrary integers $a, b \in \mathbb{Z}$ we have $0 \neq |\psi(a/b)| = |a^2 + ab - (p-1)b^2|b^{-2}$. Hence, $|a^2 + ab - (p-1)b^2| \geq 1$. By the formulas (6),

$$\begin{aligned} |\operatorname{wt}(a,b) \cdot \operatorname{swt}(a,b)| &= |\lambda(a + \lambda b)\lambda_1^{-1}(a + \lambda_1 b)| \\ &= |\lambda\lambda_1^{-1}||a^2 + (\lambda + \lambda_1)ab + \lambda\lambda_1 b^2| \\ &= \frac{\lambda^2}{p-1}|a^2 + ab - (p-1)b^2| \geq \frac{\lambda^2}{p-1}. \end{aligned}$$

□

Lemma 3.4. *Suppose that* $p \geq 3$. *Let* $w = t_0^{\alpha_0} t_1^{\alpha_1} \dots t_{n-2}^{\alpha_{n-2}} v_n$, $n \geq 1$, *be a monomial of the subalgebra* H *above, namely,* $0 \leq \alpha_i \leq p-1$, $\alpha_{n-2} \leq p-2$. *Then*

(1) $\lambda^{n-2} \leq \operatorname{wt}(w) \leq \lambda^n$;

(2) $|\operatorname{swt}(w)| \leq C|\lambda_1|^{n-2}$ *in case* $p \geq 5$;

(3) $|\operatorname{swt}(w)| \leq pn$ *in case* $p = 3$.

Proof. Clearly, $\operatorname{wt} w \leq \operatorname{wt} v_n \leq \lambda^n$. In case $n = 1$ we have only one monomial $w = v_1$ and our estimates are valid. Let $n \geq 2$, we obtain the bounds

$$\begin{aligned} \operatorname{wt}(w) &= \operatorname{wt}(v_n) + \sum_{i=0}^{n-2} \alpha_i \operatorname{wt} t_i \\ &= \lambda^n - \sum_{i=0}^{n-2} \alpha_i \lambda^i \\ &\geq \lambda^n - (p-1)\sum_{i=0}^{n-2} \lambda^i + \lambda^{n-2} \\ &\geq \lambda^n\left(1 - \tfrac{(p-1)\lambda^{-2}}{1-1/\lambda} + \tfrac{1}{\lambda^2}\right) \\ &= \lambda^n\left(\tfrac{\lambda^2-\lambda-(p-1)}{\lambda^2-\lambda} + \tfrac{1}{\lambda^2}\right) = \lambda^{n-2}. \end{aligned}$$

Claims 2 and 3. In case $p \geq 5$ we have $|\lambda_1| > 1$ and get the bound

$$|\mathrm{swt}(w)| = \left| \mathrm{swt}(v_n) + \sum_{i=0}^{n-2} \alpha_i \,\mathrm{swt}\, t_i \right| \leq |\lambda_1|^n + (p-1)\frac{|\lambda_1|^{n-2}}{1-1/|\lambda_1|} = C|\lambda_1|^{n-2}.$$

If $p = 3$ then $\lambda_1 = -1$; in this case we have the bound

$$|\,\mathrm{swt}(w)| = \left| \mathrm{swt}(v_n) + \sum_{i=0}^{n-2} \alpha_i \,\mathrm{swt}\, t_i \right| \leq 1 + (p-1)(n-1) \leq pn. \qquad \square$$

In [24] it is shown that

$$\mathrm{GKdim}\, \boldsymbol{L} = \frac{\ln p}{\ln \lambda}, \quad \mathrm{GKdim}\, \boldsymbol{A} \leq \frac{2\ln p}{\ln \lambda},$$

where $1 < \ln p/\ln \lambda < 2$. This and the theory of M. Smith [25] imply that the growth of $u(\boldsymbol{L})$ is subexponential and therefore intermediate. Let us determine the growth of $u(\boldsymbol{L})$ more precisely. We will need some definitions. Consider two series of functions $\Phi_\alpha^q(n)$, $q = 2, 3$, of natural argument with the parameter $\alpha \in \mathbb{R}^+$:

$$\Phi_\alpha^2(n) = n^\alpha, \quad \Phi_\alpha^3(n) = \exp(n^{\alpha/(\alpha+1)}).$$

We compare functions $f : \mathbb{N} \to \mathbb{R}^+$ by means of the partial order: $f(n) \leq^{\mathrm{a}} g(n)$ if and only if there exists $N > 0$ such that $f(n) \leq g(n)$, $n \geq N$. Suppose that A is a finitely generated algebra and $\gamma_A(n)$ is its growth function. We define the *dimension of level* q, $q \in \{2, 3\}$, and the *lower dimension of level* q by

$$\mathrm{Dim}^q A = \inf\{\alpha \in \mathbb{R}^+ \mid \gamma_A(n) \leq^{\mathrm{a}} \Phi_\alpha^q(n)\},$$
$$\underline{\mathrm{Dim}}^q A = \sup\{\alpha \in \mathbb{R}^+ \mid \gamma_A(n) \geq^{\mathrm{a}} \Phi_\alpha^q(n)\}.$$

The q-dimensions for arbitrary level $q \in \mathbb{N}$ were introduced by the first author in order to specify the subexponential growth of universal enveloping algebras [17]. They generalize the Gelfand–Kirillov dimensions. The condition $\mathrm{Dim}^q A = \underline{\mathrm{Dim}}^q A = \alpha$ means that the growth function $\gamma_A(n)$ behaves like $\Phi_\alpha^q(n)$. The dimensions of level 2 are exactly the upper and lower Gelfand–Kirillov dimensions [6], [14]. The dimensions of level 3 correspond to the superdimensions of [4] up to normalization (see [18]). We describe the growth of $u(\boldsymbol{L})$ in terms of $\mathrm{Dim}^3 A$.

Lemma 3.5. *Let* $\theta = \ln p/\ln \lambda$. *The growth of the restricted enveloping algebra* $u(\boldsymbol{L})$ *is intermediate and*

$$1 \leq \mathrm{Dim}^3 u(\boldsymbol{L}) \leq \theta.$$

Proof. We have $\mathrm{Dim}^2 \boldsymbol{L} = \mathrm{GKdim}\, \boldsymbol{L} = \theta$. Now the claim follows from (the proof) of Proposition 1 in [18]. That proposition deals with the growth of the universal enveloping algebra, some minor changes are needed to modify the proof for the restricted enveloping algebra. $\square$

4. The first example: weight structure

The weights in the case $p = 2$ were studied in [21]. In that case the weights of the algebras $\boldsymbol{L} = \mathrm{Lie}_p(v_1, v_2)$ and $\boldsymbol{A} = \mathrm{Alg}(v_1, v_2)$ lie in the strips $|\eta| < \text{const}$, while the weights of the restricted enveloping algebra $\boldsymbol{u} = \boldsymbol{u}(L)$ are bounded by a parabola-like curve $|\eta| \le C\xi^{\theta}$, for some constant $0 < \theta < 1$.

Now we assume that $p \ge 3$. We shall show that the weights of all three algebras $\boldsymbol{L}, \boldsymbol{A}, \boldsymbol{u}$ belong to a region bounded by a parabola-like curve as well.

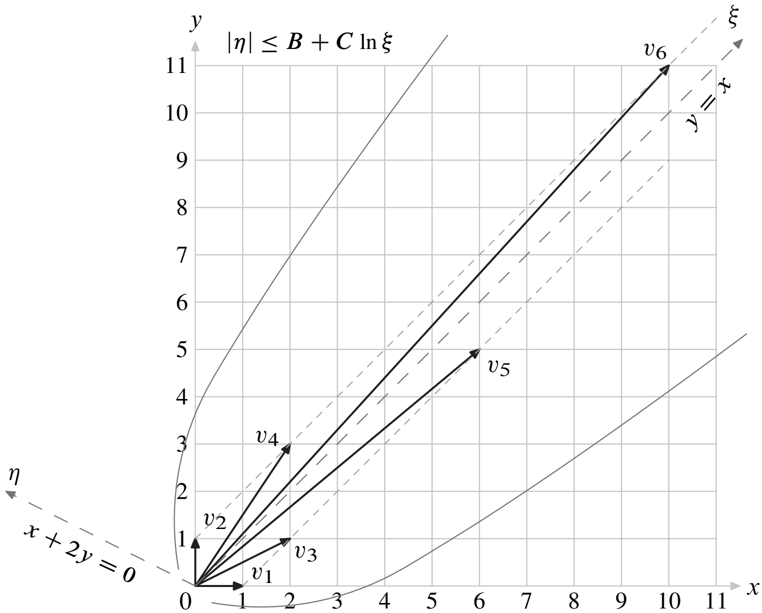

Figure 1. $p = 3$, weights of $\boldsymbol{L}$.

Theorem 4.1. *Let $p \ge 3$. Consider subalgebras the H, $\boldsymbol{L}$ of* Der R. *Then in terms of the new coordinates (ξ, η), homogeneous elements of these algebras belong to the following plane regions.*

(1) *For $p \ge 5$ we have $|\eta| \le C\xi^{\theta}$, where $0 < \theta = \frac{\ln|\lambda_1|}{\ln \lambda} < 1$.*
(2) *For $p = 3$ we have $|\eta| \le B + C \ln \xi$,*

for some positive constants B, C.

Proof. Take a basis monomial $w = t_0^{\alpha_0} t_1^{\alpha_1} \dots t_{n-2}^{\alpha_{n-2}} v_n \in H$. Consider the new coordinates (ξ, η) of w. By Lemma 3.4 we have $\xi = \mathrm{wt}(w) \ge \lambda^{n-2}$. Hence, $n \le 2 + \ln \xi / \ln \lambda$. Consider the case $p \ge 5$, we apply the second estimate of

Lemma 3.4:

$$|\eta| = |\text{swt}(w)| \le C|\lambda_1|^{n-2} \le C|\lambda_1|^{\ln \xi/\ln \lambda} = C\xi^{\ln|\lambda_1|/\ln \lambda}.$$

In the case $p = 3$ we use the third estimate of Lemma 3.4 to get

$$|\eta| = |\text{swt}(w)| \le pn \le p(2 + \ln \xi/\ln \lambda). \qquad \square$$

Theorem 4.2. *Consider the restricted enveloping algebra* $\boldsymbol{u} = u(\boldsymbol{L})$. *Then there exist constants* $C > 0$ *and* $0 < \theta < 1$ *such that homogeneous elements of* $\boldsymbol{u}$ *belong to the plane region*

$$|\eta| \le C\xi^{\theta}.$$

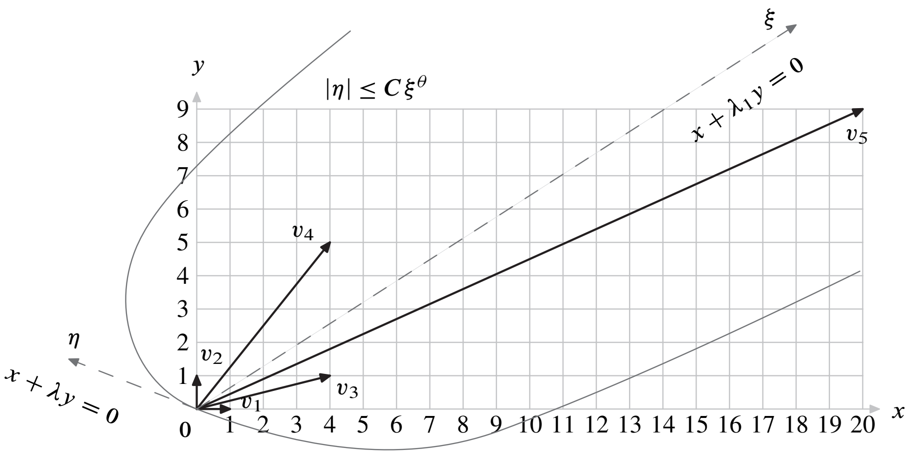

Figure 2. $p = 5$, weights of $\boldsymbol{L}$.

Proof. The case $p = 2$ was settled in [21]. First, consider the case $p \ge 5$.

We shall consider coordinates of homogeneous elements of the bigger algebra $u(H) \supset \boldsymbol{u} = u(\boldsymbol{L})$. Let $\{w_i \mid i \in \mathbb{N}\}$ be the ordered basis of H, which consists of the elements $w = t_0^{\alpha_0} t_1^{\alpha_1} \dots t_{n-2}^{\alpha_{n-2}} v_n$, where $0 \le \alpha_i \le p-1$ and $0 \le \alpha_{n-2} \le p-2$. Consider the function $l\colon \{w_i \mid i \in \mathbb{N}\} \to \mathbb{N}$, $l(t_0^{\alpha_0} t_1^{\alpha_1} \dots t_{n-2}^{\alpha_{n-2}} v_n) = n$. By Lemma 3.4 we have the estimates

$$\lambda^{n-2} \le \text{wt}(w) \le \lambda^n, \quad |\text{swt}(w)| \le C|\lambda_1|^{n-2}, \quad n \in \mathbb{N}. \tag{7}$$

Consider a standard basis element of $u(H)$ of type $v = w_{i_1} \dots w_{i_s}$, where the w_{i_j} enter the product in ordered way and each w_j occurs at most $p-1$ times. Let $N = \max\{l(w_{i_j}) \mid 1 \le j \le s\}$. Denote by μ_k the number of w_{i_j} such that

$k = l(w_{i_j})$, for all $k = 1, \dots, N$. Consider the new coordinates (ξ, η), where $\xi = \mathrm{wt}(v)$ and $\eta = \mathrm{swt}(v)$. We apply (7) and obtain the estimates

$$\frac{1}{\lambda^2} \sum_{k=1}^{N} \mu_k \lambda^k \le \xi \le \sum_{k=1}^{N} \mu_k \lambda^k, \quad |\eta| \le \frac{C}{|\lambda_1|^2} \sum_{k=1}^{N} \mu_k |\lambda_1|^k. \tag{8}$$

Let $\alpha = \alpha(v)$ be the number such that $|\eta| = \xi^\alpha$. Then

$$\alpha(v) = \frac{\ln |\eta|}{\ln \xi} \le \frac{\ln\left(\frac{C}{|\lambda_1|^2} \sum_{k=1}^{N} \mu_k |\lambda_1|^k\right)}{\ln\left(\frac{1}{\lambda^2} \sum_{k=1}^{N} \mu_k \lambda^k\right)}. \tag{9}$$

The number of different basis elements w_i such that $l(w_i) = k$ equals $p^{k-2}(p-1) < p^{k-1}$ for all $k \ge 2$. Each of them can enter v at most $(p-1)$ times. Hence we get

$$\mu_k \le p^{k-1}(p-1) < p^k, \quad k = 1, \dots, N. \tag{10}$$

Let us evaluate the maximal value of $\alpha(v)$ among all v's with the fixed value $\xi(v) = \xi_0$. From (8) we have $\xi_0 \le \sum_{k=1}^{N} \mu_k \lambda^k \le \lambda^2 \xi_0$, this estimate yields the range of values for the denominator of (9). To estimate the numerator of (9) we consider the maximum of the linear function

$$f(x_1, \dots, x_N) = \sum_{k=1}^{N} x_k |\lambda_1|^k, \quad 0 \le x_k \le p^k,\ k = 1, \dots, N,$$

subject to a constraint of the form of a hyperplane $\sum_{k=1}^{N} x_k \lambda^k = A$, where the constant A is such that $\xi_0 \le A \le \lambda^2 \xi_0$. Note that the denominator of (9) is fixed on each hyperplane. Since $|\lambda_1| < \lambda$, the maximum on each hyperplane is achieved when we assign the biggest possible values for x_k with the smallest k's. By (10), we have the bounds $0 \le x_k \le p^k$, $k = 1, \dots, N$. Thus, we take the point on the hyperplane $x_k = p^k$, $k = 1, \dots, m$, for some $m \le N$, the appropriate value $x_{k+1} \in [0, p^{k+1})$, and $x_{k+2} = \dots = x_N = 0$. This point yields the upper bound

$$\begin{aligned} \alpha(v) &\le \frac{\ln\left(\frac{C}{|\lambda_1|^2}\left(\sum_{k=1}^{m} p^k |\lambda_1|^k + x_{k+1} |\lambda_1|^{k+1}\right)\right)}{\ln\left(\frac{1}{\lambda^2}\left(\sum_{k=1}^{m} (p\lambda)^k + x_{k+1} \lambda^{k+1}\right)\right)} \\ &\le \frac{\ln\left(C_1 p^m |\lambda_1|^m\right)}{\ln\left(C_2 p^m \lambda^m\right)} = \frac{\frac{1}{m} \ln C_1 + \ln(p|\lambda_1|)}{\frac{1}{m} \ln C_2 + \ln(p\lambda)}. \end{aligned}$$

When ξ_0 increases, the number m increases as well. Let us choose the number θ such that $\ln(p|\lambda_1|)/\ln(p\lambda) < \theta < 1$. Then for sufficiently large ξ we have $\alpha(v) \le \theta$, hence $|\eta| \le \xi^\theta$. By choosing an appropriate constant C we get $|\eta| \le C\xi^\theta$ for all $\xi > 0$.

It remains to consider the case $p = 3$. For elements of the Lie algebra we have the bound $|\eta| \le B + C \ln \xi$. Recall that $\lambda_1 = -1$ in this case. Take $\lambda_2 = 3/2 < \lambda = 2$.

Let $\{w_i \mid i \in \mathbb{N}\}$ be the ordered basis of H as above. Then we have $|\mathrm{swt}(w_i)| \le \lambda_2^n$, where $n = l(w_i)$, for $i \ge N$, provided that the number N is sufficiently large. We find constant C_1 such that $|\mathrm{swt}(w_i)| \le C_1 \lambda_2^{l(w_i)}$ for all w_i. Now we can formally apply the arguments above. □

Consider the triangular decompositions of Corollary 3.2.

Corollary 4.3. *Let $\boldsymbol{L}$, $\boldsymbol{A}$, $\boldsymbol{u}$ be as above and consider the decompositions given by the superweight*

$$\boldsymbol{L} = \boldsymbol{L}_+ \oplus \boldsymbol{L}_0 \oplus \boldsymbol{L}_-, \quad \boldsymbol{A} = \boldsymbol{A}_+ \oplus \boldsymbol{A}_0 \oplus \boldsymbol{A}_-, \quad \boldsymbol{u} = \boldsymbol{u}_+ \oplus \boldsymbol{u}_0 \oplus \boldsymbol{u}_-.$$

(1) *Then the upper and lower components $\boldsymbol{L}_\pm$, $\boldsymbol{A}_\pm$, $\boldsymbol{u}_\pm$, are locally nilpotent.*

(2) *In the irrational case, the zero components above are trivial and we obtain decompositions into a direct sum of two locally nilpotent subalgebras.*

Proof. Consider, for example, $\boldsymbol{u}_+$. The line $\eta = \mathrm{swt}(x, y) = 0$ separates the upper and lower components. By (6), this line is given by the equation $x + \lambda_1 y = 0$. Consider homogeneous monomials $u_1, \dots, u_k \in \boldsymbol{u}_+$ above this line and the subalgebra $A = \mathrm{Alg}(u_1, \dots, u_k)$ generated by these elements. Let $N \in \mathbb{N}$ and consider $u = \sum_{j;\, n \ge N} \alpha_j u_{j_1} \dots u_{j_n}$, $\alpha_j \in K$. Then it is geometrically clear (see Figure 3) that the respective vectors of all homogeneous components belong to the

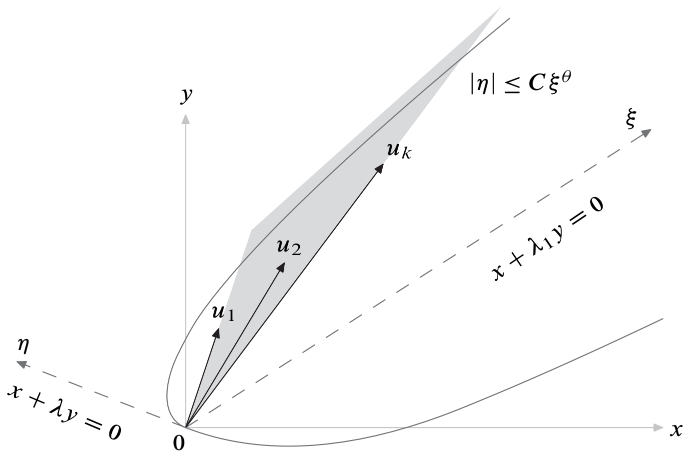

Figure 3. $p \ge 2$, weights of $\boldsymbol{u}$.

shaded angle. All components go out of the region $|\eta| < C\xi^\theta$ provided that N is sufficiently large. Hence, $A^N = 0$. □

In irrational case, we obtain some more examples of finitely generated infinite-dimensional associative algebras that are direct sums of two locally nilpotent subalgebras. Those examples were constructed in [13], [5]. Our examples, $\boldsymbol{A}$ and $\boldsymbol{u}$, are of polynomial and intermediate growth.

5. Second example

Now we turn to the study of the *second example* suggested in [24]. We keep the same notations I, R, $\boldsymbol{L}$, $\boldsymbol{A}$, H as in the first example, but now they denote another objects.

We add some negative indices to the index set $I = \{2-p, 2-p+1, \dots, 0, 1, \dots\}$ and consider the truncated polynomial ring $R = K[t_i \mid i \in I]/(t_i^p \mid i \in I)$. Then we introduce the derivations

$$v_m = \partial_m + t_{m-p+1}^{p-1}(\partial_{m+1} + t_{m-p+2}^{p-1}(\partial_{m+2} + t_{m-p+3}^{p-1}(\partial_{m+3} + \dots))), \quad m \geq 1.$$

As above, $\tau\colon R \to R$ is the endomorphism given by $\tau(t_i) = t_{i+1}$, $i \in I$. Observe that

$$v_m = \partial_m + t_{m-p+1}^{p-1} v_{m+1} = \tau^{m-1}(v_1), \quad m \geq 1.$$

Now let $\boldsymbol{L} = \mathrm{Lie}_p(v_1, \dots, v_p) \subset \mathrm{Der}\, R$ denote the restricted subalgebra generated by $\{v_1, v_2, \dots, v_p\}$. In the case of characteristic $p = 2$, this example coincides with the *Fibonacci restricted Lie algebra* [19]. In what follows we assume that $p \geq 3$.

We also can consider a slight modification $\tilde{\boldsymbol{L}} = \mathrm{Lie}_p(\partial_1, \dots, \partial_{p-1}, v_p) \subset \mathrm{Der}\, R$.

Let us make the convention that if the upper index of a product (sum) is less than the lower index, then the product is empty. Similarly, if we list a set as $\{i, i+1, \dots, j\}$ and $i > j$, then the set is assumed to be empty.

Lemma 5.1. *Let $p \geq 3$. The following commutation relations hold:*

(1) $[v_m, v_{m+1}] = -\big(\prod_{j=m-p+2}^{m-1} t_j^{p-1}\big) t_m^{p-2} v_{m+p}$ *for* $m \geq 1$;

(2) *for* $n \geq 1$, $k \geq 2$ *we have* (*both sets in the product below may be empty*)

$$[v_n, v_{n+k}] = -\sum_{j=\max\{0,k-p+1\}}^{k-1} \Big(\prod_{\substack{l \in \{1,\dots,j\} \cup \\ \{k+1,\dots,j+p-1\}}} t_{n-p+l}^{p-1}\Big) t_{n+j}^{p-2} v_{n+j+p};$$

(3) *for all* $n, m \geq 1$

$$[\partial_n, v_m] = \begin{cases} -\big(\prod_{j=m-p+1}^{n-1} t_j^{p-1}\big) t_n^{p-2} v_{n+p}, & m < n+p-1, \\ -t_n^{p-2} v_{n+p}, & m = n+p-1, \\ 0, & m > n+p-1; \end{cases}$$

(4) *for all* $n \geq 1$, $j \geq 0$ *we have the action*

$$v_m(t_j) = \begin{cases} \prod_{i=m-p+1}^{j-p} t_i^{p-1}, & m < j, \\ 1, & m = j, \\ 0, & m > j; \end{cases}$$

(5) $v_n^p = -(t_{n-(p-1)} \dots t_{n-1})^{p-1} v_{n+p}$ *for all* $n \geq 1$.

Proof. Let us check the first claim:

$$\begin{aligned}[v_m, v_{m+1}] &= [\partial_m + t_{m-p+1}^{p-1} v_{m+1}, v_{m+1}] = [\partial_m, v_{m+1}] \\ &= [\partial_m, \partial_{m+1} + t_{m-p+2}^{p-1}(\partial_{m+2} + \cdots \\ &\qquad \cdots + t_{m-1}^{p-1}(\partial_{m+p-1} + t_m^{p-1} v_{m+p}) \dots)] \\ &= -t_{m-p+2}^{p-1} \dots t_{m-1}^{p-1} \cdot t_m^{p-2} v_{m+p}.\end{aligned}$$

To prove the claim (3), observe that the product is nontrivial only for $m \leq n+p-1$. In this case we get

$$\begin{aligned}[\partial_n, v_m] &= [\partial_n, \partial_m + t_{m-p+1}^{p-1}(\cdots + t_{n-1}^{p-1}(\partial_{n+p-1} + t_n^{p-1} v_{n+p}) \dots)] \\ &= -\Big(\prod_{j=m-p+1}^{n-1} t_j^{p-1}\Big) t_n^{p-2} v_{n+p}.\end{aligned}$$

Now we prove claim (2). Let $k \geq 2$. We have

$$\begin{aligned}[v_n, v_{n+k}] &= [\partial_n + t_{n-p+1}^{p-1}(\partial_{n+1} + \cdots \\ &\qquad \cdots + t_{n+k-p-1}^{p-1}(\partial_{n+k-1} + t_{n+k-p}^{p-1} v_{n+k}) \dots), v_{n+k}] \\ &= [\partial_n + t_{n-p+1}^{p-1}(\partial_{n+1} + \cdots + t_{n+k-p-1}^{p-1} \partial_{n+k-1}), v_{n+k}] \\ &= \sum_{j=0}^{k-1} \Big(\prod_{l=1}^{j} t_{n-p+l}^{p-1}\Big)[\partial_{n+j}, v_{n+k}] \\ &= \sum_{j=\max\{0,k-p+1\}}^{k-1} \Big(\prod_{l=1}^{j} t_{n-p+l}^{p-1}\Big)[\partial_{n+j}, \partial_{n+k} \\ &\qquad + t_{n+k-p+1}^{p-1}(\cdots + t_{n+j}^{p-1} v_{n+j+p})] \\ &= -\sum_{j=\max\{0,k-p+1\}}^{k-1} \Big(\prod_{l=1}^{j} t_{n-p+l}^{p-1}\Big)\Big(\prod_{q=k+1}^{j+p-1} t_{n-p+q}^{p-1}\Big) t_{n+j}^{p-2} v_{n+j+p}.\end{aligned}$$

Claim 4 is proved as follows:

$$v_m(t_j) = (\partial_m + t_{m-p+1}^{p-1}(\cdots + t_{j-p}^{p-1}(\partial_j + \dots) \dots))(t_j) = \prod_{i=m-p+1}^{j-p} t_i^{p-1}.$$

Consider the last claim.

$$v_n^p = ((\partial_n + t_{n+1-p}^{p-1} \partial_{n+1}) + t_{n+1-p}^{p-1} t_{n+2-p}^{p-1} v_{n+2})^p = (x+y)^p.$$

We have

$$\begin{aligned}\operatorname{ad} x(y) &= \operatorname{ad}(\partial_n + t_{n+1-p}^{p-1}\partial_{n+1})(t_{n+1-p}^{p-1}t_{n+2-p}^{p-1}v_{n+2})\\ &= t_{n+1-p}^{p-1}t_{n+2-p}^{p-1}\operatorname{ad}(\partial_n)(v_{n+2})\\ &= -t_{n-(p-1)}^{p-1}t_{n-(p-2)}^{p-1}(t_{n-(p-3)}\dots t_{n-1})^{p-1}t_n^{p-2}v_{n+p}\\ &= -(t_{n-(p-1)}\dots t_{n-1})^{p-1}t_n^{p-2}v_{n+p},\end{aligned}$$

where the factor $(t_{n-(p-1)}t_{n-(p-2)})^{p-1}$ is present for all $p \geq 3$. We consider all Lie polynomials in x and y. We observe that a further multiplication by y is zero due to the total power of the letter t_{n+1-p}, which cannot be killed by derivations involved. Thus, only one term in (1) is nontrivial, namely $s_{p-1}(x, y) = (\operatorname{ad} x)^{p-1}(y)$. We get

$$\begin{aligned}v_n^p &= (x+y)^p\\ &= (\operatorname{ad} x)^{p-1}(y)\\ &= (\operatorname{ad} x)^{p-2}([x, y])\\ &= (\operatorname{ad}(\partial_n + t_{n-(p-1)}^{p-1}\partial_{n+1}))^{p-2}(-(t_{n-(p-1)}\dots t_{n-1})^{p-1}t_n^{p-2}v_{n+p})\\ &= (\operatorname{ad}\partial_n)^{p-2}(-(t_{n-(p-1)}\dots t_{n-1})^{p-1}t_n^{p-2}v_{n+p})\\ &= -(t_{n-(p-1)}\dots t_{n-1})^{p-1}v_{n+p}. \qquad \square\end{aligned}$$

Lemma 5.2. *Let H be the K-linear span of the set*

$$\{t_{2-p}^{\alpha_{2-p}}\dots t_{n-p-1}^{\alpha_{n-p-1}}t_{n-p}^{\alpha_{n-p}}v_n \mid 0 \leq \alpha_i \leq p-1,\ \alpha_{n-p} \leq p-2,\ n \geq 1\}.$$

Then H is a restricted subalgebra of $\operatorname{Der} R$ *and $\boldsymbol{L} \subset H$.*

Proof. We use Lemma 5.1 and proceed as in Lemma 2.2. □

6. Second example: weights

As above we will define a gradation on the Lie algebra $\boldsymbol{L}$ by presenting it as a direct sum of weight spaces, all weights being complex numbers. We assume that $\operatorname{wt}(\partial_i) = -\operatorname{wt}(t_i) = a_i \in \mathbb{C}$ for all $i \in \mathbb{N}$. Let us choose numbers a_i so that all terms in the expression of v_i are homogeneous. We obtain $a_i = \operatorname{wt}\partial_i = \operatorname{wt}\partial_{i+1} + (p-1)\operatorname{wt} t_{i-p+1} = a_{i+1} - (p-1)a_{i-p+1}$, which implies the recurrence relation

$$a_i = a_{i-1} + (p-1)a_{i-p}, \quad i \geq p.$$

Let us study its characteristic polynomial $\phi(x) = x^p - x^{p-1} - (p-1)$.

Lemma 6.1. *Consider a prime $p \geq 3$ and the polynomial $\phi(x) = x^p - x^{p-1} - (p-1)$. Then:*

(1) *$\phi(x)$ is irreducible and has distinct roots.*
(2) *The equation $\phi(x) = 0$ has a unique real root λ_0.*
(3) *Let $\lambda_1, \ldots, \lambda_{p-1}$ be the remaining complex roots. Then $1 < |\lambda_i| < \lambda_0 < 2$ for all $i = 1, \ldots, p-1$.*
(4) *Let $|\lambda_i| = |\lambda_j|$. Then $i = j$, or $\lambda_i = \bar{\lambda}_j$.*

Proof. Let us prove that $\phi(x)$ is irreducible modulo p. Making the substitution $x = 1/y$, we see that it is sufficient to prove that $g(y) = y^p - y + 1$ is irreducible over $\mathbb{Z}_p$. Let a be a root of g, then so is $a+1$ and hence $a+i$, $i = 0, \ldots, p-1$, are all roots of $g(y)$. If $m(x)$ is the minimal polynomial of a then $m_i(x) = m(x-i)$ is the minimal polynomial of the root $a+i$. Therefore, the minimal polynomials of all roots of $g(y)$ have the same degree k, and hence any polynomial that has a common root with $g(y)$ has an irreducible factor of degree k. If $g(y)$ is not irreducible then it is a product of several polynomials of degree k, which implies that k is a proper factor of $p = \deg g$, a contradiction.

Since the equation is over $\mathbb{Q}$, the roots are distinct.

We consider the derivative $\phi(x)' = x^{p-2}(px - (p-1))$. It has two roots $x_0 = 0$ and $x_1 = 1 - 1/p$. Observe that $\phi(0) = \phi(1) = -(p-1)$. Also, $\phi(2) = 2^{p-1} - (p-1) > 0$. We conclude that $\phi(x) = 0$ has a unique real root λ_0, moreover $1 < \lambda_0 < 2$.

Now let λ_1 be a root. Recall that $\lambda_1 \notin \mathbb{R}$. Suppose that $|\lambda_1| \geq \lambda_0$. We have two equalities $\lambda_0^p - \lambda_0^{p-1} = p-1$ and $\lambda_1^p - \lambda_1^{p-1} = p-1$, the latter can be depicted as a non-degenerate triangle in $\mathbb{C}$. By the triangle inequality, $p-1 > |\lambda_1|^p - |\lambda_1|^{p-1}$. Consider the function $f(x) = x^p - x^{p-1}$, $x \in \mathbb{R}$. Using the derivative $f'(x) = x^{p-2}(px - (p-1))$ we see that $f(x)$ is increasing for $x > 1 - 1/p$. We obtain

$$p - 1 = \lambda_0^p - \lambda_0^{p-1} = f(\lambda_0) \leq f(|\lambda_1|) = |\lambda_1|^p - |\lambda_1|^{p-1} < p - 1.$$

This contradiction proves that $|\lambda_1| < \lambda_0$. Suppose that $|\lambda_1| \leq 1$ for a root $\lambda_1 \notin \mathbb{R}$. Then $p - 1 = \lambda_1^p - \lambda_1^{p-1} = |\lambda_1^p - \lambda_1^{p-1}| < 2$, a contradiction. Thus $|\lambda_1| > 1$ and the third claim is proved.

To prove claim (4), let λ_1, λ_2 be two different complex roots of our equation such that $|\lambda_1| = |\lambda_2|$. Consider the triangle in $\mathbb{C}$ given by $p - 1 = \lambda_1^p - \lambda_1^{p-1}$, where $p-1$, λ_1^p start from the origin. Consider all triangles on the plane with the same side $p-1$, with the other sides of lengths $|\lambda_1|^p$, $|\lambda_1|^{p-1}$ and with the longest side starting from the origin. There are only two such triangles. They correspond to λ_1 and $\bar{\lambda}_1$. We have two possibilities. a) $\lambda_1^p = \lambda_2^p$, $\lambda_1^{p-1} = \lambda_2^{p-1}$, and so $\lambda_1 = \lambda_2$. b) $\lambda_1^p = \bar{\lambda}_2^p$, $\lambda_1^{p-1} = \bar{\lambda}_2^{p-1}$, and we get $\lambda_1 = \bar{\lambda}_2$. □

Denote $s = (p-1)/2$. For simplicity, order the roots so that $\lambda_{i+s} = \bar{\lambda}_i$ for $i = 1, \ldots s$. We introduce the *p weight functions*

$$\mathrm{wt}_j(\partial_n) = \lambda_j^n, \quad n \in \mathbb{N},\ j = 0, \ldots, p-1.$$

By Lemma 1.1, these weight functions define a gradation on the subalgebra $H \subset \operatorname{Der} R$ defined above. For a homogeneous element $v \in H$ let

$$\operatorname{Wt}(v) = (\operatorname{wt}_0 v, \operatorname{wt}_1 v, \dots, \operatorname{wt}_{p-1} v), \quad v \in H.$$

Theorem 6.2. *Let $\boldsymbol{L} = \operatorname{Lie}_p(v_1, \dots, v_p) \subset H \subset \operatorname{Der} R$ be the restricted subalgebras defined above. Then:*

(1) *The weight functions are additive on products of homogeneous elements of H and $\boldsymbol{L}$.*

(2) *We have the $\mathbb{Z}^p$-gradation*

$$\boldsymbol{L} = \bigoplus_{a_1,\dots,a_p \geq 0} \boldsymbol{L}_{a_1,\dots,a_p},$$

where $\boldsymbol{L}_{a_1,\dots,a_p}$ is spanned by products with a_i factors v_i, $i = 1, \dots, p$.

(3) *Let $v \in \boldsymbol{L}_{a_1,\dots,a_p}$, where $a_i \geq 0$. Then*

$$\operatorname{wt}_j v = \sum_{k=1}^{p} a_k \lambda_j^k, \quad j = 0, 1, \dots, p-1.$$

Proof. The additivity follows from Lemma 1.1 and our construction.

Also, by our construction all components of v_n, $n \in \mathbb{N}$, have the same weights, namely, $\operatorname{wt}_j(v_n) = \operatorname{wt}_j \partial_n = \lambda_j^n$, $j = 0, 1, \dots, p-1$, $n \in \mathbb{N}$. Let $v \in \boldsymbol{L}$ be a monomial that contains a_i factors v_i for $i = 1, \dots, p$. From additivity of the weight functions we get

$$\begin{aligned}(\operatorname{wt}_0 v, \dots, \operatorname{wt}_{p-1} v) &= \operatorname{Wt} v \\ &= \sum_{k=1}^{p} a_k \operatorname{Wt}(v_k) \\ &= \sum_{k=1}^{p} a_k (\lambda_0^k, \dots, \lambda_{p-1}^k) \\ &= \Big(\sum_{k=1}^{p} a_k \lambda_0^k, \dots, \sum_{k=1}^{p} a_k \lambda_{p-1}^k \Big).\end{aligned}$$

The vectors $\operatorname{Wt}(v_k) = (\lambda_0^k, \dots, \lambda_{p-1}^k)$, $k = 1, \dots, p$, are linearly independent by Vandermonde's argument. Thus we get the claimed $\mathbb{Z}^p$-grading and the third claim as well. □

This example also has a nil p-mapping.

Theorem 6.3. *Let $\boldsymbol{L} = \operatorname{Lie}_p(v_1, v_2, \dots, v_p) \subset \operatorname{Der} R$ be the restricted Lie algebra as above. Then $\boldsymbol{L}$ has a nil p-mapping.*

Proof. We refer the reader to the arguments in [24], where it was proved that the p-mapping is nil for a class of restricted Lie algebras. □

7. Second example: triangular decomposition

Now we want to introduce new coordinates in $\mathbb{R}^p$. Let $\bar{x} = (x_1, \dots, x_p) \in \mathbb{R}^p$ and set

$$\begin{aligned}
\xi_0(\bar{x}) &= x_1\lambda_0^1 + x_2\lambda_0^2 + \cdots + x_p\lambda_0^p,\\
\xi_1(\bar{x}) &= x_1\lambda_1^1 + x_2\lambda_1^2 + \cdots + x_p\lambda_1^p,\\
&\ \vdots\\
\xi_{p-1}(\bar{x}) &= x_1\lambda_{p-1}^1 + x_2\lambda_{p-1}^2 + \cdots + x_p\lambda_{p-1}^p.
\end{aligned}$$

Since $\xi_j(\bar{x})$, $\xi_{j+s}(\bar{x})$ are conjugate complex numbers for $j = 1, \dots, s$, we get real coordinates $(\eta_0, \eta_1, \dots, \eta_{p-1}) \in \mathbb{R}^p$ as follows (recall that $s = (p-1)/2$). Let $\bar{x} = (x_1, \dots, x_p) \in \mathbb{R}^p$ and define

$$\eta_k(\bar{x}) = \begin{cases} x_1\lambda_0^1 + x_2\lambda_0^2 + \cdots + x_p\lambda_0^p, & k = 0,\\ \operatorname{Re}(x_1 + x_2\lambda_k^1 + \cdots + x_p\lambda_k^{p-1}), & k = 1, \dots, s,\\ \operatorname{Im}(x_1 + x_2\lambda_k^1 + \cdots + x_p\lambda_k^{p-1}), & k = s+1, \dots, p-1. \end{cases}$$

We also consider these functions on homogeneous elements $v \in \boldsymbol{L}$. Suppose that $v \in \boldsymbol{L}_{a_1,\dots,a_p}$. Then we take $\bar{x} = (a_1, \dots, a_p) \in \mathbb{R}^p$ and define

$$\xi_j(v) = \xi_j(\bar{x}), \quad \eta_j(v) = \eta_j(\bar{x}), \quad j = 0, \dots, p-1.$$

Lemma 7.1. *The introduced weight functions have the following properties:*

(1) *Let $v \in \boldsymbol{L}_{a_1,\dots,a_p}$. Then*

$$\begin{aligned}
\xi_j(v) &= \mathrm{wt}_j\, v, & j &= 0, \dots, p-1,\\
\eta_0(v) &= \xi_0(v) = \mathrm{wt}_0\, v, & &\\
\eta_j(v) &= \operatorname{Re}(\mathrm{wt}_j(v)/\lambda_j), & j &= 1, \dots, s,\\
\eta_j(v) &= \operatorname{Im}(\mathrm{wt}_j(v)/\lambda_j), & j &= s+1, \dots, p-1.
\end{aligned}$$

(2) *These functions are additive on products of homogeneous elements of $\boldsymbol{L}$.*

(3) *Consider a lattice point $\bar{0} \neq \bar{x} = (n_1, \dots, n_p) \in \mathbb{Z}^p \subset \mathbb{R}^p$. Then $\eta_j(\bar{x}) \neq 0$ for all $j = 1, \dots, s$.*

(4) *Denote $\bar{x} = (n_1, \dots, n_p) \in \mathbb{Z}^p \subset \mathbb{R}^p$, and let $\eta_j(\bar{x}) = 0$ for some $j \in \{s+1, \dots, p-1\}$. Then $\bar{x} = (n_1, 0, \dots, 0)$.*

Proof. The first and second claims are obvious.

Let us prove the third claim. Fix $\bar{0} \neq \bar{x} = (n_1, \dots, n_p) \in \mathbb{Z}^p$ and $j \in \{1, \dots, s\}$. Suppose that

$$\eta_j(\bar{x}) = \operatorname{Re}(n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1}) = 0. \tag{11}$$

We have the field extension $\mathbb{Q} \subset \mathbb{Q}(\lambda_j)$. Denote $r = n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1}$. Suppose that $r \neq 0$. From (11) it follows that $r = iq$, where $q \in \mathbb{R}$. Consider $r^2 \in \mathbb{R} \cap \mathbb{Q}(\lambda_j) \neq \mathbb{Q}(\lambda_j)$. Since $|\mathbb{Q}(\lambda_j) : \mathbb{Q}| = p$ is a prime, we obtain $r^2 \in \mathbb{Q}$. Then $|\mathbb{Q}(r) : \mathbb{Q}| = 2$ divides p, a contradiction. Therefore, $r = n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1} = 0$, which is a contradiction to the fact that λ_j satisfies an irreducible polynomial of degree p.

We now turn to claim (4). Fix $\bar{x} = (n_1, \dots, n_p) \in \mathbb{Z}^p$ and $j \in \{s+1, \dots, p-1\}$. Suppose that

$$\eta_j(\bar{x}) = \operatorname{Im}(n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1}) = 0.$$

Denote $r = n_1 + n_2\lambda_j + \cdots + n_p\lambda_j^{p-1}$. Then $r \in \mathbb{R} \cap \mathbb{Q}(\lambda_j) \neq \mathbb{Q}(\lambda_j)$. Since $|\mathbb{Q}(\lambda_j) : \mathbb{Q}| = p$ is a prime, we get $r \in \mathbb{Q}$. We obtain $(n_1 - r) + n_2\lambda_j + \cdots + n_p\lambda_p^{p-1} = 0$, which is possible only in the case $n_1 = r, n_2 = \cdots = n_p = 0$. □

Now we get triangular decompositions where the zero component is always trivial.

Corollary 7.2. *Let* $\boldsymbol{L} = \mathrm{Lie}_p(v_1, \dots, v_p)$, *let* $\boldsymbol{A} = \mathrm{Alg}(v_1, \dots, v_p)$ *be the restricted Lie algebra and associative algebra generated by* $\{v_1, \dots, v_p\}$. *Let* $U = U(\boldsymbol{L})$, $\boldsymbol{u} = u(\boldsymbol{L})$ *be the universal enveloping algebra and the restricted enveloping algebra. Then all these algebras have decompositions into direct sums of two subalgebras as follows:*

$$\boldsymbol{L} = \boldsymbol{L}_+ \oplus \boldsymbol{L}_-, \quad \boldsymbol{A} = \boldsymbol{A}_+ \oplus \boldsymbol{A}_-, \quad U = U_+ \oplus U_-, \quad \boldsymbol{u} = \boldsymbol{u}_+ \oplus \boldsymbol{u}_-.$$

Proof. Fix $j \in \{1, \dots, s\}$ and set, for example,

$$\boldsymbol{L}_+ = \langle v \in \boldsymbol{L} \mid \eta_j(v) > 0 \rangle, \quad \boldsymbol{L}_- = \langle v \in \boldsymbol{L} \mid \eta_j(v) < 0 \rangle. \qquad \square$$

Observe that the weight functions η_j, $j \in \{s+1, \dots, p-1\}$, also yield triangular decompositions, but in this case the components $\boldsymbol{L}_0$ and $\boldsymbol{A}_0$ are nontrivial and finite dimensional. Indeed, consider $\boldsymbol{L}_0 = \langle v \in \boldsymbol{L} \mid \eta_j(v) = 0 \rangle$. By claim (4) of Lemma 7.1, $\boldsymbol{L}_0$ is spanned by products of the element v_1 only. Since $v_1^{p^2} = 0$, we conclude that $\boldsymbol{L}_0 = \langle v_1, v_1^p \rangle$, similarly, $\boldsymbol{A}_0 = \langle v_1^j \mid 1 \le j < p^2 \rangle$ is of dimension at most p^2.

Lemma 7.3. *Let* $v = t_{2-p}^{\alpha_{2-p}} \dots t_{n-p}^{\alpha_{n-p}} v_n \in H$, $n \in \mathbb{N}$, *be as in Lemma 5.2. Then*

(1) $\lambda_0^{n-p} \le \mathrm{wt}_0\, v \le \lambda_0^n$;

(2) $|\mathrm{wt}_j\, v| \le C|\lambda_j|^n$ *for all* $j = 1, \dots, p-1$, *where* C *is some constant;*

(3) $|\eta_j(v)| \le C|\lambda_j|^n$ *for all* $j = 1, \dots, p-1$.

Proof. The upper bound $\mathrm{wt}_0\, v \le \lambda_0^n$ is obvious. We check the lower bound. Recall that $\alpha_{n-p} \le p-2$. Then

$$\begin{aligned}\mathrm{wt}_0(v) &= \lambda_0^n - \sum_{i=2-p}^{n-p} \alpha_i \lambda_0^i \\ &\ge \lambda_0^n - (p-1) \sum_{i=2-p}^{n-p} \lambda_0^i + \lambda_0^{n-p} \\ &\ge \lambda_0^n - (p-1) \frac{\lambda_0^{n-p}}{1-1/\lambda_0} + \lambda_0^{n-p} \\ &= \frac{\lambda_0^{n-p}(\lambda_0^p - \lambda_0^{p-1} - (p-1))}{1-1/\lambda_0} + \lambda_0^{n-p} = \lambda_0^{n-p}.\end{aligned}$$

Similarly,

$$\begin{aligned}|\mathrm{wt}_j(v)| &= \Big|\lambda_j^n - \sum_{i=2-p}^{n-p} \alpha_i \lambda_j^i\Big| \\ &\le |\lambda_j|^n + (p-1) \sum_{i=2-p}^{n-p} |\lambda_j|^i \\ &\le |\lambda_j|^n + (p-1) \frac{|\lambda_j|^{n-p}}{1-1/|\lambda_j|} \le C|\lambda_j^n|.\end{aligned}$$

The third claim follows by the previous lemma. □

Now we are going to show that the weights of all three algebras $\boldsymbol{L}$, $\boldsymbol{A}$, $\boldsymbol{u}$ again belong to a paraboloid-like region of $\mathbb{R}^p$ stretched along the axis η_0.

Theorem 7.4. *Let $p \ge 3$ and $\boldsymbol{L} = \mathrm{Lie}_p(v_1, \dots, v_p)$, H be the subalgebras of* Der R *as above. Then the new coordinates $(\eta_0, \eta_1, \dots, \eta_{p-1})$ of homogeneous elements of these algebras belong to the following region of $\mathbb{R}^p$:*

$$|\eta_j| \le C \eta_0^{\theta_j}, \quad \theta_j = \frac{\ln|\lambda_j|}{\ln \lambda_0} < 1, \quad j = 1, \dots, p-1,$$

where C is a positive constant.

Proof. Take a basic monomial $w = t_{2-p}^{\alpha_{2-p}} \dots t_{n-2}^{\alpha_{n-2}} v_n \in H$ and consider its new coordinates $(\eta_0, \eta_1, \dots, \eta_{p-1})$. By Lemma 7.3, we have $\eta_0 = \mathrm{wt}_0(w) \ge \lambda_0^{n-p}$. Hence, $n \le p + \ln \eta_0 / \ln \lambda_0$. We apply the third estimate of Lemma 7.3

$$|\eta_j| \le C|\lambda_j|^n \le C|\lambda_j|^{p+\ln\eta_0/\ln\lambda_0} = \tilde{C} \eta_0^{\ln|\lambda_j|/\ln\lambda_0}, \quad j = 1, \dots, p-1. \quad \square$$

Theorem 7.5. *Let $p \geq 3$. Consider $\boldsymbol{A} = \mathrm{Alg}(v_1, \dots, v_p)$ and $\boldsymbol{u} = u(\boldsymbol{L})$. Then the new coordinates of homogeneous elements of these algebras also belong to the following region of $\mathbb{R}^p$:*

$$|\eta_j| \leq C\eta_0^{\theta}, \quad j = 1, \dots, p-1,$$

for some constants $C > 0$ and $0 < \theta < 1$.

Proof. Let $v = t_{2-p}^{\alpha_{2-p}} \dots t_{n-p}^{\alpha_{n-p}} v_n \in H$, $n \in \mathbb{N}$. By Lemma 7.3 we have bounds similar to (7),

$$\lambda_0^{n-p} \leq \mathrm{wt}_0\, v \leq \lambda_0^n, \quad |\eta_j(v)| \leq C|\lambda_j|^n, \quad |\lambda_j| < \lambda_0,\ j = 1, \dots, p-1.$$

It remains to repeat the arguments of Theorem 4.2. □

Corollary 7.6. *Consider the triangular decompositions of Corollary 7.2,*

$$\boldsymbol{L} = \boldsymbol{L}_+ \oplus \boldsymbol{L}_-, \quad \boldsymbol{A} = \boldsymbol{A}_+ \oplus \boldsymbol{A}_-, \quad \boldsymbol{u} = \boldsymbol{u}_+ \oplus \boldsymbol{u}_-.$$

Then all the components $\boldsymbol{L}_\pm$, $\boldsymbol{A}_\pm$, and $\boldsymbol{u}_\pm$ are locally nilpotent subalgebras.

Proof. The arguments of Corollary 4.3 apply. □

8. Second example: growth

In this section we study the growth of the algebras that appear in the second example. In particular we check that $\boldsymbol{L}$ is infinite-dimensional.

Theorem 8.1. *Let $\boldsymbol{L} = \mathrm{Lie}_p(v_1, \dots, v_p)$, and let λ_0 be the root of the characteristic polynomial above. Then* $\mathrm{GKdim}\, \boldsymbol{L} \leq \ln p / \ln \lambda_0$.

Proof. We use the embedding of Lemma 5.2. Fix a number m. Consider a homogeneous element $g \in \boldsymbol{L} \subset H$ such that $\mathrm{wt}_0(g) \leq m$. Then it is a sum of monomials $v = t_{2-p}^{\alpha_{2-p}} \dots t_{n-p}^{\alpha_{n-p}} v_n$, where $0 \leq \alpha_i \leq p-1$ and $\alpha_{n-p} \leq p-2$. By Lemma 7.3, $m \geq \mathrm{wt}_0(g) \geq \lambda_0^{n-p}$. Hence, $n \leq n_0 = p + [\ln m / \ln \lambda_0]$.

We estimate the number of monomials v of weight not exceeding m and obtain the bound

$$\tilde{\gamma}_{\boldsymbol{L}}(m) \leq \sum_{n=1}^{n_0} p^{n-1} \leq \frac{p^{n_0-1}}{1-1/p} \leq \frac{p^{p-1+\ln m/\ln \lambda_0}}{1-1/p} \approx C_0 m^{\ln p/\ln \lambda_0}. \qquad \square$$

Corollary 8.2. *Let $\boldsymbol{L} = \mathrm{Lie}_p(v_1, \dots, v_p)$, λ_0 as above and $\theta = \ln p / \ln \lambda_0$. Then the growth of the restricted enveloping algebra $u(\boldsymbol{L})$ is intermediate and*

$$1 \leq \mathrm{Dim}^3\, u(\boldsymbol{L}) \leq \theta.$$

Proof. The result follows by Proposition 1 of [18]. □

Theorem 8.3. *Let* $A = \operatorname{Alg}(v_1, \dots, v_p)$*. Then* $\operatorname{GKdim} A \le 2 \ln p / \ln \lambda_0$*, where* λ_0 *is as above.*

Proof. We embed our algebra into a bigger associative subalgebra $A \subset \operatorname{Alg}(H) \subset \operatorname{End}(R)$, where H was defined in Lemma 5.2. We claim that elements of $\operatorname{Alg}(H)$ can be expressed as linear combinations of the monomials

$$w = t_{2-p}^{\alpha_{2-p}} \dots t_{n-p}^{\alpha_{n-p}} v_1^{\beta_1} \dots v_n^{\beta_n}, \quad 0 \le \alpha_i, \beta_i \le p-1,\ \beta_n \ge 1,\ n \in \mathbb{N}, \tag{12}$$

where in case $\beta_n = 1$ we additionally assume that $\alpha_{n-p} \le p-2$.

Indeed, let us consider a product $w = u_1 \dots u_s$ of basis monomials $u_i = t_{2-p}^{\gamma_{2-p}} \dots t_{m_i-p}^{\gamma_{m_i-p}} v_{m_i} \in H$, where $\gamma_{m_i-p} \le p-2$, $i = 1, \dots, s$. Consider the largest index $M(w) = \max\{m_i \mid i = 1, \dots, s\}$. Then the highest t_i is t_{M-p}. Our product satisfies the following property VTmax: if the highest v_M is unique in the product, then the highest variable t_{M-p} has the total occurrence at most $p-2$. We straighten the product to the form (12). Let us check that VTmax is kept under the process. We perform the following transformations.

Case 1. $v_n v_m = v_m v_n + [v_n, v_m]$ if $n > m$. Consider the terms of the product $[v_n, v_m]$, see Lemma 5.1, claims (1), (2). If we get a new highest $v_{M'}$, we obtain the highest $t_{M'-p}$ in degree $p-2$ as well, the property VTmax is kept. If we get one more term v_M, then there is nothing to check. If we obtain v_j such that $j < M$, then we get at most t_{j-p}, and the total degree of the highest t_{M-p} is not changed, as required.

Case 2. v_n^p is expressed as in claim (5) of Lemma 5.1. We can only get a new highest $v_{M'}$ with no occurrence of $t_{M'-p}$ at all.

Case 3. The remaining operation is $v_n t_i = t_i v_n + v_n(t_i)$. Observe the second term. This operation cannot kill the highest v_M since $i \le M-p < M$. Also, t_i is replaced by a product of smaller t_js only. Thus, VTmax is kept.

Finally, we arrive at a monomial of type (12), the property VTmax means that in the case $\beta_n = 1$ we have $\alpha_{n-p} \le p-2$. Thus, $\operatorname{Alg}(H)$ is spanned by the claimed monomials.

Let us estimate the weight of a monomial (12). In case $\beta_n = 1$, we use the fact that $\alpha_{n-p} \le p-2$ and obtain, as in Lemma 7.3, the estimate

$$\begin{aligned} \operatorname{wt}_0(w) &\ge \lambda_0^n - \sum_{i=2-p}^{n-p} \alpha_i \lambda_0^i \ge \lambda_0^n - (p-1) \sum_{i=2-p}^{n-p} \lambda_0^i + \lambda_0^{n-p} \\ &\ge \lambda_0^n - (p-1) \frac{\lambda_0^{n-p}}{1 - 1/\lambda_0} + \lambda_0^{n-p} \\ &= \frac{\lambda_0^{n-p}(\lambda_0^p - \lambda_0^{p-1} - (p-1))}{1 - 1/\lambda_0} + \lambda_0^{n-p} = \lambda_0^{n-p}. \end{aligned}$$

In the case $\beta_n > 1$ we have

$$\mathrm{wt}_0(w) \geq 2\lambda_0^n - (p-1)\sum_{i=2-p}^{n-p} \lambda_0^i \geq \lambda_0^n - (p-1)\sum_{i=2-p}^{n-p} \lambda_0^i + \lambda_0^{n-p} \geq \lambda_0^{n-p}.$$

Fix a number m. Consider all monomials w of type (12) such that $\mathrm{wt}_0(w) \leq m$. Both cases above yield the estimate $m \geq \mathrm{wt}_0(w) \geq \lambda_0^{n-p}$. Then $n \leq n_0 = p + [\ln m/\ln \lambda_0]$.

Now we can estimate the number of monomials w of weight not exceeding m and obtain the bound

$$\tilde{\gamma}_{\boldsymbol{A}}(m) \leq \sum_{n=1}^{n_0} p^{2n-1} \leq \frac{p^{2n_0-1}}{1-1/p^2} \leq \frac{p^{2\ln m/\ln\lambda_0+2p-1}}{1-1/p^2} \approx C_0 m^{2\ln p/\ln\lambda_0}. \qquad \square$$

Let us prove the following commutation relation.

Lemma 8.4. *For all $n \geq 1$ we have*

$$\begin{aligned}
&(\mathrm{ad}\, v_n)^{p-1}(v_{n+p-1}) \\
&\quad = -v_{n+p} \\
&\qquad - t_n(t_{n-p+1})^{p-1}\cdot t_{n+1}^{p-2} v_{n+p+1} \\
&\qquad - t_n(t_{n-p+1}t_{n-p+2}\cdot t_{n+1})^{p-1}\cdot t_{n+2}^{p-2} v_{n+p+2} \\
&\qquad \vdots \\
&\qquad - t_n(t_{n-p+1}\dots t_{n-2}\cdot t_{n+1}\dots t_{n+p-3})^{p-1}\cdot t_{n+p-2}^{p-2} v_{n+2p-2} \\
&\qquad - 2t_n(t_{n-p+1}\dots t_{n-1}\cdot t_{n+1}\dots t_{n+p-2})^{p-1}\cdot t_{n+p-1}^{p-2} v_{n+2p-1}.
\end{aligned}$$

Proof. In claim (2) of Lemma 5.1 we take $k = p-1$

$$[v_n, v_{n+p-1}] = -\sum_{j=0}^{p-2}\Big(\prod_{\substack{l\in\{1,\dots,j\}\cup\\ \{p,\dots,p+j-1\}}} t_{n-p+l}^{p-1}\Big) t_{n+j}^{p-2} v_{n+j+p} \tag{13}$$

$$\begin{aligned}
&= -t_n^{p-2} v_{n+p} \\
&\quad - (t_{n-p+1}\cdot t_n)^{p-1}\cdot t_{n+1}^{p-2} v_{n+p+1} \\
&\quad - (t_{n-p+1}t_{n-p+2}\cdot t_n t_{n+1})^{p-1}\cdot t_{n+2}^{p-2} v_{n+p+2} \\
&\quad \vdots \\
&\quad - (t_{n-p+1}t_{n-p+2}\dots t_{n-2}\cdot t_n t_{n+1}\dots t_{n+p-3})^{p-1}\cdot t_{n+p-2}^{p-2} v_{n+2p-2}.
\end{aligned} \tag{14}$$

Let us further commute this expression with v_n. Recall that v_n acts trivially on $t_{n-p+1}, \dots t_{n-2}$. By Lemma 5.1 all elements $v_n(t_j)$, where $j \geq n+1$, contain the factor t_{n-p+1}^{p-1} and we get zero due to the other factor t_{n-p+1}^{p-1}. The same argument applies to $[v_n, v_j]$, where $j \geq n + p + 1$. Thus, we get a nontrivial action only in the cases $v_n(t_n) = 1$ and $[v_n, v_{n+p}]$. Therefore when commuted with v_n all terms in the sum above except the first one change only the power of t_n. Considering the first term we take into account that

$$\begin{aligned}[v_n, v_{n+p}] = &-(t_{n-p+1})^{p-1} \cdot t_{n+1}^{p-2} v_{n+p+1} \\ &- (t_{n-p+1}t_{n-p+2} \cdot t_{n+1})^{p-1} \cdot t_{n+2}^{p-2} v_{n+p+2} \\ &- (t_{n-p+1}t_{n-p+2}t_{n-p+3} \cdot t_{n+1}t_{n+2})^{p-1} \cdot t_{n+3}^{p-2} v_{n+p+3} \\ &\vdots \\ &- (t_{n-p+1}t_{n-p+2} \ldots t_{n-1} \cdot t_{n+1}t_{n+2} \ldots t_{n+p-2})^{p-1} \cdot t_{n+p-1}^{p-2} v_{n+2p-1}.\end{aligned}$$

Each time when commuting the first term of (14) with v_n, these summands add to the existing ones. As a result, there exist some scalars $B_{s,j}$ for $s = 1, \ldots, p-1$, $j = 1, \ldots, p-1$ such that

$$\begin{aligned}&(\operatorname{ad} v_n)^s (v_{n+p-1}) \\ &\quad = (-1)(-2)\ldots(-s) t_n^{p-s-1} v_{n+p} \\ &\qquad + B_{s,1} t_n^{p-s} (t_{n-p+1})^{p-1} t_{n+1}^{p-2} v_{n+p+1} \\ &\qquad + B_{s,2} t_n^{p-s} (t_{n-p+1}t_{n-p+2} \cdot t_{n+1})^{p-1} \cdot t_{n+2}^{p-2} v_{n+p+2} \\ &\qquad \vdots \\ &\qquad + B_{s,p-2} t_n^{p-s} (t_{n-p+1}t_{n-p+2} \ldots t_{n-2} \cdot t_{n+1} \ldots t_{n+p-3})^{p-1} \cdot t_{n+p-2}^{p-2} v_{n+2p-2} \\ &\qquad + B_{s,p-1} t_n^{p-s} (t_{n-p+1}t_{n-p+2} \ldots t_{n-1} \cdot t_{n+1} \ldots t_{n+p-2})^{p-1} \cdot t_{n+p-1}^{p-2} v_{n+2p-1}.\end{aligned}$$

We have the recurrence relations $B_{s+1,j} = -sB_{s,j} - (-1)^s s!$, $s \geq 1$ for all $j = 1, \ldots, p-1$ and the original conditions $B_{1,1} = B_{1,2} = \cdots = B_{1,p-2} = -1$ and $B_{1,p-1} = 0$. We check that for all $j = 1, \ldots, p-2$ we get $B_{s,j} = (-1)^s s!$, $s \geq 1$; in particular, $B_{p-1,j} = -1$. For $j = p-1$ we have $B_{s,p-1} = (-1)^s (s-1)(s-1)!$, $s \geq 1$, in particular $B_{p-1,p-1} = -2$. □

Let us introduce the following convenient notations. Let $v = \sum_{i \geq m} a_i v_i \in H$, where $a_i \in R$. Then we write $v = O(v_m)$. Also suppose that $r_1, \ldots, r_s \in R$. Then denote by $O((r_1, \ldots, r_s)v_m)$ an element $h \in H$ of the form

$$h = \sum_{i=1}^{s} r_i g_i, \quad g_i = O(v_m).$$

Lemma 8.5. *For all $m \geq 1$ we have $[H, O(v_m)] = O(v_m)$.*

Proof. Follows from the commutation relations of Lemma 5.1. □

Lemma 8.6. *Let* $\boldsymbol{L} = \mathrm{Lie}_p(v_1, \ldots, v_p)$. *Then there exist homogeneous elements of the form*

$$\tilde{v}_n = v_n + O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+1}) \in \boldsymbol{L}, \quad n = 1, 2, \ldots.$$

Proof. We begin with $\tilde{v}_1 = v_1, \ldots, \tilde{v}_p = v_p$. Assume that all elements $\tilde{v}_i$, with $i \le n + p - 1$, are defined. By assumption we have elements

$$\tilde{v}_n = v_n + O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+1}) \in \boldsymbol{L},$$
$$\tilde{v}_{n+p-1} = v_{n+p-1} + O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p}) \in \boldsymbol{L}.$$

Consider $[\tilde{v}_n, \tilde{v}_{n+p-1}]$. We use (13) and the commutation relations of Lemma 5.1 to get

$$\begin{aligned}
&[v_n, v_{n+p-1}] = -t_n^{p-2} v_{n+p} + t_{n-p+1}^{p-1} O(v_{n+p+1}),\\
&[v_n, O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p})] = O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p}),\\
&[v_{n+p-1}, O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+1})] = O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+p+1}),\\
&[O((t_{2-p}^{p-1}, \ldots, t_{n-2p+1}^{p-1})v_{n+1}), O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p})]\\
&\qquad = O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p+1}).
\end{aligned}$$

Let us explain the third relation. The action $v_{n+p-1}(t_j)$ for some v_m inside $O(v_{n+1})$ can appear only for $m \ge j + p \ge n + 2p - 1$. On the other hand, by Lemma 5.1 $[v_{n+p-1}, v_{n+1}] = O(v_{n+p+1})$. The second and forth equations are obtained by similar arguments. Thus,

$$[\tilde{v}_n, \tilde{v}_{n+p-1}] = -t_n^{p-2} v_{n+p} + t_{n-p+1}^{p-1} O(v_{n+p+1}) + O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p}).$$

We repeat this process and observe that our additional factors cannot disappear:

$$(\mathrm{ad}\, \tilde{v}_n)^{p-1}(\tilde{v}_{n+p-1}) = -v_{n+p} + t_{n-p+1}^{p-1} O(v_{n+p+1}) + O((t_{2-p}^{p-1}, \ldots, t_{n-p}^{p-1})v_{n+p}). \tag{15}$$

The last term can contain a summand with v_{n+p}, it is of the form $t_i^{p-1} r v_{n+p}$, where $r \in R$ and $2-p \le i \le n-p$. But, by construction, the element (15) is homogeneous. Then

$$\begin{aligned}
\mathrm{wt}_0(v_{n+p}) &= \mathrm{wt}_0(t_i^{p-1} r v_{n+p})\\
&= \mathrm{wt}_0(v_{n+p}) + \mathrm{wt}_0(t_i^{p-1} r)\\
&\le \mathrm{wt}_0(v_{n+p}) - (p-1)\lambda_0^i,
\end{aligned}$$

a contradiction. Therefore, the last term (15) contains only v_m with $m \ge n + p + 1$. Then we set

$$\tilde{v}_{n+p} = -(\mathrm{ad}\, \tilde{v}_n)^{p-1}(\tilde{v}_{n+p-1}) = v_{n+p} + O((t_{2-p}^{p-1}, \ldots, t_{n-p+1}^{p-1})v_{n+p+1}),$$

and the induction step is proved. □

Corollary 8.7. *The Lie algebra* $\boldsymbol{L} = \mathrm{Lie}_p(v_1, \ldots, v_p)$ *is infinite-dimensional.*

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V. M. Petrogradsky, Faculty of Mathematics and Computer Science, Ulyanovsk State University, Lev Tolstoy 42, Ulyanovsk, 432970, Russia

E-mail: petrogradsky@rambler.ru

I. P. Shestakov, Instituto de Mathemática e Estatística, Universidade de Saõ Paulo, Caixa postal 66281, CEP 05315-970, Saõ Paulo, Brazil

E-mail: shestak@ime.usp.br

E. Zelmanov, Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, U.S.A.

E-mail: ezelmano@math.ucsd.edu