Comment. Math. Helv. 82 (2007), 175–195 **Commentarii Mathematici Helvetici**


# A pinching theorem for the first eigenvalue of the Laplacian on hypersurfaces of the Euclidean space

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**Abstract.** In this paper, we give pinching theorems for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian on the compact hypersurfaces of the Euclidean space. Indeed, we prove that if the volume of $M$ is 1 then, for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending on the dimension $n$ of $M$ and the $L_\infty$-norm of the mean curvature $H$, so that if the $L_{2p}$-norm $\|H\|_{2p}$ $(p \geq 2)$ of $H$ satisfies $n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$, then the Hausdorff-distance between $M$ and a round sphere of radius $(n/\lambda_1(M))^{1/2}$ is smaller than $\varepsilon$. Furthermore, we prove that if $C$ is a small enough constant depending on $n$ and the $L_\infty$-norm of the second fundamental form, then the pinching condition $n\|H\|_{2p}^2 - C < \lambda_1(M)$ implies that $M$ is diffeomorphic to an $n$-dimensional sphere.

**Mathematics Subject Classification (2000).** 53A07, 53C21.

**Keywords.** Spectrum, Laplacian, pinching results, hypersurfaces.

## 1. Introduction and preliminaries

Let $(M^n, g)$ be a compact, connected and oriented $n$-dimensional Riemannian manifold without boundary isometrically immersed by $\phi$ into the $n+1$-dimensional euclidean space $(\mathbb{R}^{n+1}, can)$ (i.e. $\phi^\star can = g$). A well-known inequality due to Reilly ([11]) gives an extrinsic upper bound for the first nonzero eigenvalue $\lambda_1(M)$ of the Laplacian of $(M^n, g)$ in terms of the square of the length of the mean curvature. Indeed, we have

$$\lambda_1(M) \leq \frac{n}{V(M)} \int_M |H|^2 \, dv \tag{1}$$

where $dv$ and $V(M)$ denote respectively the Riemannian volume element and the volume of $(M^n, g)$. Moreover the equality holds if and only if $(M^n, g)$ is a geodesic hypersphere of $\mathbb{R}^{n+1}$.

*Supported by European Commission through Human Potential Programme.

By using Hölder inequality, we obtain some other similar estimates for the $L_{2p}$-norm ($p \geq 1$) with $H$ denoted by $\|H\|_{2p}^2$

$$\lambda_1(M) \leq \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2, \tag{2}$$

and as for the inequality (1), the equality case is characterized by the geodesic hyperspheres of $\mathbb{R}^{n+1}$.

A first natural question is to know if there exists a pinching result as the one we state now: does a constant $C$ depending on minimum geometric invariants exist so that if we have the pinching condition

$$(\mathrm{P}_C) \qquad \frac{n}{V(M)^{1/p}} \|H\|_{2p}^2 - C < \lambda_1(M)$$

then $M$ is close to a sphere in a certain sense?

Such questions are known for the intrinsic lower bound of Lichnerowicz–Obata ([9]) of $\lambda_1(M)$ in terms of the lower bound of the Ricci curvature (see [4], [8], [10]). Other pinching results have been proved for Riemannian manifolds with positive Ricci curvature, with a pinching condition on the $n+1$-st eigenvalue ([10]), the diameter ([5], [8], [15]), the volume or the radius (see for instance [2] and [3]).

For instance, S. Ilias proved in [8] that there exists $\varepsilon$ depending on $n$ and an upper bound of the sectional curvature so that if the Ricci curvature Ric of $M$ satisfies $\mathrm{Ric} \geq n-1$ and $\lambda_1(M) \leq \lambda_1(\mathbb{S}^n) + \varepsilon$, then $M$ is homeomorphic to $\mathbb{S}^n$.

In this article, we investigate the case of hypersurfaces where, as far as we know, very little is known about pinching and stability results (see however [12], [13]).

More precisely, in our paper, the hypothesis made in [8] that $M$ has a positive Ricci curvature is replaced by the fact that $M$ is isometrically immersed as a hypersurface in $\mathbb{R}^{n+1}$, and the bound on the sectional curvature by an $L^\infty$-bound on the mean curvature or on the second fundamental form. Note that we do not know if such bounds are sharp, or if a bound on the $L^q$-norm (for some $q$) of the mean curvature would be enough.

We get the following results

**Theorem 1.1.** *Let $(M^n, g)$ be a compact, connected and oriented $n$-dimensional Riemannian manifold without boundary isometrically immersed by $\phi$ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$ and let $x_0$ be the center of mass of $M$. Then for any $p \geq 2$ and for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending only on $n$, $\varepsilon > 0$ and on the $L_\infty$-norm of $H$ so that if*

$$(\mathrm{P}_{C_\varepsilon}) \qquad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

*then the Hausdorff-distance $d_H$ of $M$ to the sphere $S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ of center $x_0$ and radius $\sqrt{\frac{n}{\lambda_1(M)}}$ satisfies $d_H\left(\phi(M), S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)\right) < \varepsilon$.*

We recall that the Hausdorff-distance between two compact subsets $A$ and $B$ of a metric space is given by

$$d_H(A, B) = \inf\{\eta | V_\eta(A) \supset B \text{ and } V_\eta(B) \supset A\}$$

where for any subset $A$, $V_\eta(A)$ is the tubular neighborhood of $A$ defined by $V_\eta(A) = \{x \mid \operatorname{dist}(x, A) < \eta\}$.

**Remark.** We will see in the proof that $C_\varepsilon(n, \|H\|_\infty) \to 0$ when $\|H\|_\infty \to \infty$ or $\varepsilon \to 0$.

In fact the previous theorem is a consequence of the above definition and the following theorem

**Theorem 1.2.** *Let $(M^n, g)$ be a compact, connected and oriented $n$-dimensional Riemannian manifold without boundary isometrically immersed by $\phi$ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$ and let $x_0$ be the center of mass of $M$. Then for any $p \geq 2$ and for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ depending only on $n$, $\varepsilon > 0$ and on the $L_\infty$-norm of $H$ so that if*

$$(\mathrm{P}_{C_\varepsilon}) \qquad n\|H\|_{2p}^2 - C_\varepsilon < \lambda_1(M)$$

*then*

(1) $\phi(M) \subset B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon\right) \setminus B\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right)$;

(2) $B(x, \varepsilon) \cap \phi(M) \neq \emptyset$ *for all* $x \in S\left(x_0, \sqrt{\frac{n}{\lambda_1(M)}}\right)$.

In the following theorem, if the pinching is strong enough, with a control on $n$ and the $L_\infty$-norm of the second fundamental form, we obtain that $M$ is diffeomorphic to a sphere and even almost isometric with a round sphere in a sense we will make precise.

**Theorem 1.3.** *Let $(M^n, g)$ be a compact, connected and oriented $n$-dimensional Riemannian manifold $(n \geq 2)$ without boundary isometrically immersed by $\phi$ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$. Then for any $p \geq 2$, there exists a constant $C$ depending only on $n$ and the $L_\infty$-norm of the second fundamental form $B$ so that if*

$$(\mathrm{P}_C) \qquad n\|H\|_{2p}^2 - C < \lambda_1(M).$$

*Then $M$ is diffeomorphic to $\mathbb{S}^n$.*

*More precisely, there exists a diffeomorphism $F$ from $M$ into the sphere $\mathbb{S}^n\left(\sqrt{\frac{n}{\lambda_1(M)}}\right)$ of radius $\sqrt{\frac{n}{\lambda_1(M)}}$ which is a quasi-isometry. Namely, for any $\theta$,*

$0 < \theta < 1$, *there exists a constant* $C$ *depending only on* $n$, *the* $L_\infty$*-norm of* $B$ *and* $\theta$, *so that the pinching condition* $(\mathrm{P}_C)$ *implies*

$$\left||dF_x(u)|^2 - 1\right| \leq \theta$$

*for any* $x \in M$ *and* $u \in T_x M$ *so that* $|u| = 1$.

Now we will give some preliminaries for the proof of these theorems. Throughout the paper, we consider a compact, connected and oriented $n$-dimensional Riemannian manifold $(M^n, g)$ without boundary isometrically immersed by $\phi$ into $(\mathbb{R}^{n+1}, can)$ (i.e. $\phi^\star can = g$). Let $\nu$ be the outward normal vector field. Then the second fundamental form of the immersion will be defined by $B(X, Y) = \left\langle \nabla^0_X \nu, Y \right\rangle$, where $\nabla^0$ and $\langle\ ,\ \rangle$ are respectively the Riemannian connection and the inner product of $\mathbb{R}^{n+1}$. Moreover the mean curvature $H$ will be given by $H = (1/n)\,\mathrm{trace}(B)$.

Now let $\partial_i$ be an orthonormal frame of $\mathbb{R}^{n+1}$ and let $x_i : \mathbb{R}^{n+1} \to \mathbb{R}$ be the associated component functions. Putting $X_i = x_i \circ \phi$, a straightforward calculation shows us that

$$B \otimes \nu = -\sum_{i \leq n+1} \nabla d X_i \otimes \partial_i$$

and

$$nH\nu = \sum_{i \leq n+1} \Delta X_i \, \partial_i,$$

where $\nabla$ and $\Delta$ denote respectively the Riemannian connection and the Laplace–Beltrami operator of $(M^n, g)$. On the other hand, we have the well-known formula

$$\frac{1}{2}\Delta |X|^2 = nH\,\langle \nu, X\rangle - n \tag{3}$$

where $X$ is the position vector given by $X = \sum_{i \leq n+1} X_i \partial_i$.

We recall that to prove the Reilly inequality, we use the functions $X_i$ as test functions (cf. [11]). Indeed, doing a translation if necessary, we can assume that $\int_M X_i \, dv = 0$ for all $i \leq n+1$ and we can apply the variational characterization of $\lambda_1(M)$ to $X_i$. If the equality holds in (1) or (2), then the functions are nothing but eigenfunctions of $\lambda_1(M)$ and from the Takahashi Theorem ([14]) $M$ is immersed isometrically in $\mathbb{R}^{n+1}$ as a geodesic sphere of radius $\sqrt{\frac{n}{\lambda_1(M)}}$.

Throughout the paper we use some notations. From now on, the inner product and the norm induced by $g$ and $can$ on a tensor $T$ will be denoted respectively by $\langle\ ,\ \rangle$ and $|\ |^2$, and the $L_p$-norm will be given by

$$\|T\|_p = \left(\int_M |T|^p \, dv\right)^{1/p}$$

and

$$\|T\|_\infty = \sup_M |T|.$$

We end these preliminaries by a convenient result.

**Lemma 1.1.** *Let $(M^n, g)$ be a compact, connected and oriented $n$-dimensional Riemannian manifold ($n \geq 2$) without boundary isometrically immersed by $\phi$ in $\mathbb{R}^{n+1}$. Assume that $V(M) = 1$. Then there exist constants $c_n$ and $d_n$ depending only on $n$ so that for any $p \geq 2$, if ($P_C$) is true with $C < c_n$ then*

$$\frac{n}{\lambda_1(M)} \leq d_n. \tag{4}$$

*Proof.* We recall the standard Sobolev inequality (cf. [6], [7], [16] and p. 216 in [1]). If $f$ is a smooth function and $f \geq 0$, then

$$\left(\int_M f^{\frac{n}{n-1}}\, dv\right)^{1-(1/n)} \leq K(n) \int_M (|df| + |H| f)\, dv \tag{5}$$

where $K(n)$ is a constant depending on $n$ and the volume of the unit ball in $\mathbb{R}^n$. Taking $f = 1$ on $M$, and using the fact that $V(M) = 1$, we deduce that

$$\|H\|_{2p} \geq \frac{1}{K(n)}$$

and if ($P_C$) is satisfied and $C \leq \frac{n}{2K(n)^2} = c_n$, then

$$\frac{n}{\lambda_1(M)} \leq \frac{1}{n\|H\|_{2p}^2 - C} \leq 2K(n)^2 = d_n. \qquad \square$$

Throughout the paper, we will assume that $V(M) = 1$ and $\int_M X_i\, dv = 0$ for all $i \leq n+1$. The last assertion implies that the center of mass of $M$ is the origin of $\mathbb{R}^{n+1}$.

## 2. An $L^2$-approach of the problem

A first step in the proof of Theorem 1.2 is to prove that if the pinching condition ($P_C$) is satisfied, then $M$ is close to a sphere in an $L^2$-sense.

In the following lemma, we prove that the $L^2$-norm of the position vector is close to $\sqrt{\frac{n}{\lambda_1(M)}}$.

**Lemma 2.1.** *If we have the pinching condition* ($\mathrm{P}_C$) *with* $C < c_n$*, then*

$$\frac{n\lambda_1(M)}{(C+\lambda_1(M))^2} \le \|X\|_2^2 \le \frac{n}{\lambda_1(M)} \le d_n.$$

*Proof.* Since $\int_M X_i\, dv = 0$, we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M)\int_M \sum_{i\le n+1} |X_i|^2\, dv \le \int_M \sum_{i\le n+1} |dX_i|^2\, dv = n$$

which gives the inequality of the right-hand side.

Let us prove now the inequality of the left-hand side.

$$\begin{aligned}\lambda_1(M)\int_M |X|^2\, dv &\le \frac{\left(\int_M \sum_{i\le n+1} |dX_i|^2\, dv\right)^4}{\left(\int_M \sum_{i\le n+1} |dX_i|^2\, dv\right)^3} = \frac{\left(\int_M \sum_{i\le n+1} (\Delta X_i)X_i\, dv\right)^4}{n^3}\\ &\le \frac{\left(\int_M \sum_{i\le n+1} (\Delta X_i)^2\, dv\right)^2 \left(\int_M |X|^2\, dv\right)^2}{n^3}\\ &= n\left(\int_M H^2\, dv\right)^2 \left(\int_M |X|^2\, dv\right)^2\end{aligned}$$

then using again the Hölder inequality, we get

$$\lambda_1(M) \le \frac{1}{n}\left(n\|H\|_{2p}^2\right)^2 \int_M |X|^2\, dv \le \frac{(C+\lambda_1(M))^2}{n}\int_M |X|^2\, dv.$$

This completes the proof. □

From now on, we will denote by $X^T$ the orthogonal tangential projection on $M$. In fact, at $x \in M$, $X^T$ is nothing but the vector of $T_xM$ defined by $X^T = \sum_{1\le i\le n} \langle X, e_i\rangle\, e_i$ where $(e_i)_{1\le i\le n}$ is an orthonormal basis of $T_xM$. In the following lemma, we will show that the condition ($\mathrm{P}_C$) implies that the $L^2$-norm of $X^T$ of $X$ on $M$ is close to 0.

**Lemma 2.2.** *If we have the pinching condition* ($\mathrm{P}_C$)*, then*

$$\|X^T\|_2^2 \le A(n)C.$$

*Proof.* From Lemma 2.1 and the relation (3), we have

$$\lambda_1(M)\int_M |X|^2\, dv \le n = n\left(\int_M H\, \langle X, \nu\rangle\, dv\right)^2$$

$$\leq \left(\int_M |H||\langle X,\nu\rangle|\,dv\right)^2 \leq n\|H\|_{2p}^2\left(\int_M |\langle X,\nu\rangle|^{\frac{2p}{2p-1}}\,dv\right)^{\frac{2p-1}{p}}$$
$$\leq n\|H\|_{2p}^2\left(\int_M |\langle X,\nu\rangle|^2\,dv\right) = n\|H\|_{2p}^2\int_M |X|^2\,dv.$$

Then we deduce that

$$n\|H\|_{2p}^2\|X^T\|_2^2 = n\|H\|_{2p}^2\left(\int_M \left(|X|^2 - |\langle X,\nu\rangle|^2\right)dv\right)$$
$$\leq (n\|H\|_{2p}^2 - \lambda_1(M))\|X\|_2^2 \leq d_n C$$

where in the last inequality we have used the pinching condition and Lemma 2.1. □

Next we will show that the condition ($P_C$) implies that the component functions are almost eigenfunctions in an $L^2$-sense. For this, let us consider the vector field $Y$ on $M$ defined by

$$Y = \sum_{i\leq n+1}(\Delta X_i - \lambda_1(M)X_i)\,\partial_i = nH\nu - \lambda_1(M)X.$$

**Lemma 2.3.** *If* ($P_C$) *is satisfied, then*

$$\|Y\|_2^2 \leq nC.$$

*Proof.* We have

$$\int_M |Y|^2\,dv = \int_M \left(n^2H^2 - 2n\lambda_1(M)H\langle\nu,X\rangle + \lambda_1(M)^2|X|^2\right)dv.$$

Now by integrating the relation (3) we deduce that

$$\int_M H\langle\nu,X\rangle\,dv = 1.$$

Furthermore, since $\int_M X_i\,dv = 0$, we can apply the variational characterization of the eigenvalues to obtain

$$\lambda_1(M)\int_M |X|^2\,dv = \lambda_1(M)\int_M\sum_{i\leq n+1}|X_i|^2\,dv \leq \int_M\sum_{i\leq n+1}|dX_i|^2\,dv = n.$$

Then

$$\int_M |Y|^2\,dv \leq n^2\int_M |H|^2\,dv - n\lambda_1(M) \leq n\left(n\|H\|_{2p}^2 - \lambda_1(M)\right) \leq nC$$

where in this last inequality we have used the Hölder inequality. □

To prove Assertion 1 of Theorem 1.2, we will show that $\left\| |X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \right\|_\infty \le \varepsilon$. For this we need an $L^2$-upper bound on the function $\varphi = |X|\Big(|X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\Big)^2$.

Before giving such estimate, we will introduce the vector field $Z$ on $M$ defined by

$$Z = \left(\frac{n}{\lambda_1(M)}\right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}}.$$

We have

**Lemma 2.4.** *If* $(\mathrm{P}_C)$ *is satisfied with* $C < c_n$, *then*

$$\|Z\|_2^2 \le B(n)C.$$

*Proof.* We have

$$\begin{aligned}
\|Z\|_2^2 &= \left\| \left(\frac{n}{\lambda_1(M)}\right)^{1/2} |X|^{1/2} H\nu - \frac{X}{|X|^{1/2}} \right\|_2^2 \\
&= \int_M \left( \frac{n}{\lambda_1(M)} |X| H^2 - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2} H\,\langle \nu, X\rangle + |X| \right) dv \\
&\le \frac{n}{\lambda_1(M)} \left(\int_M |X|^2\,dv\right)^{1/2} \left(\int_M H^4\,dv\right)^{1/2} \\
&\qquad - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \left(\int_M |X|^2\,dv\right)^{1/2}.
\end{aligned}$$

Note that we have used the relation (3). Finally for $p \ge 2$, we get

$$\begin{aligned}
\|Z\|_2^2 &\le \left(\int_M |X|^2\,dv\right)^{1/2} \left(\frac{n}{\lambda_1(M)} \|H\|_{2p}^2 + 1\right) - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2} \\
&\le \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \left(\frac{C}{\lambda_1(M)} + 2\right) - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2} \\
&= \left(\frac{n}{\lambda_1(M)}\right)^{1/2} \frac{C}{\lambda_1(M)} \le \frac{d_n^{3/2}}{n} C.
\end{aligned}$$

This concludes the proof of the lemma. □

Now we give an $L^2$-upper bound of $\varphi$.

**Lemma 2.5.** *Let* $p \ge 2$ *and* $C \le c_n$. *If we have the pinching condition* $(\mathrm{P}_C)$, *then*

$$\|\varphi\|_2 \le D(n) \|\varphi\|_\infty^{3/4} C^{1/4}.$$

*Proof.* We have

$$\|\varphi\|_2 = \left(\int_M \varphi^{3/2}\varphi^{1/2}\,dv\right)^{1/2} \le \|\varphi\|_\infty^{3/4}\|\varphi^{1/2}\|_1^{1/2},$$

and noting that

$$|X|\left(|X| - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right)^2 = \left||X|^{1/2}X - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\frac{X}{|X|^{1/2}}\right|^2$$

we get

$$\begin{aligned}\int_M \varphi^{1/2}\,dv &= \left\||X|^{1/2}X - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\frac{X}{|X|^{1/2}}\right\|_1\\ &= \left\| -\frac{|X|^{1/2}}{\lambda_1(M)}Y + \frac{n}{\lambda_1(M)}|X|^{1/2}H\nu - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\frac{X}{|X|^{1/2}}\right\|_1\\ &\le \left\|\frac{|X|^{1/2}}{\lambda_1(M)}Y\right\|_1 + \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\|Z\|_1\,. \end{aligned} \tag{6}$$

From Lemmas 2.3 and 1.1 we get

$$\begin{aligned}\left\|\frac{|X|^{1/2}}{\lambda_1(M)}Y\right\|_1 &\le \frac{1}{\lambda_1(M)}\left(\int_M |X|\,dv\right)^{1/2}\|Y\|_2\\ &\le \frac{1}{\lambda_1(M)}\left(\int_M |X|^2\,dv\right)^{1/4}\|Y\|_2 \le \frac{d_n^{3/4}}{n^{1/2}}C^{1/2}.\end{aligned}$$

Moreover, using Lemmas 2.4 and 1.1 again it is easy to see that the last term of (6) is bounded by $d_n^{1/2}B(n)^{1/2}C^{1/2}$. Then $\|\varphi^{1/2}\|_1^{1/2} \le D(n)C^{1/4}$. □

## 3. Proof of Theorem 1.2

The proof of Theorem 1.2 is immediate from the two following technical lemmas which we state below.

**Lemma 3.1.** *For $p \ge 2$ and for any $\eta > 0$, there exists $K_\eta(n, \|H\|_\infty) \le c_n$ so that if $(\mathrm{P}_{K_\eta})$ is true, then $\|\varphi\|_\infty \le \eta$. Moreover, $K_\eta \to 0$ when $\|H\|_\infty \to \infty$ or $\eta \to 0$.*

**Lemma 3.2.** *Let $x_0$ be a point of the sphere $S(O, R)$ of $\mathbb{R}^{n+1}$ with the center at the origin and of radius $R$. Assume that $x_0 = Re$ where $e \in \mathbb{S}^n$. Now let $(M^n, g)$ be a compact oriented $n$-dimensional Riemannian manifold without boundary isometrically* *immersed by* $\phi$ *in* $\mathbb{R}^{n+1}$ *so that* $\phi(M) \subset (B(O, R+\eta)\backslash B(O, R-\eta)) \backslash B(x_0, \rho)$ *with* $\rho = 4(2n-1)\eta$ *and suppose that there exists a point* $p \in M$ *so that* $\langle X, e\rangle > 0$. *Then there exists* $y_0 \in M$ *so that the mean curvature* $H(y_0)$ *at* $y_0$ *satisfies* $|H(y_0)| \geq \frac{1}{4n\eta}$.

Now, let us see how to use these lemmas to prove Theorem 1.2.

*Proof of Theorem* 1.2. We consider the function $f(t) = t\left(t - \left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right)^2$. For $\varepsilon > 0$ let us put

$$\begin{aligned}\eta(\varepsilon) &= \min\left(\left(\frac{1}{\|H\|_\infty} - \varepsilon\right)\varepsilon^2, \left(\frac{1}{\|H\|_\infty} + \varepsilon\right)\varepsilon^2, \frac{1}{27\|H\|_\infty^3}\right)\\ &\leq \min\left(f\left(\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon\right), f\left(\left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon\right), \frac{1}{27\|H\|_\infty^3}\right).\end{aligned}$$

Then, as $\eta(\varepsilon) > 0$ and from Lemma 3.1, it follows that if the pinching condition $(\mathrm{P}_{K_{\eta(\varepsilon)}})$ is satisfied with $K_{\eta(\varepsilon)} \leq c_n$, then for any $x \in M$, we have

$$f(|X|) \leq \eta(\varepsilon). \tag{7}$$

Now to prove Theorem 1.2, it is sufficient to assume $\varepsilon < \frac{2}{3\|H\|_\infty}$. Let us show that either

$$\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon \leq |X| \leq \left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon \quad \text{or} \quad |X| < \frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}. \tag{8}$$

By studying the function $f$ it is easy to see that $f$ has a unique local maximum in $\frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}$ and from the definition of $\eta(\varepsilon)$ it follows that $\eta(\varepsilon) < \frac{4}{27}\frac{1}{\|H\|_\infty^3} \leq \frac{4}{27}\left(\frac{n}{\lambda_1(M)}\right)^{3/2} = f\left(\frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}\right)$.

Since $\varepsilon < \frac{2}{3\|H\|_\infty}$, we have $\varepsilon < \frac{2}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}$ and $\frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2} < \left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon$. This and (7) yield (8).

Now, from Lemma 2.1 we deduce that there exists a point $y_0 \in M$ so that $|X(y_0)| \geq \frac{n^{1/2}\lambda_1(M)^{1/2}}{(K_{\eta(\varepsilon)}+\lambda_1(M))}$ and since $K_{\eta(\varepsilon)} \leq c_n = \frac{n}{d_n} \leq \lambda_1(M) \leq 2\lambda_1(M)$ (see the proof of Lemma 1.1), we obtain $|X(y_0)| \geq \frac{1}{3}\left(\frac{n}{\lambda_1(M)}\right)^{1/2}$.

By the connectedness of $M$, it follows that $\left(\frac{n}{\lambda_1(M)}\right)^{1/2} - \varepsilon \leq |X| \leq \left(\frac{n}{\lambda_1(M)}\right)^{1/2} + \varepsilon$ for any point of $M$ and Assertion 1 of Theorem 1.2 is shown for the condition $(\mathrm{P}_{K_{\eta(\varepsilon)}})$.

In order to prove the second assertion, let us consider the pinching condition $(\mathrm{P}_{C_\varepsilon})$ with $C_\varepsilon = K_{\eta(\frac{\varepsilon}{4(2n-1)})}$. Then Assertion 1 is still valid. Let $x = \left(\frac{n}{\lambda_1(M)}\right)^{1/2} e \in S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$, with $e \in \mathbb{S}^n$ and suppose that $B(x, \varepsilon) \cap M = \emptyset$. Since $\int_M X_i \, dv = 0$

for any $i \leq n+1$, there exists a point $p \in M$ so that $\langle X, e\rangle > 0$ and we can apply Lemma 3.2. Therefore there is a point $y_0 \in M$ so that $H(y_0) \geq \frac{2n-1}{n\varepsilon} > \|H\|_\infty$ since we have assumed $\varepsilon < \frac{2}{3\|H\|_\infty} \leq \frac{2n-1}{2n\|H\|_\infty}$. Then we obtain a contradiction which implies $B(x,\varepsilon) \cap M \neq \emptyset$ and Assertion 2 is satisfied. Furthermore, $C_\varepsilon \to 0$ when $\|H\|_\infty \to \infty$ or $\varepsilon \to 0$. □

## 4. Proof of Theorem 1.3

From Theorem 1.2, we know that for any $\varepsilon > 0$, there exists $C_\varepsilon$ depending only on $n$ and $\|H\|_\infty$ so that if $(\mathrm{P}_{C_\varepsilon})$ is true then

$$\left| |X|_x - \sqrt{\frac{n}{\lambda_1(M)}} \right| \leq \varepsilon$$

for any $x \in M$. Now, since $\sqrt{n}\|H\|_\infty \leq \|B\|_\infty$, it is easy to see from the previous proofs that we can assume that $C_\varepsilon$ is depending only on $n$ and $\|B\|_\infty$.

The proof of Theorem 1.3 is a consequence of the following lemma on the $L_\infty$-norm of $\psi = |X^T|$.

**Lemma 4.1.** *For $p \geq 2$ and for any $\eta > 0$, there exists $K_\eta(n, \|B\|_\infty)$ so that if $(\mathrm{P}_{K_\eta})$ is true, then $\|\psi\|_\infty \leq \eta$. Moreover, $K_\eta \to 0$ when $\|B\|_\infty \to \infty$ or $\eta \to 0$.*

This lemma will be proved in the Section 5.

*Proof of Theorem* 1.3. Let $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{\|B\|_\infty}} \leq \sqrt{\frac{n}{\lambda_1(M)}}$. From the choice of $\varepsilon$, we deduce that the condition $(\mathrm{P}_{C_\varepsilon})$ implies that $|X_x|$ is nonzero for any $x \in M$ (see the proof of Theorem 1.2) and we can consider the differential application

$$\begin{aligned} F\colon M &\longrightarrow S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right), \\ x &\longmapsto \sqrt{\frac{n}{\lambda_1(M)}}\frac{X_x}{|X_x|}. \end{aligned}$$

We will prove that $F$ is a quasi-isometry. Indeed, for any $0 < \theta < 1$, we can choose a constant $\varepsilon(n, \|B\|_\infty, \theta)$ so that for any $x \in M$ and any unit vector $u \in T_xM$, the pinching condition $(\mathrm{P}_{C_{\varepsilon(n,\|B\|_\infty,\theta)}})$ implies

$$\left| |dF_x(u)|^2 - 1 \right| \leq \theta.$$

For this, let us compute $dF_x(u)$. We have

$$dF_x(u) = \sqrt{\frac{n}{\lambda_1(M)}}\nabla^0_u\left(\frac{X}{|X|}\right)\Big|_x = \sqrt{\frac{n}{\lambda_1(M)}}u\left(\frac{1}{|X|}\right)X + \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}\nabla^0_u X =$$

$$
\begin{aligned}
&= -\frac{1}{2}\sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|^3}u(|X|^2)X + \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}u \\
&= -\sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|^3}\langle u, X\rangle X + \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}u \\
&= \sqrt{\frac{n}{\lambda_1(M)}}\frac{1}{|X|}\left(-\frac{\langle u, X\rangle}{|X|^2}X + u\right).
\end{aligned}
$$

By a straightforward computation, we obtain

$$
\begin{aligned}
\left||dF_x(u)|^2 - 1\right| &= \left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2}\left(1 - \frac{\langle u, X\rangle^2}{|X|^2}\right) - 1\right| \\
&\leq \left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2} - 1\right| + \frac{n}{\lambda_1(M)}\frac{1}{|X|^4}\langle u, X\rangle^2.
\end{aligned}
\tag{9}
$$

Now

$$
\begin{aligned}
\left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2} - 1\right| &= \frac{1}{|X|^2}\left|\frac{n}{\lambda_1(M)} - |X|^2\right| \\
&\leq \varepsilon\frac{\left|\sqrt{\frac{n}{\lambda_1(M)}} + |X|\right|}{|X|^2} \leq \varepsilon\frac{2\sqrt{\frac{n}{\lambda_1(M)}} + \varepsilon}{\left(\sqrt{\frac{n}{\lambda_1(M)}} - \varepsilon\right)^2}.
\end{aligned}
$$

Let us recall that $\frac{n}{d_n} \leq \lambda_1(M) \leq \|B\|_\infty^2$ (see (4) for the first inequality). Since we assume $\varepsilon < \frac{1}{2}\sqrt{\frac{n}{\|B\|_\infty}}$, the right-hand side is bounded above by a constant depending only on $n$ and $\|B\|_\infty$ and we have

$$
\left|\frac{n}{\lambda_1(M)}\frac{1}{|X|^2} - 1\right| \leq \varepsilon\gamma(n, \|B\|_\infty). \tag{10}
$$

On the other hand, since $C_\varepsilon(n, \|B\|_\infty) \to 0$ when $\varepsilon \to 0$, there exists $\varepsilon(n, \|B\|_\infty, \eta)$ so that $C_{\varepsilon(n,\|B\|_\infty,\eta)} \leq K_\eta(n, \|B\|_\infty)$ (where $K_\eta$ is the constant of the lemma) and then by Lemma 4.1, $\|\psi\|_\infty^2 \leq \eta^2$. Thus there exists a constant $\delta$ depending only on $n$ and $\|B\|_\infty$ so that

$$
\frac{n}{\lambda_1(M)}\frac{1}{|X|^4}\langle u, X\rangle^2 \leq \frac{n}{\lambda_1(M)}\frac{1}{|X|^4}\|\psi\|_\infty^2 \leq \eta^2\delta(n, \|B\|_\infty), \tag{11}
$$

and from (9), (10) and (11) we deduce that the condition $(\mathrm{P}_{C_{\varepsilon(n,\|B\|_\infty,\eta)}})$ implies

$$
\left||dF_x(u)|^2 - 1\right| \leq \varepsilon\gamma(n, \|B\|_\infty) + \eta^2\delta(n, \|B\|_\infty).
$$

Now let us choose $\eta = \left(\frac{\theta}{2\delta}\right)^{1/2}$. Then we can assume that $\varepsilon(n, \|B\|_\infty, \eta)$ is small enough in order to have $\varepsilon(n, \|B\|_\infty, \eta)\gamma(n\|B\|_\infty) \le \frac{\theta}{2}$. In this case we have

$$\left| |dF_x(u)|^2 - 1 \right| \le \theta.$$

Now let us fix $\theta$, $0 < \theta < 1$. It follows that $F$ is a local diffeomorphism from $M$ to $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$. Since $S\left(O, \sqrt{\frac{n}{\lambda_1(M)}}\right)$ is simply connected for $n \ge 2$, $F$ is a diffeomorphism. □

## 5. Proof of the technical lemmas

The proofs of Lemmas 3.1 and 4.1 are providing from a result stated in the following proposition using a Nirenberg–Moser type of proof.

**Proposition 5.1.** *Let $(M^n, g)$ be a compact, connected and oriented $n$-dimensional Riemannian manifold without boundary isometrically immersed into the $n+1$-dimensional euclidean space $(\mathbb{R}^{n+1}, can)$. Let $\xi$ be a nonnegative continuous function so that $\xi^k$ is smooth for $k \ge 2$. Let $0 \le r < s \le 2$ so that*

$$\frac{1}{2}\Delta\xi^2\xi^{2k-2} \le \delta\omega + (A_1 + kA_2)\xi^{2k-r} + (B_1 + kB_2)\xi^{2k-s}$$

*where $\delta\omega$ is the codifferential of a 1-form and $A_1$, $A_2$, $B_1$, $B_2$ are nonnegative constants. Then for any $\eta > 0$, there exists a constant $L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)$ depending only on $n$, $A_1$, $A_2$, $B_1$, $B_2$, $\|H\|_\infty$ and $\eta$ so that if $\|\xi\|_\infty > \eta$ then*

$$\|\xi\|_\infty \le L(n, A_1, A_2, B_1, B_2, \|H\|_\infty, \eta)\|\xi\|_2.$$

*Moreover, $L$ is bounded when $\eta \to \infty$, and if $B_1 > 0$, $L \to \infty$ when $\|H\|_\infty \to \infty$ or $\eta \to 0$.*

This proposition will be proved at the end of the paper.

Before giving the proofs of Lemmas 3.1 and 4.1, we will show that under the pinching condition $(\mathrm{P}_C)$ with $C$ small enough, the $L_\infty$-norm of $X$ is bounded by a constant depending only on $n$ and $\|H\|_\infty$.

**Lemma 5.1.** *If we have the pinching condition $(\mathrm{P}_C)$ with $C < c_n$, then there exists $E(n, \|H\|_\infty)$ depending only on $n$ and $\|H\|_\infty$ so that $\|X\|_\infty \le E(n, \|H\|_\infty)$.*

*Proof.* From the relation (3), we have

$$\frac{1}{2}\Delta|X|^2|X|^{2k-2} \le n\|H\|_\infty|X|^{2k-1}.$$

Then applying Proposition 5.1 to the function $\xi = |X|$ with $r = 0$ and $s = 1$, we obtain that if $\|X\|_\infty > E$, then there exists a constant $L(n, \|H\|_\infty, E)$ depending only on $n$, $\|H\|_\infty$ and $E$ so that

$$\|X\|_\infty \le L(n, \|H\|_\infty, E)\|X\|_2,$$

and under the pinching condition ($\mathrm{P}_C$) with $C < c_n$ we have from Lemma 2.1 that

$$\|X\|_\infty \le L(n, \|H\|_\infty, E)d_n^{1/2}.$$

Now since $L$ is bounded when $E \to \infty$, we can choose $E = E(n, \|H\|_\infty)$ large enough so that

$$L(n, \|H\|_\infty, E)d_n^{1/2} < E.$$

In this case, we have $\|X\|_\infty \le E(n, \|H\|_\infty)$. □

*Proof of Lemma* 3.1. First we compute the Laplacian of the square of $\varphi^2$. We have

$$\begin{aligned}\Delta\varphi^2 &= \Delta\left(|X|^4 - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2}|X|^3 + \frac{n}{\lambda_1(M)}|X|^2\right)\\ &= -2|X|^2|d|X|^2|^2 + 2|X|^2\Delta|X|^2\\ &\quad - 2\left(\frac{n}{\lambda_1(M)}\right)^{1/2}\left(-\frac{3}{4}|X|^{-1}|d|X|^2|^2 + \frac{3}{2}|X|\Delta|X|^2\right) + \frac{n}{\lambda_1(M)}\Delta|X|^2.\end{aligned}$$

Now by a direct computation one gets $|d|X|^2|^2 \le 4|X|^2$. Moreover by the relation (3) we have $|\Delta|X|^2| \le 2n\|H\|_\infty|X| + n$. Then applying Lemmas 1.1 and 5.1 we get

$$\Delta\varphi^2 \le \alpha(n, \|H\|_\infty)$$

and

$$\frac{1}{2}\Delta\varphi^2\varphi^{2k-2} \le \alpha(n, \|H\|_\infty)\varphi^{2k-2}.$$

Now, we apply Proposition 5.1 with $r = 0$ and $s = 2$. Then if $\|\varphi\|_\infty > \eta$, there exists a constant $L(n, \|H\|_\infty)$ depending only on $n$ and $\|H\|_\infty$ so that

$$\|\varphi\|_\infty \le L\|\varphi\|_2.$$

From Lemma 2.5, if $C \le c_n$ and ($\mathrm{P}_C$) is true, we have $\|\varphi\|_2 \le D(n)\|\varphi\|_\infty^{3/4}C^{1/4}$. Therefore

$$\|\varphi\|_\infty \le (LD)^4C.$$

Consequently, if we choose $C = K_\eta = \inf\left(\frac{\eta}{(LD)^4}, c_n\right)$, then we obtain $\|\varphi\|_\infty \le \eta$. □

*Proof of Lemma* 4.1. First we will prove that for any $C < c_n$, if $(\mathrm{P}_C)$ is true, then

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \le \delta\omega + (\alpha_1(n, \|B\|_\infty) + k\alpha_2(n, \|B\|_\infty))\,\psi^{2k-2} \tag{12}$$

where $\delta\omega$ is the codifferential of a 1-form $\omega$.

First observe that the gradient $\nabla^M|X|^2$ of $|X|^2$ satisfies $\nabla^M|X|^2 = 2X^T$. Then by the Bochner formula we get

$$\begin{aligned}\frac{1}{2}\Delta|X^T|^2 &= \frac{1}{4}\left\langle \Delta d|X|^2, d|X|^2\right\rangle - \frac{1}{4}|\nabla d|X|^2|^2 - \frac{1}{4}\operatorname{Ric}(\nabla^M|X|^2, \nabla^M|X|^2)\\ &\le \frac{1}{4}\left\langle d\Delta|X|^2, d|X|^2\right\rangle - \frac{1}{4}\operatorname{Ric}(\nabla^M|X|^2, \nabla^M|X|^2)\end{aligned}$$

and by the Gauss formula we obtain

$$\begin{aligned}\frac{1}{2}\Delta|X^T|^2 &\le \frac{1}{4}\left\langle d\Delta|X|^2, d|X|^2\right\rangle - \frac{1}{4}nH\left\langle B\nabla^M|X|^2, \nabla^M|X|^2\right\rangle + \frac{1}{4}|B\nabla^M|X|^2|^2\\ &= \frac{1}{4}\left\langle d\Delta|X|^2, d|X|^2\right\rangle - nH\left\langle BX^T, X^T\right\rangle + |BX^T|^2.\end{aligned}$$

By Lemma 5.1 we know that $\|X\|_\infty \le E(n, \|B\|_\infty)$ (the dependance in $\|H\|_\infty$ can be replaced by $\|B\|_\infty$). Then it follows that

$$\frac{1}{2}(\Delta\psi^2)\psi^{2k-2} \le \frac{1}{4}\left\langle d\Delta|X|^2, d|X|^2\right\rangle\psi^{2k-2} + \alpha'(n, \|B\|_\infty)\psi^{2k-2}. \tag{13}$$

Now, let us compute the term $\left\langle d\Delta|X|^2, d|X|^2\right\rangle\psi^{2k-2}$. We have

$$\begin{aligned}&\left\langle d\Delta|X|^2, d|X|^2\right\rangle\psi^{2k-2}\\ &\qquad= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - (2k-2)\Delta|X|^2\left\langle d|X|^2, d\psi\right\rangle\psi^{2k-3}\\ &\qquad= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - 2(2k-2)\Delta|X|^2\left\langle X^T, \nabla^M\psi\right\rangle\psi^{2k-3}\end{aligned}$$

where $\omega = -\Delta|X|^2\psi^{2k-2}d|X|^2$. Now,

$$e_i(\psi) = \frac{e_i|X^T|^2}{2|X^T|} = \frac{e_i|X|^2 - e_i\left\langle X, \nu\right\rangle^2}{2|X^T|} = \frac{\langle e_i, X\rangle - B_{ij}\left\langle X, e_j\right\rangle\langle X, \nu\rangle}{|X^T|}.$$

Then

$$\begin{aligned}\left\langle d\Delta|X|^2, d|X|^2\right\rangle\psi^{2k-2} &= \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} - 2(2k-2)\Delta|X|^2|X^T|\psi^{2k-3}\\ &\qquad + 2(2k-2)\Delta|X|^2\frac{\left\langle BX^T, X^T\right\rangle}{|X^T|}\langle X, \nu\rangle\,\psi^{2k-3}\\ &\le \delta\omega + (\Delta|X|^2)^2\psi^{2k-2} + 2(2k-2)|\Delta|X|^2|\psi^{2k-2}\\ &\qquad + 2(2k-2)|\Delta|X|^2|\,|B|\,|X|\psi^{2k-2}.\end{aligned}$$

Now by relation (3) and Lemma 5.1 we have

$$\left\langle d\Delta|X|^2, d|X|^2\right\rangle \psi^{2k-2} \leq \delta\omega + \left(\alpha_1''(n, \|B\|_\infty) + k\alpha_2''(n, \|B\|_\infty)\right) \psi^{2k-2}.$$

Inserting this in (13), we obtain the desired inequality (12).

Now applying again Proposition 5.1, we get that there exists $L(n, \|B\|_\infty, \eta)$ so that if $\|\psi\|_\infty > \eta$ then

$$\|\psi\|_\infty \leq L\|\psi\|_2.$$

From Lemma 2.2 we deduce that if the pinching condition ($\mathrm{P}_C$) holds then $\|\psi\|_2 \leq A(n)^{1/2}C^{1/2}$. Then taking $C = K_\eta = \inf\left(\frac{\eta}{LA^{1/2}}, c_n\right)$, then $\|\psi\|_\infty \leq \eta$. □

*Proof of Lemma* 3.2. The idea of the proof consists in foliating the region $B(O, R+\eta)\backslash B(O, R-\eta)$ with hypersurfaces of large mean curvature and to show that one of these hypersurfaces is tangent to $\phi(M)$. This will imply that $\phi(M)$ has a large mean curvature at the contact point.

Consider $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}e$. Let $a$, $L > l > 0$ and

$$\begin{aligned} \Phi_{L,l,a} \colon \mathbb{S}^{n-1} \times \mathbb{S}^1 &\longrightarrow \mathbb{R}^{n+1} \\ (\xi, \theta) &\longmapsto L\xi - l\cos\theta\xi + l\sin\theta e + ae. \end{aligned}$$

Then $\Phi_{L,l,a}$ is a family of embeddings from $\mathbb{S}^{n-1} \times \mathbb{S}^1$ in $\mathbb{R}^{n+1}$. If we orient the family of hypersurfaces $\Phi_{L,l,a}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$ by the unit outward normal vector field, a straightforward computation shows that the mean curvature $H(\theta)$ depends only on $\theta$ and we have

$$H(\theta) = \frac{1}{n}\left(\frac{1}{l} - \frac{(n-1)\cos\theta}{L - l\cos\theta}\right) \geq \frac{1}{n}\left(\frac{1}{l} - \frac{n-1}{L-l}\right). \tag{14}$$

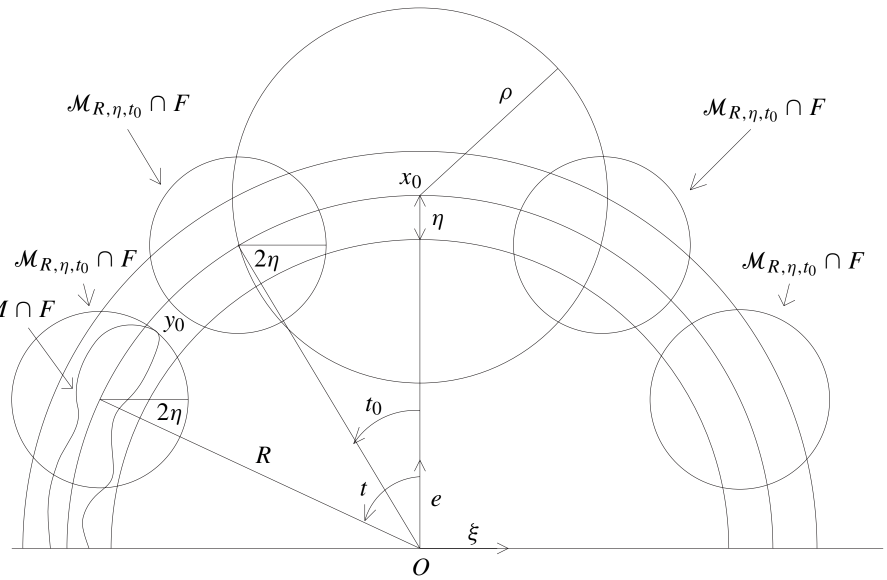
Now, let us consider the hypotheses of the lemma and for $t_0 = 2\arcsin\left(\frac{\rho}{2R}\right) \leq t \leq \frac{\pi}{2}$, put $L = R\sin t$, $l = 2\eta$ and $a = R\cos t$. Then $L > l$ and we can consider for $t_0 \leq t \leq \frac{\pi}{2}$ the family $\mathcal{M}_{R,\eta,t}$ of hypersurfaces defined by $\mathcal{M}_{R,\eta,t} = \Phi_{R\sin t, 2\eta, R\cos t}(\mathbb{S}^{n-1} \times \mathbb{S}^1)$.

From the relation (14), the mean curvature $H_{R,\eta,t}$ of $\mathcal{M}_{R,\eta,t}$ satisfies

$$\begin{aligned} H_{R,\eta,t} &\geq \frac{1}{n}\left(\frac{1}{2\eta} - \frac{n-1}{R\sin t - 2\eta}\right) \geq \frac{1}{n}\left(\frac{1}{2\eta} - \frac{n-1}{R\sin t_0 - 2\eta}\right) \\ &\geq \frac{1}{n}\left(\frac{1}{2\eta} - \frac{n-1}{R\sin(t_0/2) - 2\eta}\right) = \frac{1}{n}\left(\frac{1}{2\eta} - \frac{n-1}{\frac{\rho}{2} - 2\eta}\right) = \frac{1}{4n\eta} \end{aligned}$$

where we have used in this last equality the fact that $\rho = 4(2n-1)\eta$.

Since there exists a point $p \in M$ so that $\langle X(p), e\rangle > 0$, we can find $t \in [t_0, \pi/2]$ and a point $y_0 \in M$ which is a contact point with $\mathcal{M}_{R,\eta,t}$. Therefore $|H(y_0)| \geq \frac{1}{4n\eta}$.




$F$ is the vector space spanned by $e$ and $\xi$.

□

*Proof of Proposition* 5.1. Integrating by parts we have

$$\int_M \frac{1}{2}\Delta\xi^2\xi^{2k-2}\,dv = \frac{1}{2}\int_M \left\langle d\xi^2, d\xi^{2k-2}\right\rangle dv = 2\left(\frac{k-1}{k^2}\right)\int_M |d\xi^k|^2\,dv$$
$$\le (A_1+kA_2)\int_M \xi^{2k-r}\,dv + (B_1+kB_2)\int_M \xi^{2k-s}\,dv.$$

Now, given a smooth function $f$ and applying the Sobolev inequality (5) to $f^2$, we get

$$\left(\int_M f^{\frac{2n}{n-1}}\,dv\right)^{1-(1/n)} \le K(n)\int_M \left(2|f||df| + |H|f^2\right)dv$$
$$\le 2K(n)\left(\int_M f^2\,dv\right)^{1/2}\left(\int_M |df|^2\,dv\right)^{1/2} + K(n)\|H\|_\infty\int_M f^2\,dv$$
$$= K(n)\left(\int_M f^2\,dv\right)^{1/2}\left(2\left(\int_M |df|^2\,dv\right)^{1/2} + \|H\|_\infty\left(\int_M f^2\,dv\right)^{1/2}\right)$$

where in the second inequality, we have used the Hölder inequality. Using it again, by assuming that $V(M) = 1$, we have

$$\left(\int_M f^2\,dv\right)^{1/2} \le \left(\int_M f^{\frac{2n}{n-1}}\,dv\right)^{\frac{n-1}{2n}}.$$

And finally, we obtain

$$\|f\|_{\frac{2n}{n-1}} \le K(n)(2\|df\|_2 + \|H\|_\infty \|f\|_2).$$

For $k \ge 2$, $\xi^k$ is smooth and we apply the above inequality to $f = \xi^k$. Then we get

$$\begin{aligned}
\|\xi\|^k_{\frac{2kn}{n-1}} &\le K(n)\left[2\left(\int_M |d\xi^k|^2\,dv\right)^{1/2} + \|H\|_\infty \left(\int_M \xi^{2k}\,dv\right)^{1/2}\right] \\
&\le K(n)\Bigg[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\Bigg((A_1 + kA_2)\int_M \xi^{2k-r}\,dv \\
&\qquad + (B_1 + kB_2)\int_M \xi^{2k-s}\,dv\Bigg)^{1/2} + \|H\|_\infty\left(\int_M \xi^{2k}\,dv\right)^{1/2}\Bigg] \\
&\le K(n)\Bigg[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\big((A_1 + kA_2)\|\xi\|_\infty^{2-r} \\
&\qquad + (B_1 + kB_2)\|\xi\|_\infty^{2-s}\big)^{1/2}\|\xi\|^{k-1}_{2k-2} + \|H\|_\infty\|\xi\|_\infty\|\xi\|^{k-1}_{2k-2}\Bigg] \\
&\le K(n)\Bigg[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\left(\frac{A_1 + kA_2}{\|\xi\|^r_\infty} + \frac{B_1 + kB_2}{\|\xi\|^s_\infty}\right)^{1/2} \\
&\qquad + \|H\|_\infty\Bigg]\|\xi\|_\infty\|\xi\|^{k-1}_{2k-2} \\
&\le K(n)\Bigg[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}\left(\frac{A_1^{1/2} + k^{1/2}A_2^{1/2}}{\|\xi\|^{r/2}_\infty} + \frac{B_1^{1/2} + k^{1/2}B_2^{1/2}}{\|\xi\|^{s/2}_\infty}\right) \\
&\qquad + \|H\|_\infty\Bigg]\|\xi\|_\infty\|\xi\|^{k-1}_{2k-2}.
\end{aligned}$$

If we assume that $\|\xi\|_\infty > \eta$, the last inequality becomes

$$\begin{aligned}
\|\xi\|^k_{\frac{2kn}{n-1}} \le K(n)\Bigg[2\left(\frac{k^2}{2(k-1)}\right)^{1/2}&\left(\frac{A_1^{1/2} + k^{1/2}A_2^{1/2}}{\eta^{r/2}} + \frac{B_1^{1/2} + k^{1/2}B_2^{1/2}}{\eta^{s/2}}\right) \\
&+ \|H\|_\infty\Bigg]\|\xi\|_\infty\|\xi\|^{k-1}_{2k-2}
\end{aligned}$$

$$= \left[ (K_1 + k^{1/2} K_2) \left( \frac{k^2}{k-1} \right)^{1/2} + K' \right] \|\xi\|_\infty \|\xi\|_{2k-2}^{k-1}.$$

Now let $q = \frac{n}{n-1} > 1$ and for $i \geq 0$ let $k = q^i + 1 \geq 2$. Then

$$\begin{aligned} \|\xi\|_{2(q^{i+1}+q)} &\leq \left( \left( K_1 + (q^i+1)^{1/2} K_2 \right) \left( \frac{q^i+1}{q^{i/2}} \right) + K'' \right)^{\frac{1}{q^i+1}} \|\xi\|_\infty^{\frac{1}{q^i+1}} \|\xi\|_{2q^i}^{1-\frac{1}{q^i+1}} \\ &\leq \left( \tilde{K} q^i \right)^{\frac{1}{q^i+1}} \|\xi\|_\infty^{\frac{1}{q^i+1}} \|\xi\|_{2q^i}^{1-\frac{1}{q^i+1}} \end{aligned}$$

where $\tilde{K} = 2K_1 + 2^{3/2} K_2 + K'$. We see that $\tilde{K}$ has a finite limit when $\eta \to \infty$ and if $B_1 > 0$, $\tilde{K} \to \infty$ when $\|H\|_\infty \to \infty$ or $\eta \to 0$. Moreover the Hölder inequality gives

$$\|\xi\|_{2q^{i+1}} \leq \|\xi\|_{2(q^{i+1}+q)}$$

which implies

$$\|\xi\|_{2q^{i+1}} \leq \left( \tilde{K} q^i \right)^{\frac{1}{q^i+1}} \|\xi\|_\infty^{\frac{1}{q^i+1}} \|\xi\|_{2q^i}^{1-\frac{1}{q^i+1}}.$$

Now, by iterating from 0 to $i$, we get

$$\begin{aligned} &\|\xi\|_{2q^{i+1}} \\ &\leq \tilde{K}^{\left(1-\prod_{k=i-j}^{i}\left(1-\frac{1}{q^k+1}\right)\right)} q^{\sum_{k=i-j}^{i} \frac{k}{q^k+1}} \|\xi\|_\infty^{\left(1-\prod_{k=i-j}^{i}\left(1-\frac{1}{q^k+1}\right)\right)} \|\xi\|_{2q^{i-j}}^{\prod_{k=i-j}^{i}\left(1-\frac{1}{q^k+1}\right)} \\ &\leq \tilde{K}^{\left(1-\prod_{k=0}^{i}\left(1-\frac{1}{q^k+1}\right)\right)} q^{\sum_{k=0}^{i} \frac{k}{q^k+1}} \|\xi\|_\infty^{\left(1-\prod_{k=0}^{i}\left(1-\frac{1}{q^k+1}\right)\right)} \|\xi\|_{2}^{\prod_{k=0}^{i}\left(1-\frac{1}{q^k+1}\right)}. \end{aligned}$$

Let $\alpha = \sum_{k=0}^{\infty} \frac{k}{q^k+1}$ and $\beta = \prod_{k=0}^{\infty} \left(1 - \frac{1}{q^k+1}\right) = \prod_{k=0}^{\infty} \left( \frac{1}{1+(1/q)^k} \right)$. Then

$$\|\xi\|_\infty \leq \tilde{K}^{1-\beta} q^\alpha \|\xi\|_\infty^{(1-\beta)} \|\xi\|_2^\beta,$$

and finally

$$\|\xi\|_\infty \leq L \|\xi\|_2$$

where $L = \tilde{K}^{\frac{1-\beta}{\beta}} q^{\alpha/\beta}$ is a constant depending only on $n$, $A_1$, $A_2$, $B_1$, $B_2$, $\|H\|_\infty$ and $\eta$. From classical methods we show that $\beta \in [e^{-n}, e^{-n/2}]$. In particular, $0 < \beta < 1$ and we deduce that $L$ is bounded when $\eta \to \infty$ and $L \to \infty$ when $\|H\|_\infty \to \infty$ or $\eta \to 0$ with $B_1 > 0$. □

**Remark.** In [12] and [13] Shihohama and Xu have proved that if $(M^n, g)$ is a compact $n$-dimensional Riemannian manifold without boundary isometrically immersed in

$\mathbb{R}^{n+1}$ and if $\int_M \left(|B|^2 - n|H|^2\right) < D_n$ where $D_n$ is a constant depending on $n$, then all Betti numbers are zero. For $n = 2$, $D_2 = 4\pi$, and it follows that if

$$\int_M |B|^2\, dv - 4\pi < \lambda_1(M)V(M)$$

then we deduce from the Reilly inequality $\lambda_1(M)V(M) \leq 2\int_M H^2\, dv$ that $\int_M (|B|^2 - 2|H|^2)\, dv < 4\pi$ and by the result of Shihohama and Xu $M$ is diffeomorphic to $\mathbb{S}^2$.

## References


[1] D. Burago, V. A. Zalgaller, *Geometric Inequalities*. Grundlehren Math. Wiss. 285, Springer-Verlag, Berlin 1988. Zbl 0633.53002 MR 0936419

[2] T. H. Colding, Shape of manifolds with positive Ricci curvature. *Invent. Math.* **124** (1996), 175–191. Zbl 0871.53027 MR 1369414

[3] T. H. Colding, Large manifolds with positive Ricci curvature. *Invent. Math.* **124** (1996), 193–214. Zbl 0871.53028 MR 1369415

[4] C. B. Croke, An eigenvalue Pinching Theorem. *Invent. Math.* **68** (1982), 253–256. Zbl 0505.53018 MR 0666162

[5] J. H. Eschenburg, Diameter, volume and topology for positive Ricci curvature. *J. Differential Geom.* **33** (1991), 743–747. Zbl 0724.53026 MR 1100210

[6] D. Hoffman, J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds. *Comm. Pure Appl. Math.* **27** (1974), 715–727. Zbl 0295.53025 MR 0365424

[7] D. Hoffman, J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds. Erratum *Comm. Pure Appl. Math.* **28** (1975), 765–766. Zbl 0327.53035 MR 0397625

[8] Ilias S., Un nouveau résultat de pincement de la première valeur propre du Laplacien et conjecture du diamètre pincé. *Ann. Inst. Fourier* **43** (3) (1993), 843–863. Zbl 0783.53024 MR 1242618

[9] A. Lichnerowicz, *Géométrie des groupes de transformations*. Travaux et Recherches Mathématiques III, Dunod, Paris 1958. Zbl 0096.16001 MR 0124009

[10] P. Petersen, On eigenvalue pinching in positive Ricci curvature. *Invent. Math.* **138** (1999), 1–21. Zbl 0988.53011 MR 1714334

[11] R. Reilly, On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space. *Comment. Math. Helv.* **52** (1977), 525–533. Zbl 0382.53038 MR 0482597

[12] K. Shiohama, H. Xu, Rigidity and sphere theorems for submanifolds. *Kyushu J. Math.* **48** (2) (1994), 291–306. Zbl 0826.53045 MR 1294532

[13] K. Shiohama, H. Xu, Rigidity and sphere theorems for submanifolds II. *Kyushu J. Math.* **54** (1) (2000), 103–109. Zbl 1005.53028 MR 1762795

[14] T. Takahashi, Minimal immersions of Riemannian manifolds. *J. Math. Soc. Japan* **18** (1966), 380–385. Zbl 0145.18601 MR 0198393

[15] J. Y. Wu, A diameter pinching sphere theorem for positive Ricci curvature. *Proc. Amer. Math. Soc.* **107** (3) (1989), 797–802. Zbl 0676.53045 MR 0984822

[16] H. W. Xu, Some results on geometry of Riemannian submanifolds. Ph.D. Dissertation, Fudan University, 1990.

Received October 21, 2003; revised November 4, 2005

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