Kähler–Einstein metrics on orbifolds and Einstein metrics on spheres

Alessandro Ghigi and János Kollár

Abstract. A construction of Kähler–Einstein metrics using Galois coverings, studied by Arezzo–Ghigi–Pirola, is generalized to orbifolds. By applying it to certain orbifold covers of $\mathbb{CP}^n$ which are trivial set theoretically, one obtains new Einstein metrics on odd-dimensional spheres. The method also gives Kähler–Einstein metrics on degree 2 Del Pezzo surfaces with A_1- or A_2-singularities.

1. Introduction

The aim of this paper is to explain how the methods of Arezzo, Ghigi, and Pirola [1] can be applied to construct Kähler–Einstein metrics on compact complex orbifolds with positive first Chern class, and then use the approach of Boyer, Galicki, and Kollár [10] to obtain new Einstein metrics on odd dimensional spheres.

The somewhat unusual aspect is that we work with orbifolds $\mathcal{X}$ that admit a map $\pi : \mathcal{X} \to \mathbb{P}^n$ which is the *identity* map set theoretically. Nonetheless, in the orbifold category π is a nontrivial Galois cover, although with trivial Galois group.

The existence of Kähler–Einstein metrics on compact complex manifolds with positive first Chern class is still a difficult problem. For surfaces and toric manifolds a complete solution is known, due respectively to Tian [28] and Wang–Zhu [31]. Apart from these cases, there are two large classes of examples. The simplest are homogeneous spaces, for instance $\mathbb{P}^n$, quadrics, Grassmannians. In all these cases, the first Chern class is large, meaning for instance, that it is a large multiple of a generator of $H_2(X, \mathbb{Z})$. The opposite case, when the first Chern class is a small multiple of a generator of $H_2(X, \mathbb{Z})$ is also understood in many instances; see [8] for a good overview.

A blending of these two approaches was developed in Arezzo, Ghigi, and Pirola [1] to yield Kähler–Einstein metrics on certain manifolds X which can be realized

as Galois covers of another manifold Y with a Kähler–Einstein metric. Since the method relies on finite group actions, it is most successful when symmetries form a natural part of the complex structure, for instance for double covers of $\mathbb{P}^n$.

A construction of Einstein metrics on odd dimensional spheres was studied in Boyer, Galicki, and Kollár [10]. The idea is that the quotient of an odd dimensional sphere by a circle action is frequently a complex orbifold, and a result of Kobayashi [18] allows one to lift a Kähler–Einstein orbifold metric from the quotient to an Einstein metric on the sphere.

A frequently occurring case, studied by Orlik and Wagreich [24] and Boyer, Galicki, and Kollár [10], appears when the quotient S^{2n+1}/S^1 is $\mathbb{P}^n$ as a manifold, and the orbifold structure is given by a $\mathbb{Q}$-divisor

$$\Delta = \sum_{i=0}^{n+1}\left(1 - \tfrac{1}{m_i}\right)D_i,$$

where

$$D_i = \{z_i = 0\} \quad \text{for } i = 0, \ldots, n, \quad D_{n+1} = \{z_0 + \cdots + z_n = 0\},$$

and the $m_0, \ldots, m_{n+1}$ are pairwise relatively prime ramification indices. (See Section 4 for precise definitions.) The orbifold first Chern class is

$$c_1(\mathbb{P}^n, \Delta) = (n+1) - \sum_{i=0}^{n+1}\left(1 - \tfrac{1}{m_i}\right) = \sum_{i=0}^{n+1}\tfrac{1}{m_i} - 1,$$

where we have identified $H^2(\mathbb{P}^n, \mathbb{Q})$ with $\mathbb{Q}$. Thus $c_1(\mathbb{P}^n, \Delta)$ is positive iff

$$\sum_{i=0}^{n+1}\tfrac{1}{m_i} - 1 > 0. \tag{1}$$

The existence result [10, Theorem 34] shows that $(\mathbb{P}^n, \Delta)$ has an orbifold Kähler–Einstein metric if in addition the following inequality is also satisfied:

$$\sum_{i=0}^{n+1}\tfrac{1}{m_i} - 1 < \tfrac{n+1}{n}\min_i\left\{\tfrac{1}{m_i}\right\}. \tag{2}$$

This paper started with the observation that one can apply the method of [1] to the *identity* map $(\mathbb{P}^n, \Delta) \to \mathbb{P}^n$ which is a Galois cover (with trivial Galois group). On the other hand, over the affine chart $\mathbb{P}^n \setminus \{D_i \cup D_j\}$ the same map can be viewed as having cyclic Galois group of order $\prod_{k \neq i, j} m_k$. This approach improves the bound of [10] by a factor of n, and we obtain

Theorem 1. *Let $D_0, \ldots, D_{n+1} \subset \mathbb{P}^n$ be hyperplanes in general position, and let $m_0, \ldots, m_{n+1}$ be pairwise relatively prime natural numbers. Assume[1] that*

$$0 < \sum_{i=0}^{n+1} \frac{1}{m_i} - 1 < (n+1) \min_i \{ \frac{1}{m_i} \}. \tag{3}$$

Then there is an orbifold Kähler–Einstein metric on $\left(\mathbb{P}^n, \sum_{i=0}^{n+1} \left(1 - \frac{1}{m_i} \right) D_i \right)$.

Set $M = \prod_i m_i$ and $w_i = M/m_i$. As shown in [10] the intersection of the unit sphere with the Brieskorn–Pham singularity

$$L(m_0, \ldots, m_{n+1}) := S^{2n+3} \cap \left(\sum_{i=0}^{n+1} z_i^{m_i} = 0 \right) \subset \mathbb{C}^{n+2}$$

is homeomorphic to S^{2n+1} and a Kähler–Einstein metric on the corresponding projective orbifold

$$(X, \Delta_X) := \left(\left(\sum_{i=0}^{n+1} z_i^{m_i} = 0 \right), \sum_{i=0}^{n+1} (1 - \frac{1}{m_i})[z_i = 0] \right) \subset \mathbb{P}(w_0, \ldots, w_{n+1})$$

lifts to a positive Ricci curvature Einstein metric on $L(m_0, \ldots, m_{n+1})$. The weighted projective space $\mathbb{P}(w_0, \ldots, w_{n+1})$ is not well formed and it is isomorphic to the ordinary projective space $\mathbb{P}^{n+1}$ by the map

$$(z_0, \ldots, z_{n+1}) \mapsto (x_0 = z_0^{m_0}, \ldots, x_{n+1} = z_{n+1}^{m_{n+1}}).$$

Under this isomorphism we get that

$$(X, \Delta_X) \cong \left(\left(\sum_{i=0}^{n+1} x_i = 0 \right), \sum_{i=0}^{n+1} (1 - \frac{1}{m_i})[x_i = 0] \right) \subset \mathbb{P}^{n+1}.$$

By eliminating the variable x_{n+1} we get that

$$(X, \Delta_X) \cong (\mathbb{P}^n, \Delta).$$

The isometry class of the metric on the sphere determines the complex orbifold $(\mathbb{P}^n, \sum_{i=0}^{n+1} (1 - \frac{1}{m_i}) D_i)$, except possibly when $(\mathbb{P}^n, \sum_{i=0}^{n+1} (1 - \frac{1}{m_i}) D_i)$ has a holomorphic contact structure. The latter can happen only when n is odd; see [10, Lemma 17]

---

[1] Recent results of Gauntlett, Martelli, Sparks and Yau (Obstructions to the Existence of Sasaki–Einstein Metrics, *Comm. Math. Phys.* **273** (3) (2007), 803–827, see esp. (3.23)) show that (3) is also necessary for the existence of an orbifold Kähler–Einstein metric with positive Ricci curvature. Equivalently, if the m_i's are pairwise relatively prime, then there is a Sasaki–Einstein metric on the link of the singularity $z_0^{m_0} + \cdots + z_{n+1}^{m_{n+1}} = 0$, if and only if (3) holds.

for another necessary condition. (Note that $n + 2$ hyperplanes in general position do not have moduli, so the numbers $m_0, \ldots, m_{n+1}$ alone determine the complex orbifold.)

Even with the improved bounds, the equations (3) are not easy to satisfy. Still, as in Example 45, we get 12 new Einstein metrics on S^5 corresponding to the ramification indices

$$m_0 = 2, m_1 = 3, m_2 = 5, m_3 \in \{17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59\},$$

$\geq 10^3$ new Einstein metrics on S^7, $\geq 10^6$ new Einstein metrics on S^9, ….

The above construction can be varied in many ways. For instance, one can take more than $n + 2$ hyperplanes and quadrics. In all of these cases one gets an improvement by a factor roughly n compared to the bounds in [10], but this gives many new cases only for n large. (As shown by Orlik and Wagreich [24], taking higher degree hypersurfaces for the D_i yields Einstein metrics on various rational homology spheres.)

As another application, we consider singular degree 2 Del Pezzo surfaces. These are all double covers of $\mathbb{P}^2$ ramified along a quartic curve. In the smooth case the existence of Kähler–Einstein metrics was proved by Tian [28]. For singular surfaces we get the following.

Theorem 2. *Let S be a degree 2 Del Pezzo surface with only A_1- or A_2-singularities. Then S has an orbifold Kähler–Einstein metric.*

Remark 3. It is known that for a Fano manifold M the asymptotic Chow stability of $(M, -K_M)$ is a necessary condition for the existence of a Kähler–Einstein metric on M. (This idea goes back to Yau and was proved by Tian, Donaldson, Mabuchi and others in different settings. See e.g. [29] and [16].) This may also explain why our method breaks down for a degree 2 Del Pezzo with an A_n-singularity for $n \geq 3$. A plane quartic with an A_3-singularity is not stable as a plane curve (see [22], p. 80). Mapping a quartic C to the double cover $S \to \mathbb{P}^2$ branched over C yields an isomorphism between the space of plane quartics and the family of degree 2 Del Pezzo surfaces. If one chooses the same polarization, C is stable iff S is. Thus a degree 2 Del Pezzo surface S with an A_3-singularity (which is a double cover branched along a quartic with an A_3-singularity) is not stable. Although one should really consider asymptotic stability to get an actual obstruction, this suggests that S might not admit an orbifold Kähler–Einstein metric.

Remark 4. The orbifolds that we consider can not be viewed as limits of smooth manifolds. The obstruction is in fact completely local. Deforming an orbifold which is locally $\mathbb{C}^n/G$ needs deformations of $\mathbb{C}^n$ together with the G-action. Every such deformation is, however, locally trivial. Even in the case of A_n-singularities we have

orbifold rigidity. These are given by $X_0 = \{(x, y, z) : xy - z^{n+1} = 0\}$, which can be smoothed as $X_t = \{(x, y, z) : xy - z^{n+1} = t^r\}$ for any r. However the link of any 3-dimensional isolated hypersurface singularity is simply connected, so no contractible neighborhood of the origin in the threefold $X = \{(x, y, z, t) : xy - z^{n+1} = t^r\}$ can be written as a nontrivial quotient of anything. Therefore the family X_t can not be viewed as an *orbifold* deformation of X_0.

Anyone well versed in orbifolds, stacks and in the theory of Monge–Ampère equations should have no problem developing the theory of [1] in the orbifold setting. Nonetheless, since the theory of orbifolds has too many "well-known" but never proved theorems and not quite correct definitions and proofs, we felt that it makes sense to write down the arguments in some detail.

2. Analytic coverings

Let X and Y be reduced complex spaces. A map $\pi : X \to Y$ is called *finite* if it is proper and has finite fibres. Since X is locally compact a finite to one map is proper if and only if it is closed. Therefore a map is finite if and only if it is closed and has finite fibres. (By contrast note that $\pi : \mathbb{C} \setminus \{-1\} \to \{y^2 = x^3 + x^2\} \subset \mathbb{C}^2$ given by $t \mapsto (t^2 - 1, t^3 - t)$ is a closed map of algebraic varieties with finite fibers but π is not proper.)

The fundamental theorem on finite maps (see [17, p. 179]) states that when X and Y are irreducible any finite surjective map $\pi : X \to Y$ is an *analytic covering*. This means that there is a thin subset $T \subset Y$ such that

a) $\pi^{-1}(T)$ is thin in X, and

b) the restriction $\pi^{-1}(Y \setminus T) \to Y \setminus T$ is locally biholomorphic (étale).

Put $Y_0 = Y \setminus T$ and $X_0 = \pi^{-1}(Y_0)$. Then $\pi : X_0 \to Y_0$ is a topological covering. We call it a *regular subcover* of π.

We assume that our spaces are irreducible so that "analytic covering" and "finite holomorphic surjection" can be regarded as synonyms.

Another important fact is that an analytic covering $\pi : X \to Y$ with X and Y normal is an open map (see [17, p. 135]).

Let now $\pi : X \to Y$ be an analytic covering among connected *normal* complex spaces. Put $Y' = \{y \in Y_{\text{reg}} : \pi^{-1}(y) \subset X_{\text{reg}}\}$ and $X' = \pi^{-1}(Y')$. Then X' and Y' are open sets with complements of codimension at least 2. Now $\pi : X' \to Y'$ is a finite surjective map between complex manifolds. Pick local coordinates $z_1, \ldots, z_n$ on a neighbourhood U of a point in X' and let $w_1, \ldots, w_n$ be coordinates around its image in Y'. Let $w_i = \pi_i(z)$ be the local expression of π. The divisors locally

defined by the equation

$$\det\left(\frac{\partial \pi_i}{\partial z_j}\right) = 0$$

glue together yielding a well-defined divisor on X'. Since the complement of X' has codimension at least 2, the Remmert–Stein extension theorem (see e.g. [17, p. 181]) ensures that the topological closure of this divisor is a divisor in X, called the *ramification divisor* of π, and denoted by $R = R(\pi)$. It satisfies the Hurwitz formula $K_{Y'} = \pi^* K_{X'} + R$. Write $R = \sum_j r_j R_j$ with R_j distinct prime divisors on X'. The reduced divisor $R_{\text{red}} = \sum_j R_j$ is called the *ramification locus*. By the implicit function theorem $R_{\text{red}} \cap X'$ is the set of points $x \in X'$ such that π is not étale at x, that is the set of critical points of π. Since π is finite, the image $B = \pi(R_{\text{red}})$ is a divisor on Y, called the *branch divisor* of π.

Consider now the sets $Y'' = Y' \setminus \big(B_{\text{sing}} \cup \pi(R_{\text{sing}})\big)$ and $X'' = \pi^{-1}(Y'')$. Both are open and have complements of codimension at least 2 in X and Y respectively. We use this notation often in the sequel. When we want to stress the dependence on π, we write $X''(\pi)$ and $Y''(\pi)$. If $x \in X''$ either $x \notin R_{\text{red}}$ or x belongs to one and only one component R_j. In the first case we say that π is *unramified* at x, in the latter case we say that the *ramification order of π at x is $r_j + 1$*. The ramification order of π at x will be denoted by $\text{ord}_\pi(x)$. When π is unramified at x, we put $\text{ord}_\pi(x) = 1$. If $D \subset X$ is an irreducible divisor, then there is an open dense subset $D'' \subset D$ such that $\text{ord}_\pi(x)$ does not depend on $x \in D''$. This common value is denoted by $\text{ord}_\pi(D)$ and it is called the *ramification order of π along D*.

We use some basic properties of analytic coverings and maps between them (see, for instance, [6, Lemma 16.1]).

Lemma 5. *Let $x \in X''$. If π is unramified at x, then π is a local biholomorphism at x. If it has ramification order $m > 1$, let R_j be the component of R_{red} passing through x. Then there are local coordinates $z_1, \dots, z_n$ on X'' and $w_1, \dots, w_n$ on Y'' centred at x and $y = \pi(x)$ respectively, such that locally $R_j = \{z_1 = 0\}$, $B = \{w_1 = 0\}$ and $\pi(z_1, \dots, z_n) = (z_1^m, z_2, \dots, z_n)$.*

Since the complement of X'' has codimension 2, R_{red} is the closure of $R_{\text{red}} \cap X''$, that is the closure of the set of points where π has ramification order > 1.

The next lemma considers the problem of lifting in the simplest case. Denote by $D(r)$ the disc of radius r centred at the origin, by $D^*(r)$ the complement of $\{0\}$ in $D(r)$, and by $P(r_1, \dots, r_n)$ the polydisc centred at the origin with polyradius $(r_1, \dots, r_n)$.

Lemma 6. *Let $P_1 = P(r_1, \dots, r_n)$, $P_2 = P(\rho_1, \dots, \rho_n)$, $Q_1 = P(r_1^{m_1}, r_2, \dots, r_n)$, $Q_2 = P(\rho_1^{m_2}, \rho_2, \dots, \rho_n)$. Set $P_1^* = D^*(r_1) \times P(r_2, \dots, r_n)$ and similarly for P_2^*, Q_1^*, Q_2^*. Let $\pi_i \colon P_i \to Q_i$ be the maps $\pi_1(z_1, \dots, z_n) = (z_1^{m_1}, z_2, \dots, z_n)$,*

$\pi_2(z_1, .., z_n) = (z_1^{m_2}, z_2, \ldots, z_n)$. Let $f : Q_1 \to Q_2$ be a holomorphic map such that $f(Q_1^*) \subset Q_2^*$. If $m_2|m_1$ there are exactly m_2 liftings of f (that is maps $\tilde{f} : P_1 \to P_2$ such that $\pi_2 \tilde{f} = f\pi_1$). Any local lifting of f defined in a neighbourhood of some point $x \in P_1$ extends to one of these liftings defined on P_1.

Lemma 7. Let $\pi_1 : X_1 \to Y$ and $\pi_2 : X_2 \to Y$ be analytic coverings. For $U \subset X_1$ set

$$\mathfrak{F}(U) = \{holomorphic\ maps\ s : U \to X_2\ such\ that\ \pi_1 = \pi_2 \circ s\}.$$

Then $\mathfrak{F}$ is a Hausdorff sheaf (of sets) over X_1. Assume that for any $x_1 \in X_1''$, $x_2 \in X_2''$ with $\pi_1(x_1) = \pi_2(x_2)$,

$$\mathrm{ord}_{\pi_2}(x_2)|\,\mathrm{ord}_{\pi_1}(x_1).$$

Then the restriction of $\mathfrak{F}$ to $X_1'' \cap \pi_1^{-1}Y_2''(\pi_2)$ is a finite topological covering. In particular, if X_1'' is simply connected, then for every $x_1 \in X_1'' \cap \pi_1^{-1}Y_2''(\pi_2)$ and $x_2 \in X_2''$ such that $\pi_1(x_1) = \pi_2(x_2)$ there is an analytic map $f : X_1'' \to X_2$ such that $f(x_1) = x_2$ and $\pi_1 = \pi_2 \circ f$.

In fact, the above f extends to X_1 by the following immediate consequence of the Riemann Extension Theorem (see e.g. [17, p. 144])

Lemma 8. Let $\pi_1 : X_1 \to Y$ and $\pi_2 : X_2 \to Y$ be analytic coverings, X_1 normal and $T \subset X_1$ a thin set. Let $f^0 : X_1 \setminus T \to X_2$ be an analytic map such that $\pi_1 = \pi_2 \circ f^0$. Then f^0 extends to $f : X_1 \to X_2$ such that $\pi_1 = \pi_2 \circ f$.

3. The Galois group of coverings

Let $\pi : X \to Y$ be an analytic covering of normal complex spaces. Put $\mathrm{Gal}(\pi) = \{f \in \mathrm{Aut}(X) : \pi \circ f = \pi\}$. $\mathrm{Gal}(\pi)$ is a finite subgroup of $\mathrm{Aut}(X)$. In fact fix $x \in X'' \setminus R$, $y = \pi(x)$, and let V be a neighbourhood of y in Y such that $\pi^{-1}(V) = \bigsqcup_{i=1}^{k} U_i$ with $\pi : U_i \to V$ a biholomorphism and $x \in U_1$. Then the stabiliser $\mathrm{Gal}(\pi)_x$ is a subgroup of finite index in $\mathrm{Gal}(\pi)$. Moreover any $f \in \mathrm{Gal}(\pi)_x$ maps U_1 to itself. Since $\pi_{|U_1}$ is injective, the restriction of f to U_1 is the identity. By the connectedness of X, $f = \mathrm{id}_X$, so $\mathrm{Gal}(\pi)_x = \{1\}$ and $\mathrm{Gal}(\pi)$ is finite.

Since π is $\mathrm{Gal}(\pi)$-invariant, the $\mathrm{Gal}(\pi)$-orbit of $x \in X$ is contained in $\pi^{-1}(\pi(x))$. We say that an analytic covering $\pi : X \to Y$ is *Galois* if the converse holds, that is two points of X lie on the same fibre of π only if they belong to the same $\mathrm{Gal}(\pi)$-orbit.

Lemma 9. Let X and Y be normal complex spaces, $\pi : X \to Y$ an analytic covering and $Y_0 \subset Y$ an open subset with thin complement. Put $X_0 = \pi^{-1}(Y_0)$ and $\pi_0 = \pi_{|X_0} : X_0 \to Y_0$. Then the elements of $\mathrm{Gal}(\pi_0)$ extend to elements of $\mathrm{Gal}(\pi)$, and if π_0 is Galois, then π is Galois too.

Proof. The first part follows from Lemma 8. For the second part, let $x, x' \in X$ be such that $\pi(x) = \pi(x') = y$. If $y \in Y_0$ there is some $g \in \mathrm{Gal}(\pi_0)$ such that $g.x = x'$. Since we have just proved that $\mathrm{Gal}(\pi_0) = \mathrm{Gal}(\pi)$ the Galois condition is satisfied for these points. If instead $y \in Y \setminus Y_0$, let $\pi^{-1}(y) = \{x_1, \ldots, x_k\}$. Choose neighbourhoods U_i and V of x_i and y respectively such that $\pi^{-1}(V) = \bigsqcup_{i=1}^{k} U_i$. Assume $x = x_1 \in U_1$ and $x' = x_2 \in U_2$. Let $\{z_n\}$ be a sequence of points in $X_0 \cap U_1$ converging to x. Then $y_n = \pi(z_n)$ converge to y. Since π is open, $\pi(U_2) = V$. Therefore there are points $z'_n \in U_2 \cap X_0$ such that $\pi(z'_n) = y_n$. By the Galois condition on X_0, there are $g_n \in \mathrm{Gal}(\pi)$ such that $z'_n = g_n.z_n$. As $\mathrm{Gal}(\pi)$ is finite, we can extract a subsequence with $g_n \equiv g$. Since $\lim z'_n = x_2$ as $\pi^{-1}(y) \cap U_2 = \{x_2\}$, we get $x_2 = g.x_1$. $\qquad\square$

If $\pi \colon X \to Y$ is a Galois covering, then $\mathrm{Gal}(\pi)$ acts freely on any regular subcover X_0. Therefore if $x, x' \in X_0$ and $\pi(x) = \pi(x')$, then there is a unique $g \in \mathrm{Gal}(\pi)$ such that $g.x = x'$. In particular the cardinality of $\mathrm{Gal}(\pi)$ equals that of the generic fibre. This condition is also sufficient: π is Galois iff $|\mathrm{Gal}(\pi)|$ equals the cardinality of the general fibre iff $\mathrm{Gal}(\pi)$ is transitive on the general fibre.

For later reference we state the following simple lemma.

Lemma 10. *Let X, Y and Z be irreducible complex spaces, and let $f \colon X \to Z$, $g \colon Y \to Z$, $h \colon X \to Y$ be analytic coverings such that $gh = f$. If f is Galois, then h is Galois too.*

Proof. Thanks to Lemma 9 it is enough to consider the unramified case. Fix $x \in X$ and put $y = h(x)$, $z = f(x) = g(y)$. We need to show that $h_*\pi_1(X, x)$ is a normal subgroup of $\pi_1(Y, y)$. Since $g_* \colon \pi_1(Y, y) \to \pi_1(Z, z)$ is injective it is enough to check that $g_*h_*\pi_1(X, x)$ is a normal subgroup of $g_*\pi_1(Y, y)$. But f being Galois $f_*\pi_1(X, x) = g_*h_*\pi_1(X, x)$ is normal in $\pi_1(Z, z)$, hence a fortiori in $g_*\pi_1(Y, y)$. $\qquad\square$

For a general analytic covering $\pi \colon X \to Y$ it is not possible to assign multiplicity to the branching divisor in any reasonable way. In fact, different points in the preimage of a point $y \in B$ have different branching orders. A typical example is $X = \{z^3 - 3yz + 2x = 0\} \subset \mathbb{C}^3$ projecting on $\mathbb{C}^2_{x,y}$. Even shrinking the domain around the origin, one cannot separate the branches with different orders.

On the other hand, when the covering is Galois, for any $y \in Y''$ all points in $\pi^{-1}(y)$ have the same branching order. Therefore we can assign multiplicities to the branch divisor according to the following rule. Let $y \in Y'' \cap B$ and let x be any point in $\pi^{-1}(y)$. Then we define the multiplicity of B in y to be $1 - 1/\mathrm{ord}_\pi(x)$. We still denote by B the $\mathbb{Q}$-divisor given by the branching locus provided with these multiplicities. Note that with this convention $R = \pi^*B$, that is, the ramification divisor is the pull back of the branch divisor.

The branching divisor of a Galois cover can be described also in the following way. Given a prime divisor D in X, set $\Gamma(D) = \{\gamma \in \mathrm{Gal}(\pi) : D \subset \mathrm{Fix}(\gamma)\}$. For each prime divisor D the image $\pi(D)$ is a prime divisor in Y. The prime divisors for which $\Gamma(D) \neq 0$ are exactly the R_j. Set $B_j = \pi(R_j)$. In general different R_j's can have the same image. Assume that $\{B_i\}_{i \in I}$ is the set of all images of the R_j's (that is $B_i \neq B_k$ if $i \neq k$). Then

$$B(\pi) = \sum_{i \in I} \left(1 - \frac{1}{|\Gamma(R_i)|} \right) B_i. \tag{4}$$

4. Orbifolds as pairs

As in [10], we look at orbifolds as a particular type of log pairs. (X, Δ) is a log pair if X is a normal algebraic variety (or a normal complex space) and $\Delta = \sum_i d_i D_i$ is an effective $\mathbb{Q}$-divisor where the D_i are distinct, irreducible divisors and $d_i \in \mathbb{Q}$. The number d_i is called the multiplicity of Δ along D_i, it is denoted by $\mathrm{mult}_{D_i} \Delta$. We set $\mathrm{mult}_D \Delta = 0$ for every other irreducible divisor $D \neq D_i$ for all i.

Let $X''(\Delta)$ (or simply X'') be the complement of $X_{\mathrm{sing}} \cup \Delta_{\mathrm{sing}}$. For $x \in X''$ the multiplicity of Δ at x is a well defined rational number. For orbifolds, we need to consider only pairs (X, Δ) such that Δ has the form

$$\Delta = \sum_i \left(1 - \tfrac{1}{m_i} \right) D_i,$$

where the D_i are prime divisors and $m_i \in \mathbb{N}$. If (X, Δ) is such a pair then for any divisor $D \subset X$ we put

$$\mathrm{ord}_\Delta(D) = \frac{1}{1 - \mathrm{mult}_D \Delta}.$$

The assumption on the multiplicities of Δ amounts to saying that the order is always a nonnegative integer.

Definition 11. An *orbifold chart* on X compatible with Δ is a Galois covering $\varphi\colon U \to \varphi(U) \subset X$ such that

(1) U is a domain in $\mathbb{C}^n$ and $\varphi(U)$ is open in X;

(2) the branch locus of φ is $\Delta_{\mathrm{red}} \cap \varphi(U)$;

(3) for any $x \in U''(\varphi)$ such that $\varphi(x) \in D_i$, $\mathrm{ord}_\varphi(x) = m_i$.

Conditions (2) and (3) are equivalent to

$$B(\varphi) = \Delta \cap \varphi(U). \tag{5}$$

Definition 12. An orbifold is a log pair (X, Δ) such that X is covered by orbifold charts compatible with Δ.

(For a slightly more general approach, see [13, §14].)

Let X be a normal complex space and $\pi \colon U \to X$ a Galois cover where U is smooth. As discussed earlier, the branch divisor $B(\pi)$ of π is defined and we get a log pair $(X, B(\pi))$. If U is simply connected, (which we can always assume by shrinking U suitably) then by Lemma 7 the log pair $(X, B(\pi))$ determines $\pi \colon U \to X$ up to biholomorhisms. Thus we recover the classical definition of orbifolds (as in [4] for example).

Example 13. Let X be a complex manifold and $D = \sum_{i \in I} D_i$ a divisor with *local* normal crossing. By this we mean that for any point $x \in X$ there is a holomorphic coordinate system $(V, z_1, \ldots, z_n)$ such that $D \cap V = \{z \in V : z_1 \ldots z_k = 0\}$. If $D_i \cap V \neq \emptyset$ then $D_i \cap V$ is the union of some of the hypersurfaces $\{z_j = 0\}$. (D is said to be a divisor with *global* normal crossing if, in addition, each D_i is smooth.) For any $i \in I$, fix an integer $m_i > 1$ and put $\Delta = \sum_i (1 - 1/m_i) D_i$. We claim that (X, Δ) is an orbifold. Indeed, fix a coordinate system as above and put $m'_j = m_i$ if $\{z_j = 0\} \subset D_i \cap V$. Set

$$\varphi \colon U \to V, \quad \varphi(x_1, \ldots, x_n) = (x_1^{m'_1}, \ldots, x_k^{m'_k}, x_{k+1}, \ldots, x_n). \tag{6}$$

Then (U, φ) is an orbifold chart on X compatible with Δ and so (X, Δ) is an orbifold.

In the same way, the usual definition of orbifold map is equivalent to the following one.

Definition 14. For a finite holomorphic map $f \colon X \to Y$ the map $f \colon (X, \Delta_X) \to (Y, \Delta_Y)$ is an *orbifold map* if

$$\operatorname{ord}_{\Delta_Y}(f(D)) \mid \operatorname{ord}_{\Delta_X}(D) \cdot \operatorname{ord}_f D \tag{7}$$

for every divisor $D \subset X$.

An *orbifold automorphism* is an orbifold map that is invertible with inverse an orbifold map. The group of automorphisms of (X, Δ) is denoted by $\operatorname{Aut}(X, \Delta)$.

Definition 15. An orbifold Galois covering $f \colon (X, \Delta_X) \to (Y, \Delta_Y)$ is an orbifold map such that $f \colon X \to Y$ is a Galois analytic cover and $\operatorname{Gal}(f) \subset \operatorname{Aut}(X, \Delta_X)$.

By the *degree* of an orbifold Galois cover we mean its *degree* as an analytic cover.

Lemma 16. *Let $f \colon (X, \Delta_X) \to (Y, \Delta_Y)$ be an orbifold map. Then given $x \in X$ and $y = f(x) \in Y$ there are orbifold charts (U, φ) and (V, ψ) around x and y respectively such that f has a lifting $\tilde{f} \colon U \to V$. If, in addition, $f \colon X \to Y$ is an orbifold Galois covering then $\tilde{f} \colon U \to V$ is also a Galois covering.*

Proof. Choose the chart (U, φ) such that U is simply connected and $f\big(\varphi(U)\big) \subset \psi(V)$. If $D \subset U$ is any divisor then

$$\operatorname{ord}_{f \circ \varphi} D = \operatorname{ord}_f \varphi(D) \cdot \operatorname{ord}_\varphi D = \operatorname{ord}_f \varphi(D) \cdot \operatorname{ord}_{\Delta_X} \varphi(D).$$

By the definition of orbifold maps,

$$\operatorname{ord}_{\Delta_Y} (f \circ \varphi)(D) \mid \operatorname{ord}_f \varphi(D) \cdot \operatorname{ord}_{\Delta_X} \varphi(D),$$

hence we conclude that $\operatorname{ord}_{\Delta_Y} (f \circ \varphi)(D)$ divides $\operatorname{ord}_{f \circ \varphi} D$. Thus the assumption of Lemma 7 is satisfied and so $f \circ \varphi$ lifts to $\tilde{f} \colon U \to V$. Assume next that $f \colon (X, \Delta_X) \to (Y, \Delta_Y)$ is an orbifold Galois covering. By restricting U we can assume that for any $\sigma \in \operatorname{Gal}(f)$ either $\sigma \varphi(U) = \varphi(U)$ or $\sigma \varphi(U) \cap \varphi(U) = \emptyset$. Pick $u_1, u_2 \in U$ such that $f \varphi(u_1) = f \varphi(u_2)$. Then there is a Galois automorphism σ of f such that $\varphi(u_1) = \sigma(\varphi(u_2))$ and $\sigma \varphi(U) = \varphi(U)$. Since $\operatorname{Gal}(f) \subset \operatorname{Aut}(X, \Delta_X)$, $\operatorname{ord}_{\Delta_X} D = \operatorname{ord}_{\Delta_X} \sigma(D)$ for any divisor D. Hence applying Lemma 7 we conclude that $\sigma \colon \varphi(U) \to \varphi(U)$ lifts to a biholomorphism $\tilde{\sigma}$ of U such that $\tilde{\sigma}(u_2) = u_1$. Moreover $f \varphi \tilde{\sigma} = f \sigma \varphi = f \varphi$. Therefore $\tilde{\sigma} \in \operatorname{Gal}(f \varphi)$. This shows that in the commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ \tilde{f}\ } & V \\
\varphi \downarrow & & \downarrow \psi \\
\varphi(U) & \xrightarrow[\ f\ ]{} & \psi(V),
\end{array}
\tag{8}
$$

the composite $f \circ \varphi$ is Galois. But $f \varphi = \psi \tilde{f}$ and by Lemma 10 $\tilde{f}$ is a Galois cover. $\qquad \square$

Example 17. Let (X, Δ) be any orbifold, and let $(X, 0)$ denote the orbifold structure on X with trivial branching divisor. It is a nontrivial result that $(X, 0)$ is an orbifold, that is, X has quotient singularities (see [25]). (We use mainly the case when X is smooth, and then the orbifold charts of $(X, 0)$ are simply the manifold charts of X.)

The identity map $\operatorname{id}_X \colon (X, \Delta) \to (X, 0)$ is trivially an orbifold Galois covering. In fact it is both an orbifold map and a Galois analytic cover, and $\operatorname{Gal}(\operatorname{id}_X) = \{\operatorname{id}_X\} \subset \operatorname{Aut}(X, \Delta)$.

If $f \colon (X, \Delta) \to (Y, \Delta_Y)$ is an orbifold Galois covering the *orbifold ramification divisor* of f is defined as

$$R^{\mathrm{orb}}(\Delta_X, \Delta_Y, f) = R(f) + \Delta_X - f^* \Delta_Y.$$

With this definition the logarithmic ramification formula

$$K_X + \Delta_X = f^*(K_Y + \Delta_Y) + R^{\mathrm{orb}}(\Delta_X, \Delta_Y, f)$$

is automatically satisfied. To understand the geometric meaning of R^{orb} it is useful to look at the open set

$$X''(\Delta_X, \Delta_Y, f) = X_{\mathrm{reg}} \cap f^{-1}\big(Y_{\mathrm{reg}} \setminus (\Delta_Y \cup B(f))_{\mathrm{sing}}\big) \setminus (\Delta_X \cup R(f))_{\mathrm{sing}}.$$

This means that $x \in X'' = X''(\Delta_X, \Delta_Y, f)$ if (a) X is smooth at x, (b) Y is smooth at $y = f(x)$, (c) x belongs to at most one component D of $\Delta_X + R(f)$ and in this case x is a smooth point of D, (d) y belongs to at most one component D' of $\Delta_Y + B(f)$ and in this case it is a smooth point of D'. As usual the complement of this set has codimension 2. Let D be any smooth divisor passing through x and D' a smooth component passing through y. Assume first that f is unbranched at x and that locally $\Delta_X = (1 - 1/p)D$ and $\Delta_Y = (1 - 1/q)D'$. Then there is a local diagram like (8), with $p = \deg \varphi$ and $q = \deg \psi$. Put $k = \deg \tilde{f}$. Since f is unbranched we can assume that its restriction to $\varphi(U)$ is a biholomorphism onto $\psi(V)$. Therefore $p = qk$. If $p = 1$, then $q = k = 1$, and as expected $\mathrm{mult}_x R^{\mathrm{orb}} = 0$. If $p > 1$, then necessarily $D' = f(D)$ because of (7) and $f^*D' = D$, since f is étale. Therefore $R^{\mathrm{orb}} = (1/q - 1/p)D = (k-1)/p \cdot D$. If instead $\mathrm{ord}_x(f) = m > 1$, then again $D' = f(D)$, $R(f) = (m-1)D$, $f^*D' = mD$, $pm = qk$ and $R^{\mathrm{orb}} = (m/q - 1/p)D = (k-1)/p \cdot D$ once more. Roughly the orbifold ramification divisor is the ramification of the lifting $\tilde{f}$ divided the degree of the local chart φ.

Let (X, Δ) be an orbifold and $\Gamma \subset \mathrm{Aut}(X, \Delta)$ a finite subgroup. We want to define a quotient orbifold (Y, Δ'). By Cartan's lemma [11] $Y = X/\Gamma$ is a normal analytic space and the canonical projection $\pi : X \to Y$ is an analytic covering. The support of the branch divisor Δ' is defined to be $\pi(\Delta) \cup B(\pi)$, while the multiplicities are specified as follows. Let D be an irreducible component of $\pi(\Delta) \cup B(\pi)$. If D is a component of $\pi(\Delta)$ and not of $B(\pi)$, then we assign to D the multiplicity $\mathrm{mult}_x(\Delta)$, where x is any point in $X''(\Delta)$ such that $\pi(x) \in D$ is a smooth point of $\pi(\Delta) \cup B(\pi)$. If D is a component of $B(\pi)$ and not of $\pi(\Delta)$ then we assign to D the same multiplicity it has as a component of $B(\pi)$, that is $1 - 1/\mathrm{ord}_\pi(x)$ for any $x \in X''(\pi)$ such that $\pi(x) \in D$ is a smooth point of $\pi(\Delta) \cup B(\pi)$. Finally, if D is a common component of $\pi(\Delta)$ and $B(\pi)$ then we assign to it the multiplicity

$$1 - \frac{1 - \mathrm{mult}_x \Delta}{\mathrm{ord}_\pi(x)}$$

for any $x \in X''(\Delta) \cap X''(\pi)$ such that $\pi(x) \in D$ is a smooth point of $\pi(\Delta) \cup B(\pi)$.

Proposition 18. *Let (X, Δ) be an orbifold, and $\Gamma \subset \mathrm{Aut}(X, \Delta)$ a finite subgroup. Let $Y = X/\Gamma$ be the quotient analytic space, and Δ' the $\mathbb{Q}$-divisor defined above. Then (Y, Δ') is an orbifold and the canonical projection*

$$\pi : (X, \Delta_X) \longrightarrow (Y, \Delta') \tag{9}$$

is an orbifold Galois covering.

Proof. We need to show that Y is covered by orbifold charts compatible with Δ'. Fix $y \in Y$, $x \in \pi^{-1}(y)$ and let $\varphi \colon U \to \varphi(U)$ be an orbifold chart with $x \in \varphi(U)$. If the stabiliser Γ_x is trivial we can assume that $\gamma \varphi(U) \cap \varphi(U) = \emptyset$ for any $\gamma \neq e$. Then $\pi \colon \varphi(U) \to Y$ is a biholomorphism onto its image. Put $\psi = \pi \varphi \colon U \to Y$. We claim that ψ is an orbifold chart on Y compatible with Δ'. In fact ψ is Galois since π is a biholomorphism on $\varphi(U)$, and $\pi^* B(\psi) = B(\varphi) = \Delta \cap \varphi(U)$. On the other hand $B(\pi) \cap \psi(U) = \emptyset$ since $\pi \colon \varphi(U) \to \psi(U)$ is biholomorphic. Therefore on $\psi(U)$ the divisor Δ' coincides with $B(\psi)$. This proves that $\psi \colon U \to Y$ is an orbifold chart. If $\Gamma_x \neq \{e\}$ take a chart $\varphi \colon U \to \varphi(U) \subset X$ such that $\varphi(U)$ be a Γ_x-invariant neighbourhood of x. Lemma 16 ensures that also in this case $\psi = \pi \varphi \colon U \to \psi(U) \cong \varphi(U)/\Gamma_x$ is a Galois covering. It is easy to verify that $B(\psi) = \Delta'$ on $\psi(U)$. Finally that π is an orbifold Galois covering is clear: a lifting of $\pi \colon \varphi(U) \to \psi(U)$ is given by the identity map $U \to U$, so π is an orbifold map, while $\mathrm{Gal}(\pi) = \Gamma \subset \mathrm{Aut}(X, \Delta)$ by assumption. $\qquad\square$

5. Basic estimates for orbifold Kähler–Einstein metrics

In this section we collect the orbifold versions of some fundamental results due to Aubin, Bando–Mabuchi and Tian, that are needed in the existence criteria in the next section. Most of the proofs are the same as in the case of a manifold and we just give appropriate references. For the basic definitions of differential geometry on orbifolds see [4], [3], [9] and [7] . Some information on Sobolev spaces and Laplace operators on orbifolds can be found e.g. in [12].

Remark 19. Note that if X is a complex manifold and Δ is a non trivial branching divisor, then smoothness in the orbifold sense is rather different from ordinary smoothness. For example, $f(z) = |z|$ is not smooth in the ordinary sense, but it belongs to $C^\infty(\mathbb{C}, \Delta)$, where Δ is the divisor concentrated at the origin with multiplicity $1/2$. In fact the inclusions $C^\infty(X) \subsetneq C^\infty(X, \Delta)$ and $\wedge^k(X) \subsetneq \wedge^k(X, \Delta)$ are in general strict.

Definition 20. A *Fano orbifold* is a compact complex orbifold (X, Δ) such that $-(K_X + \Delta)$ is ample.

By the Baily–Kodaira imbedding theorem [3] this is equivalent to the fact that $c_1(X, \Delta)$ contains an orbifold Kähler metric.

The following is the orbifold analogue of Bonnet–Myers Theorem. It follows, for example, from the Bishop volume comparison Theorem for orbifolds, see [7, Proposition 20, Corollary 21].

Theorem 21. *Let X be an m-dimensional orbifold and g a Riemannian orbifold metric on X with $\mathrm{Ric}(g) \geq \varepsilon(m-1)g$ for some $\varepsilon > 0$. Then $\mathrm{diam}(X, g) \leq \pi/\sqrt{\varepsilon}$.*

Theorem 22 ([23, Theorem B]). *Let (X, g) be a Riemannian orbifold of dimension $m > 2$ with $\mathrm{Ric}(g) \geq -(m-1)\varepsilon^2 g$ for some $\varepsilon \geq 0$. Then there is a constant $C > 0$ depending only on m and $\varepsilon \cdot \mathrm{diam}(X, g)$ such that*

$$\|\nabla u\|_{L^2} \geq C \frac{\mathrm{vol}(X, g)^{1/m}}{\mathrm{diam}(X, g)} \|u\|_{L^{2m/(m-2)}} \tag{10}$$

for any $u \in W^{1,2}(X)$ with $\int_X u \, \mathrm{dvol}_g = 0$.

Combining the last two theorems one gets the following uniform Sobolev embedding.

Corollary 23. *Let (X, Δ) be an n-dimensional Fano orbifold. For any $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that for any metric ω in the class $2\pi c_1(X, \Delta)$ with $\mathrm{Ric}(\omega) \geq \varepsilon\omega$ and any $u \in W^{1,2}(X, \Delta)$*

$$\|u\|_{L^{2n/(n-1)}} \leq C \|u\|^2_{W^{1,2}}. \tag{11}$$

If (X, Δ) is a Kähler orbifold, $\omega \in \wedge^{1,1}(X, \Delta)$ is a closed smooth form and $\varphi \in C^\infty(X, \Delta)$, put $\omega_\varphi = \omega + i\,\partial\bar\partial\varphi$. We write $\omega_\varphi > 0$ to mean that it is a Kähler metric. If ω is such that

$$\langle[\omega]^n, [X]\rangle = \int_X \omega^n > 0$$

and $\varphi \in C^\infty(X, \Delta)$, put

$$I_\omega(\varphi) = \frac{1}{\langle[\omega]^n, [X]\rangle} \int \varphi(\omega^n - \omega_\varphi^n), \tag{12}$$

$$J_\omega(\varphi) = \int_0^1 \frac{I_\omega(s\varphi)}{s} \, ds, \tag{13}$$

$$F_\omega^0(\varphi) = J_\omega(\varphi) - \frac{1}{\langle[\omega]^n, [X]\rangle} \int \varphi\omega^n. \tag{14}$$

Lemma 24.

$$J_\omega(\varphi) = \frac{1}{\langle[\omega]^n, [X]\rangle} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_M i\,\partial\varphi \wedge \bar\partial\varphi \wedge \omega^k \wedge \omega_\varphi^{n-k-1}, \tag{15}$$

$$I_\omega(\varphi) - J_\omega(\varphi) = \frac{1}{\langle[\omega]^n, [X]\rangle} \sum_{k=0}^{n-1} \frac{n-k}{n+1} \int_X i\,\partial\varphi \wedge \bar\partial\varphi \wedge \omega^k \wedge \omega_\varphi^{n-k-1}. \tag{16}$$

If $\omega > 0$ and $\omega_\varphi > 0$, then $I_\omega(\varphi)$, $J_\omega(\varphi)$ and $I_\omega(\varphi) - J_\omega(\varphi)$ are nonnegative and vanish only if φ is constant. Moreover $J_\omega \le I_\omega \le (n+1)J_\omega$.

Proof. For (15) see [27, Lemma 2.2] or [1, Lemma 2.1]. For (16) expand $\omega^n - \omega_\varphi^n$. The last statements follow diagonalising simultaneously ω and $i\,\partial\bar\partial\varphi$.   □

Lemma 25. *If λ is a positive constant then*

$$F_{\lambda\omega}^0(\lambda\varphi) = \lambda F_\omega^0(\varphi). \tag{17}$$

Let ω_0 be a closed $(1,1)$-form with $\langle[\omega_0]^n, [X]\rangle > 0$. Given $\varphi_{01}, \varphi_{12} \in C^\infty(X, \Delta)$, put $\omega_1 = \omega_0 + i\,\partial\bar\partial\varphi_{01}$, $\varphi_{02} = \varphi_{01} + \varphi_{12}$. Then

$$F_{\omega_0}^0(\varphi_{02}) = F_{\omega_0}^0(\varphi_{01}) + F_{\omega_1}^0(\varphi_{12}). \tag{18}$$

(Same proof as in [30, pp. 60f].)

Lemma 26 ([30, p. 59]). *If φ_t is a differentiable family of smooth functions on (X, Δ) then*

$$\frac{d}{dt}\, J_\omega(\varphi_t) = \frac{1}{\langle[\omega]^n, [X]\rangle} \int\limits_X \dot\varphi_t(\omega^n - \omega_t^n), \tag{19}$$

$$\frac{d}{dt}\, F_\omega^0(\varphi_t) = -\frac{1}{\langle[\omega]^n, [X]\rangle} \int\limits_X \dot\varphi_t \omega_t^n. \tag{20}$$

Assume now that ω is a Kähler orbifold metric in the canonical class, that is $\omega \in 2\pi c_1(X, \Delta)$. Let $f = f(\omega) \in C^\infty(X, \Delta)$ be the unique function such that

$$\mathrm{Ric}(\omega) - \omega = i\,\partial\bar\partial f(\omega), \quad \int\limits_X e^{f(\omega)} = \int\limits_X \omega^n. \tag{21}$$

Put $V = \langle[\omega]^n, [X]\rangle = n!\,\mathrm{vol}(X)$ and define $A_\omega, F_\omega \colon C^\infty(X, \Delta) \to \mathbb{R}$ by

$$A_\omega(\varphi) = \log\left[\frac{1}{V} \int\limits_X e^{f(\omega)-\varphi}\omega^n\right], \quad F_\omega(\varphi) = F_\omega^0(\varphi) - A_\omega(\varphi). \tag{22}$$

Using the notation of Lemma 25 if ω_0, ω_1 and ω_2 are Kähler metrics, then

$$F_{\omega_0}(\varphi_{02}) = F_{\omega_0}(\varphi_{01}) + F_{\omega_1}(\varphi_{12}). \tag{23}$$

For $G \subset \mathrm{Aut}(X, \Delta)$ a subgroup of isometries of (X, Δ, ω) put

$$P_G(X, \Delta, \omega) = \{\varphi \in C^\infty(X, \Delta) : \omega_\varphi > 0, \text{ and } \varphi \text{ is } G\text{-invariant}\}. \tag{24}$$

If $G = \{1\}$ we simply write $P(X, \Delta, \omega)$.

In order to construct a Kähler–Einstein metric on (X, Δ) the continuity method is applied: fix a Kähler metric ω in the canonical class and consider the well-known equations

$$(\omega + i \partial \bar{\partial} \varphi_t)^n = e^{f - t\varphi_t} \omega^n \qquad (*)_t$$

for a smooth family of functions in $C^\infty(X, \Delta)$. Yau's estimates hold for orbifold metrics, and in particular the Calabi conjecture is true, which implies that $(*)_0$ admits a unique solution. Denote by Δ the negative definite $\bar{\partial}$-Laplacian on functions (that is $\Delta = -\bar{\partial}^* \bar{\partial}$) and by $-\lambda_j$ its eigenvalues.

Lemma 27 ([2, Theorem 4.20, p. 116]). *Let ω be a Kähler metric on the compact orbifold (X, Δ). If $\mathrm{Ric}(\omega) \geq \varepsilon > 0$, then $\lambda_1 \geq 1$.*

It follows that the times t for which $(*)_t$ is solvable form an open subset $S \subset [0, 1]$ and that solutions φ_t are smooth in t, see [30, pp. 63–66]. Given a C^0-estimate for the solutions, Yau's estimates ensure that S is closed, thus yielding the solution up to $t = 1$, which is a Kähler–Einstein metric.

Proposition 28. *Let φ_t be a solution to $(*)_t$ for $t \in [0, T_0)$. Then $I_\omega(\varphi_t) - J_\omega(\varphi_t)$ is nondecreasing and $F_\omega^0(\varphi_t) \leq 0$.*

Proof. Differentiating $(*)_t$ with respect to t one gets

$$(\Delta_t + t)\dot{\varphi}_t = -\varphi_t. \qquad (25)$$

Therefore

$$\frac{d}{dt}\big(I_\omega(\varphi_t) - J_\omega(\varphi_t)\big)$$
$$= \frac{1}{V} \int_X \varphi_t(\varphi_t + t\dot{\varphi}_t)\omega_t^n = (1 - t^2)\frac{1}{V} \int_X \varphi_t^2 \omega_t^n + \frac{1}{V} \int_X |\bar{\partial}\dot{\varphi}_t|^2 \omega_t.$$

This gives the first result. For the second use (20) and (25):

$$\frac{d}{dt} t F_\omega^0(\varphi_t) = F_\omega^0(\varphi_t) - \frac{t}{V} \int_X \dot{\varphi}_t \omega_t^n = F_\omega^0(\varphi_t) + \frac{1}{V} \int_X (\Delta_t \dot{\varphi}_t + \varphi_t)\omega_t^n = J_\omega(\varphi_t).$$

Since $J_\omega \geq 0$, the result follows. $\qquad \square$

The following estimates depend on the uniform Sobolev embedding (Lemma 23) and their proof uses Moser iteration.

Theorem 29 ([30, p. 67ff]). *If φ_t is a family of solutions to $(*)_t$ on the time interval $[0, T_0)$, then there is a constant $C = C(T_0) > 0$ such that for any $t < T_0$*

$$\|\varphi_t\|_\infty \leq C\big(1 + J_\omega(\varphi_t)\big), \tag{26}$$

$$0 \leq -\inf_X \varphi_t \leq C\left(\frac{1}{V}\int_X (-\varphi_t)\omega_t^n + C\right), \tag{27}$$

$$F_\omega(\varphi_t) \leq -A_\omega(\varphi_t) \leq C(1-t) \leq C. \tag{28}$$

Lemma 30 ([5, §6]). *Let (X, Δ) be a Fano orbifold, ω_{KE} a Kähler–Einstein metric and ω a metric in the canonical class. Then there is $g \in \mathrm{Aut}(X, \Delta)$ such that $\omega = g^*\omega_{KE} + \mathrm{i}\,\partial\bar\partial\psi$ with ψ orthogonal to $\ker(\Delta_{g^*\omega_{KE}} + 1)$ in $L^2(X, \omega_{KE}^n)$.*

Proposition 31 ([29, Proposition 5.3]). *Let (X, Δ) be a Fano orbifold and ω_{KE} a Kähler–Einstein metric in the canonical class. If $\omega = \omega_{KE} + \mathrm{i}\,\partial\bar\partial\psi$ is a Kähler metric, with $\psi \perp \ker(\Delta_{KE} + 1)$ and $\int_X e^{-\psi}\omega_{KE}{}^n = 0$, there is a solution $\{\varphi_t\}_{t\in[0,1]}$ of $(*)_t$ with $\varphi_0 = 0$ and $\varphi_1 = -\psi$.*

Theorem 32 ([15, Theorem 2.2]). *If a Fano orbifold (X, Δ) admits a Kähler–Einstein metric ω_{KE}, then F_ω is bounded from below on $P(X, \Delta, \omega)$ for any ω in the canonical class.*

Proof. Thanks to (23) it is enough to bound $F_{\omega_{KE}}$. Given $\varphi \in P(X, \Delta, \omega_{KE})$ put $\omega = \omega_{KE} + \mathrm{i}\,\partial\bar\partial\varphi$ and let g and ψ be as in Lemma 30. Using again (23) it is enough to bound $F_{g^*\omega_{KE}}(\psi)$. Take a path as in Lemma 31. Thanks to Proposition 28 $F_{g^*\omega_{KE}}(\psi) = -F_\omega(-\psi) = -F_\omega(\varphi_1) = F_\omega^0(\varphi_1) \geq 0$. $\qquad\square$

Remark 33. These estimates are enough to prove one half of Tian's fundamental theorem, namely that properness of F_ω implies the existence of a Kähler–Einstein metric (see [30, p. 63]).

The following normalisation of potentials is useful:

$$Q_G(X, \Delta, \omega) = \{\varphi \in P_G(X, \Delta, \omega) : A_\omega(\varphi) = 0\}. \tag{29}$$

For any $\varphi \in P_G(X, \Delta, \omega)$, $\varphi + A_\omega(\varphi) \in Q_G(X, \Delta, \omega)$.

Proposition 34. *Let (X, Δ) be a Fano orbifold, $\omega \in 2\pi c_1(X, \Delta)$ a Kähler metric and G a compact group of isometries of (X, Δ, ω). If there are constants $C_1, C_2 > 0$ such that*

$$F_\omega(\varphi) \geq C_1 \sup_X \varphi - C_2 \tag{30}$$

for any $\varphi \in Q_G(X, \Delta, \omega)$, then (X, Δ) admits a Kähler–Einstein metric.

Proof. Let φ_t be a solution of $(*)_t$ on $[0, T_0)$. Since $\varphi_t + A_\omega(\varphi_t) \in Q_G(X, \Delta, \omega)$

$$
\begin{aligned}
F_\omega(\varphi_t) &= F_\omega\big(\varphi_t + A_\omega(\varphi_t)\big) \\
&\geq C_1 \sup_X \big(\varphi_t + A_\omega(\varphi_t)\big) - C_2 \\
&= C_1 \sup_X \varphi_t + C_1 A_\omega(\varphi_t) - C_2.
\end{aligned}
\tag{31}
$$

Using (28)

$$
C_1 \sup_X \varphi_t \leq F_\omega(\varphi_t) - C_1 A_\omega(\varphi_t) + C_2 \leq C_3 + C_2 + C_1 C_3.
$$

Hence $\sup_X \varphi_t$ is uniformly bounded. But $F^0(\varphi_t) \leq 0$, so $J_\omega(\varphi_t) \leq F_\omega^0(\varphi_t) + \sup \varphi_t$ is bounded and (26) yields the required bound of the C^0 norm. $\square$

Lemma 35 ([1, Lemma 2.3]). *Let (X, Δ) be a Fano orbifold, and $\omega \in 2\pi c_1(M)$ a Kähler metric. Then for any $\beta > 0$ there are constants $C_1, C_2 > 0$ such that for any $\varphi \in Q(X, \Delta, \omega)$*

$$
\log\left[\frac{1}{V} \int_X e^{-(1+\beta)\varphi} \omega^n\right] \geq C_1 \sup_X \varphi - C_2.
\tag{32}
$$

Corollary 36. *If there are constants $C_1, C_2 > 0$ and $\beta > 0$ such that*

$$
F_\omega(\varphi) \geq C_1 \log\left[\frac{1}{V} \int_X e^{-(1+\beta)\varphi} \omega^n\right] - C_2
\tag{33}
$$

for any $\varphi \in Q_G(X, \Delta, \omega)$, then (X, Δ) admits a Kähler–Einstein metric.

6. Existence theorems

A *current* on an orbifold (X, Δ) is a collection of $\mathrm{Gal}(\varphi)$-invariant currents on any uniformiser (U, φ), satisfying the usual compatibility condition with respect to injections of uniformisers. In case X is smooth, orbifold differential forms on (X, Δ) are *more* than ordinary differential forms on X. By duality orbifold currents on (X, Δ) are *less* than ordinary currents on X: they are the continuous functionals on $\wedge^k(X)$ that can be extended to the larger space $\wedge^k(X, \Delta)$. For *positive* (p, p)-currents there is no difference between the two notions, since every positive current has measure coefficients, and every orbifold differential form has continuous coefficients. If γ is a continuous hermitian form on a compact orbifold (X, Δ), an orbifold *Kähler current* is a closed positive (orbifold) current T of bidegree $(1,1)$ such that for some positive

constant c, $T \geq c\gamma$ in the sense of orbifold currents, that is $\langle T - c\gamma, \eta \rangle \geq 0$ for any positive $\eta \in \wedge^{n-1,n-1}(X, \Delta)$. The definition does not depend on the choice of γ, since X is compact.

If (X, Δ) is a Fano orbifold, $G \subset \mathrm{Aut}(X, \Delta)$ is a compact subgroup and ω is a G-invariant Kähler form in $2\pi c_1(X, \Delta)$, put

$$P_G^0(X, \Delta, \omega) = \{\chi \in C^0(X) : \omega + i\partial\bar{\partial}\chi \text{ is a Kähler orbifold current}\}.$$

Proposition 37. (a) *Any* $\chi \in P_G^0(X, \Delta, \omega)$ *is the C^0-limit of a sequence* $\varphi_n \in P_G(X, \Delta, \omega)$.

(b) *The functionals* I_ω, J_ω, F_ω^0 *and* F_ω *can be extended to* $P_G^0(X, \Delta, \omega)$ *and the extensions are continuous with respect to the C^0-topology.*

(See Propositions 2.2 and 2.3 in [1].)

Lemma 38 ([1, Lemma 2.6]). *If* $\pi : (X, \Delta_X) \to (Y, \Delta_Y)$ *is an orbifold map between compact orbifolds, the direct image* $\pi_* T$ *of a Kähler current* T *on* (X, Δ) *is a Kähler current on* (Y, Δ_Y).

Proof. First of all observe that if $f : (X, \Delta_X) \to (Y, \Delta_Y)$ is an orbifold map of degree d and $\alpha \in \wedge^{2n}(Y, \Delta_Y)$, then $\int_X f^*\alpha = d \cdot \int_Y \alpha$. Next let γ_X and γ_Y be continuous hermitian forms on (X, Δ_X) and (Y, Δ_Y) respectively. Since $\pi^*\gamma_Y$ is continuous and γ_X is positive definite, there is $c_1 > 0$ such that $\gamma_X \geq c_1\pi^*\gamma_Y$. If T is a Kähler current on (X, Δ), by definition $T \geq c_2\gamma_X$ for some $c_2 > 0$, so $T \geq c\pi^*\gamma_Y$ with $c = c_1 c_2 > 0$. We want to prove that for any positive form $\eta \in \wedge^{n-1,n-1}(Y, \Delta_Y)$, $\langle \pi_* T, \eta \rangle \geq c \cdot \deg \pi \cdot \langle \gamma_Y, \eta \rangle$. Choose orbifold charts (V, ψ) on (Y, Δ_Y) and (U_i, φ_i) on (X, Δ) such that $\pi^{-1}(\psi(V)) = \bigsqcup_i \varphi_i(U_i)$. Denote by $\tilde{T}_i$, $\tilde{\eta}$ and $\tilde{\gamma}_Y$ the local representations in the orbifold charts and by $\tilde{\pi}_i : U_i \to V$ the liftings of π. We can assume $\mathrm{supp}(\eta) \subset \psi(V)$. Then

$$\langle \pi_* T, \eta \rangle = \langle T, \pi^*\eta \rangle = \sum_i \frac{\langle \tilde{T}_i, \tilde{\pi}_i^*\tilde{\eta} \rangle}{|\mathrm{Gal}(\varphi_i)|}$$

$$\geq \sum_i \frac{c \cdot \langle \tilde{\pi}_i^*\tilde{\gamma}_Y, \tilde{\pi}_i^*\tilde{\eta} \rangle}{|\mathrm{Gal}(\varphi_i)|}$$

$$= \sum_i \frac{c}{|\mathrm{Gal}(\varphi_i)|} \int_{U_i} \tilde{\pi}_i^*(\tilde{\gamma}_Y \wedge \tilde{\eta})$$

$$= c \cdot \left(\sum_i \frac{\deg \tilde{\pi}_i}{|\mathrm{Gal}(\varphi_i)|}\right) \cdot \int_V (\tilde{\gamma}_Y \wedge \tilde{\eta}).$$

Since

$$\sum_i \frac{\deg \tilde{\pi}_i}{|\operatorname{Gal}(\varphi_i)|} = \frac{\deg \pi}{|\operatorname{Gal}(\psi)|}$$

we finally get

$$\langle \pi_* T, \eta \rangle \geq c \cdot \deg \pi \int_{\psi(V)} (\gamma_Y \wedge \eta),$$

and this proves the lemma. □

Lemma 39 ([1, Lemma 2.7]). *Let* $\pi : (X, \Delta) \to (Y, \Delta_Y)$ *be an orbifold map between* n-*dimensional Kähler orbifolds. Let* ω_Y *be a Kähler metric on* (Y, Δ_Y) *and* $\chi \in P^0(Y, \Delta_Y, \omega_Y)$ *a continuous* potential *such that* $\pi^* \chi \in C^\infty(X, \Delta)$. *Then*

$$F^0_{\pi^* \omega_Y}(\pi^* \chi) = F^0_{\omega_Y}(\chi). \tag{34}$$

Theorem 40. *Let* (X, Δ_X) *and* (Y, Δ_Y) *be Fano orbifolds,* $\pi : (X, \Delta) \to (Y, \Delta_Y)$ *an orbifold Galois covering of degree* d *with* $G = \operatorname{Gal}(\pi)$, ω_Y *a Kähler–Einstein metric on* (Y, Δ_Y) *and* $\omega \in 2\pi c_1(X, \Delta)$ *a* G-*invariant Kähler metric. Assume that numerically* $R^{\mathrm{orb}}(\pi) \equiv -\beta(K_X + \Delta_X)$ *for some* $\beta \in \mathbb{Q}_+$. *Then there is a constant* C *such that for any* $\varphi \in P_G(X, \Delta, \omega)$

$$F^0_\omega(\varphi) \geq \frac{1}{1 + \beta} \log \left[\frac{1}{V} \int_X e^{-(1+\beta)\varphi} \pi^* \omega_Y^n \right] - C. \tag{35}$$

The proof is identical to that of Theorem 2.2 in [1] and depends on the previous lemmata. Notice that a G-invariant orbifold Kähler metric ω always exists since, according to Definition 15, $G \subset \operatorname{Aut}(X, \Delta)$.

Theorem 41. *Let* $(X, \Delta), (X_1, \Delta_1), \ldots, (X_k, \Delta_k)$ *be* n-*dimensional Fano orbifolds. Assume that each* (X_i, Δ_i) *admits a Kähler–Einstein metric and that* $\pi_i : (X, \Delta) \to (X_i, \Delta_i)$ *are orbifold Galois coverings such that*

(1) *the groups* $\operatorname{Gal}(\pi_i)$ *are all contained in some* compact *subgroup of* $\operatorname{Aut}(X, \Delta)$;

(2) $R^{\mathrm{orb}}(\pi_i) \equiv -\beta_i(K_X + \Delta)$ *for some* $\beta_i \in \mathbb{Q}_+$.

Define $\eta \in C^\infty(X, \Delta)$ *by*

$$\frac{1}{k} \sum_{i=1}^k \pi_i^* \omega_i^n = \eta \, \omega^n, \tag{36}$$

put $c := \sup\{\lambda \geq 0 : \eta^{-\lambda} \in L^1(X, \omega^n)\}$ *and* $\beta := \min \beta_i$. *If*

$$\frac{1}{c} < \beta, \tag{37}$$

then (X, Δ) *admits a Kähler–Einstein metric.*

The proof is the same as that of Theorem 2.3 and Proposition 2.4 in [1].

Remark 42. If $k = 1$, then c is the infimum of the *complex singularity exponents* (that is of the *log canonical thresholds*, see [14] and [19]) of the pairs $(U, \varphi^* R^{\mathrm{orb}})$, where (U, φ) runs over all orbifold charts. On the other hand if there are enough coverings and the intersection of the ramification divisors $R^{\mathrm{orb}}(\pi_i)$ is empty, then $c = +\infty$ and (37) is automatically satisfied.

7. Applications

Here we exhibit some concrete examples where Theorem 41 can be used to prove the existence of Kähler–Einstein metrics on orbifolds.

Theorem 43. *Let X be a Fano manifold, $\sum_{i=1}^{N} D_i$ a divisor with local normal crossing and ω a Kähler–Einstein metric on X. Given integers $m_i > 1$ put $\Delta = \sum_i (1 - 1/m_i) D_i$. If $\Delta \equiv -\delta K_X$ with $\delta \in (0, 1)$ and*

$$m_i - 1 < \frac{\delta}{1 - \delta} \tag{38}$$

for any $i = 1, \ldots, N$, then (X, Δ) is a Fano orbifold and has an orbifold Kähler–Einstein metric.

Proof. (X, Δ) is a Fano orbifold because $K_X + \Delta \equiv (1 - \delta) K_X$ and $\delta < 1$. As observed in Example 17 the map $\mathrm{id}\colon (X, \Delta) \to X$ is an orbifold Galois cover and we want to apply Proposition 41 to it. The ramification divisor is just $R^{\mathrm{orb}} = \Delta$ so

$$R^{\mathrm{orb}}(\mathrm{id}) \equiv -\beta(K_X + \Delta)$$

with $\beta = \delta/(1 - \delta)$. It remains to check that (38) implies (37). Let x be any point in X. Choose a system of coordinates $(V, z^1, \ldots, z^n)$ on X as in Example 13 and let (U, φ) be the corresponding orbifold chart for (X, Δ) as in (6). Then on $\varphi(U) = V$

$$R^{\mathrm{orb}} = \Delta = \sum_{j=1}^{k} \left(1 - \frac{1}{m'_j}\right) \{z_j = 0\} \tag{39}$$

so that in the notation of (36), $\eta(z) = \gamma(z)|f(z)|^2$ on U, where $f(z) = z_1^{m'_1 - 1} \ldots z_k^{m'_k - 1}$ and γ is a smooth positive function. Set $c_x = \sup\{\lambda \geq 0 : \int_U |f|^{-2\lambda} < +\infty\}$. Since

$$\int_U |f|^{-2\lambda} = \mathrm{const} \cdot \prod_{j=1}^{k} \int_D |z|^{-2\lambda(m'_j - 1)} \tag{40}$$

where D is the disk in $\mathbb{C}$, we get that $|f|^{-2\lambda} \in L^1_{\text{loc}}$ on U iff $\lambda < 1/(m'_j - 1)$. So $c_x = \min\{1/(m'_j - 1) : 1 \le j \le k\}$,

$$c = \inf_{x \in X} c_x = \min_i \frac{1}{m_i - 1} \tag{41}$$

and

$$\frac{1}{c} = \max(m_i - 1) < \frac{\delta}{1 - \delta} = \beta. \tag{42}$$

$\square$

Example 44. Let some divisors $D_i \in |\mathcal{O}_{\mathbb{P}^n}(d_i)|$, and some integers $m_i > 1$ be given for $i = 1, \ldots, N$. Let m_1 be the greatest of the m_i's. Put $\Delta = \sum_i (1 - 1/m_i) D_i$ and

$$\delta = \frac{\sum_i d_i \left(1 - \frac{1}{m_i}\right)}{n + 1}. \tag{43}$$

Assume that

(1) $\sum_i D_i$ is local normal crossing;

(2) $\delta < 1$;

(3) $m_1(1 - \delta) < 1$.

Then $(\mathbb{P}^n, \Delta)$ admits an orbifold Kähler–Einstein metric of positive scalar curvature.

Example 45 (Compare [10, Note 36]). Let D_i be $n+2$ hyperplanes in general position in $\mathbb{P}^n$: $D_i = \{z_i = 0\}$ for $i = 0, \ldots, n$, $D_{n+1} = \{z_0 + \cdots + z_n = 0\}$. Set

$$\Delta = \sum_{i=0}^{n+1} \left(1 - \frac{1}{m_i}\right) D_i.$$

Then $(\mathbb{P}^n, \Delta)$ has an orbifold Kähler–Einstein metric as soon as

$$1 < \sum_{i=0}^{n+1} \frac{1}{m_i} < 1 + (n + 1) \min_i \frac{1}{m_i} \tag{44}$$

As in [10], many numerical examples come from Euclid's or Sylvester's sequence (cf. [26, A000058]). This is defined by the recursion relation

$$c_{k+1} = c_1 \ldots c_k + 1 = c_k^2 - c_k + 1$$

beginning with $c_1 = 2$. The sequence grows doubly exponentially, and it starts as

$$2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots.$$

It is easy to see that

$$\sum_{i=1}^{n} \frac{1}{c_i} = 1 - \frac{1}{c_{n+1} - 1} = 1 - \frac{1}{c_1 \ldots c_n}.$$

We get many new examples by taking

$$(m_0 = c_1, m_1 = c_2, \ldots, m_n = c_{n+1} - 2, m_{n+1}).$$

Then

$$\sum_{i=0}^{n} \frac{1}{m_i} = 1 + \frac{1}{(c_{n+1} - 1)(c_{n+2} - 2)}.$$

Thus our conditions are satisfied as long as

$$c_{n+1} - 2 < m_{n+1} < n(c_{n+1} - 1)(c_{n+2} - 2)$$

and m_{n+1} is relatively prime to the other m_i.

Another case when Theorem 41 works is for degree 2 Del Pezzo surfaces S. Here we consider the case when S is allowed to have cyclic quotient singularities. These are necessarily of the form $\mathbb{C}^2/\mathbb{Z}_n$ where the group action is given by $(u, v) \mapsto (\epsilon u, \epsilon^{-1} v)$ where ϵ is a primitive n-th root of unity. The $\mathbb{Z}_n$-invariant functions are generated by u^n, v^n, uv. This singularity is denoted by A_{n-1}.

For any degree 2 Del Pezzo surface S the anticanonical class is ample and it gives a degree 2 cover $\pi \colon S \to \mathbb{P}^2$. If H denotes the hyperplane class on $\mathbb{P}^2$, then $-K_S = \pi^* H$. The double cover π ramifies along a quartic curve C, thus $R = \frac{1}{2}\pi^* C = \pi^* 2H$, $\beta = 2$ and to apply Theorem 41 we need to ensure that $\eta^{-\lambda}$ be integrable for $\lambda \leq \frac{1}{2}$. The singularities of π lie over the singularities of C, an A_{n-1}-singularity of S lies over an A_{n-1}-singularity of C (cf. [6, p. 87]) and we can find local coordinates (x, y) on $\mathbb{P}^2$ such that S is locally isomorphic to some neighbourhood of the origin in the affine surface $\{(x, y, t) \in \mathbb{C}^3 : t^2 = x^2 + 4y^n\}$, the map π being given simply by $\pi(x, y, t) = (x, y)$. An orbifold chart is given by $\varphi \colon U \subset \mathbb{C}^2 \to S$ where $\varphi(u, v) = (u^n - v^n, uv, u^n + v^n)$. Thus $\varphi^* \pi^* (dx \wedge dy) = n(u^n + v^n) \cdot du \wedge dv$ and $\eta(u, v) = \text{const} \cdot |u^n + v^n|^2$. It is easy to see by direct integration or by blowing up (see e.g. [20, Proposition 6.39, p. 168]) that for $n \geq 2$, $|u^n + v^n|^{-2\lambda}$ is integrable if and only if $\lambda < \frac{2}{n}$. Thus Theorem 41 applies as long as $\frac{1}{2} < c = \frac{2}{n}$, that is for $n < 4$. This proves Theorem 2.

One can also give a different proof of the following result of Mabuchi and Mukai [21, Corollary C].

Theorem 46. *A diagonalizable singular Del Pezzo surface of degree 4 admits an orbifold Kähler–Einstein metric.*

A quartic Del Pezzo surface S is the intersection of two quadrics in $\mathbb{P}^4$, $S = Q_1 \cap Q_2$. It is said to be diagonalizable if both Q_1 and Q_2 can be put simultaneously in diagonal form. If S is singular then in suitable coordinates it is given by equations

$$h_0 := x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \quad \text{and} \quad h_1 := \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 = 0.$$

If two of the λ_i coincide then S is a quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ and so has an orbifold Kähler–Einstein metric (see [21, p. 136]). Thus assume that the λ_i are distinct nonzero complex numbers. For $i = 2, 3, 4$, the equation $\lambda_i h_0 - h_1 = 0$ does not involve x_i, and by dropping the x_i variable we get smooth quadrics

$$Q_i = \{(\lambda_i h_0 - h_i = 0)\} \subset \mathbb{P}^3.$$

The map $\pi_i \colon S \to Q_i$ given by forgetting x_i is a double cover ramified over the hyperplane section $S \cap \{x_i = 0\}$. Since the Q_i are smooth two-dimensional quadrics, they are Kähler–Einstein. On the other hand, the divisors $R^{\mathrm{orb}}(\pi_i)$ are disjoint, so η is strictly positive on all S, $c = \infty$ and Theorem 41 yields that S admits an orbifold Kähler–Einstein metric.

References

[1] C. Arezzo, A. Ghigi, and G. P. Pirola, Symmetries, Quotients and Kähler-Einstein metrics. *J. Reine Angew. Math.* **591** (2006), 177–200. Zbl 1089.32013 MR 2212883

[2] Thierry Aubin, *Some nonlinear problems in Riemannian geometry*. Springer Monogr. in Math., Springer-Verlag, Berlin 1998. Zbl 0896.53003 MR 1636569

[3] W. L. Baily, On the imbedding of V-manifolds in projective space. *Amer. J. Math.* **79** (1957), 403–430. Zbl 0173.22706 MR 0100104

[4] Walter L. Baily, Jr., The decomposition theorem for V-manifolds. *Amer. J. Math.* **78** (1956), 862–888. Zbl 0173.22705 MR 0100103

[5] Shigetoshi Bando and Toshiki Mabuchi, Uniqueness of Einstein Kähler metrics modulo connected group actions. In *Algebraic geometry* (Sendai, 1985), North-Holland, Amsterdam 1987, 11–40. Zbl 0641.53065 MR 0946233

[6] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*. Ergeb. Math. Grenzgeb. (3) 4, Springer-Verlag, Berlin 1984. Zbl 0718.14023 MR 0749574

[7] Joseph E. Borzellino, Orbifolds of maximal diameter. *Indiana Univ. Math. J.* **42** (1) (1993), 37–53. Zbl 0801.53031 MR 1218706

[8] Jean Pierre Bourguignon, Métriques d'Einstein-Kähler sur les variétés de Fano: obstructions et existence (d'après Y. Matsushima, A. Futaki, S. T. Yau, A. Nadel et G. Tian). Séminaire Bourbaki, Vol. 1996/97; *Astérisque* **245** (1997), 277–305. Zbl 0935.32019 MR 1627115

[9] Charles P. Boyer and Krzysztof Galicki, The twistor space of a 3-Sasakian manifold. *Internat. J. Math.* **8** (1) (1997), 31–60. Zbl 0887.53043 MR 1433200

[10] Charles P. Boyer, Krzysztof Galicki, and János Kollár, Einstein metrics on spheres. *Ann. of Math.* **162** (2005), 557–580. Zbl 05011502 MR 2178969

[11] Henri Cartan, Quotient d'un espace analytique par un groupe d'automorphismes. In *Algebraic geometry and topology*, A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, N. J., 1957, 90–102. Zbl 0084.07202 MR 0084174

[12] Yuan-Jen Chiang, Spectral geometry of V-manifolds and its application to harmonic maps. In *Differential geometry: partial differential equations on manifolds* (Los Angeles, CA, 1990), Proc. Sympos. Pure Math. 54, Amer. Math. Soc., Providence, R.I., 1993, 93–99. Zbl 0806.58005 MR 1216577

[13] Pierre Deligne and G. Daniel Mostow, *Commensurabilities among lattices in* PU$(1, n)$. Ann. of Math. Stud. 132, Princeton University Press, Princeton, N.J., 1993. Zbl 0826.22011 MR 1216577

[14] Jean-Pierre Demailly and János Kollár, Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. *Ann. Sci. École Norm. Sup.* (4) **34** (4) (2001), 525–556. Zbl 0994.32021 MR 1852009

[15] Wei Yue Ding and Gang Tian, Kähler-Einstein metrics and the generalized Futaki invariant. *Invent. Math.* **110** (1992), 315–335. Zbl 0779.53044 MR 1185586

[16] S. K. Donaldson, Scalar curvature and projective embeddings. I. *J. Differential Geom.* **59** (3) (2001), 479–522. Zbl 1052.32017 MR 1916953

[17] Hans Grauert and Reinhold Remmert, *Coherent analytic sheaves*. Grundlehren Math. Wiss. 265, Springer-Verlag, Berlin 1984. Zbl 0537.32001 MR 0755331

[18] Shoshichi Kobayashi, Topology of positively pinched Kaehler manifolds *Tôhoku Math. J.* (2) **15** (1963), 121–139. Zbl 0114.37601 MR 0154235

[19] János Kollár, Singularities of pairs. In *Algebraic geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math. 62, Amer. Math. Soc., Providence, R.I., 1997, 221–287. Zbl 0905.14002 MR 1492525

[20] János Kollár and Karen E. Smith, and Alessio Corti, *Rational and nearly rational varieties*. Cambridge Stud. Adv. Math. 92, Cambridge University Press, Cambridge 2004. Zbl 1060.14073 MR 2062787

[21] Toshiki Mabuchi and Shigeru Mukai, Stability and Einstein-Kähler metric of a quartic del Pezzo surface. In *Einstein metrics and Yang-Mills connections* (Sanda, 1990), Lecture Notes in Pure Appl. Math. 145, Dekker, New York 1993, 133–160. Zbl 0809.53070 MR 1215285

[22] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*. Third ed., Ergeb. Math. Grenzgeb. (2) 34, Springer-Verlag, Berlin 1994. Zbl 0797.14004 MR 1304906

[23] Yasuhiro Nakagawa, An isoperimetric inequality for orbifolds. *Osaka J. Math.* **30** (4) (1993), 733–739. Zbl 0815.53079 MR 1250779

[24] Peter Orlik and Philip Wagreich, Seifert n-manifolds. *Invent. Math.* **28** (1975), 137–159. Zbl 0295.57021 MR 0361150

[25] David Prill, Local classification of quotients of complex manifolds by discontinuous groups. *Duke Math. J.* **34** (1967), 375–386. Zbl 0179.12301 MR 0210944

[26] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. AT&T Research, 2003. http://www.research.att.com/~njas/sequences/

[27] Gang Tian, On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. *Invent. Math.* **89** (2) (1987), 225–246. Zbl 0599.53046 MR 0894378

[28] Gang Tian, On Calabi's conjecture for complex surfaces with positive first Chern class. *Invent. Math.* **101** (1990), 101–172. Zbl 0716.32019 MR 1055713

[29] Gang Tian, Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.* **130** (1) (1997), 1–37. Zbl 0892.53027 MR 1471884

[30] Gang Tian, *Canonical metrics in Kähler geometry*. Lectures Math. ETH Zurich, Birkhäuser, Basel 2000. Zbl 0978.53002 MR 1787650

[31] Xu-Jia Wang and Xiaohua Zhu, Kähler-Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.* **188** (1) (2004), 87–103. Zbl 1086.53067 MR 2084775

Alessandro Ghigi, Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, via Cozzi 53, 20125 Milano, Italy

E-mail: alessandro.ghigi@unimib.it

János Kollár, Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, U.S.A.

E-mail: kollar@math.princeton.edu