Dense subgroups with Property (T) in Lie groups

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Abstract. We characterize connected Lie groups that have a dense, finitely generated subgroup with Property (T).

1. Introduction

Not much is known about the structure of dense subgroups in connected Lie groups, in contrast to discrete subgroups. However, given a class of groups, it is natural, and sometimes possible, to study which connected Lie groups (more generally, which locally compact groups) contain a dense embedded copy of a group in this class. For the class of non-abelian free groups of finite rank, this study has been carried out in [Kur], [BrG], and in [BGSS] for surface groups and, more generally, fully residually groups. In this paper, we study the existence of dense finitely generated subgroups of a very different type, namely with Kazhdan's Property (T).

We begin by recalling relevant definitions. If G is a locally compact group and π is a unitary representation into a Hilbert space $\mathcal{H}$, and $X \subset G$ is any subset and $\varepsilon > 0$, the representation π is said to have (X, ε)-invariant vectors if there exists $v \in \mathcal{H}$ such that $\|v\| = 1$ and $\sup_{g \in X} \|\pi(g)v - v\| \leq \varepsilon$. The subset X is said to be a *Kazhdan subset* of G if there exists $\varepsilon > 0$ such that every continuous unitary representation having (X, ε)-invariant vectors actually has nonzero invariant vectors. The locally compact group G has *Property* (T) [Kaz], [HV], [BeHV] if it has a compact Kazhdan subset. The Lie algebra of a Lie group or an algebraic group is denoted by the corresponding Gothic letter.

In this paper, we characterize connected Lie groups that have a dense finitely generated subgroup Γ with Property (T) (when viewed as a discrete group). The existence of such a dense subgroup is a strengthening of Property (T); this has been used by Margulis and Sullivan [Mar1], [Sul] to solve the Ruziewicz Problem in dimension

$n \geq 4$, namely that the Lebesgue measure is the only mean on the measurable subsets of the n-sphere, invariant under SO_{n+1}.

We begin by a result that characterizes connected Lie groups with Property (T). This is essentially due to S. P. Wang [Wang2], but we give a different formulation. Recall that a connected Lie group is amenable if and only if its radical is cocompact.

Proposition 1. *Let G be a connected Lie group. Then G has Property* (T) *if and only if*

(i) *every amenable quotient of G is compact, and*

(ii) *no simple quotient of G is locally isomorphic to* $SO(n, 1)$ *or* $SU(n, 1)$ *for some* $n \geq 2$.

In Proposition 1, Condition (i) can be shown [Cor2, Proposition 4.5.4] to be equivalent to the following: Every isometric action of G on a Euclidean space has a fixed point. As we have embeddings

$$\text{Isom}(\mathbb{R}^{n-1}) \subseteq \text{Isom}(\mathbb{H}_{\mathbb{R}}^n) \subseteq \text{Isom}(\mathbb{H}_{\mathbb{C}}^n),$$

we get, as a consequence of Proposition 1, the following geometric characterization of Property (T) for connected Lie groups.

Proposition 2. *Let G be a connected Lie group. Then G has Property* (T) *if and only if every isometric action of G on a finite-dimensional complex hyperbolic space has a fixed point.* $\square$

Here is the main result of the paper.

Theorem 3. *Let G be a connected Lie group. Then G has a dense, finitely generated subgroup with Property* (T) *if and only if G has Property* (T) *(i.e. satisfies* (i) *and* (ii) *of Proposition 1), and*

(iii) $\mathbb{R}/\mathbb{Z}$ *is not a quotient of G (that is, $\overline{[G, G]} = G$);*

(iv) $SO_3(\mathbb{R})$ *is not a quotient of G.*

Remark 4. It is easy to check that, for a connected Lie group with Property (T), (iii) together with (iv) is equivalent to $\text{Hom}(G, \text{PSL}_2(\mathbb{C})) = \{1\}$, which means, geometrically, that every isometric action on the three-dimensional real hyperbolic space is identically trivial.

Theorem 3 can be compared to the following result.

Proposition 5. *Let G be a connected Lie group. Then G has an infinite, finitely generated subgroup with Property* (T) *if and only if G has at least one simple factor not locally isomorphic to* $SO(3)$, $SL_2(\mathbb{R})$, $SL_2(\mathbb{C})$, $SO(4, 1)$, $SU(2, 1)$.

Remark 6. In contrast, it is proved in [Cor2, Theorem 4.6.1] that, in $SO_0(4, 1)$ and $SU(2, 1)$, there exist infinite finitely generated subgroups $\Lambda \subset \Gamma$, such that (Γ, Λ) has *relative* Property (T). Moreover, Λ cannot be chosen normal, and Γ is necessarily dense.

In some "minimal" cases, an infinite subgroup with Property (T) is necessarily dense or Zariski dense. For simplicity, let us focus on the case of non-compact simple Lie groups with Property (T).

Proposition 7. *Let G be a simple, non-compact connected Lie group, and Γ an infinite, finitely generated subgroup with Property* (T).

- *If G is locally isomorphic to $Sp_4(\mathbb{R})$ or $SL_3(\mathbb{R})$, then Γ is either dense, or discrete and Zariski dense.*

- *If G is locally isomorphic to $Sp(2, 1)$, then Γ is either relatively compact, or dense, or discrete and Zariski dense*[1].

- *Otherwise, and also excluding the groups given in Proposition 5, G has an infinite discrete subgroup with Property* (T)*, that is not Zariski dense.*

This motivates the following question, which has already circulated among the specialists for $SL_3(\mathbb{R})$ and seems to be hard to handle.

Question 8. Does there exist an infinite, discrete subgroup of $SL_3(\mathbb{R})$, $Sp_4(\mathbb{R})$, or $Sp(2, 1)$ that has Property (T), but is not a lattice?

Remark 9. Following Shalom [Sha], a locally compact group has *strong Property* (T) if it has a *finite* Kazhdan subset.

The following implications are immediate: G has a dense finitely generated subgroup with Property (T) $\Rightarrow$ G has strong Property (T) $\Rightarrow$ G has Property (T).

Shalom proves that a connected Lie group G with Property (T) has strong Property (T) if and only if $\mathbb{R}/\mathbb{Z}$ is not a quotient of G, i.e. if G is topologically perfect. The most remarkable result is that $SO_3(\mathbb{R})$ has strong Property (T); this is actually a reformulation of a deep result of Drinfeld [Dri].

Remark 10. There is no obvious generalization of Theorem 3 to all connected locally compact groups. For instance, does $\prod_{d \geq 5} SO_d(\mathbb{R})$ have a finitely generated dense subgroup with Property (T)? The question makes sense more generally for the product of any sequence of simple, connected compact Lie group of dimensions tending to infinity. On the other hand, the infinite product $K^{\mathbb{N}}$ of a *fixed* compact Lie group

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[1]Note that G is not necessarily algebraic; however this statement makes sense if we define a Zariski dense subset as a subset that is Zariski dense modulo the centre.

$K \neq \{1\}$ cannot have any dense finitely generated subgroup with Property (T). Let us sketch the argument (for which we thank A. Lubotzky). If K is not connected, then $K^{\mathbb{N}}$ maps onto the infinite, locally finite group $(K/K_0)^{\mathbb{N}}$ and therefore has no dense finitely generated subgroup at all. Suppose now that K is connected, and let Γ be a dense, finitely generated subgroup of $K^{\mathbb{N}}$. The density of Γ implies that the projections p_n of Γ on each factor are pairwise non-conjugate. Then Weil's Rigidity Theorem [Weil] implies that, for some (actually all but finitely many) of those projections p_n, we have $H^1(\Gamma, \rho_n) \neq \{0\}$, where ρ_n denotes the adjoint action of Γ on the Lie algebra of K, through the projection p_n. By a result of S. P. Wang [Wang1] (see also [HV], [BeHV]), this implies that Γ does not have Property (T).

The only nontrivial point as regards the necessary condition in Theorem 3 is due to Zimmer [Zim], who shows that $SO_3(\mathbb{R})$ has no infinite finitely generated subgroup with Property (T). The sufficient condition, constructing a dense subgroup with Property (T), was proved by Margulis [Mar2, Chapter III, Proposition 5.7] for G compact.

Let us sketch the proof of the sufficient condition in Theorem 3. We proceed in six steps. In the first step, we suppose that G is actually algebraic over $\mathbb{Q}$; then we use a standard argument to project densely a lattice into G, which is similar to that in [Mar2].

In the second step, we reduce to the case where G has a perfect Lie algebra, and then show, in the third step, that this implies that the subalgebra obtained by removing simple compact factors is also perfect.

In the fourth step, we prove the following result, which is perhaps of independent interest.

Proposition 11. *Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Lie algebra over $\mathbb{R}$, with $\mathfrak{s}$ semisimple and $\mathfrak{r}$ nilpotent. Then there exists a Lie algebra $\mathfrak{h} = \mathfrak{s} \ltimes \mathfrak{n}$, defined over $\mathbb{Q}$, with $\mathfrak{n}$ nilpotent, and a surjection $p \colon \mathfrak{h} \to \mathfrak{g}$ which is the identity on the Levi factor $\mathfrak{s}$, and maps $\mathfrak{n}$ onto $\mathfrak{r}$. If, moreover, $[\mathfrak{s}_{nc}, \mathfrak{r}] = \mathfrak{r}$, we can impose $[\mathfrak{s}_{nc}, \mathfrak{n}] = \mathfrak{n}$.*

The main ingredient for this proposition is the following result.

Theorem 12 (Witte [Wit]). *Every real finite-dimensional representation of a real semisimple Lie algebra has a $\mathbb{Q}$-form.*

Remark 13. Theorem 12 is equivalent to the following statement: if $\mathfrak{g}$ is a perfect Lie algebra over $\mathbb{R}$ with abelian radical, then $\mathfrak{g}$ has a $\mathbb{Q}$-form. The corresponding assertion is false if we replace "abelian" by "2-nilpotent", as there exist $2^{\aleph_0}$ non-isomorphic real Lie algebras with 2-nilpotent radical [Cor1, Proposition 1.12].

In the fifth step, we prove Theorem 3 in the particular case when G is algebraic over $\mathbb{R}$.

Finally, in the sixth step, we prove the general case; we actually have to deal with an extension of a real algebraic group by an infinite discrete centre.

2. Proofs of the results

The definition of Property (T) will not appear in the proofs below: what we will need are the following standard properties.

- If G, H are locally compact groups, $f\colon G \to H$ is a continuous morphism with dense image and if G has Property (T), then H has Property (T). This is immediate from the definition.

- If G is a locally compact group with Property (T) and Γ is a closed subgroup of finite covolume (e.g. a lattice), then Γ has Property (T). This is due to Kazhdan [Kaz], see also [HV, §3.a], [BeHV].

- If G is a locally compact group with Property (T), and if $\tilde{G}$ is another locally compact group lying in a central extension $1 \to Z \to \tilde{G} \to G \to 1$, then $\tilde{G}$ has Property (T) if and only if its abelianization $\tilde{G}_{\mathrm{ab}}$ is compact. The "only if" part follows from the fact that non-compact amenable groups do not have Property (T). The "if" part is due to Serre, see [HV, §2.c], [BeHV].

We will also use S. P. Wang's characterization of connected Lie groups with Property (T), encoded in Proposition 1.

Proof of Proposition 1. If the connected Lie group G has Property (T), then Conditions (i) and (ii) are satisfied, since non-compact amenable groups, and connected Lie groups locally isomorphic to $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ for some $n \geq 2$ do not have Property (T) (see [HV, §6.d]).

Conversely suppose that G does not have Property (T). Denote by R its radical, and by S_{nc} the sum of all noncompact simple factors in a Levi factor. If G does not have Property (T), then by S. P. Wang's characterization [Wang2], either (1) S_{nc} does not have Property (T), or (2) $W = \overline{S_{nc}[S_{nc}, R]} \cap R$ is not cocompact in R. In Case (1), (ii) is not satisfied. On the other hand, it is easily seen that W is a normal subgroup of G. So, in Case (2), taking the quotient, we can suppose that $W = 1$. So G is locally isomorphic to $S_{nc} \times R_m$, where R_m denotes the amenable radical $RS_c = \{rs \mid r \in R, s \in S_c\}$, and $\overline{S_{nc}} \cap R = 1$. This implies that G is actually the direct product of R and S_{nc}. So either R or S_{nc} does not have Property (T), giving either the negation of (i) or (ii). □

Proof of Theorem 3. If G has a finitely generated dense subgroup Γ with Property (T), then G has Property (T) (indeed, Property (T) is inherited by morphisms with dense

image, as follows immediately from the definition); (iii) is also clearly satisfied (because Γ has finite abelianization), and also (iv) by [Zim] (see also [HV, Chapter 6, 26]). We must show that, conversely, these conditions are sufficient.

First step. Suppose that $G = H(\mathbb{R})_0$, where H is a linear algebraic group defined over $\mathbb{Q}$ (the subscript 0 means the connected component in the Hausdorff topology). It is well known that $H(\mathbb{R})_0$ is an open subgroup of finite index in $H(\mathbb{R})$ [BoT, Corollaire 14.5]. Consider the normal subgroup $W = S_{nc}[S_{nc}, R]$ of H, where S_{nc} denotes the sum of all simple $\mathbb{R}$-isotropic factors in a Levi factor S. Then $W(\mathbb{R})$ is cocompact in $H(\mathbb{R})$ (since $H(\mathbb{R})$ has Property (T)). The hypotheses (iii) and (iv) then imply that H/W is, modulo its finite centre, a product of simple factors of $\mathbb{C}$-rank ≥ 2. This implies that $S[S, R] = H$, and that $(H/R)(\mathbb{C})$ has Property (T). By [Wang2], this implies that $H(\mathbb{C})$ has Property (T).

Now fix a number field of degree 3 over $\mathbb{Q}$, not totally real, and $\mathcal{O}$ its ring of integers: for instance, $\mathcal{O} = \mathbb{Z}[2^{1/3}]$. Then, since H is perfect, by the Borel–Harish-Chandra Theorem [BoHC], $H(\mathcal{O})$ embeds as an irreducible lattice in $H(\mathbb{R}) \times H(\mathbb{C})$, which has Property (T). So its projection on $G = H(\mathbb{R})$ is a dense subgroup with Property (T). This proves the case of the first step.

Let $\mathfrak{g}$ be the Lie algebra of $\mathfrak{g}$. Set $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$, where $\mathfrak{r}$ is the radical. Let $\mathfrak{s}_{nc}$ be the sum of all factors of positive $\mathbb{R}$-rank of $\mathfrak{s}$, and let $\mathfrak{g}_{nc}$ be the preimage of $\mathfrak{s}_{nc}$ in $\mathfrak{g}$: this is an ideal of $\mathfrak{g}$.

Second step. We reduce to the case where the Lie algebra $\mathfrak{g}$ is perfect.

Set $\mathfrak{h} = \bigcap_{n \geq 0} D^n \mathfrak{g}$, where $D\mathfrak{g}$ means the derived subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, generating a normal Lie subgroup H (not necessarily closed) of G. Moreover, $G/\overline{H}$ is solvable, hence trivial by the assumption (iii). This means that H is dense in G. Accordingly, since any dense subgroup of H is dense in G, we can replace G by H and thus suppose that $\mathfrak{g}$ is perfect.

Third step. Let us show that if $\mathfrak{g}$ is perfect, and if (i) and (iii) are satisfied, then $\mathfrak{g}_{nc}$ is also perfect, that is, $[\mathfrak{s}_{nc}, \mathfrak{r}] = \mathfrak{r}$.

Consider the adjoint action of G on the quotient $\mathfrak{g}/D\mathfrak{g}_{nc}$. This defines a morphism $f : G \rightarrow \mathrm{GL}(\mathfrak{g}/D\mathfrak{g}_{nc})$, such that $\overline{f(G)}$ is amenable. Therefore, the Lie group $f(G)$ is also amenable, hence compact. This implies that $\mathfrak{g}/D\mathfrak{g}_{nc}$ is a compact Lie algebra [Hel, Chap. 2, §5], that is, the direct product of an abelian Lie algebra and a semisimple compact Lie algebra. Since $\mathfrak{g}$ is perfect, this implies that $\mathfrak{g}/D\mathfrak{g}_{nc}$ is semisimple. Since $\mathfrak{g}_{nc}/D\mathfrak{g}_{nc}$ is an abelian ideal in $\mathfrak{g}/D\mathfrak{g}_{nc}$, we conclude that $D\mathfrak{g}_{nc} = \mathfrak{g}_{nc}$.

Fourth step. We begin with the following standard lemma.

Lemma 14. *Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{n}$ a nilpotent ideal. Let π denote the projection $\mathfrak{g} \rightarrow \mathfrak{g}/[\mathfrak{n}, \mathfrak{n}]$. Let $X \subseteq \mathfrak{g}$ satisfy: $\pi(X)$ generates $\mathfrak{g}/[\mathfrak{n}, \mathfrak{n}]$. Then X generates $\mathfrak{g}$.*

Proof. Argue by induction on the length of the descending central series of $\mathfrak{n}$. If $\mathfrak{n}$ is abelian, the result is trivial. Otherwise, let $\mathfrak{z}$ be the least nonzero term of the descending central series of $\mathfrak{n}$. By induction hypothesis, X generates $\mathfrak{g}$ modulo $\mathfrak{z}$. On the other hand, $\mathfrak{z}$ is contained in $[\mathfrak{n}, \mathfrak{n}]$, that is, $\mathfrak{z}$ is generated by some elements of the form $[n, n']$, for some $n, n' \in \mathfrak{n}$. Since $\mathfrak{z}$ is central in $\mathfrak{n}$, these elements can be chosen modulo $\mathfrak{z}$, so that they can be taken in the subalgebra generated by X. This implies that $\mathfrak{z}$ is contained in the subalgebra generated by X, so that X generates $\mathfrak{g}$.   □

Proof of Proposition 11. Let $\mathfrak{v}$ be a complementary $\mathfrak{s}$-subspace of $[\mathfrak{r}, \mathfrak{r}]$ in $\mathfrak{r}$. By [Wit], we can fix a $\mathbb{Q}$-form of $\mathfrak{s}$, together with a $\mathbb{Q}$-form of $\mathfrak{v}$, so that the representation of $\mathfrak{s}$ on $\mathfrak{v}$ is defined over $\mathbb{Q}$. By Lemma 14, $\mathfrak{r}$ is generated by $\mathfrak{v}$ as a Lie algebra. Let m be such that $\mathfrak{r}$ is m-nilpotent, and let $\mathfrak{n}$ be the free m-nilpotent Lie algebra generated by the vector space $\mathfrak{v}$. The action of $\mathfrak{s}$ on $\mathfrak{v}$, which is defined over $\mathbb{Q}$, extends naturally to an action on $\mathfrak{n}$, also defined over $\mathbb{Q}$. On the other hand, by the universal property of the embedding $\mathfrak{v} \to \mathfrak{n}$, the identity map $\mathfrak{v} \to \mathfrak{v}$ extends to a Lie algebra morphism of $\mathfrak{n}$ onto $\mathfrak{r}$, which is actually a morphism of $\mathfrak{s}$-modules: indeed, every $s \in \mathfrak{s}$ gives a derivation on $\mathfrak{n}$, whose image is a derivation of $\mathfrak{r}$ which coincides, in restriction to $\mathfrak{v}$, with $\mathfrak{ad}(s)$. Since $\mathfrak{r}$ is generated by $\mathfrak{v}$, this implies that they coincide on all of $\mathfrak{r}$. Therefore, the surjection $\mathfrak{n} \to \mathfrak{r}$ extends to a surjection $\mathfrak{s} \ltimes \mathfrak{n} \to \mathfrak{s} \ltimes \mathfrak{r}$.

If $[\mathfrak{s}, \mathfrak{r}] = \mathfrak{r}$, the condition $[\mathfrak{s}_{nc}, \mathfrak{n}] = \mathfrak{n}$ is immediate, since then $[\mathfrak{s}_{nc}, \mathfrak{n}]$ contains $\mathfrak{v}$.
□

In view of the second and third steps, the hypotheses are now: $\mathfrak{g}_{nc}$ is perfect, and $\mathfrak{g}$ has no simple factor $\mathfrak{s}$ isomorphic to $\mathfrak{so}(3)$, $\mathfrak{so}(n, 1)$ or $\mathfrak{su}(n, 1)$.

Fifth step. Suppose that $G = H(\mathbb{R})_0$, where H is a connected linear algebraic group defined over $\mathbb{R}$ such that $\mathfrak{h} = \mathfrak{g}$ is perfect. Choose $p: \hat{\mathfrak{h}} \to \mathfrak{h}$ as in Proposition 11, and let $\mathfrak{p} \subset \hat{\mathfrak{h}} \times \mathfrak{h}$ be its graph. Using [Bo, Corollary 7.9] twice, $\hat{\mathfrak{h}}$ is the Lie algebra of a simply connected linear algebraic group $\hat{H}$ defined over $\mathbb{Q}$, and $\mathfrak{p}$ is the Lie algebra of a $\mathbb{R}$-closed subgroup $P \subset \hat{H} \times H$. Since $\mathfrak{p} \cap \mathfrak{h} = \{0\}$, by [Bo, Corollary 6.12], $P \cap H$ is finite. Since p is onto, the projection of $P(\mathbb{R})$ into $H(\mathbb{R})$ is Zariski dense; thus $W = P \cap H$ is normal in H. Replacing H by H/W if necessary, we assume that $W = \{1\}$. Since the projection $\mathfrak{p} \to \hat{\mathfrak{h}}$ is onto, the projection of $P(\mathbb{R})$ on $\hat{H}(\mathbb{R})$ contains an open subgroup for the Hausdorff topology, but this topology is connected since we have chosen $\hat{H}$ simply connected. Hence P is the graph of a morphism of $\hat{H}$ onto H, still denoted by P.

By the first step, $\hat{H}(\mathbb{R})$ has a finitely generated dense subgroup $\hat{\Gamma}$ with Property (T). It follows that $P(\hat{\Gamma}) \cap G$ is a dense subgroup with Property (T) in G.

Sixth step. We now conclude. We have reduced to the case where $\mathfrak{g}$ is perfect. Therefore, we have to show that every connected Lie group satisfying the hypotheses recalled before the fifth step has a dense finitely generated subgroup with Property (T). As these hypotheses only depend on the Lie algebra $\mathfrak{g}$, we can suppose that G is simply

connected. Indeed, otherwise we can project a dense subgroup with Property (T) from its universal covering. Since $\mathfrak{g}$ is perfect, there exists a linear algebraic $\mathbb{R}$-group H with Lie algebra $\mathfrak{g}$, so that there exists a discrete, central subgroup Z of G such that G/Z is isomorphic to $H(\mathbb{R})_0$. By the fifth step, $H(\mathbb{R})_0 = G/Z$ has a dense subgroup Γ with Property (T).

Let $\tilde{\Gamma}$ be the preimage of Γ in G. Define Z_n as the kernel of the natural morphism $D^n(\tilde{\Gamma}) \to D^n(\Gamma)$, so that we have, for all n, an exact sequence:

$$1 \to Z_n \to D^n(\tilde{\Gamma}) \to D^n(\Gamma) \to 1.$$

Then (Z_n) is a decreasing sequence of subgroups of Z. Moreover, since Γ has Property (T), for every n, $D^n(\Gamma)$ has finite index in Γ. Accordingly, for each n such that $D^n(\tilde{\Gamma})/D^{n+1}(\tilde{\Gamma})$ is infinite, we have $\mathrm{rk}(Z_{n+1}) < \mathrm{rk}(Z_n)$ (where the rank of an abelian group A is by definition the dimension of the vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$). This implies the existence of n such that $D^n(\tilde{\Gamma})$ has finite abelianization. Therefore, by Serre's Theorem on central extensions [HV, Théorème 12], $D^n(\tilde{\Gamma})$ has Property (T). We finally claim that $D^n(\tilde{\Gamma})$ is dense in G: this follows from the fact that $\tilde{\Gamma}$ is dense in G and G is topologically perfect. The proof of Theorem 3 is now complete. □

Proof of Proposition 5. Suppose that G has such a simple factor S; through a Levi factor, S embeds in G as a (non-necessarily closed) subgroup of G. If S has Property (T), then it has a dense (hence infinite) subgroup with Property (T) (this follows from Theorem 3, since we have excluded SO(3)). Otherwise, S is locally isomorphic to SU$(n, 1)$ $(n \geq 3)$ or SO$(n, 1)$ $(n \geq 5)$. Then S has a compact subgroup K locally isomorphic to SU(n) $(n \geq 3)$ or SO(n) $(n \geq 5)$. By [Mar2, chap. III, Proposition 5.7] (or Theorem 3), K has a dense (hence infinite) finitely generated subgroup with Property (T).

Conversely, if G contains an infinite subgroup Γ with Property (T), then, since Γ is not virtually solvable, the projection of Γ modulo the radical is infinite, so that we are reduced to the case when G is semisimple; we assume this now. Similarly, the projection of Γ modulo the centre is infinite. So now we suppose that G is a connected, centre-free semisimple Lie group, hence a direct product of simple factors. The projection into at least one factor, say, S, must be infinite. It then suffices to show that S cannot be locally isomorphic to one of the five groups quoted in the proposition. Since each of these five groups has the Haagerup Property [HV, §6.d], i.e. acts properly on a Hilbert space, Γ must be contained in a maximal compact subgroup. Thus Γ maps into SO$_3(R)$ with infinite image, and this is in contradiction with Zimmer's result already used above [Zim]. □

Proof of Proposition 7. Let G be a simple connected Lie group, locally isomorphic to SL$_3(\mathbb{R})$, Sp$_4(\mathbb{R})$, or Sp$(2, 1)$, and Γ an infinite finitely generated subgroup with Property (T). Projecting modulo the centre, we can suppose that G is center-free and

thus is the Hausdorff unit component of an algebraic group. Let H be the Zariski closure of Γ.

First case: suppose that $H \neq G$. Then H has a simple factor S that is not one of the five groups quoted in Proposition 5.

Observe that $\dim(S) < \dim(G)$. If G is $\mathrm{SL}_3(\mathbb{R})$, then this implies $\dim(S) < 8$ and thus S is one of the five groups quoted in Proposition 5, contradiction. If G is $\mathrm{PSp}_4(\mathbb{R})$, then $\dim(G) = 10$ and we must have $\dim(S) = 8$, otherwise we contradict again Proposition 5. But passing to the complexification, we get an embedding of the simple 8-dimensional subalgebra $\mathfrak{sl}_3$ into the simple 10-dimensional simple Lie algebra $\mathfrak{sp}_4(\simeq \mathfrak{so}_5)$, and this does not exist (the root system A_2 does not embed in the root system B_2), a contradiction. If G is $\mathrm{PSp}(2, 1)$, then H has the Haagerup Property (see [Cor1, Theorem 1.10 and Remark 4.5]), i.e. has a unitary representation with almost invariant vectors, whose coefficients vanish at infinity. This forces Γ to be relatively compact.

Second case: suppose that Γ is Zariski dense. Then the Lie algebra of its Hausdorff closure is normalized by all of G, hence is either trivial or all of $\mathfrak{g}$, i.e. Γ is either discrete or dense.

Finally, suppose that G is non-compact with Property (T), and is not locally isomorphic to $\mathrm{Sp}(2, 1)$, $\mathrm{SL}_3(\mathbb{R})$, or $\mathrm{Sp}_4(\mathbb{R})$. If G has $\mathbb{R}$-rank one, then it is locally isomorphic to $\mathrm{Sp}(n, 1)$ ($n \geq 3$) or $F_{4(-20)}$ and therefore contains a proper, closed subgroup H locally isomorphic to $\mathrm{Sp}(2, 1)$. If G has rank at least 2, then it follows from the classification of root systems that G contains a closed subgroup H locally isomorphic to either $\mathrm{SL}_3(\mathbb{R})$ or $\mathrm{Sp}_4(\mathbb{R})$. In all cases, H contains a lattice Γ: this is an infinite non-Zariski-dense, discrete subgroup with Property (T) of G.   □

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