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Logarithmic plurigenera of smooth affine surfaces with finite Picard groups

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Dedicated to Professor Hisao Yoshihara on the occasion of his sixtieth birthday

Abstract. Let S be a smooth complex affine surface with finite Picard group. We prove that if $\bar{\kappa}(S) = 1$ (resp. $\bar{\kappa}(S) = 2$), then $\bar{P}_2(S) > 0$ (resp. $\bar{P}_6(S) > 0$) and determine the surface S when $\bar{\kappa}(S) \geq 0$ and $\bar{P}_6(S) = 0$. Moreover, we prove that if $\mathrm{Pic}(S) = (0)$, $\Gamma(S, \mathcal{O}_S)^* = \mathbb{C}^*$ and $\bar{P}_2(S) = 0$, then $S \cong \mathbb{C}^2$.

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1. Introduction

Throughout the present article, we work over the field of complex numbers $\mathbb{C}$.

In [19] and [30], Kuramoto and Tsunoda considered the problem finding the smallest positive integer m such that $\overline{P}_m(X) > 0$ for a smooth open algebraic surface X with $\overline{\kappa}(X) \geq 0$, where $\overline{P}_m(X)$ and $\overline{\kappa}(X)$ denote respectively the logarithmic m-genus and the logarithmic Kodaira dimension of X. It follows from [30, Theorem 3.3] that a smooth affine surface S has non-negative logarithmic Kodaira dimension if and only if $\overline{P}_{12}(S) > 0$. Recently, in [17] and [18], the author studied the problem for $\mathbb{Q}$-homology planes minus non-empty reduced algebraic curves, homology planes and complements of projective plane curves. In particular, we have the following results.

Theorem A (cf. [18, Theorem 1.1]). *Let X be a $\mathbb{Q}$-homology plane* (*for the definition, see Definition* 2.6) *and C a non-empty reduced algebraic curve on X. Then $\overline{\kappa}(X - C) = -\infty$ if and only if $\overline{P}_2(X - C) = 0$.*

Theorem B (cf. [18, Theorem 1.3]). *Let S be a homology plane* (*for the definition, see Definition* 2.6). *Then $S \cong \mathbb{C}^2$ if and only if $\overline{P}_2(S) = 0$.*

Theorem C (cf. [17, Theorem 1.1] and [18, Theorem 1.2]). *Let $B \subset \mathbb{P}^2$ be a (not necessarily irreducible) plane curve. Then $\bar{\kappa}(\mathbb{P}^2 - B) = -\infty$ if and only if $\bar{P}_6(\mathbb{P}^2 - B) = 0$. Moreover, if $\bar{\kappa}(\mathbb{P}^2 - B) \geq 0$ and $\bar{P}_3(\mathbb{P}^2 - B) = 0$, then B can be constructed as either Orevkov's curve C_4 or Orevkov's curve C_4^* (for the definitions, see* [26], [29]).

In the present article, we shall study logarithmic plurigenera of smooth affine surfaces with finite Picard groups. In Section 2, we recall some results on open algebraic surfaces which will be used later. Moreover, we prove that every smooth affine surface with $\bar{\kappa} \geq 0$ and $\bar{P}_2 = 0$ is rational (cf. Lemma 2.9). In Sections 3 and 4, we shall prove the following result.

Theorem 1.1. *Let S be a smooth affine surface with finite Picard group. Then the following assertions hold true.*

(1) *If $\bar{\kappa}(S) = 1$, then $\bar{P}_2(S) > 0$.*
(2) *If $\bar{\kappa}(S) = 2$, then $\bar{P}_6(S) > 0$.*
(3) *The surface S is isomorphic to the surface $Y\{2, 4, 4\}$ (see* [2, (8.53), (8.54)]) *if and only if $\bar{\kappa}(S) \geq 0$ and $\bar{P}_6(S) = 0$.*

Here we recall the surface $Y\{2, 4, 4\}$. Let $V_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let ℓ_1, ℓ_2 and ℓ_3 be three distinct irreducible curves with $\ell_i \sim \ell$, where ℓ is a fiber of a fixed ruling on V_0, and let $\bar{\ell}_1$, $\bar{\ell}_2$, and $\bar{\ell}_3$ be three distinct curves with $\bar{\ell}_i \sim M_0$, where M_0 is a minimal section of V_0. Set $P_1 := \ell_1 \cap \bar{\ell}_1$, $P_2 := \ell_2 \cap \bar{\ell}_1$, $P_3 := \ell_2 \cap \bar{\ell}_3$ and $P_4 := \ell_3 \cap \bar{\ell}_3$. Let $\mu_0 \colon V_1 \to V_0$ be the blowing-up with centers $P_1, \ldots, P_4$. Set $E_1 := \mu_0^{-1}(P_1)$ and $E_4 := \mu_0^{-1}(P_4)$. Let $\mu_1 \colon V_2 \to V_1$ be the blowing-up with centers $Q_1 := E_1 \cap \mu_0'(\ell_1)$ and $Q_2 := E_4 \cap \mu_0'(\ell_3)$. Set $V := V_2$ and $D := \mu_1'(E_1 + E_4 + \mu_0'(\sum_{i=1}^{3}(\ell_i + \bar{\ell}_i)))$. Then the surface $Y\{2, 4, 4\}$ is the surface $V - D$.

In Section 5, we shall prove the following results.

Theorem 1.2. *Let X be a smooth affine surface with finite Picard group and C a non-empty reduced algebraic curve on X. Then $\bar{\kappa}(X - C) = -\infty$ if and only if $\bar{P}_2(X - C) = 0$.*

Theorem 1.3. *Let $S =$ Spec A be a smooth affine surface with* $\mathrm{Pic}(S) = (0)$. *Then the following assertions hold true.*

(1) *$\bar{\kappa}(S) = -\infty$ if and only if $\bar{P}_2(S) = 0$.*
(2) *Assume further that $A^* = \mathbb{C}^*$, where A^* denotes the multiplicative group consisting of invertible elements of A and $\mathbb{C}^* = \mathbb{C} - \{0\}$. Then $S \cong \mathbb{C}^2$ if and only if $\bar{P}_2(S) = 0$.*

By [23, Lemma 1.1 (1)], the Picard group of every $\mathbb{Q}$-homology plane is finite. In particular, the Picard group of every homology plane is trivial. So, Theorem 1.2 (resp. Theorem 1.3) includes Theorem A (resp. Theorem B).

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2. Preliminary results

Given a connected smooth quasi-projective variety S, we denote by $\bar{p}_g(S)$ (resp. $\bar{P}_n(S)$ ($n \geq 2$), $\bar{\kappa}(S)$) the logarithmic geometric genus of S (resp. the logarithmic n-genus of S, the logarithmic Kodaira dimension of S). For the definitions, see [9], [21]. By a $(-n)$-curve, we mean a smooth complete rational curve with self-intersection number $(-n)$. A reduced effective divisor D is called an SNC-divisor (resp. NC-divisor) if D has only simple normal crossings (resp. normal crossings). For a $\mathbb{Q}$-divisor $G = \sum \alpha_i C_i$, where the C_i are irreducible components of G and $\alpha_i \in \mathbb{Q}$, we write as $\lfloor G \rfloor = \sum \lfloor \alpha_i \rfloor C_i$, where $\lfloor \alpha_i \rfloor$ is the greatest integer $\leq \alpha_i$. For an effective divisor F, we denote by $\# F$ the number of all irreducible components in $\operatorname{Supp} F$.

We recall some basic notions in the theory of peeling (cf. [21, Chapter 2]). Let (V, D) be a pair of a smooth projective surface V and an SNC-divisor D on V. We call such a pair (V, D) an *SNC-pair*. A connected curve T consisting of irreducible components of D (a connected curve in D, for short) is a twig if the dual graph of T is a linear chain and T meets $D - T$ in a single point at one of the end components of T. A connected curve R (resp. F) in D is a rod (resp. fork) if R (resp. F) is a connected component of D and the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of the minimal resolution of a non-cyclic quotient singularity). A connected curve E in D is rational (resp. admissible) if each irreducible component of E is rational (resp. if there are no (-1)-curves in $\operatorname{Supp} E$ and the intersection matrix of E is negative definite). An admissible rational twig T in D is maximal if T is not extended to an admissible rational twig with more irreducible components of D.

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$, $\{F_\nu\}$) be the set of all admissible rational maximal twigs (resp. all admissible rational rods, all admissible rational forks), where no irreducible components of T_λ's belong to R_μ's or F_ν's. Then there exists a unique decomposition of D as a sum of effective $\mathbb{Q}$-divisors $D = D^\# + \operatorname{Bk}(D)$ such that the following two conditions (1) and (2) are satisfied:

(1) $\operatorname{Supp}(\operatorname{Bk}(D)) = \left(\bigcup_\lambda T_\lambda\right) \cup \left(\bigcup_\mu R_\mu\right) \cup \left(\bigcup_\nu F_\nu\right)$.

(2) $(D^{\#} + K_V \cdot Z) = 0$ for every irreducible component Z of $\mathrm{Supp}(\mathrm{Bk}(D))$.

We call the $\mathbb{Q}$-divisor $\mathrm{Bk}(D)$ the bark of D and say that $D^{\#}$ is produced by the peeling of D.

Lemma 2.1. *Let (V, D) be an SNC-pair. Then we have*

$$h^0(V, n(D + K_V)) = h^0(V, \lfloor n(D^{\#} + K_V)\rfloor)$$

for every integer $n \geq 0$.

Proof. See [21, Lemma 3.10.1 (p. 106)]. □

Definition 2.2. A morphism ϕ from a smooth algebraic surface to a smooth algebraic curve is called a $\mathbb{P}^1$-*fibration* if a general fiber of ϕ is isomorphic to $\mathbb{P}^1$. Similarly, an $\mathbb{A}^1$-*fibration* and a $\mathbb{C}^*$-*fibration* are defined, where $\mathbb{C}^* = \mathbb{A}^1 - \{0\}$. A $\mathbb{C}^*$-fibration is said to be *untwisted* if it is a Zariski-locally trivial fibration on a non-empty Zariski open subset of the base. Otherwise, it is said to be *twisted*.

Lemma 2.3. *Let S be a smooth affine rational surface with $\bar{\kappa}(S) = 1$. Then there exists a $\mathbb{C}^*$-fibration $\phi\colon S \to T$ onto a smooth rational curve T.*

Proof. See [10, Theorem (2.3)]. (See also [21, Theorem 1.7.1 (p. 201)].) □

For a topological space T, $e(T)$ denotes the topological Euler characteristic of T. We recall some well-known results on the topological Euler characteristics of some affine surfaces (cf. Lemmas 2.4 and 2.5).

Lemma 2.4. *Let S be a smooth affine surface. Then the following assertions hold true.*

(1) *If $\bar{\kappa}(S) = 2$, then $e(S) > 0$.*

(2) *If $\bar{\kappa}(S) = 0$ or 1, then $e(S) \geq 0$.*

Proof. (1) See [24, Theorem 1.4]. (See also [5].)

(2) It follows from [5, Section 5] that if $e(S) < 0$, then $\bar{\kappa}(S) = -\infty$. So the assertion follows. □

The following result is usually called the Suzuki–Zaidenberg formula (cf. [32, Lemma 3.2] and [3]).

Lemma 2.5. *Let S be a smooth affine surface and $\phi\colon S \to T$ a morphism onto a smooth curve T. Then*

$$e(S) = e(T)e(f) + \sum_i (e(f_i) - e(f)),$$

where f is a general fiber of ϕ and the summation is over all the singular fibers of ϕ. Further, $e(f_i) \geq e(f)$ for all i and the equality holds if and only if either $f \cong \mathbb{A}^1$ or $f \cong \mathbb{C}^*$ and $(f_i)_{\mathrm{red}} \cong f$.

We recall some results on $\mathbb{Q}$-homology planes.

Definition 2.6. A smooth algebraic surface S is called a $\mathbb{Q}$-homology plane (resp. a homology plane) if $H_i(S; \mathbb{Q}) = (0)$ (resp. $H_i(S; \mathbb{Z}) = (0)$) for any positive integer i.

It is well-known that every $\mathbb{Q}$-homology plane is affine and rational (see [2], [27], [7], [6]).

Lemma 2.7. *Let S be a $\mathbb{Q}$-homology plane and $\phi\colon S \to T$ a $\mathbb{C}^*$-fibration onto a smooth curve T. Then T is isomorphic to $\mathbb{P}^1$ or $\mathbb{A}^1$. Moreover, the following assertions hold true.*

(1) *If $T \cong \mathbb{P}^1$, then ϕ is untwisted, all fibers of ϕ are irreducible and there exists exactly one fiber f with $f_{\mathrm{red}} \cong \mathbb{A}^1$ (all the other fibers are isomorphic to $\mathbb{C}^*$, if taken with reduced structure).*

(2) *If $T \cong \mathbb{A}^1$ and ϕ is untwisted, then all fibers of ϕ are irreducible except for one singular fiber which consists of two irreducible components. If $T \cong \mathbb{A}^1$ and ϕ is twisted, then all fibers are irreducible and there exists exactly one fiber f with $f_{\mathrm{red}} \cong \mathbb{A}^1$ (all the other fibers are isomorphic to $\mathbb{C}^*$, if taken with reduced structure).*

(3) *If S is a homology plane, then $T \cong \mathbb{P}^1$.*

Proof. The assertions (1) and (2) follow from [23, Lemma 1.4]. The assertion (3) follows from [4, Theorems 3 and 4]. □

We shall prove some results on smooth affine surfaces with $\bar{p}_g = 0$ or $\bar{P}_2 = 0$ (cf. Lemmas 2.8 and 2.9).

Lemma 2.8. *Let S be a smooth affine rational surface with $\bar{p}_g(S) = 0$ and with finite Picard group. Then the following assertions hold true.*

(1) *If $\bar{\kappa}(S) \geq 0$, then $e(S) = 0$ or 1. Moreover, $e(S) = 1$ if and only if S is a $\mathbb{Q}$-homology plane.*

(2) *If $\bar{\kappa}(S) = 2$, then $e(S) = 1$. In particular, S is a $\mathbb{Q}$-homology plane.*

Proof. Let (V, D) be an SNC-pair such that $V - D \cong S$ and let $D = \sum_{i=1}^{r} D_i$ be the decomposition of D into irreducible components, where $r = \#D$. Since $\mathrm{Pic}(S) = \mathrm{Pic}(V - D)$ is finite, we have $r \geq \rho(V)$, where $\rho(V)$ denotes the Picard number of V. Since $\bar{p}_g(S) = h^0(V, D + K_V) = 0$ and V is a rational surface, we infer from [20, Lemma I.2.1.3] that $D_i \cong \mathbb{P}^1$ for $1 \leq i \leq r$ and the dual graph of D is a tree. So, D is simply connected and $e(D) = 1 + r$. Thus,

$$e(S) = e(V) - e(D) = \rho(V) - r + 1 \leq 1.$$

Lemma 2.4 implies that $e(S) = 0$ or 1 (resp. $e(S) = 1$) provided $\bar{\kappa}(S) = 0$ or 1 (resp. $\bar{\kappa}(S) = 2$). Suppose that $e(S) = 1$. Then $r = \rho(V)$ and so the natural homomorphism $H^2(V; \mathbb{Q}) \to H^2(D; \mathbb{Q})$ is an isomorphism. By [22, Lemma 2.1 (3)], S is a $\mathbb{Q}$-homology plane. □

Lemma 2.9. *Let S be a smooth affine surface with $\bar{\kappa}(S) \geq 0$ and $\bar{P}_2(S) = 0$. Then S is a rational surface.*

Proof. Let S be a smooth affine surface with $\bar{\kappa}(S) \geq 0$. It suffices to show that $\bar{P}_2(S) > 0$ if S is not a rational surface. Let (V, D) be an SNC-pair such that $V - D \cong S$. We treat the following four cases separately.

Case 1. $\kappa(V) = 2$. It then follows that $P_2(V) > 0$ (see [1, Theorem 9.1]). Hence, $\bar{P}_2(S) \geq P_2(V) > 0$.

Case 2. V is an irrational ruled surface. In this case, there exists an $\mathbb{P}^1$-fibration $p\colon V \to B$ onto a smooth projective curve B with $g(B) = h^1(V, \mathcal{O}_V)(> 0)$, where $g(B)$ denotes the genus of B. Let $D' = \sum_{i=1}^{\ell} D_i$ $(\ell \geq 0)$ be the sum of all components of D that are not fiber components of p. Since $\bar{\kappa}(S) = \bar{\kappa}(V - D) \geq 0$, we have $(F \cdot D) = (F \cdot D') \geq 2$ for a fiber F of p. It then follows from [20, Lemmas I.2.3.1 and I.2.3.2] that $\bar{\kappa}(V - D') \geq 0$. By [28, Proposition 2.2], we have $\bar{P}_2(V - D') > 0$, here we note that the divisor D' is semi-stable in the sense of [28] because it is an SNC-divisor and contains no rational curves. Hence, $\bar{P}_2(S) = \bar{P}_2(V - D) \geq \bar{P}_2(V - D') > 0$.

Case 3. $\kappa(V) = 0$. If $P_2(V) > 0$, then $\bar{P}_2(S) \geq P_2(V) > 0$. So we may assume that $P_2(V) = 0$. Then V is a hyperelliptic surface and so there exists an elliptic fibration $f\colon V \to E$ onto a smooth projective elliptic curve E. Since $V - D = S$ is affine, D contains an irreducible curve D_1 that is not a fiber component of f. Then $g(D_1) > 0$, i.e., D_1 is semi-stable in the sense of [28]. Since $\bar{\kappa}(V - D_1) \geq \kappa(V) = 0$, we have $\bar{P}_2(V - D_1) > 0$ by [28, Proposition 2.2]. Hence, $\bar{P}_2(S) \geq \bar{P}_2(V - D_1) > 0$.

Case 4. $\kappa(V) = 1$. Then, there exists an elliptic fibration $f\colon V \to B$ onto a smooth projective curve B. We may assume that $P_2(V) = 0$ (cf. Case 3). Then $p_g(V) = 0$ and $g(B) \leq 1$. We note that D contains an irreducible component D_1 that is not a fiber component of f because $S = V - D$ is affine. Assume that $g(B) = 1$. Then, $g(D_1) \geq 1$. Hence, by using the same argument as in Case 3, we know that $\bar{P}_2(S) \geq \bar{P}_2(V - D_1) > 0$.

Assume that $g(B) = 0$. By the canonical bundle formula for elliptic fibrations, we know that $P_2(V) > 0$ if $\chi(\mathcal{O}_V) \geq 1$. So, we may assume further that $\chi(\mathcal{O}_V) = 0$. Then $h^1(V, \mathcal{O}_V) = 1$. Let $\alpha\colon V \to E$ be the Albanese mapping of V, where E is a smooth projective elliptic curve. Then, by using the same argument as in the case $g(B) = 1$, we know that $\bar{P}_2(S) > 0$.

The proof of Lemma 2.9 is thus completed. □

Now, we recall some results on log del Pezzo surfaces of rank one. Let $\overline{V}$ be a normal projective surface with only quotient singular points, let $\pi\colon V \to \overline{V}$ be the minimal resolution of the singularities on $\overline{V}$ and let D be the reduced exceptional divisor with respect to π. We often denote (V, D) and $\overline{V}$ interchangeably. Since $\overline{V}$ has only quotient singular points, D is an SNC-divisor and $D^\# + K_V \equiv \pi^*(K_{\overline{V}})$ (for the definition of $D^\#$, see before Lemma 2.1).

Definition 2.10. The above surface $\overline{V}$ (or the above pair (V, D)) is called a log del Pezzo surface if the anticanonical divisor $-K_{\overline{V}}$ is ample. A log del Pezzo surface is said to have rank one if its Picard number equals one. In the present article, we call a log del Pezzo surface of rank one an *LDP*1*-surface*.

Hereafter in the present section, we assume that $\overline{V}$ is an LDP1-surface and we use the same notation as above.

Lemma 2.11. *With the same notation and assumptions as above, the following assertions hold true.*

(1) $-(D^\# + K_V)$ *is a nef and big* $\mathbb{Q}$*-Cartier divisor. Moreover, for any irreducible curve* F, $-(D^\# + K_V \cdot F) = 0$ *if and only if* F *is a component of* D.

(2) *Any* $(-n)$*-curve with* $n \geq 2$ *on* V *is a component of* D.

(3) V *is a rational surface.*

Proof. See [34, Lemma 1.1]. □

Lemma 2.12. *There is no* (-1)*-curve* E *on* V *such that the divisor* $E + D$ *has negative definite intersection matrix.*

Proof. See [33, Lemma 1.4]. □

By Lemma 2.11 (1), if C is an irreducible curve not contained in $\operatorname{Supp} D$, then $-(C \cdot D^\# + K_V)$ takes value in $\{n/p \mid n \in \mathbb{N}\}$, where p is the smallest positive integer such that $pD^\#$ is an integral divisor. So we can find an irreducible curve C such that $-(C \cdot D^\# + K_V)$ attains the smallest positive value. We denote the set of such irreducible curves by $\mathrm{MV}(V, D)$. The pair (V, D) is said to be of the first kind if there exits an irreducible curve $C \in \mathrm{MV}(V, D)$ such that $|C + D + K_V| \neq \emptyset$. Otherwise, the pair (V, D) is said to be of the second kind.

Lemma 2.13. *Assume that (V, D) is of the first kind and that $\overline{V}$ has a singular point P that is not a rational double point. Then the following assertions hold true.*

(1) *Let $C \in \mathrm{MV}(V, D)$ be an irreducible curve such that $|C + D + K_V| \neq \emptyset$. Then there exists a unique decomposition of D as a sum of effective integral divisors $D = D' + D''$ such that the following two conditions are satisfied*:

 (i) $(C \cdot D_i) = (D'' \cdot D_i) = (K_V \cdot D_i) = 0$ *for any component D_i of D'.*

 (ii) $C + D'' + K_V \sim 0$.

(2) *The singular point P is a cyclic quotient singular point and the other singular points on $\overline{V}$ are rational double points.*

Proof. The assertion (1) follows from [33, Lemma 2.1]. We prove the assertion (2). With the same notation as in the assertion (1), we know that $\mathrm{Supp}\, D' \cap \mathrm{Supp}\, D'' = \emptyset$ and each connected component of D' can be contracted to a rational double point. By the hypothesis that P is not a rational double point, we have $D'' \neq 0$. Since $C + D'' + K_V \sim 0$, $|C + K_V| = |-D''| = \emptyset$. So C is a smooth rational curve and $(C \cdot D'') = (C \cdot -C - K_V) = 2$. Further, for every irreducible component D_i of D'', we have $(D_i \cdot C + D'' - D_i) = (D_i \cdot -K_V - D_i) = 2$. Hence we know that $D'' = \pi^{-1}(P)$ and D'' is a linear chain of smooth rational curves. □

Lemma 2.14. *Assume that (V, D) is of the second kind and $\rho(V) \geq 3$. Then every irreducible curve $C \in \mathrm{MV}(V, D)$ is a (-1)-curve.*

Proof. See [33, Lemma 2.2] and [8, Proposition 3.6]. □

Lemma 2.15. *Let $\Phi\colon V \to \mathbb{P}^1$ be a $\mathbb{P}^1$-fibration. Assume that there exists a singular fiber F whose configuration is given as one of* (i) *and* (ii) *in Figure* 1 *and that $C \in$* $\mathrm{MV}(V, D)$*, where C is the unique (-1)-curve in* Supp F*. Then each singular fiber of Φ consists of (-2)-curves and (-1)-curves, say E_1 and E_2 (possibly $E_1 = E_2$), and $E_i \in$* $\mathrm{MV}(V, D)$ *for $i = 1, 2$.*

Proof. See [33, Lemma 1.6 (3)]. □

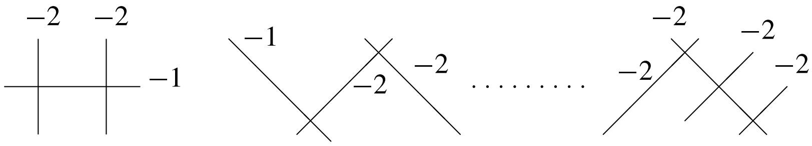


Figure 1

3. Proof of the assertion (1) of Theorem 1.1

In this section, we shall prove the assertion (1) of Theorem 1.1.

Let S be a smooth affine surface with $\bar{\kappa}(S) = 1$ and with finite Picard group. If S is not a rational surface, then $\bar{P}_2(S) \geq 1$ by Lemma 2.9. So we may assume that S is a rational surface. Further, we may assume that $\bar{p}_g(S) = 0$. Lemma 2.8 (1) then implies that $e(S) = 0$ or 1 and $e(S) = 1$ if and only if S is a $\mathbb{Q}$-homology plane. By Lemma 2.3, there exists a $\mathbb{C}^*$-fibration $\phi\colon S \to T$ onto a smooth rational curve T. The $\mathbb{C}^*$-fibration ϕ is extended to a $\mathbb{P}^1$-fibration $\Phi\colon V \to \mathbb{P}^1$, where V is a smooth projective rational surface such that $D := V - S$ is an SNC-divisor. Since $\bar{p}_g(S) = 0$ and S is a rational surface, D is a tree of smooth rational curves by [20, Lemma I.2.1.3].

The following lemma can be proved by using the same argument as in [18, Section 3]. For the sake of completeness, we shall reproduce the proof.

Lemma 3.1. *With the same notation and assumptions as above, assume further that* $e(S) = 0$. *Then* $\bar{P}_2(S) > 0$.

Proof. Since $e(S) = 0$, it follows from Lemma 2.5 that every fiber of ϕ is isomorphic to $\mathbb{C}^*$ if taken with reduced structure. We shall consider the following two cases separately.

Case 1. ϕ is twisted. In this case, D contains exactly one irreducible component H that is not a fiber component of Φ. The curve H is then a 2-section of Φ and hence $\Phi|_H\colon H \to \mathbb{P}^1$ is a double covering. Since $H \cong \mathbb{P}^1$, there exist two branch points Q_1, Q_2 ($\in \mathbb{P}^1$) of $\Phi|_H$. Set $F_i := \Phi^{-1}(Q_i)$ for $i = 1, 2$.

Suppose that $\mathrm{Supp}\,(F_i) \cap S \neq \emptyset$ for $i = 1$ or 2. Since D is connected and $\#(\mathrm{Supp}\,(F_i) \cap H) = 1$, $(F_i|_S)_{\mathrm{red}}$ contains an affine line. This is a contradiction. So, $\mathrm{Supp}\,(F_i) \subset \mathrm{Supp}\,D$ for $i = 1, 2$ and hence T is contained in $\mathbb{C}^*$ as a Zariski open subset. Since $\bar{\kappa}(T) \geq \bar{\kappa}(\mathbb{C}^*) = 0$, it follows from [19, Proposition 1] that $\bar{P}_2(S) > 0$.

Case 2. ϕ is untwisted. In this case, D contains exactly two sections H_1 and H_2 of Φ and each component of $D - (H_1 + H_2)$ is a fiber component of Φ. Since D is a tree of smooth rational curves, we may assume that $(E \cdot D - E) \geq 3$ for any (-1)-curve $E \subset \mathrm{Supp}\,(D - (H_1 + H_2))$ (i.e., (V, D) is minimal along fibers (cf. [4, p. 87])).

Claim 1. $(H_1 \cdot H_2) = 0$.

Proof. Suppose that $(H_1 \cdot H_2) > 0$. Let P be a point of $H_1 \cap H_2$ and set $F_P := \Phi^{-1}(\Phi(P))$. Since D is a connected SNC-divisor and $\mathrm{Supp}\, F_P \cap H_1 \cap H_2 = \{P\}$, $\mathrm{Supp}\,(F_P)$ contains no components of D. Then, $\mathrm{Supp}\, F_P \cap S \cong \mathbb{A}^1$, a contradiction. Hence, $(H_1 \cdot H_2) = 0$. □

Claim 2. *Let F be a fiber of Φ. Then F is reducible if and only if $F|_S (\neq \emptyset)$ is a singular fiber of ϕ.*

Proof. The "if" part is clear because every singular fiber of ϕ is multiple. We shall prove the "only if" part. Suppose that F is a reducible fiber of Φ. Then F contains a (-1)-curve E. We note that, if the coefficient of E in F equals one, then [2, Lemma (7.3)] implies that $(E \cdot F_{\mathrm{red}} - E) = 1$ and F contains another (-1)-curve. Suppose that $\mathrm{Supp}\, F \subset \mathrm{Supp}\, D$. Then E meets both H_1 and H_2 because H_1 and H_2 are sections of Φ and $(E \cdot D - E) \geq 3$. Then the coefficient of E in F equals one and hence F has another (-1)-curve E'. Then $(E' \cdot D - E') = (E' \cdot F_{\mathrm{red}} - E') \leq 2$, a contradiction. Suppose that $F|_S (\neq \emptyset)$ is not a singular fiber of ϕ. Let F_0 be the component of F with $F_0 \cap S \neq \emptyset$. Since the coefficient of F_0 in F equals one, F has a (-1)-curve other than F_0. So we may assume that $E \neq F_0$. Then E is a component of D. Since H_1 and H_2 are sections of Φ and $(E \cdot D - E) \geq 3$, we know that E meets both H_1 and H_2. So $(F_0 \cdot H_1) = (F_0 \cdot H_2) = 0$. Since $F_0|_S \cong \mathbb{C}^*$ and $\mathrm{Supp}\,(F_{\mathrm{red}} - F_0) \subset \mathrm{Supp}\, D$, $\mathrm{Supp}\, D$ is not connected, a contradiction. Therefore, we know that $F|_S (\neq \emptyset)$ is a singular fiber of ϕ. □

Claim 3. *Let f be a singular fiber of ϕ and let F be the fiber of Φ containing f. Then the weighted dual graph of F_{red} is linear. Moreover, F has exactly one (-1)-curve, say E,* $\mathrm{Supp}\,(F_{\mathrm{red}} - E)$ *consists of two connected components and $(F|_S)_{\mathrm{red}} = E|_S$.*

Proof. The fiber F has exactly one irreducible component, say F_0, with $F_0 \cap S \neq \emptyset$. If F contains no components meeting both H_1 and H_2, then the assertions follow from [2, Lemma (7.6)]. Suppose that F contains a component F_1 meeting both H_1 and H_2. Then $F_0 \neq F_1$. Since $F_0|_S \cong \mathbb{C}^*$, $F_{\mathrm{red}} - F_0$ is not connected. This is a contradiction because D is connected. □

Let $m_1 f_1, \ldots, m_r f_r$ be all the singular fibers of ϕ with respective multiplicities $m_1, \ldots, m_r$, where $f_i \cong \mathbb{C}^*$ $(1 \leq i \leq r)$, and let F_i $(1 \leq i \leq r)$ be the fiber of Φ containing $m_i f_i$. Let $E_i = \bar{f_i}$ $(1 \leq i \leq r)$ be the closure of $f_i = (m_i f_i)_{\mathrm{red}}$ in V.

Claim 4. (1) $T \cong \mathbb{A}^1$.

(2) $r \geq 2$.

Proof. (1) If $T \cong \mathbb{P}^1$, then D cannot be connected by Claims 1 and 3. This is a contradiction. If $T \cong \mathbb{P}^1 \setminus \{s \text{ points}\}$ $(s \geq 2)$, then D contains a loop of (smooth rational) curves. So, $\bar{p}_g(S) = h^0(V, D + K_V) > 0$, which contradicts our assumption.

(2) Suppose that $r \leq 1$. Since ϕ is untwisted, S contains $\mathbb{C}^* \times \mathbb{C}^*$ as a Zariski open subset. Then

$$0 = \bar{\kappa}(\mathbb{C}^* \times \mathbb{C}^*) \geq \bar{\kappa}(S) = 1,$$

which is a contradiction. Hence, $r \geq 2$. □

Set $P_0 := \mathbb{P}^1 \setminus T$ and $F_0 := \Phi^{-1}(P_0)$. By Claim 2, F_0 is irreducible. Let $(F_i)_{\mathrm{red}} - E_i = \sum_{j=1}^{r_i} C_{ij}$ $(1 \leq i \leq r)$ be the decomposition of $(F_i)_{\mathrm{red}} - E_i$ into irreducible components. Then,

$$D = F_0 + H_1 + H_2 + \sum_{i=1}^{r} \Big(\sum_{j=1}^{r_i} C_{ij} \Big).$$

Since $(H_1 \cdot H_2) = 0$ by Claim 1, we obtain a birational morphism $\rho\colon V \to \mathbb{F}_a$ onto a Hirzebruch surface $\mathbb{F}_a$ of degree a such that $\overline{H}_1 := \rho(H_1)$ and $\overline{H}_2 := \rho(H_2)$ are sections of a $\mathbb{P}^1$-fibration $\Phi \circ \rho^{-1}\colon \mathbb{F}_a \to \mathbb{P}^1$ and $(\overline{H}_1 \cdot \overline{H}_2) = 0$. Since $(\overline{H}_1 \cdot \overline{H}_2) = 0$, we may assume that $(\overline{H}_1^2) \leq 0$. Then, $\overline{H}_1$ is a minimal section and $\overline{H}_2 \sim \overline{H}_1 + a\ell$, where ℓ is a fiber of the $\mathbb{P}^1$-fibration $\Phi \circ \rho^{-1}$. Moreover, we may assume that $(H_1^2) = (\overline{H}_1^2) = -a$, namely, V is obtained from $\mathbb{F}_a$ by starting the blowing-ups with centers at points on $\overline{H}_2$ or fibers of $\Phi \circ \rho^{-1}$, while no points on $\overline{H}_1$ are blown up.

Since $K_{\mathbb{F}_a} \sim -2\overline{H}_1 - (a+2)\ell \sim -\overline{H}_1 - \overline{H}_2 - 2\ell$, we have

$$K_V \sim -H_1 - \rho^*(\overline{H}_2) - 2F_0 + \sum_{i=1}^{r} \Big(\sum_{j=1}^{r_i} \lambda_{ij} C_{ij} \Big) + \sum_{i=1}^{r} \lambda_i E_i,$$

where $\lambda_i \geq 0$ and $\lambda_{ij} \geq 0$ for $1 \leq i \leq r$ and $1 \leq j \leq r_i$. We set

$$\rho^*(\overline{H}_2) = H_2 + \sum_{i=1}^{r} \Big(\sum_{j=1}^{r_i} \mu_{ij} C_{ij} \Big) + \sum_{i=1}^{r} \mu_i E_i$$

and

$$F_i = \sum_{j=1}^{r_i} \alpha_{ij} C_{ij} + m_i E_i$$

for $1 \leq i \leq r$. Since $D = F_0 + H_1 + H_2 + \sum_{i=1}^{r} (\sum_{j=1}^{r_i} C_{ij})$, we have

$$D + K_V \sim \sum_{i=1}^{r} \Big(\sum_{j=1}^{r_1} (\lambda_{ij} - \mu_{ij} + 1) C_{ij} \Big) + \sum_{i=1}^{r} (\lambda_i - \mu_i) E_i - F_0.$$

By Claim 2, the weighted dual graph of F_i ($1 \le i \le r$) looks like the one given in [20, p. 79]. As seen from [20, I.4.9.3 (p. 80)], we know that

$$F_i = \sum_{j=1}^{r_i} (\lambda_{ij} - \mu_{ij} + 1) C_{ij} + (\lambda_i - \mu_i + 1) E_i$$

for $1 \le i \le r$. Then,

$$D + K_V \sim \sum_{i=1}^{r} F_i - \sum_{i=1}^{r} E_i - F_0 \sim (r-1) F_0 - \sum_{i=1}^{r} E_i.$$

Since $r \ge 2$ by Claim 4 (2) and $2E_i \le m_i E_i \le F_i$ for $i = 1, \dots, r$, we have

$$2(D + K_V) \sim 2(r-1) F_0 - 2 \sum_{i=1}^{r} E_i \sim (r-2) F_0 + \sum_{i=1}^{r} (F_i - 2E_i) \ge 0.$$

Therefore, $\bar{P}_2(S) > 0$. □

Remark 3.2. As seen from the proof of Lemma 3.1, we know that a smooth affine surface with $\bar{\kappa} = 1$ and $e = 0$ has positive logarithmic bigenus.

In the following lemma, we shall treat the case $e(S) = 1$.

Lemma 3.3. *With the same notation and assumptions as above, assume further that $e(S) = 1$. Then $\bar{P}_2(S) > 0$.*

Proof. We shall consider the following two cases separately.

Case 1. ϕ is untwisted. In this case, D contains exactly two sections H_1 and H_2 of Φ and each component of $D - (H_1 + H_2)$ is a fiber component of Φ. As seen from [4, Section 3], we may assume that:

(i) $(H_1 \cdot H_2) = 0$.

(ii) If G is a (-1)-curve contained in $\mathrm{Supp}\,(D - (H_1 + H_2))$, then $(G \cdot D - G) \ge 2$. Moreover, if $(G \cdot D - G) = 2$, then $(G \cdot H_1) = (G \cdot H_2) = 1$.

By Lemma 2.7, ϕ has a singular fiber f satisfying one of the following two conditions (a) and (b):

(a) f is irreducible and $f_{\mathrm{red}} \cong \mathbb{A}^1$.

(b) f_{red} consists of two irreducible components f_1 and f_2, where $f_1 \cong \mathbb{A}^1$ and $f_2 \cong \mathbb{A}^1$ or $\mathbb{C}^*$, and $\#(f_1 \cap f_2) \le 1$.

Let E (resp. E_i $(i = 1, 2)$) be the closure of f (resp. f_i $(i = 1, 2)$) in V if the condition (a) (resp. the condition (b)) holds.

If the condition (a) holds, then E is a smooth rational curve. Moreover, by the assumptions (i) and (ii), E must be a (-1)-curve and $(E \cdot D) = 1$. Hence, $\bar{P}_n(S) = \bar{P}_n(V - (E + D))$ for any integer $n > 0$. If the condition (b) holds, then E_1 is a smooth rational curve. By the assumptions (i) and (ii), we may assume that E_1 is a (-1)-curve and $(E_1 \cdot D) = 1$. Then, $\bar{P}_n(S) = \bar{P}_n(V - (E_1 + D))$ for any integer $n > 0$.

Set $S' := S \setminus f$ (resp. $S' := S \setminus f_1$) if the condition (a) (resp. the condition (b)) holds. Then $e(S') = 0$, $\mathrm{Pic}(S')$ is finite and $\bar{\kappa}(S') = \bar{\kappa}(S) = 1$. We infer from Lemma 3.1 that $\bar{P}_2(S') > 0$. Hence, $\bar{P}_2(S) = \bar{P}_2(S') > 0$.

Case 2. ϕ is twisted. In this case, D contains exactly one component H that is not a fiber component of Φ. The curve H is a 2-section of Φ and so $\Phi|_H : H \to \mathbb{P}^1$ is a double covering. Since $H \cong \mathbb{P}^1$, there exist two branch points Q_0, $Q_\infty (\in \mathbb{P}^1)$ of $\Phi|_H$. Set $F_0 := \Phi^{-1}(Q_0)$ and $F_\infty := \Phi^{-1}(Q_\infty)$. By Lemma 2.7, we may assume that $\mathrm{Supp}\,(F_\infty) \subset \mathrm{Supp}\, D$ and $\mathrm{Supp}\,(F_0) \not\subseteq \mathrm{Supp}\, D$. Then $F_0 \cap S$ is written as $m_0 f_0$ with $f_0 \cong \mathbb{A}^1$. Let $m_i f_i$ $(1 \le i \le r)$ exhaust all singular fibers with $f_i \cong \mathbb{C}^*$ and $m_i \ge 2$. Let E_i be the closure of f_i in V and set $F_i := \Phi^{-1}(\Phi(E_i))$, $1 \le i \le r$. As seen from [21, p. 241], we may assume that the following conditions are satisfied:

(1) The dual graph of F_i $(1 \le i \le r)$ is a linear chain and E_i is a unique (-1)-curve in $\mathrm{Supp}\,(F_i)$. The fiber F_i meets the 2-section H at the terminal components.

(2) The dual graph of $F_\infty + H$ is given as in Figure 2.

(3) The dual graph of the fiber F_0 is that for F_∞ with the corresponding components A_0, B_0, C_0 and with A_0 either contained in $\mathrm{Supp}\, D$ or not.

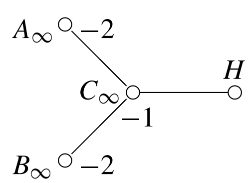


Figure 2

We need some explanation about the condition (3) as above (cf. [21, p. 241]). If A_0, which is the closure of f_0 in V, is a (-1)-curve, then $\bar{P}_n(S) = \bar{P}_n(S - f_0)$ for any integer $n > 0$. Since $e(S - f_0) = 0$ and $\bar{\kappa}(S - f_0) = 1$, it follows from Lemma 3.1 that $\bar{P}_2(S) > 0$. In this case, Lemma 3.3 follows. So we may assume that $(A_0^2) \le -2$ and that S is NC-minimal (for the definition, see [21, Definition 4.4.1 (p. 232)]). Then the contractions of $\mathrm{Supp}\,(F_0)$ except for A_0 makes its dual graph look like that for F_∞ with the component A_0 not contained in $\mathrm{Supp}\, D$.

By virtue of the proof of [21, Theorem 4.6.5 (p. 243)], we know that

$$D + K_V \sim_{\mathbb{Q}} -F + (B_0 + C_0) + \frac{1}{2}F + \frac{1}{2}(A_\infty + B_\infty) + \sum_{i=1}^{r}(F_i - E_i),$$

where F is a general fiber of Φ. So,

$$2(D + K_V) \sim -F + 2(B_0 + C_0) + (A_\infty + B_\infty) + \sum_{i=1}^{r}(2F_i - 2E_i),$$

here we note that V is a rational surface. Since $2F_i - 2E_i = F_i + (F_i - 2E_i) \geq F_i$ for $1 \leq i \leq r$, we have $2(D + K_V) \geq 0$ if $r \geq 1$. If $r = 0$, then

$$N := r - \frac{1}{2} - \sum_{i=1}^{r}\frac{1}{m_i} = -\frac{1}{2} < 0.$$

Then [21, Theorem 4.6.5 (1) (p. 243)] implies that $\bar{\kappa}(S) = -\infty$, which is a contradiction. Therefore, we know that $\bar{P}_2(S) > 0$. □

The proof of the assertion (1) of Theorem 1.1 is thus completed.

4. Proof of the assertions (2) and (3) of Theorem 1.1

The assertion (3) of Theorem 1.1 easily follows from [2, Theorem (8.70) 1)] and the assertions (1) and (2) of Theorem 1.1. Indeed, if S is a smooth affine surface with $\bar{\kappa}(S) \geq 0$, with $\bar{P}_6(S) = 0$ and with finite Picard group, then $\bar{\kappa}(S) = 0$ by the assertions (1) and (2) of Theorem 1.1. By [2, Theorem (8.70) 1)], S is then isomorphic to the surface $Y\{2, 4, 4\}$, here we note that the surface $Y\{3, 3, 3\}$ and the surface $Y\{2, 3, 6\}$ have positive logarithmic 6-genera. From now on, we shall prove the assertion (2) of Theorem 1.1.

Let S be a smooth affine surface with $\bar{\kappa}(S) = 2$ and with finite Picard group. By virtue of Lemma 2.9, we may assume that S is a rational surface. Moreover, by virtue of Lemma 2.8 (2), we may assume further that S is a $\mathbb{Q}$-homology plane.

Let (V, D) be a pair of a smooth projective rational surface and an NC-divisor D on V such that $V - D \cong S$ and $(E \cdot D - E) \geq 3$ for any (-1)-curve $E \subset \operatorname{Supp} D$. Since S is a $\mathbb{Q}$-homology plane, the divisor D is a tree of smooth rational curves by [23, Lemma 1.1 (1)]. In particular, D is an SNC-divisor.

Lemma 4.1. *The pair* (V, D) *is almost minimal* (*for the definition, see* [21, Chapter 2, Section 3]).

Proof. The assertion can be verified by using the same argument as in the proof of [17, Lemma 4.3]. □

Let $D+K_V \equiv (D+K_V)^+ + (D+K_V)^-$ be the Zariski–Fujita decomposition of $D+K_V$ (see [21, p. 69]), where $(D+K_V)^+$ is the nef part of $D+K_V$. By Lemma 4.1, $\mathrm{Supp}\,((D+K_V)^-) \subset \mathrm{Supp}\, D$ and $D-(D+K_V)^-$ is an effective $\mathbb{Q}$-Cartier divisor (for more details, see [21, Chapter 2, Section 3]). Moreover, $D^\# = D - (D+K_V)^-$ (for the definition of $D^\#$, see Section 2). Since $\bar{\kappa}(S) = 2$, it follows from Lemma 4.1 that $D^\# + K_V$ is a nef and big $\mathbb{Q}$-Cartier divisor.

Lemma 4.2. *For an integer $n \geq 2$, set $K_n := (n-1)K_V - \lfloor -(n-1)D^\# \rfloor + \lfloor D^\# \rfloor$. Then we have*

$$\bar{P}_n(S) \geq \frac{1}{2}(K_V + K_n \cdot K_n) + 1. \tag{4.1}$$

Proof. Since V is a rational surface and the SNC-pair (V, D) is almost minimal, the assertion follows from [30, Proposition 3.1], where we note that the $\mathbb{Q}$-divisor D_m in [30, Proposition 3.1] is $D^\#$. □

Since $\mathrm{Supp}\, D$ is connected, so is $\mathrm{Supp}\,(\lfloor D^\# \rfloor)$. Set $\ell := (\lfloor D^\# \rfloor \cdot D - \lfloor D^\# \rfloor)$ and let $C_1, C_2, \ldots, C_\ell$ exhaust all irreducible components of $D - \lfloor D^\# \rfloor$ meeting $\lfloor D^\# \rfloor$. Since D is a tree of smooth rational curves and $\bar{\kappa}(V-D) = 2$, it follows from [21, Corollary 2.11.1 (p. 82)] that ℓ equals the number of the maximal admissible rational twigs in D. For each i, $1 \leq i \leq \ell$, the coefficient of the curve C_i in $D^\#$ equals $(a_i - 1)/a_i$, where a_i is an integer ≥ 2. Let $D^{(i)}$ $(1 \leq i \leq \ell)$ be the maximal admissible rational twig in D with $C_i \subset \mathrm{Supp}\,(D^{(i)})$.

Since $D^\# + K_V$ is a nef and big $\mathbb{Q}$-Cartier divisor, it follows from the Kawamata–Viehweg vanishing theorem (see [11] and [31]) and the Riemann–Roch theorem that

$$h^0(V, K_V + K_n - \lfloor D^\# \rfloor) = \frac{1}{2}(K_V + K_n - \lfloor D^\# \rfloor \cdot K_n - \lfloor D^\# \rfloor) + 1 \tag{4.2}$$

for any integer $n \geq 2$. Since

$$n(D+K_V) \geq nK_V - \lfloor -(n-1)D^\# \rfloor + \lfloor D^\# \rfloor = K_V + K_n,$$

we have $\bar{P}_n(S) = h^0(V, n(D+K_V)) \geq h^0(V, K_V + K_n - \lfloor D^\# \rfloor)$ for any integer $n \geq 2$. By (4.1) and (4.2), we know that if $\bar{P}_n(S) = 0$, then

$$(\lfloor D^\# \rfloor \cdot (2n-1)K_V - 2\lfloor -(n-1)D^\# \rfloor + \lfloor D^\# \rfloor) \leq 0. \tag{4.3}$$

Lemma 4.3. *With the same notation and assumptions as above, assume further that $\bar{P}_2(S) = 0$. Then the following assertions hold true.*

(1) $\ell = 3$.

(2) $\lfloor D^{\#} \rfloor$ *is irreducible.*

Proof. If $\ell \leq 2$, then the dual graph of D is linear. Since $\overline{\kappa}(S) = 2$, it follows from [21, Corollary 2.11.1 (p. 82)] that the intersection matrix of D is negative definite, which is a contradiction. So, $\ell \geq 3$. Since $\lfloor D^{\#} \rfloor$ is a tree of smooth rational curves, we know that $(\lfloor D^{\#} \rfloor \cdot \lfloor D^{\#} \rfloor + K_V) = -2$. Then, by (4.3) for $n = 2$, we have

$$0 \geq (\lfloor D^{\#} \rfloor \cdot 3K_V + 2D + \lfloor D^{\#} \rfloor) = 2\ell - 6.$$

Hence, $\ell = 3$. This proves the assertion (1). The assertion (2) easily follows from the assertion (1). □

From now on, we assume that $\bar{P}_6(S) = 0$. Then $\bar{P}_2(S) = \bar{P}_3(S) = 0$. Set $D_0 := \lfloor D^{\#} \rfloor$, which is a smooth irreducible rational curve by Lemma 4.3 (2). Then, for any integer $n \geq 2$, we have

$$\begin{aligned} &(D_0 \cdot (2n-1)K_V - 2\lfloor -(n-1)D^{\#} \rfloor + D_0) \\ &\quad = 2(1-2n) - 2\sum_{i=1}^{3} \Big\lfloor -(n-1)\Big(1 - \frac{1}{a_i}\Big)\Big\rfloor. \end{aligned} \tag{4.4}$$

Lemma 4.4. *With the same notation and assumptions as above, the weighted dual graph of D is given as one of* (1)–(18) *in Figure* 3*, where the weights of the vertices corresponding to D_0 and (-2)-curves of D are omitted.*

Proof. We assume that $a_1 \leq a_2 \leq a_3$. By (4.3) and (4.4) for $n = 3$, we have

$$5 \geq -\sum_{i=1}^{3} \Big\lfloor -2\Big(1 - \frac{1}{a_i}\Big)\Big\rfloor.$$

Since $2 \leq a_1 \leq a_2 \leq a_3$, it follows that $a_1 = 2$.

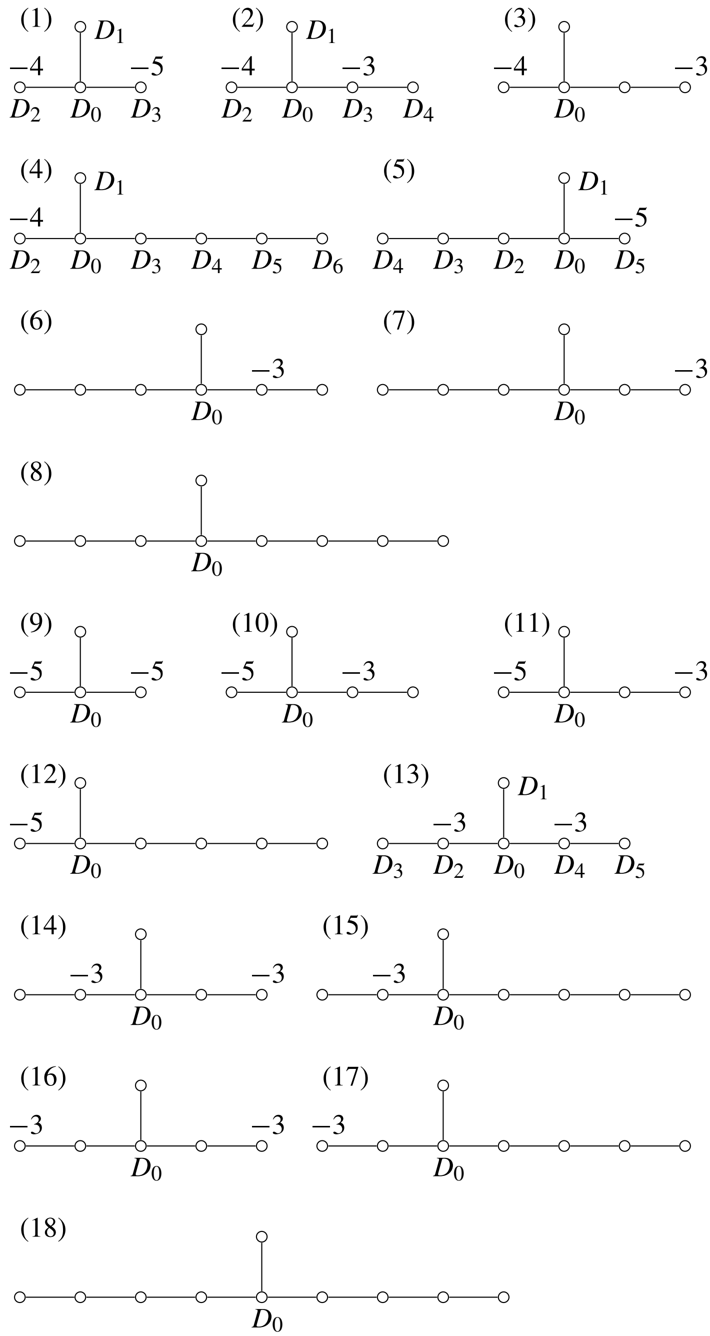
Since $(D \cdot D^{\#} + K_V) = (D_0 \cdot D^{\#} + K_V)$ and D and $D^{\#} + K_V$ are big $\mathbb{Q}$-Cartier divisors, we infer from the Hodge index theorem that

$$0 < (D_0 \cdot D^{\#} + K_V).$$

Since $(D_0 \cdot D^{\#} + K_V) = (D_0 \cdot D_0 + K_V) + 3 - \sum_{i=1}^{3} 1/a_i$ and $a_1 = 2$, we have

$$\frac{1}{a_2} + \frac{1}{a_3} < \frac{1}{2}.$$

In particular, we know that $3 \leq a_2 \leq a_3$ and that if $a_2 = 3$ (resp. $a_2 = 4$), then $a_3 \geq 7$ (resp. $a_3 \geq 5$).


(1)
D_1
-4
-5
D_2 D_0 D_3
(2)
D_1
-4
-3
D_2 D_0 D_3 D_4
(3)
-4
-3
D_0
(4)
D_1
-4
D_2 D_0 D_3 D_4 D_5 D_6
(5)
D_1
-5
D_4 D_3 D_2 D_0 D_5
(6)
-3
D_0
(7)
-3
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(8)
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-3
-3
D_3 D_2 D_0 D_4 D_5
(14)
-3
-3
D_0
(15)
-3
D_0
(16)
-3
-3
D_0
(17)
-3
D_0
(18)
D_0

Figure 3

By (4.3) and (4.4) for $n = 6$, we know that

$$11 \geq -\sum_{i=1}^{3} \left\lfloor -5\left(1 - \frac{1}{a_i}\right)\right\rfloor.$$

Since $a_1 = 2$ and $3 \leq a_2 \leq a_3$, we have

$$8 \geq -\sum_{i=2}^{3} \left\lfloor -5\left(1 - \frac{1}{a_i}\right)\right\rfloor$$

and

$$\left\lfloor -5\left(1 - \frac{1}{a_i}\right)\right\rfloor \leq -4$$

for $i = 2, 3$. Then $\lfloor -5(1-(1/a_i))\rfloor = -4$ for $i = 1, 2$ and hence $a_3 \leq 5$. Therefore, $(a_2, a_3) = (4, 5)$ or $(5, 5)$.

By using [21, Lemma 3.4.2 (p. 92)], we can determine the possible weighted dual graphs of $D - D_0$. This proves Lemma 4.4. □

Set $D' := D - D_0$. Then D' can be contracted to three cyclic quotient singular points.

Lemma 4.5. *With the same notation and assumptions as above, the pair (V, D') is an LDP1-surface.*

Proof. Since $S = V - D$ is a $\mathbb{Q}$-homology plane, we know that $\rho(V) = 1 + \#D'$. If $\bar{\kappa}(V - D') = -\infty$, then (V, D') is an LDP1-surface by [33, Remark 1.2 (2)]. So we shall prove that $\bar{\kappa}(V - D') = -\infty$.

Set $D_1 := D^{(1)} (= C_1)$. Since D_1 is a (-2)-curve and $(D_1 \cdot D' - D_1) = 0$, the coefficient of D_1 in $(D')^{\#}$ equals zero. So, $\bar{\kappa}(V - D') = \bar{\kappa}(V - (D' - D_1))$ by Lemma 2.1. We treat three cases (4), (5) and (18) as in Figure 3.

Case (4). In this case, $\rho(V) = 7$, $(K_V^2) = 3$ and

$$D^{\#} = D_0 + \frac{1}{2}D_1 + \frac{3}{4}D_2 + \frac{4}{5}D_3 + \frac{3}{5}D_4 + \frac{2}{5}D_2 + \frac{1}{5}D_6.$$

So $(D_0 \cdot D^{\#} + K_V) = 1/20$ and $(D^{\#} + K_V)^2 = (K_V \cdot D_0) + 91/20$. By the log Miyaoka–Yau inequality (see [13], [12]), we know that

$$(0 <)(D^{\#} + K_V)^2 = (K_V \cdot D_0) + \frac{91}{20} \leq 3e(V - D) = 3.$$

Hence, $(D_0^2) \geq 0$. By virtue of [21, Corollary 2.11.1 (p. 82)], we know that $\bar{\kappa}(V - (D - D_1)) = -\infty$. Therefore,

$$\bar{\kappa}(V - D') = \bar{\kappa}(V - (D' - D_1)) = \bar{\kappa}(V - (D - D_1)) = -\infty.$$

Case (5). Since $S = V - D$ is affine, the intersection matrix of D is not negative definite. Then $(D_0^2) \geq -1$ and so $\bar{\kappa}(V - (D - D_1)) = -\infty$ by [21, Corollary 2.11.1 (p. 82)]. Therefore,

$$\bar{\kappa}(V - D') = \bar{\kappa}(V - (D' - D_1)) = \bar{\kappa}(V - (D - D_1)) = -\infty.$$

Case (18). In this case, $(D')^\# = 0$. So, $\bar{\kappa}(V - D') = \kappa(V) = -\infty$ by Lemma 2.1.

The other cases can be treated similarly. □

Now, we shall consider the cases (1)–(18) as in Figure 3 separately. We shall use the same notation as in Figure 3.

Cases (1) *and* (9). We consider Case (1); Case (9) can be treated similarly. In this case, $\rho(V) = 4$ and $(D')^\# = (1/2)D_2 + (3/5)D_3$. Let C be an irreducible curve in $\mathrm{MV}(V, D')$ (for the definition, see Section 2). By virtue of Lemmas 2.13 and 2.14, we know that C is a (-1)-curve and $|C + D' + K_V| = \emptyset$. So, $(C \cdot D_i) = 0$ or 1 for $i = 1, 2, 3$. Since $-(C \cdot (D')^\# + K_V) = 1 - (1/2)(D_2 \cdot C) - (3/5)(C \cdot D_3) > 0$, it follows that $(C \cdot D_2) = 0$ or $(C \cdot D_3) = 0$. Then the intersection matrix of $C + D'$ is negative definite, which contradicts Lemma 2.12. Therefore, Case (1) does not take place.

Cases (2)*,* (3)*,* (10) *and* (11). We consider Case (2); Cases (3), (10) and (11) can be treated similarly. In this case, $\rho(V) = 5$ and $(D')^\# = (1/2)D_2 + (2/5)D_3 + (1/5)D_4$. Let C be an irreducible curve in $\mathrm{MV}(V, D')$. By virtue of Lemmas 2.13 and 2.14, we know that C is a (-1)-curve and $|C + D' + K_V| = \emptyset$. So, $(C \cdot D_i) = 0$ or 1 for $i = 1, 2, 3, 4$ and $(C \cdot D_3 + D_4) = 0$ or 1. If $(C \cdot D_1) = 0$, then we know that the intersection matrix of $C + D'$ is negative definite, which contradicts Lemma 2.12. So, $(C \cdot D_1) = 1$.

Suppose that $(C \cdot D_4) = 0$. Since the intersection matrix of $C + D'$ is not negative definite, we know that $(C \cdot D_2) = (C \cdot D_3) = 1$. Let $\mu\colon V \to V'$ be the contraction of C, D_1, D_3, D_4. Then $V' \cong \mathbb{P}^2$ and $(\mu_*(D_2)^2) = -4 + 2 + 4 + 4 = 6$. This is a contradiction. Hence, $(C \cdot D_4) = 1$ and so $(C \cdot D_3) = 0$. Then a divisor $F := 2C + D_1 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|}\colon V \to \mathbb{P}^1$ and D_3 is a section of Φ. By virtue of Lemma 2.15, we know that $(C \cdot D_2) = 1$ (i.e., D_2 is not a fiber component of Φ). In particular, D_2 is a 2-section of Φ. Since $\rho(V) = 5$, Φ has a singular fiber G other than F. Lemma 2.11 (2) implies that $G = E_1 + E_2$, where E_1 and E_2 are (-1)-curves and $(E_1 \cdot E_2) = 1$. Then we can easily see that either $E_1 + D'$ or $E_2 + D'$ has negative definite intersection matrix. This contradicts Lemma 2.12. Therefore, Case (2) does not take place.

Cases (4)*,* (5) *and* (12). We consider Case (4); Cases (5) and (12) can be treated similarly. In this case, $\rho(V) = 7$ and $(D')^\# = (1/2)D_2$. Let C be an irreducible curve in $\mathrm{MV}(V, D')$.

Assume that $|C + D' + K_V| \neq \emptyset$. Then Lemma 2.13 implies that $(C \cdot D_2) = 2$, $C + D_2 + K_V \sim 0$ and C is a smooth rational curve. Then

$$(0 <) - (C \cdot (D')^{\#} + K_V) = -(C \cdot K_V) - 1$$

and so $(C^2) \geq 0$ and $-(C \cdot (D')^{\#} + K_V) \geq 1$. Since V contains a (-1)-curve E and $-(E \cdot (D')^{\#} + K_V) \leq 1$, we have $(C^2) = 0$. So, C defines a $\mathbb{P}^1$-fibration $\Phi_{|C|}\colon V \to \mathbb{P}^1$. Let G_1 and G_2 be the fibers of $\Phi_{|C|}$ containing D_1 and $D_3 + D_4 + D_5 + D_6$, respectively. Then $G_1 \neq G_2$, $\#G_1 \geq 3$ and $\#G_2 \geq 6$. Then we have

$$7 = \rho(V) \geq 2 + (\#G_1 - 1) + (\#G_2 - 1) \geq 9,$$

which is a contradiction. Therefore, $|C + D' + K_V| = \emptyset$. In particular, $(C \cdot D_i) = 0$ or 1 for $i = 1, 2, \ldots, 6$. By Lemma 2.14, C is a (-1)-curve.

If $(C \cdot D_2) = 0$, then there exists an effective divisor Δ with $\operatorname{Supp} \Delta \subset \operatorname{Supp}(D_1 + D_3 + D_4 + D_5 + D_6)$ such that $2C + \Delta$ defines a $\mathbb{P}^1$-fibration $\Phi_{|2C+\Delta|}\colon V \to \mathbb{P}^1$ and D_2 is a fiber component of $\Phi_{|2C+\Delta|}$. This contradicts Lemma 2.15. So, $(C \cdot D_2) = 1$.

Suppose that $(C \cdot D_1) = 1$. Since the intersection matrix of $C + D'$ is not negative definite, we know that $(C \cdot D_3 + D_4 + D_5 + D_6) = (C \cdot D_j) = 1$ for some j, $3 \leq j \leq 6$. Then $F := 2C + D_1 + D_j$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|}\colon V \to \mathbb{P}^1$ and D_2 is a 2-section of Φ. We may assume that $j = 3$ or 4. Consider the case $j = 3$. Then D_4 is a section of Φ. Let F' be the fiber of Φ containing D_5 and D_6. By considering Lemma 2.11 (2), we know that $F' = E + D_5 + D_6 + E'$, where E and E' are (-1)-curves and $(E \cdot D_5) = (E' \cdot D_6) = 1$. Since $-(E \cdot (D')^{\#} + K_V), -(E' \cdot (D')^{\#} + K_V) > 0$ and $(D')^{\#} = (1/2)D_2$, we have $(E \cdot D_2) = (E' \cdot D_2) = 1$. Then $G := 4E' + 3D_6 + 2D_5 + D_2 + D_4$ defines a $\mathbb{P}^1$-fibration $\Phi' := \Phi_{|G|}\colon V \to \mathbb{P}^1$ and D_3 is a section of Φ'. Let G' be the fiber of Φ' containing D_1. Then $\#G' \geq 3$. However, this contradicts $\rho(V) = 7$ because $\rho(V) \geq \#G + \#G' \geq 8$. Consider the case $j = 4$. Then D_3 and D_5 are sections of Φ. Let F' be a fiber of Φ containing D_6. By considering Lemma 2.11 (2), we know that $F' = E + D_6 + E'$, where E and E' are (-1)-curves. By using the same argument as in the case $j = 3$, we can derive a contradiction. Thus, we see that the case $(C \cdot D_1) = 1$ does not take place.

Suppose that $(C \cdot D_1) = 0$. Then, $(C \cdot D_3 + D_4 + D_5 + D_6) = (C \cdot D_j) = 1$ for some j, $3 \leq j \leq 6$. If $j = 3$ or 6, then we can derive a contradiction by using the same argument as in the previous paragraph. So, we may assume that $j = 4$. Then $F := 2(C + D_4) + D_3 + D_5$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|}\colon V \to \mathbb{P}^1$, D_6 is a section of Φ and D_2 is a 2-section of Φ. Let F' be the fiber of Φ containing D_1. By considering Lemma 2.11 (2), we know that $F' = E_2 + D_1 + E_2'$, where E_2 and E_2' are (-1)-curves. Since D_2 is a 2-section of Φ and $-(E_2 \cdot (D')^{\#} + K_V), -(E_2' \cdot (D')^{\#} + K_V) > 0$, we know that $(E_2 \cdot D_2) = (E_2' \cdot D_2) = 1$. We may assume further that $(E_2' \cdot D_6) = 1$ since D_6 is a section of Φ. Then $(E_2 \cdot D') = (E_2 \cdot D_1 + D_2) = 2$ and so $E_2 + D'$ has negative definite intersection matrix. This contradicts Lemma 2.12.

Therefore, Case (4) does not take place.

Cases (6), (7), (15) *and* (17). Let $\pi : V \to \overline{V}$ be the contraction of D'. Then $\overline{V}$ has a unique rational triple point and two rational double points. Namely, $\overline{V}$ is a dP3-surface in the sense of [34]. However, by [34, Main Theorem], these cases do not take place.

Cases (8) *and* (18). Let $\pi : V \to \overline{V}$ be the contraction of D'. The $\overline{V}$ is a Gorenstein log del Pezzo surface. However, by [25, Lemma 3], these cases do not take place.

Cases (13), (14) *and* (16). We treat Case (13); Cases (14) and (16) can be treated similarly. In this case, $\rho(V) = 6$ and

$$(D')^{\#} = \frac{2}{5}(D_2 + D_4) + \frac{1}{5}(D_3 + D_5).$$

Let C be an irreducible curve in $\mathrm{MV}(V, D')$. By virtue of Lemmas 2.13 and 2.14, we know that C is a (-1)-curve and $|C + D' + K_V| = \emptyset$. So, $(C \cdot D_i) = 0$ or 1 ($i = 1, \ldots, 5$), $(C \cdot D_2 + D_3) = 0$ or 1 and $(C \cdot D_4 + D_5) = 0$ or 1. Since the intersection matrix of $C + D'$ is not negative definite, we may assume that one of the following six cases (i)–(vi) takes place.

(i) $(C \cdot D_1) = (C \cdot D_3) = 1$ and $(C \cdot D_j) = 0$ if $j \neq 1, 3$.

(ii) $(C \cdot D_1) = (C \cdot D_3) = (C \cdot D_4) = 1$ and $(C \cdot D_2) = (C \cdot D_5) = 0$.

(iii) $(C \cdot D_1) = (C \cdot D_3) = (C \cdot D_5) = 1$ and $(C \cdot D_2) = (C \cdot D_4) = 0$.

(iv) $(C \cdot D_1) = (C \cdot D_2) = (C \cdot D_4) = 1$.

(v) $(C \cdot D_3) = (C \cdot D_4) = 1$ and $(C \cdot D_1) = 0$.

(vi) $(C \cdot D_3) = (C \cdot D_5) = 1$ and $(C \cdot D_1) = 0$.

Case (i). Then $F := 2C + D_1 + D_3$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$ and $D_4 + D_5$ is contained in a fiber of Φ. This contradicts Lemma 2.15.

Case (ii). Then $F := 2C + D_1 + D_3$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|} : V \to \mathbb{P}^1$ and D_5 is contained in a fiber G of Φ. By Lemma 2.11 (2), we know that $G = E + D_5 + E'$, where E and E' are (-1)-curves with $(E \cdot D_5) = (E' \cdot D_5) = 1$. Since D_4 is a 2-section of Φ, we may assume that $(E \cdot D_4) = 1$. Then $-(C \cdot (D')^{\#} + K_V) = -(E \cdot (D')^{\#} + K_V)$ and so $E \in \mathrm{MV}(V, D')$. On the other hand, since $(E \cdot D_4) = (E \cdot D_5) = 1$, we have $|E + D' + K_V| \neq \emptyset$. This is a contradiction.

Case (iii). By using the same argument as in Case (i), we know that this case does not take place.

Case (iv). Let $\mu : V \to W$ be the contraction of C, D_1, D_4 and D_5. Then $W = \mathbb{F}_2$, a Hirzebruch surface of degree two, and $M_2 := \mu_*(D_3)$ is the minimal section of $\mathbb{F}_2$. Moreover, $(\mu_*(D_2)^2) = 7$. On the other hand, since $(\mu_*(D_2) \cdot M_2) = 1$, we have $\mu_*(D_2) \sim \alpha M_2 + (2\alpha + 1)\ell$, where ℓ is a fiber of the ruling on $\mathbb{F}_2$. Hence, $(\mu_*(D_2)^2) = 2\alpha^2 + 2\alpha$ is even. This is a contradiction.

Case (v). Then $F := 5C + 3D_3 + 2D_4 + D_2 + D_5$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|}\colon V \to \mathbb{P}^1$ and D_1 is contained in a fiber G of Φ. Since $\#G \geq 3$, we have

$$\rho(V) = 6 \geq 2 + (\#F - 1) + (\#G - 1) \geq 8.$$

This is a contradiction.

Case (vi). Then $F := 2C + D_3 + D_5$ defines a $\mathbb{P}^1$-fibration $\Phi := \Phi_{|F|}\colon V \to \mathbb{P}^1$, D_2 and D_4 are sections of Φ and D_1 is contained in a fiber G of Φ. By Lemma 2.11 (2), we know that $G = E + D_1 + E'$, where E and E' are (-1)-curves and $(E \cdot D_1) = (E' \cdot D_1) = 1$. Since D_2 and D_4 are sections of Φ, we can easily see that one of $E + D'$ and $E' + D'$ has negative definite intersection matrix. This contradicts Lemma 2.12.

Therefore, Case (13) does not take place.

Thus, we know that Cases (1)–(18) do not take place. This proves the assertion (2) of Theorem 1.1.

The proof of Theorem 1.1 is thus completed.

5. Proofs of Theorems 1.2 and 1.3

In this section, we shall prove Theorems 1.2 and 1.3 by using results in previous sections.

Proof of Theorem 1.2. Let X be a smooth affine surface with finite Picard group and C a non-empty reduced algebraic curve on X. Set $S := X - C$. Then $\mathrm{Pic}(S)$ is finite. It suffices to show that $\bar{\kappa}(S) = -\infty$ provided $\bar{P}_2(S) = 0$. Let $C = \bigcup_{i=1}^{r} C_i$ be the decomposition of C into irreducible components.

Suppose to the contrary that $\bar{\kappa}(S) \geq 0$. Lemma 2.9 then implies that S is a rational surface. Let (V, D) be an SNC-pair with $V - D \cong S$ and $D = \sum_{j=1}^{s} D_j$ the decomposition of D into irreducible components. Since V is a rational surface and $\bar{p}_g(V - D) = \bar{p}_g(S) = 0$, D is a tree of smooth rational curves by [20, Lemma I.2.1.3]. Hence, we know that C is a disjoint union of topologically contractible curves. In particular, $e(S) = e(X) - r$. By virtue of Lemma 2.8, we know that $e(S) = 0$ or 1 and $e(S) = 1$ if $\bar{\kappa}(S) = 2$.

Now, let (V', D') be an SNC-pair with $V' - D' \cong X$. Since $\bar{p}_g(X) = \bar{p}_g(S) = 0$, D' is a tree of smooth rational curves. Let $D' = \sum_{t=1}^{k} D'_t$ be the decomposition of D' into irreducible components. Then

$$e(X) = e(V') - e(D') = \rho(V') - k + 1.$$

Since $\mathrm{Pic}(X)$ is finite, we have $k \geq \rho(V')$. So,

$$e(X) = \rho(V') - k + 1 \leq 1.$$

Hence, $e(S) = e(X) - e(C) \le 1 - r \le 0$ because $r \ge 1$. Lemma 2.8 (1) implies that $\bar{\kappa}(S) = 0$ or 1. By Theorem 1.1 (1), $\bar{\kappa}(S) \ne 1$. Since every smooth affine surface with $\bar{\kappa} = 0$ and $\bar{P}_2 = 0$ has positive topological Euler characteristic by [14, Theorem 0.1] (see also [2, Section 8]), we know that $\bar{\kappa}(S) \ne 0$. The proof of Theorem 1.2 is thus completed. □

Proof of Theorem 1.3. The assertion (2) is a consequence of the assertion (1) and the algebraic characterization of $\mathbb{C}^2$ due to Fujita, Miyanishi and Sugie (see [20], [21]). We shall prove the assertion (1). Let $S = \operatorname{Spec} A$ be a smooth affine surface with $\operatorname{Pic}(S) = (0)$. It suffices to show that $\bar{P}_2(S) > 0$ provided $\bar{\kappa}(S) \ge 0$.

If $\bar{\kappa}(S) = 0$, then, by virtue of [2, Theorem (8.70) 3)], we know that $\bar{p}_g(S) > 0$. Hence, $\bar{P}_2(S) > 0$. If $\bar{\kappa}(S) = 1$, then $\bar{P}_2(S) > 0$ by Theorem 1.1 (1). Suppose that $\bar{\kappa}(S) = 2$ and $\bar{P}_2(S) = 0$. By virtue of Lemmas 2.8 and 2.9, we know that S is a $\mathbb{Q}$-homology plane. In particular, S is a homology plane because $\operatorname{Pic}(S) = (0)$. However, by Theorem B (cf. [18, Theorem 1.3]), we know that $\bar{P}_2(S) > 0$. This is a contradiction. The proof of Theorem 1.3 is thus completed. □

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