Irreducibly represented groups

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Abstract. A group is *irreducibly represented* if it has a faithful irreducible unitary representation. For countable groups, a criterion for irreducible representability is given, which generalises a result obtained for finite groups by W. Gaschütz in 1954. In particular, torsionfree groups and infinite conjugacy class groups are irreducibly represented.

We indicate some consequences of this for operator algebras. In particular, we characterise up to isomorphism the countable subgroups Δ of the unitary group of a separable infinite dimensional Hilbert space $\mathcal{H}$ of which the bicommutants Δ'' (in the sense of the theory of von Neumann algebras) coincide with the algebra of all bounded linear operators on $\mathcal{H}$.

1. Gaschütz Theorem for infinite groups, and consequences

Define a group to be *irreducibly represented* if it has a faithful irreducible unitary representation and *irreducibly underrepresented*[1] if not. For example, a finite abelian group is irreducibly represented if and only if it is cyclic (because finite subgroups of multiplicative groups of fields, in particular finite subgroups of $\mathbb{C}^*$, are cyclic). It is a straightforward consequence of Schur's lemma that a group of which the centre contains a non-cyclic finite subgroup is irreducibly underrepresented. For finite groups, there are also standard examples of groups without centre which are irreducibly underrepresented (see Note F in [Burns11]); moreover, there exists a criterion due to Gaschütz who states for finite dimensional representations over algebraically closed fields of characteristic zero the equivalence of Properties (i), (iv), and (v) in Theorem 2 below (see [Gasch54], as well as [Hupp98], § 42, and [Pálfy79]).

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[1] On the day of writing, Google shows 29 000 000 entries for *represented groups,* 2 390 000 for *underrepresented groups,* 641 000 for *"represented groups",* 670 000 for *"underrepresented groups",* and zero entry for *"irreducibly underrepresented groups".* In some sense at least, what we have to say is new.

The purpose of the present paper is to extend Gaschütz' result to infinite groups and unitary representations; for the particular case of finite groups, our arguments provide a new proof of the main result of [Gasch54] (at least for complex representations). For a generalisation of Gachütz' result of a rather different kind, see [Tushe93].

Since our arguments use measure theory, it is convenient to avoid the difficulties connected with non-standard spaces, so that we assume systematically that *the groups involved are countable* (see also Example VII in Subsection 5.1 below). Moreover and from now on, *we write "representation" for "unitary representation"* and, similarly, "character" for "unitary character".

To formulate our results, we need the following preliminaries. Let Γ be a group.

Let N be a normal subgroup of Γ. A representation σ of N is said to be Γ-*faithful* if $\bigcap_{\gamma \in \Gamma} \ker(\sigma^\gamma) = \{e\}$, where e denotes the unit element of the group and where σ^γ denotes the representation $n \mapsto \sigma(\gamma n \gamma^{-1})$, namely the conjugate of σ by γ. For example, if V denotes the normal subgroup of order 4 in the symmetric group Sym(4) on four letters, any character of V distinct from the unit character is Sym(4)-faithful (even though V does not have any faithful character).

If $\{S_i\}_{i \in I}$ is a family of subsets of Γ, we denote by $\langle \{S_i\}_{i \in I} \rangle$ the subgroup of Γ generated by $\bigcup_{i \in I} S_i$. Following [Remak30], we define a *foot* of Γ to be a *minimal* normal subgroup of Γ, namely a normal subgroup M in Γ such that $M \neq \{e\}$, and any normal subgroup of Γ contained in M is either M or $\{e\}$. We denote by $\mathcal{F}_\Gamma$ the set of finite feet of Γ. The *minisocle* of Γ is the subgroup $MS(\Gamma)$ of Γ generated by the union of its finite feet; it is a characteristic subgroup of Γ. Let $\mathcal{A}_\Gamma$ denote the subset of $\mathcal{F}_\Gamma$ of abelian groups, and let $\mathcal{H}_\Gamma$ be the complement of $\mathcal{A}_\Gamma$ in $\mathcal{F}_\Gamma$. We define $MA(\Gamma)$ and $MH(\Gamma)$ to be the subgroups of Γ generated by $\bigcup_{A \in \mathcal{A}_\Gamma} A$ and $\bigcup_{H \in \mathcal{H}_\Gamma} H$ respectively; both are characteristic subgroups of Γ contained in $MS(\Gamma)$. By the usual convention, $MS(\Gamma) = \{e\}$ if $\mathcal{F}_\Gamma$ is empty, and similarly for $MA(\Gamma)$ and $MH(\Gamma)$.

Proposition 1. *Let Γ be a group, and let the notation be as above.*

(i) *Each $A \in \mathcal{A}_\Gamma$ is isomorphic to $(\mathbb{F}_p)^n$ for some prime p and some positive integer n (depending on A).*

(ii) *There exists a subset $\{A_i\}_{i \in I}$ of $\mathcal{A}_\Gamma$ such that $MA(\Gamma) = \bigoplus_{i \in I} A_i$. In particular, the group $MA(\Gamma)$ is abelian.*

(iii) *For each $H \in \mathcal{H}_\Gamma$, the feet $S_1, \ldots, S_k$ of H are conjugate in Γ, and simple. Moreover $H = S_1 \oplus \cdots \oplus S_k$.*

(iv) *We have $MH(\Gamma) = \bigoplus_{H \in \mathcal{H}_\Gamma} H$.*

(v) *We have $MS(\Gamma) = MA(\Gamma) \oplus MH(\Gamma)$.*

For some examples of minisocles, see Section 5.1. Here is our first main result.

Theorem 2. *Let Γ be a countable group. Let $MA(\Gamma) = \bigoplus_{i \in I} A_i$ and $MS(\Gamma) = MA(\Gamma) \oplus MH(\Gamma)$ be as above. The following properties are equivalent:*

(i) *Γ is irreducibly represented;*

(ii) *$MA(\Gamma)$ has a Γ-faithful character;*

(iii) *$MS(\Gamma)$ has a Γ-faithful irreducible representation;*

(iv) *for every finite subset E of I, there exists an element x_E in $MA_E(\Gamma) \doteq \bigoplus_{i \in E} A_i$ such that the Γ-conjugacy class of x_E generates $MA_E(\Gamma)$;*

(v) *for every pair of finite subsets $E \subset I$ and $F \subset \mathcal{H}_\Gamma$, there exists an element $z_{E,F}$ in $MS_{E,F}(\Gamma) \doteq \left(\bigoplus_{i \in E} A_i \right) \oplus \left(\bigoplus_{H \in F} H \right)$ such that the Γ-conjugacy class of $z_{E,F}$ generates $MS_{E,F}(\Gamma)$.*

In particular, a countable group Γ has a faithful irreducible representation as soon as $MA(\Gamma) = \{e\}$, and a fortiori as soon as $MS(\Gamma) = \{e\}$.

The next corollary is a straightforward consequence of Theorem 2. Recall that a group is *icc* if it is not reduced to one element and if all its conjugacy classes distinct from $\{e\}$ are infinite.

Corollary 3. *For a countable group to be irreducibly represented, any of the three following conditions is sufficient:* (i) *the group is torsionfree,* (ii) *the group is icc,* (iii) *the group has a faithful primitive action on an infinite set.*

The case of icc groups is well known, sometimes with a different proof. Indeed, a group is icc if and only if its von Neumann algebra is a factor of type II_1 (Lemma 5.3.4 of [ROIV]); it is then a standard fact that the reduced C*-algebra of an icc group has a faithful irreducible representation, so that *a fortiori* the group itself has a faithful irreducible representation (see for example Proposition 21 of [Harpe07]).

For a group Γ which has a faithful primitive action on an infinite set X (see [GelGl08]), observe that any normal subgroup of Γ not reduced to $\{e\}$ is transitive on X and therefore infinite, so that $MS(\Gamma) = \{e\}$.

Theorem 2 does not state anything on the dimensions of the representations which can occur in (i). Before providing some information, let us recall that a group is *virtually abelian* if it has an abelian subgroup of finite index.

Theorem 4. *For a countable group Γ, the two following properties are equivalent:*

(i) *Γ has an infinite dimensional faithful irreducible representation;*

(ii) *Γ has the properties of Theorem 2 and is not virtually abelian.*

In other words, the following properties are equivalent:

(iii) *Γ has a faithful irreducible representation, and all its faithful irreducible representations are finite dimensional;*

(iv) Γ *has the properties of Theorem* 2 *and is virtually abelian.*

Let M be a von Neumann algebra. We denote by $\mathcal{U}(M)$ the unitary group $\{X \in M \mid X^*X = XX^* = 1\}$ of M and by S'' the double commutant of a subset S of M. Recall that M is a *factor* if its centre is reduced to $\mathbb{C}$, a *factor of type I* if there exists a Hilbert space $\mathcal{H}$ such that $M = \mathcal{L}(\mathcal{H})$, and a *factor of type* I_∞ in case $\mathcal{H}$ is infinite dimensional (moreover, we assume here that Hilbert spaces are separable). For factors of type I, we write $\mathcal{U}(\mathcal{H})$ instead of $\mathcal{U}(M)$.

Corollary 5. *Let* $M = \mathcal{L}(\mathcal{H})$ *be a factor of type* I_∞. *For a countable group* Γ, *the following two properties are equivalent:*

  – *there is a subgroup* Δ *of* $\mathcal{U}(\mathcal{H})$ *isomorphic to* Γ *such that* $\Delta'' = M$;

  – Γ *has the properties of Theorem* 2 *and is not virtually abelian.*

It would be interesting to have some information of this kind for other factors. In particular, we do not know any analogue of Theorem 4 for any given finite dimension $n \geq 2$, nor of Corollary 5 for the finite dimensional factor $\mathcal{L}(\mathbb{C}^n)$. We do not know any solution to the *a priori* easier problem to characterise the countable groups which have at least one finite dimensional faithful irreducible representation.

The proof of Proposition 1 uses standard arguments (compare with Section 4.3 of [DixMo96]). For the convenience of the reader, we give details in Section 2. Theorem 2 is proved in Section 3. Theorem 4 and Corollary 5 are proved in Section 4. We formulate a few remarks in Section 5: on examples of socles and minisocles, on the comparison between minisocles and periodic FC-kernels, on a theorem of Gelfand and Raikov, on tensor products of faithful representations, and on countable groups with primitive maximal C*-algebras. The final Section 6 is devoted to a generalisation of Theorem 2 concerning a countable group Γ given together with a group of automorphisms G which contains the group of inner automorphisms.

Understanding groups of a given class includes understanding their faithful actions of various kinds, and the setting of linear (or unitary) actions is only one among several others. For example, in the case of finite groups, the questions of classifying multiply transitive actions and primitive actions which are faithful have been central in group theory for more than hundred years; faithful primitive actions for infinite groups have been addressed in [GelGl08]. Faithful amenable actions are the subject of [GlaMo07]. Our initial motivation has been to ask some of the corresponding questions for linear actions.

2. Proof of Proposition 1

We prepare the proof of Proposition 1 by recalling two lemmas.

Lemma 6. *Let Γ be a group. Let M be a minimal normal subgroup of Γ and N a normal subgroup of Γ. Then either $M \subset N$ or $\langle M, N \rangle = M \oplus N$.*

Proof. We can assume $N \neq \{e\}$. Since $M \cap N$ is both in M and normal in Γ, either $M \cap N = M$, and $M \subset N$, or $M \cap N = \{e\}$, and $\langle M, N \rangle = M \oplus N$. □

Lemma 7. *Let A be a group and let $(S_i)_{i \in I}$ be a family of nonabelian simple groups; set $S = A \oplus \left(\bigoplus_{i \in I} S_i \right)$. Let M be a minimal normal subgroup of S.*
  Then either $M = S_\ell$ for some $\ell \in I$, or $M \subset A$.

Proof. Assume that $M \neq S_\ell$ for all $\ell \in I$. Choose $i \in I$; by Lemma 6 applied to M and $N = S_i$, the groups M and S_i commute. It follows that M is a subgroup of the centraliser of $\bigoplus_{i \in I} S_i$ in S, namely a subgroup of A. □

Proof of Proposition 1. (i) Let $A \in \mathcal{A}_\Gamma$. By the structure theory of finite abelian groups, there exist a prime p and an element $a \in A$ of order p. Let A^* denote the set of elements of order p in A. Then $A^* \cup \{e\}$ is a characteristic subgroup of A, and therefore a normal subgroup of Γ. By minimality of A, we have $A^* \cup \{e\} = A$, so that A is isomorphic to $(\mathbb{F}_p)^n$ for some $n \geq 1$, as claimed.

(ii) Let $\mathcal{L}$ be the set of subsets $\{A_\ell\}_{\ell \in L}$ of $\mathcal{A}_\Gamma$ such that $\langle \{A_\ell\}_{\ell \in L} \rangle = \bigoplus_{\ell \in L} A_\ell$; we order $\mathcal{L}$ by inclusion. The crucial observation is that the ordered set $\mathcal{L}$ is inductive, so that we can choose a maximal element, say $\{A_i\}_{i \in I}$. Suppose that $\bigoplus_{i \in I} A_i$ is strictly contained in $MA(\Gamma)$; we will arrive at a contradiction.

Choose $B \in \mathcal{A}_\Gamma$ such that B is not contained in $\bigoplus_{i \in I} A_i$. By Lemma 6 applied to $M = B$ and $N = \bigoplus_{i \in I} A_i$, we have either $B \subset \bigoplus_{i \in I} A_i$, which is ruled out by the choice of B, or $\langle B, \{A_i\}_{i \in I} \rangle = B \oplus \left(\bigoplus_{i \in I} A_i \right)$, which is ruled out by the maximality of I. This is the announced contradiction.

(iii) Let $H \in \mathcal{H}_\Gamma$. Choose a minimal normal subgroup S in H (this is possible since H is finite). For each $x \in \Gamma$, the subgroup xSx^{-1} is minimal normal in H. Choose a set $S_1, \ldots, S_k$ of such conjugates of S in Γ which is such that $\langle S_1, \ldots, S_k \rangle = S_1 \oplus \cdots \oplus S_k$ and which is maximal for this property. Set $N = \langle S_1, \ldots, S_k \rangle$; it is a normal subgroup of H.

We claim that $xSs^{-1} \subset N$ for each $x \in \Gamma$, so that N is normal in Γ. Indeed, by Lemma 6 applied to $M = xSx^{-1}$ and N in H, either $\langle xSx^{-1}, S_1, \ldots, S_k \rangle = xSx^{-1} \oplus S_1 \oplus \cdots \oplus S_k$, but this is ruled out by the maximality of the set $\{S_1, \ldots, S_k\}$, or $xSx^{-1} \subset N$, and this establishes the claim.

Since N is normal in Γ and $N \subset H$, we have $N = H$ by minimality of H. Observe that, for each $i \in \{1, \ldots, k\}$, any normal subgroup of S_i is normal in H; it

follows that S_i is a simple group. Finally, the set $\{S_1, \ldots, S_k\}$ coincides with the set of all minimal normal subgroups of H by Lemma 7.

(iv) The same argument as for (ii) shows that there exists a subset $\{H_k\}_{k \in K}$ of $\mathcal{H}_\Gamma$ such that $\bigoplus_{k \in K} H_k = MH(\Gamma)$, and Lemma 7 implies that $\{H_k\}_{k \in K} = \mathcal{H}_\Gamma$.

(v) Again by the same argument as for (ii), there exists a subset $\{M_\ell\}_{\ell \in L}$ of $\mathcal{F}_\Gamma$ such that $\bigoplus_{\ell \in L} M_\ell = MS(\Gamma)$, and Lemma 7 implies that $\{M_\ell\}_{\ell \in L}$ contains $\mathcal{H}_\Gamma$. $\qquad\square$

3. Proof of Theorem 2

We will prove successively that

$$
\begin{array}{llll}
\text{(i)} & \Longrightarrow & \text{(ii) \& (iii)} & \text{(see Lemma 9),} \\
\text{(iii)} & \Longrightarrow & \text{(i)} & \text{(Lemma 10),} \\
\text{(ii)} & \Longleftrightarrow & \text{(iii)} & \text{(Lemma 13),} \\
\text{(ii)} & \Longleftrightarrow & \text{(iv)} & \text{(Lemma 14).}
\end{array}
$$

The equivalence (iv) $\Longleftrightarrow$ (v) is straightforward, since nonabelian feet are direct products of simple groups. Recall that we write "representation" for "unitary representation".

Given a representation π of a countable group Γ in a Hilbert space $\mathcal{H}$, there exist a standard Borel space Ω, a bounded positive measure μ on Ω, a measurable field $\omega \mapsto \pi_\omega$ of irreducible representations of Γ in a measurable field $\omega \mapsto \mathcal{H}_\omega$ of Hilbert spaces on Ω, and an isomorphism of $\mathcal{H}$ with $\int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega)$ which implements a unitary equivalence

$$
\pi(\gamma) \approx \int_\Omega^\oplus \pi_\omega(\gamma) d\mu(\omega)
$$

for all $\gamma \in \Gamma$. See [Dix69C*, Sections 8.5 and 18.7.6]. (Such decompositions in irreducible representations carry over to continuous representations of separable locally compact groups, and more generally of separable C*-algebras. They are applications of the *reduction theory* for von Neumann algebras [Dix69vN, Chapter II]). The following lemma is standard, but we haven't found any appropriate reference.

Lemma 8. *Let Γ be a countable group. Let Ω be a measure space with a positive measure μ. Let $\omega \mapsto \pi_\omega$ be a measurable field of representations of Γ in a measurable field of Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$ over Ω and let $\gamma \in \Gamma$.*

Then $\{\omega \in \Omega \mid \pi_\omega(\gamma) = I\}$ is a measurable subset of Ω.

Proof. Let $(\xi^{(1)}, \xi^{(2)}, \ldots)$ be a fundamental sequence of measurable vector fields (see [Dix69vN, Chapter II, Number 1.3]). For $i, j \geq 1$, consider the set

$$
\Omega_{i,j} = \{\omega \in \Omega \mid \langle \pi_\omega(\gamma) \xi^{(i)}(\omega), \xi^{(j)}(\omega) \rangle = \langle \xi^{(i)}(\omega), \xi^{(j)}(\omega) \rangle \}.
$$

Observe that

$$\{\omega \in \Omega \mid \pi_\omega(\gamma) = I\} = \bigcap_{i,j \geq 1} \Omega_{i,j}.$$

Therefore it suffices to show that each set $\Omega_{i,j}$ is measurable.

For fixed $i, j \geq 1$, the functions

$$\omega \longmapsto \langle \xi^{(i)}(\omega), \xi^{(j)}(\omega) \rangle \quad \text{and} \quad \omega \longmapsto \langle \pi_\omega(\gamma)\xi^{(i)}(\omega), \xi^{(j)}(\omega) \rangle$$

are measurable, by definition of a measurable vector field and of a measurable field of representations. Hence $\Omega_{i,j}$ is measurable and the proof is complete. $\square$

Let us now recall a general fact which can be seen as a weak form of Clifford theorem for infinite dimensional representations. (For a version of Clifford theorem concerning finite dimensional representations but possibly infinite groups, see Theorem 2.2 in [Dixon71].)

Lemma 9. *Let Γ be a countable group, N a normal subgroup, π an irreducible representation of Γ in a Hilbert space $\mathcal{H}$, and σ the restriction of π to N. Identify σ to a direct integral of irreducible representations*

$$\sigma = \pi|_N = \int_\Omega^\oplus \sigma_\omega d\mu(\omega)$$

as above.

If the representation π is faithful, then the representation σ_ω is Γ-faithful for almost all $\omega \in \Omega$.

Proof. If $N = \{e\}$, there is nothing to prove. We assume from now on that N is not reduced to one element.

Denote by $\{C_j\}_{j \in J}$ the family of Γ-conjugacy classes in N distinct from $\{e\}$. For each $j \in J$, denote by N_j the subgroup of N generated by C_j; observe that each N_j is normal in Γ, and that the family $\{N_j\}_{j \in J}$ is countable (possibly finite) and nonempty. Set

$$\Omega_j = \left\{\omega \in \Omega \mid N_j \subset \ker\left(\bigoplus_{\gamma \in \Gamma} \sigma_\omega^\gamma\right)\right\} \quad \text{and} \quad \widetilde{\Omega} = \bigcup_{j \in J} \Omega_j .$$

For $\omega \in \Omega$, observe that σ_ω is not Γ-faithful if and only if the kernel of $\bigoplus_{\gamma \in \Gamma} \sigma_\omega^\gamma$ contains one of the N_j; thus $\widetilde{\Omega}$ is the subset of Ω of the points ω such that σ_ω is not Γ-faithful. Each Ω_j is measurable in Ω as a consequence of Lemma 8; as J is countable, $\widetilde{\Omega}$ is also measurable.

To end the proof, we assume that $\mu(\widetilde{\Omega}) > 0$ and we will arrive at a contradiction.

As the family J is countable, there exists $\ell \in J$ such that $\mu(\Omega_\ell) > 0$. Hence the unit representation 1_{N_ℓ} of the group N_ℓ is strongly contained in the restriction of σ to N_ℓ, so that the subspace of $\mathcal{H}$ of N_ℓ-invariant vectors is not reduced to $\{0\}$. Since N_ℓ is normal in Γ, this subspace is invariant by $\pi(\Gamma)$; by irreducibility of π, this subspace is the whole of $\mathcal{H}$. In other words, the restriction of π to N_ℓ is the unit representation. The last statement is a contradiction, since π is faithful. $\square$

The particular case of Lemma 9 for which $N = MA(\Gamma)$ [respectively $N = MS(\Gamma)$] shows that (i) implies (ii) [respectively (iii)] in Theorem 2. The implication (iii) $\Longrightarrow$ (i) follows from the next lemma applied to $N = MS(\Gamma)$ since, by definition, there does not exist any finite foot M of Γ such that $M \cap MS(\Gamma) = \{e\}$.

Lemma 10. *Let Γ be a countable group, N a normal subgroup, σ an irreducible representation of N in a Hilbert space $\mathcal{K}$, and $\pi = \operatorname{Ind}_N^\Gamma(\sigma)$ the corresponding induced representation. Let $\pi = \int_\Omega^\oplus \pi_\omega d\mu(\omega)$ be a direct integral decomposition of π into irreducible representations. Assume that there does not exist any finite foot M in Γ such that $M \cap N = \{e\}$.*

If the representation σ is Γ-faithful, then the representation π_ω is faithful for almost all ω in Ω.

Proof. In the model we choose for induced representations, π acts on the Hilbert space $\mathcal{H}$ of mappings $f : \Gamma \to \mathcal{K}$ with the two following properties:

$$(1) \quad f(\gamma n) = \sigma(n^{-1}) f(\gamma) \quad \text{for all } \gamma \in \Gamma \text{ and } n \in N,$$

$$(2) \quad \sum_{\Gamma/N} \|f(\gamma)\|^2 < \infty.$$

(The notation of (2) indicates a summation over one representative $\gamma \in \Gamma$ of each class in Γ/N.) Then $(\pi(x)f)(\gamma) = f(x^{-1}\gamma)$ for all $x, \gamma \in \Gamma$.

Denote this time by $\{C_j\}_{j \in J}$ the family of conjugacy classes of Γ distinct from $\{e\}$. For each $j \in J$, denote by Γ_j the subgroup generated by C_j, which is a normal subgroup of Γ not reduced to $\{e\}$; set

$$\Omega_j = \{\omega \in \Omega \mid \Gamma_j \subset \ker(\pi_\omega)\} \quad \text{and} \quad \widetilde{\Omega} = \bigcup_{j \in J} \Omega_j.$$

As in the proof of Lemma 9, $\widetilde{\Omega}$ is the set of points ω such that π_ω is not faithful, and it is measurable. To end the proof, we assume that $\mu(\widetilde{\Omega}) > 0$, so that there exists $\ell \in J$ for which $\mu(\Omega_\ell) > 0$, and we will arrive at a contradiction.

Continuing as in the proof of Lemma 9, we observe that there exists a nonzero vector $f : \Gamma \to \mathcal{K}$ in $\mathcal{H} = \int_\Omega^\oplus \mathcal{H}_\omega d\mu(\omega)$ which is supported in Ω_ℓ (as a measurable

section of the field of Hilbert spaces $\omega \mapsto \mathcal{H}_\omega$ underlying the field of representations $\omega \mapsto \pi_\omega$), and which is such that $\pi(x)f = f$ for all $x \in \Gamma_\ell$.

Let $\gamma_0 \in \Gamma$ be such that $f(\gamma_0^{-1}) \neq 0$; set $\xi = f(\gamma_0^{-1})$. Using (1), we find

$$(3) \quad \xi = f(\gamma_0^{-1}) = f(x^{-1}\gamma_0^{-1}) = f\left(\gamma_0^{-1}(\gamma_0 x^{-1}\gamma_0^{-1})\right) = \sigma(\gamma_0 x \gamma_0^{-1})\xi = \sigma^{\gamma_0}(x)\xi$$

for all $x \in \Gamma_\ell \cap N$.

Claim 1. $\Gamma_\ell \cap N = \{e\}$. Denote by $\mathcal{K}^{\Gamma_\ell \cap N}$ the subspace of $\mathcal{K}$ of vectors invariant by $\sigma^{\gamma_0}(\Gamma_\ell \cap N)$. This is a $\sigma^{\gamma_0}(N)$-invariant subspace of $\mathcal{K}$, since $\Gamma_\ell \cap N$ is a normal subgroup of N. Now $\mathcal{K}^{\Gamma_\ell \cap N} \neq \{0\}$ by (3) and $\mathcal{K}^{\Gamma_\ell \cap N} = \mathcal{K}$ because σ^{γ_0} is irreducible. Thus $\Gamma_\ell \cap N$ is inside the kernel of the representation σ^{γ_0} of N; as $\Gamma_\ell \cap N$ is normal in Γ, the group $\Gamma_\ell \cap N$ is also inside the kernel of the representation σ^γ of N for all $\gamma \in \Gamma$. As σ is Γ-faithful, $\Gamma_\ell \cap N = \{e\}$, as claimed.

Claim 2. *The subgroup Γ_ℓ of Γ is finite.* Consider the function

$$\varphi \colon \Gamma \longrightarrow \mathbb{R}_+, \quad \gamma \longmapsto \|f(\gamma)\|.$$

We have

$$(4) \quad \varphi(\gamma_0^{-1}) \neq 0,$$

$$(5) \quad \varphi \quad \text{is constant under right translations by elements of } N,$$

$$(6) \quad \sum_{\Gamma/N} |\varphi(\gamma)|^2 < \infty,$$

$$(7) \quad \varphi \quad \text{is invariant under left translations by elements of } \Gamma_\ell.$$

It follows from (4) to (7) that the image of $\Gamma_\ell \gamma_0$ in Γ/N is finite. The image of $\gamma_0^{-1}\Gamma_\ell \gamma_0 = \Gamma_\ell$ in Γ/N is also finite, so that the index of N in $\Gamma_\ell N$ is finite. Claim 2 follows since $\Gamma_\ell N$ is isomorphic to the direct sum $\Gamma_\ell \oplus N$ by Claim 1.

Any subgroup M of Γ_ℓ which is normal in Γ and minimal for this property is a finite foot of Γ, and $M \cap N = \{e\}$ by Claim 1. This is in contradiction with one of the hypotheses of the lemma. $\square$

The particular case $N = \{e\}$ is of independent interest.

Proposition 11. *Let Γ be a countable infinite group which does not contain any finite foot, and let $\lambda_\Gamma = \int_\Omega^\oplus \pi_\omega d\mu(\omega)$ be a direct integral decomposition of the left regular representation λ_Γ into irreducible representations. Then π_ω is faithful for almost all $\omega \in \Omega$.*

Next, we show that (ii) $\Longleftrightarrow$ (iii) in Theorem 2. This will be a consequence of Lemma 13, for the proof of which we will call upon the following lemma.

For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{L}(\mathcal{H})$ its algebra of bounded linear operators.

Lemma 12. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces. Let $S_1 \in \mathcal{L}(\mathcal{H}_1)$, $S_2 \in \mathcal{L}(\mathcal{H}_2)$ be such that $S_1 \otimes S_2 \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a non-zero multiple of the identity operator.*
   Then S_1 and S_2 are multiples of the identity.

Proof. Let $\lambda \in \mathbb{C}^*$ be such that $S_1 \otimes S_2 = \lambda I$. Let $\{\xi_i\}_{i \in I}$ be a Hilbert space basis of $\mathcal{H}_1$. Since $S_2 \neq 0$, there exist $\eta_1, \eta_2 \in \mathcal{H}_2$ such that

$$\langle S_2(\eta_1), \eta_2 \rangle \neq 0.$$

For every $\xi \in \mathcal{H}_1$, we have

$$\langle (S_1 \otimes S_2)(\xi \otimes \eta_1), \xi_i \otimes \eta_2 \rangle = \langle S_1(\xi), \xi_i \rangle \langle S_2(\eta_1), \eta_2 \rangle$$

and hence

$$
\begin{aligned}
\langle S_1(\xi), \xi_i \rangle &= \frac{1}{\langle S_2(\eta_1), \eta_2 \rangle} \langle (S_1 \otimes S_2)(\xi \otimes \eta_1), \xi_i \otimes \eta_2 \rangle \\
&= \frac{\lambda}{\langle S_2(\eta_1), \eta_2 \rangle} \langle \xi \otimes \eta_1, \xi_i \otimes \eta_2 \rangle \\
&= \frac{\lambda \langle \eta_1, \eta_2 \rangle}{\langle S_2(\eta_1), \eta_2 \rangle} \langle \xi, \xi_i \rangle
\end{aligned}
$$

for all $i \in I$. It follows that

$$
\begin{aligned}
S_1(\xi) &= \sum_{i \in I} \langle S_1(\xi), \xi_i \rangle \xi_i \\
&= \frac{\lambda \langle \eta_1, \eta_2 \rangle}{\langle S_2(\eta_1), \eta_2 \rangle} \sum_{i \in I} \langle \xi, \xi_i \rangle \xi_i \\
&= \frac{\lambda \langle \eta_1, \eta_2 \rangle}{\langle S_2(\eta_1), \eta_2 \rangle} \xi
\end{aligned}
$$

for every $\xi \in H_1$, showing that S_1 is a multiple of the identity. A similar argument applies to S_2.   □

Lemma 13. *Let Γ be a group and let N be a normal subgroup of Γ. Assume that $N = A \oplus S$, where A is an abelian normal subgroup of Γ and where S is the direct sum of a family $(S_i)_{i \in I}$ of finite simple nonabelian normal subgroups of S. The following properties are equivalent:*

(i) *N has a Γ-faithful irreducible representation;*

(ii) *A has a Γ-faithful character.*

Proof. Assume first that there exists a Γ-faithful irreducible representation π of N. Since the factor A of $N = A \oplus S$ is abelian, and in particular a type I group, there exist a character χ of A and an irreducible representation ρ of S such that $\pi = \chi \otimes \rho$ [Dix69C*, Proposition 13.1.8]. Since $\ker(\chi^\gamma) = \ker((\pi^\gamma)|_A)$ for all $\gamma \in \Gamma$, the character χ of A is Γ-faithful.

Assume now that there exists a Γ-faithful character χ of A. We claim that there exists an irreducible representation ρ of S such that, for every $\gamma \in S$, $\gamma \neq e$, the operator $\rho(\gamma)$ is not a multiple of the identity operator. Lemma 12 will then imply that the exterior tensor product $\chi \otimes \rho$ is a Γ-faithful representation of $N = A \oplus S$.

For every $i \in I$, let ρ_i be an irreducible representation of S_i distinct from the unit representation, in some Hilbert space $\mathcal{H}_i$. Choose a unit vector $\eta_i \in \mathcal{H}_i$. Consider the infinite tensor product $\rho = \bigotimes_{i \in I} \rho_i$ of the family $(\rho_i)_{i \in I}$ with respect to the family $(\eta_i)_{i \in I}$. Recall that ρ is the representation of S defined on the infinite tensor product $\mathcal{H} = \otimes_{i \in I}(\mathcal{H}_i, \eta_i)$ of the family of Hilbert spaces $(\mathcal{H}_i)_{i \in I}$ with respect to the family $(\eta_i)_{i \in I}$ by

$$\rho\Big((\gamma_i)_{i \in I}\Big)\Big(\big(\bigotimes_{f \in F} \xi_f\big) \otimes \big(\bigotimes_{i \in I \setminus F} \eta_i\big)\Big) = \big(\bigotimes_{f \in F} \rho_i(\gamma_i)\xi_f\big) \otimes \big(\bigotimes_{i \in I \setminus F} \eta_i\big),$$

for every finite subset F of I, element $(\gamma_i)_{i \in I} \in S$ with $\gamma_i = 1$ whenever $i \in I \setminus F$, and decomposable vector $(\xi_f)_{f \in F} \in \bigotimes_{f \in F} \mathcal{H}_f$. The representation ρ is irreducible, since the ρ_i's are irreducible. For all this, see for example [Guich66], in particular Corollary 2.1.

Let us check that, for $\gamma = (\gamma_i)_{i \in I} \in S$, $\gamma \neq e$, the operator $\rho(\gamma)$ is not a multiple of the identity operator. Choose $j \in I$ such that $\gamma_j \neq e$. Observe that the set

$$\{\delta \in S_j \ : \ \rho_j(\delta) \text{ is a multiple of the identity operator}\}$$

is an abelian normal subgroup of S_j and is therefore reduced to $\{e\}$ since S_j is simple and nonabelian. The operator $\rho_j(\gamma_j)$ is therefore not a multiple of the identity operator. Denote by ρ_j' the tensor product of the family $(\rho_\ell)_{\ell \in I \setminus \{j\}}$, defined on $\mathcal{H}_j' = \bigotimes_{\ell \in I \setminus \{j\}}(\mathcal{H}_\ell, \eta_\ell)$. We can then write

$$\mathcal{H} = \mathcal{H}_j \otimes \mathcal{H}_j' \quad \text{and} \quad \rho = \rho_j \otimes \rho_j'.$$

Lemma 12 implies that $\rho(\gamma)$ is not a multiple of the identity operator. $\qquad\square$

It remains to show that (ii) $\Longleftrightarrow$ (iv) in Theorem 2. This will be a consequence of the following lemma.

We are most grateful to Roland Lötscher, who pointed out a mistake at this point in a first version of our paper; we are also grateful to Jacques Thévenaz for a helpful discussion on modular representations.

Lemma 14. *Let Γ be a countable group; set $A = MA(\Gamma)$. Let $\{A_i\}_{i \in I}$ be a set of finite abelian feet of Γ as in Proposition 1, so that $A = \bigoplus_{i \in I} A_i$. For each finite subset E of I, set $A_E = \bigoplus_{i \in E} A_i$, which is a finite abelian group. Let $\widehat{A}$, $\widehat{A}_E$ denote the dual group of A, A_E respectively. The following properties are equivalent:*

(i) *A has a Γ-faithful character;*

(ii) *there exists a character $\chi \in \widehat{A}$ such that the subgroup generated by $\chi^{\Gamma} \doteq \{\chi^{\gamma} \mid \gamma \in \Gamma\}$ is dense in $\widehat{A}$;*

(iii) *for every finite subset E of I, the finite group $\widehat{A}_E$ has a Γ-faithful character;*

(iv) *for every finite subset E of I, there exists $\chi \in \widehat{A}_E$ such that $\widehat{A}_E$ is generated by the Γ-orbit of χ;*

(v) *for every finite subset E of I, there exists $x_E \in A_E$ such that A_E is generated by the Γ-conjugacy class of x_E.*

Proof. Equivalence of (i) *and* (ii) *and equivalence of* (iii) *and* (iv). Let N be a normal abelian subgroup of Γ. Let $\chi \in \widehat{N}$. Denote by H the closed subgroup of $\widehat{N}$ generated by χ^{Γ}. By Pontrjagin duality, the unitary dual of the compact abelian group $\widehat{N}/H$ can be identified with the subgroup

$$H^{\perp} = \{a \in N \ : \ \psi(a) = 1 \text{ for all } \psi \in H\};$$

observe that

$$H^{\perp} = \{a \in N \ : \ \psi(a) = 1 \text{ for all } \psi \in \chi^{\Gamma}\} = \bigcap_{\gamma \in \Gamma} \ker(\chi^{\gamma}).$$

Thus χ^{Γ} is dense in $\widehat{N}$ if and only if $H^{\perp} = \{e\}$, namely if and only if χ is Γ-faithful.

Equivalence of (ii) *and* (iii). It is clear that (ii) implies (iii). Let us assume that (iii) holds; we have to check that this implies (ii). For every finite subset E of I, denote by $p_E : \widehat{A} \to \widehat{A}_E$ the canonical projection. Consider the subset

$$X_E = \{\chi \in \widehat{A} \mid \text{the } \Gamma\text{-orbit of } p_E(\chi) \text{ generates } \widehat{A}_E\}.$$

Since the group $\widehat{A}_E$ is finite, the subset X_E of $\widehat{A}$ is closed. For a finite family $E_1, \ldots, E_k$ of finite subsets of I, the intersection $X_{E_1} \cap \cdots \cap X_{E_k}$ contains $X_{E_1 \cup \cdots \cup E_k}$. By Condition (iii), X_E is non empty for any finite subset E of I. Since $\widehat{A}$ is compact, it follows that

$$\bigcap_E X_E \neq \emptyset,$$

where E runs over all finite subsets of I. Let $\chi \in \bigcap_E X_E$. It is easily checked that χ is Γ-faithful.

Equivalence of (iv) *and* (v). Consider a finite subset E of I. Recall that each A_i is a finite dimensional vector space over a prime field $\mathbb{F}_{p_i}$, for a prime number p_i. For each prime p, denote by V_p the direct sum of those A_i with $i \in E$ which are vector spaces over $\mathbb{F}_p$, and denote by P the set of primes p such that $V_p \neq \{0\}$. We have $A_E = \bigoplus_{p \in P} V_p$. Since the V_p's are subgroups of A_E of pairwise coprime orders, every subgroup H of A_E is a direct sum $\bigoplus_{p \in P} (H \cap V_p)$. The dual group $\widehat{V}_p$ of V_p can be identified with the dual vector space V_p^*; as before, each subgroup H^* of $\widehat{A}_E$ is a direct sum $\bigoplus_{p \in P} (H^* \cap V_p^*)$. It follows that, in order to prove the equivalence of (iv) and (v), we can assume that P consists of a single element p. We can also assume that Γ is a subgroup of $\mathrm{GL}(V_p)$.

Let $\mathbb{F}_p[\Gamma]$ denote the group algebra of Γ over $\mathbb{F}_p$. Observe that V_p is a semi-simple $\mathbb{F}_p[\Gamma]$-module, since V_p is a direct sum of minimal normal subgroups of Γ. (A module is *semi-simple* if it is a direct sum of simple modules; other authors use the terminology *completely reducible*.)

Under the identification of $\widehat{V}_p$ with V_p^*, the Γ-action on $\widehat{V}_p$ corresponds to the dual (or contragredient) action of Γ on V_p^*. Observe that V_p^* is a semi-simple $\mathbb{F}_p[\Gamma]$-module. Indeed, if W is submodule of V_p^*, then its annihilator $W^\perp$ has a complement Z in V_p and $Z^\perp$ is a complement of W in V_p^* (compare with Lemma 6.2 in [Landr83]).

Observe also that there exists $x \in V_p$ such that V_p is generated by the Γ-conjugacy class of x (respectively, there exists $\chi \in \widehat{V}_p$ such that $\widehat{V}_p$ is generated by the Γ-orbit of χ) if and only if V_p (respectively V_p^*) is isomorphic, as $\mathbb{F}_p[\Gamma]$-module, to a quotient of the left regular module $\mathbb{F}_p[\Gamma]$. To conclude the proof, we show that V_p is isomorphic to a quotient of $\mathbb{F}_p[\Gamma]$ if and only if V_p^* is isomorphic to a quotient of $\mathbb{F}_p[\Gamma]$.

We first show that every semi-simple submodule of $\mathbb{F}_p[\Gamma]$ is isomorphic to a quotient of $\mathbb{F}_p[\Gamma]$. Indeed, let $\mathbb{F}_p[\Gamma] = \bigoplus_{j \in J} P_j$ be a direct sum decomposition of $\mathbb{F}_p[\Gamma]$ into indecomposable submodules P_j. Every P_j contains a unique simple module S_j. Moreover, S_j is isomorphic to a quotient of P_j and $M = \bigoplus_{j \in J} S_j$ is the sum of all simple submodules of $\mathbb{F}_p[\Gamma]$. For the standard facts on representation theory of finite groups, see for example [Landr83], in particular Theorem 6.8. Let N be a semi-simple submodule of $\mathbb{F}_p[\Gamma]$. Then N is a submodule of M and is therefore isomorphic to $\bigoplus_{j \in J'} S_j$ for a subset J' of J. Hence, N is isomorphic to a quotient of $\bigoplus_{j \in J'} P_j$. Since $\bigoplus_{j \in J'} P_j$ is a direct summand of $\mathbb{F}_p[\Gamma]$, it follows that N is isomorphic to a quotient of $\mathbb{F}_p[\Gamma]$ and this proves our claim.

Assume that V_p is isomorphic to a quotient of $\mathbb{F}_p[\Gamma]$. Then V_p^* is isomorphic to a submodule of $\mathbb{F}_p[\Gamma]^*$. Now, it is standard that $\mathbb{F}_p[\Gamma]^*$ is isomorphic to $\mathbb{F}_p[\Gamma]$ as a $\mathbb{F}_p[\Gamma]$-module (see [Landr83, Theorem 6.3]). Hence, V_p^* is isomorphic to a submodule of $\mathbb{F}_p[\Gamma]$. By what we have seen above, it follows that V_p^* is isomorphic to a quotient of $\mathbb{F}_p[\Gamma]$. Similarly, if V_p^* is isomorphic to a quotient of $\mathbb{F}_p[\Gamma]$, then V_p is isomorphic to a quotient of $\mathbb{F}_p[\Gamma]$. $\square$

4. Finite and infinite dimensional representations

Our proof of Theorem 4 uses the following elementary lemma, which is well known. To our surprise, we haven't been able to find a convenient reference.

Lemma 15. *Let Ω a standard Borel space and μ a bounded positive measure on Ω.*

(i) *Let A be a separable C*-algebra, $\underline{\pi}$ a representation of A, and*

$$\underline{\pi} = \int_\Omega^\oplus \underline{\pi}_\omega d\mu(\omega)$$

a direct integral decomposition of $\underline{\pi}$ with respect to a measurable field $\omega \mapsto \underline{\pi}_\omega$ of representations of A. Then $\underline{\pi}_\omega$ is weakly contained in $\underline{\pi}$ for almost all ω in Ω.

(ii) *Let Γ be a countable group, π a representation of Γ, and*

$$\pi = \int_\Omega^\oplus \pi_\omega d\mu(\omega)$$

a direct integral decomposition of π with respect to a measurable field $\omega \mapsto \pi_\omega$ of representations of Γ. Then π_ω is weakly contained in π for almost all $\omega \in \Omega$.

Proof. (i) By definition of "weak containment", we have to show that $\ker(\underline{\pi}) \subset \ker(\underline{\pi}_\omega)$ for almost all $\omega \in \Omega$. Since A is separable, so is $\ker(\underline{\pi})$, and we can choose in this kernel a countable dense subset, say C. For any $x \in A$, recall from the theory of direct integrals that $\|\underline{\pi}(x)\|$ is the essential supremum (on $\omega \in \Omega$) of the norms $\|\underline{\pi}_\omega(x)\|$, so that $\|\underline{\pi}_\omega(x)\| \leq \|\underline{\pi}(x)\|$ for almost all $\omega \in \Omega$; in particular, any $x \in C$ is in $\ker(\underline{\pi}_\omega)$ for almost all $\omega \in \Omega$. Since C is countable, we have also $C \subset \ker(\underline{\pi}_\omega)$ for almost all $\omega \in \Omega$, and this implies the announced conclusion.

(ii) Any representation π of Γ corresponds to a representation $\underline{\pi}$ of the maximal C*-algebra $A = C^*_{\max}(\Gamma)$ of the group. For two representations π_1, π_2 of the group, π_1 is *weakly contained* in π_2 if and only if $\ker(\underline{\pi}_2) \subset \ker(\underline{\pi}_1)$; moreover, a direct integral decomposition $\pi = \int_\Omega^\oplus \pi_\omega d\mu(\omega)$ at the level of Γ corresponds to a direct integral decomposition $\underline{\pi} = \int_\Omega^\oplus \underline{\pi}_\omega d\mu(\omega)$ at the level of $C^*_{\max}(\Gamma)$, with the same space Ω and the same measure μ. Thus (ii) is a consequence of (i).

[More generally, both (ii) and its proof hold *verbatim* for representations of separable locally compact groups.] $\square$

To prove Theorem 4, it is clearly enough to show that Conditions (i) and (ii) there are equivalent. The implication (i) $\Longrightarrow$ (ii) is a straightforward consequence of [Thoma64, Korollar 1], according to which *every* irreducible representation of a virtually abelian group is finite dimensional.

End of proof of Theorem 4, *namely of* (ii) $\Longrightarrow$ (i). We assume that Γ has Property (ii), and we split the proof in two cases.

Assume first that Γ is not amenable. Let σ be a Γ-faithful irreducible representation of $MS(\Gamma)$; set $\pi = \mathrm{Ind}_{MS(\Gamma)}^{\Gamma} \sigma$. By Lemmas 10 and 15, some (in fact almost every) irreducible representation π_0 which occurs in some direct integral decomposition of π is faithful and is weakly contained in π. As $MS(\Gamma)$ is amenable, π is weakly contained in the left regular representation of Γ, and therefore the same holds for π_0. As Γ is not amenable, π_0 cannot be finite dimensional, so that Γ has Property (i).

Assume now that Γ is amenable. Assume furthermore, by contradiction, that Γ does not have Property (i). Then Γ has a finite dimensional faithful irreducible representation, by the first part of (ii). In particular, Γ can be viewed as a subgroup of the compact unitary group $\mathcal{U}(n)$, for some integer $n \geq 1$. By Tits' alternative [Tits79], there exists in Γ a soluble subgroup Δ of finite index. Let R denote the closure of Δ in $\mathcal{U}(n)$ and let R^0 denote its connected component; then R^0 is of finite index in R (because R is a compact Lie group, see for example [Helga62, Chapter II, Theorem 2.3]) and an abelian group (because a connected compact group is soluble if and only if it is abelian, see for example [Bourb82, Appendice I]). Thus $\Delta \cap R^0$ is an abelian subgroup of finite index in Γ; but this contradicts the hypothesis that Γ is not virtually abelian, and this ends the proof. $\square$

Proposition 16. *Let Γ be a countable group.*

(i) *If there exists a factor M and an injective homomorphism $\pi : \Gamma \to \mathcal{U}(M)$ such that $\pi(\Gamma)'' = M$, then Γ is irreducibly represented.*

(ii) *If Γ is irreducibly represented, then there exists a factor $M = \mathcal{L}(\mathcal{H})$ of type I and a faithful representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ such that $\pi(\Gamma)'' = \mathcal{L}(\mathcal{H})$.*

Proof. Let π be as in (i). If M is an algebra of operators on some Hilbert space $\mathcal{K}$, then π is in particular a factorial representation of Γ in $\mathcal{K}$. It corresponds to a C^*-representation, say $\underline{\pi} : C^*_{\max}(\Gamma) \to \mathcal{L}(\mathcal{K})$. By a result of Dixmier (Corollary 3, page 100 of [Dix60]), there exists an irreducible representation $\underline{\rho}$ of $C^*_{\max}(\Gamma)$ such that $\underline{\pi}$ and $\underline{\rho}$ have the same kernel. The restriction ρ of $\underline{\rho}$ to Γ is therefore a faithful irreducible representation.

In view of Schur's lemma, (ii) is nothing but a reformulation of the definition of "irreducibly represented". $\square$

Corollary 5 is a straightforward consequence of Theorem 4 and Proposition 16.

Short of knowing how to answer the questions which follow Corollary 5, let us record the following elementary remark.

Observation. If Γ is a countable group which has a finite dimensional faithful irreducible representation, then $MH(\Gamma)$ is a finite group.

Proof. Consider the following properties of a group Γ:

(a) Γ has a finite dimensional faithful irreducible representation;

(b) $MS(\Gamma)$ has a finite dimensional Γ-faithful irreducible representation;

(c) $MA(\Gamma)$ has a Γ-faithful character and $MH(\Gamma)$ has a finite dimensional faithful irreducible representation;

(d) $MA(\Gamma)$ has a Γ-faithful character and $MH(\Gamma)$ is a finite group.

Property (a) implies Property (b) by Lemma 9, Properties (b) and (c) are equivalent because $MS(\Gamma) = MA(\Gamma) \oplus MH(\Gamma)$, and Properties (c) and (d) are equivalent because $MH(\Gamma)$ is a direct sum of finite simple groups.

[Observe that, however, Property (b) does not imply Property (a): if Γ is an icc group which does not have any finite dimensional faithful representation, for example the group of permutations of finite support of $\mathbb{Z}$, then Γ has Property (b) since $MS(\Gamma) = \{e\}$, but does not have Property (a).] $\square$

About Conditions (ii) and (iv) of Theorem 4, let us moreover recall the following facts. For countable groups, and more generally for separable locally compact groups and for separable C^*-algebras, there is a notion of being *of type I*, defined in terms of the von Neumann algebras generated by the images of appropriate representations. It is then a theorem of Thoma that a countable group is of type I if and only if it is virtually abelian, if and only if all its irreducible representations are finite dimensional. See [Thoma64] and [Glimm61].

5. Remarks

5.1. Minisocles, socles, and examples. The *socle* of a group Γ is the subgroup $S(\Gamma)$ generated by the union of the minimal normal subgroups (finite or infinite). Here are some examples of socles and minisocles.

(I) For a prime p and an integer $n \geq 1$, the socle of the finite cyclic group $\mathbb{Z}/p^n\mathbb{Z}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. The socle of the finite symmetric group $\mathrm{Sym}(n)$ is the corresponding alternating group if $n = 3$ or $n \geq 5$, and the Vierergruppe if $n = 4$.

If Γ is a 2-transitive subgroup of $\mathrm{Sym}(n)$, then $S(\Gamma)$ is either of the form $(\mathbb{F}_p)^m$ or a finite simple group. More generally and more precisely, if Γ is a primitive subgroup of $\mathrm{Sym}(n)$, the O'Nan–Scott Theorem (1980) provides detailed information on the socle of Γ; in particular, $S(\Gamma) \approx S^m$ for some finite simple group S and some integer m. See for example Chapter 4 in [DixMo96].

(II) Free abelian groups $\mathbb{Z}^n$, $n \geq 1$, and nonabelian free groups have socles reduced to one element.

(III) For $n \geq 3$, the socle of $\mathrm{SL}_n(\mathbb{Z})$ is reduced to one element or is of order two, if n is odd or even respectively (because any noncentral normal subgroup of $\mathrm{SL}_n(\mathbb{Z})$ contains a congruence subgroup, and consequently is never minimal).

(IV) Let Γ be a lattice in a connected semisimple Lie group G with finite center $Z(G)$ and without compact factor. It is an easy consequence of the Borel density theorem that, if the centre of Γ is $\{e\}$, then Γ is icc, so that $MS(\Gamma) = \{e\}$; more generally, $MS(\Gamma) = \Gamma \cap Z(G)$.

The minisocle of a just infinite group is reduced to one element (by definition). In particular, the minisocle of the Grigorchuk group is reduced to one element.

(V) If Γ is a direct sum of a family of infinite simple groups, then $MS(\Gamma) = \{e\}$ and $S(\Gamma) = \Gamma$. If Γ is a direct sum of a family of finite simple groups, then $MS(\Gamma) = \Gamma$.

(VI) The socle of a nilpotent group Γ is contained in the centre $Z(\Gamma)$ of Γ, because $N \cap Z(\Gamma) \neq \{e\}$ for any normal subgroup $N \neq \{e\}$ of Γ.

(VII) Let Γ be an abelian torsion-free group with cardinal strictly larger than that of the real numbers, for example a direct product of copies of $\mathbb{Z}$ indexed by $\mathbb{R}$. Then $MS(\Gamma)$ is reduced to one element, and Γ does not have any faithful character, so that the equivalences of Theorem 2 do not hold for Γ.

(VIII) Let H be a group, p a prime number, U a vector space over the prime field with p elements, $\pi \colon H \to \mathrm{GL}(U)$ a faithful representation which is semi-simple (namely which is a direct sum of irreducible representations), and $\Gamma = H \ltimes U$ the corresponding semi-direct product. Then U is the socle of Γ.

Indeed, let N a minimal normal subgroup of Γ. If $N \cap U \neq \{0\}$, then $N \subset U$, and moreover N is a H-invariant subspace of U which is irreducible, by minimality; these N's generate U. If one had $N \cap U = \{0\}$, then N and U would commute (being two normal subgroups of Γ), so that N would act trivially on U, and this is ruled out by the faithfulness of π.

Let U be of the form $U = \left(\bigoplus_{i \in I} V_i\right) \oplus \left(\bigoplus_{j \in J} W_j\right)$, with each V_i a H-invariant irreducible finite-dimensional subspace of U, and each W_j a H-invariant irreducible infinite-dimensional subspace of U. Then the mini-socle of Γ is $\bigoplus_{i \in I} V_i$.

The construction carries over to the case where each V_i and W_j is a vector space over a prime field of which the number of elements depends on i and j.

5.2. Minisocles, FC-kernels, and P. Hall's theorems.

The *FC-kernel* of a group Γ is the subset Γ_{FC} of Γ of elements which have a finite conjugacy class. It is a characteristic subgroup of Γ.

The *periodic FC-kernel* of Γ is the subset $\Gamma_{\mathrm{FC}}^{\mathrm{per}}$ of Γ_{FC} of elements of finite order. It is also a subgroup of Γ, indeed a locally finite subgroup (Dicman's Lemma, see

e.g. [Tomki84]). It follows from the definitions and from Dicman's Lemma that

$$MS(\Gamma) \subset \Gamma_{\mathrm{FC}}^{\mathrm{per}}$$

(the inclusion can be strict, as it is for example the case if Γ is cyclic of order four).

Any subgroup of a restricted direct product of finite groups is a periodic FC-group which is residually finite, and any quotient of a periodic FC-group is a periodic FC-group. For countable groups, Philip Hall has established in 1959 the converse implications:

any *countable* periodic FC-group which is residually finite can be embedded in a restricted direct product of finite groups, and any *countable* periodic FC-group is isomorphic to a quotient of a subgroup of a restricted direct product of finite groups

(Theorems 2.5 and 3.2 in [Tomki84]).

A *hoof* of a group Γ is a foot of a foot. Thus, with the notation of Proposition 1, the subgroups $\mathbb{F}_p$ and S_1 are hooves of Γ.

Let Γ be a group which has a finite Jordan–Hölder sequence (for example a finite group); if a simple group S is a foot of Γ, then S is isomorphic to a quotient of some Jordan–Hölder sequence of Γ (Bourbaki, Algèbre, nouvelle édition, 1970, chap. I, § 4, no 7). But the converse does not hold: the group of order 3 is a simple quotient of a Jordan–Hölder sequence of the alternating group Alt(4) of order 12, but Alt(4) has a unique foot which is the Vierergruppe, of order 4.

5.3. Recall of a theorem of Gelfand and Raikov.

Recall the following basic result of the theory of group representations, due to Gelfand and Raikov (see [GelRa42], as well as Corollary 13.6.6 in [Dix69C*]):

for any $\gamma \in \Gamma$, $\gamma \neq e$, there exists an irreducible representation π_γ such that $\pi_\gamma(\gamma) \neq \pi_\gamma(e)$.

This holds for any group Γ, countable or not; indeed, this holds for any locally compact group, with π_γ a continuous representation. There are two main ingredients of the proof: the group has a faithful representation which is the left-regular representation, and any representation has some description in terms of irreducible representations (via functions of positive types and a theorem of the Krein–Milman type).

For a countable group Γ which has the properties (ii) to (v) of Theorem 2, we have shown that π_γ can be chosen *independently of γ*.

5.4. Recall on tensor powers of faithful representations.

Let Γ be a group and let π be a faithful representation of Γ. For integers $m, n \geq 0$, consider the tensor power $\pi_{m,n} = \pi^{\otimes m} \otimes \bar{\pi}^{\otimes n}$, where $\bar{\pi}$ denotes the representation conjugate to π and $\pi^{\otimes m}$ the tensor product of m copies of π. Then the left regular representation of Γ is weakly contained in the direct sum $\bigoplus_{m,n \geq 0} \pi_{m,n}$ (see Example 1.11 in [BeLaS92]).

This is a generalisation, with weak containment replacing strong containment, of a well-known fact about finite groups (and compact groups, see [Cheva46], Chapter VI, § VII, Proposition 3). Thus, if, in addition, Γ is amenable, then every representation of Γ is weakly contained in $\bigoplus_{m,n \geq 0} \pi_{m,n}$. (All this carries over to locally compact groups.)

For a countable group Γ which has the properties (ii) to (v) of Theorem 2, the representation π can be chosen to be faithful *and irreducible.*

5.5. Primitive group C*-algebras.

Denote by $C_{\mathrm{red}}^*(\Gamma)$ the reduced C*-algebra, and as above by $C_{\mathrm{max}}^*(\Gamma)$ the maximal C*-algebra of a group Γ. A representation of either one of these algebras is irreducible if and only if its restriction to Γ is irreducible. It follows that, *if one of* $C_{\mathrm{red}}^*(\Gamma)$, $C_{\mathrm{max}}^*(\Gamma)$ *is primitive, then* Γ *is irreducibly represented.*

Many examples of countable groups are known for which $C_{\mathrm{red}}^*(\Gamma)$ is simple [Harpe07], and *a fortiori* primitive. These groups are in particular irreducibly represented. Concerning the properties of Γ and $C_{\mathrm{red}}^*(\Gamma)$, consider the three following conditions:

(NF) Γ does not have any finite normal subgroup besides $\{e\}$;

(NA) Γ does not have any amenable normal subgroup besides $\{e\}$;

(C*S) $C_{\mathrm{red}}^*(\Gamma)$ is simple.

It is straightforward that (NF) is a rephrasing of the condition $MS(\Gamma) = \{e\}$, and that it follows from (NA). It is elementary to check that (NA) follows from (C*S), but we recall that it is not known whether the converse holds (see [BekHa00] and [Harpe07]).

If Γ is amenable, the C*-algebras $C_{\mathrm{red}}^*(\Gamma)$ and $C_{\mathrm{max}}^*(\Gamma)$ are isomorphic. They are primitive if and only if Γ is icc [Murph03].

If Γ is a nonabelian free group, it is a result of Yoshiwaza that $C_{\mathrm{max}}^*(\Gamma)$ is primitive (see [Yoshi51], as well as [Choi80]). See the discussion around Problem 25 in [Harpe07].

6. A generalisation of Theorem 2

Consider a countable group Γ and a subgroup G of the automorphism group of Γ which contains all inner automorphisms. There is an obvious notion of G-faithful representation, which coincides with that of Γ-faithful representation in case G coincides with the group of inner automorphisms. Observe that $MS(\Gamma)$, $MA(\Gamma)$, and $MH(\Gamma)$ are G-invariant subgroups of Γ, since all three are characteristic.

We define a *G-foot* to be a minimal G-invariant subgroup of Γ. Let $\mathscr{F}_{\Gamma}^{G}$ denote the set of finite G-feet of Γ; it is the union of the set $\mathscr{A}_{\Gamma}^{G}$ of abelian finite G-feet and

of its complement $\mathcal{H}_\Gamma^G$. The G-*minisocle* of Γ is the subgroup $MS^G(\Gamma)$ generated by its G-feet, and we have as in Section 1 subgroups $MA^G(\Gamma)$ and $MH^G(\Gamma)$.

Proposition 17. *Let Γ and G be as above.*

(i) *Each $B \in \mathcal{A}_\Gamma^G$ is isomorphic to $(\mathbb{F}_p)^n$ for some prime p and some positive integer n (depending on B).*

(ii) *There exists a subset $\{B_i\}_{i \in I}$ of $\mathcal{A}_\Gamma^G$ such that $MA^G(\Gamma) = \bigoplus_{i \in I} B_i$. In particular, the group $MA^G(\Gamma)$ is abelian.*

(iii) *For each $H \in \mathcal{H}_\Gamma^G$, the feet $S_1, \ldots, S_k$ of H are conjugate under G, and simple. Moreover $H = S_1 \oplus \cdots \oplus S_k$.*

(iv) *We have $MH^G(\Gamma) = \bigoplus_{H \in \mathcal{H}_\Gamma^G} H$.*

(v) *We have $MS^G(\Gamma) = MA^G(\Gamma) \oplus MH^G(\Gamma)$.*

Theorem 18. *Let Γ, G, and $MA^G(\Gamma) = \bigoplus_{i \in I} B_i$ be as above. The following properties are equivalent:*

(i) *Γ has a representation which is irreducible and G-faithful;*

(ii) *$MA(\Gamma)$ has a G-faithful character;*

(ii′) *$MA^G(\Gamma)$ has a G-faithful character;*

(iii) *$MS(\Gamma)$ has a G-faithful irreducible representation;*

(iii′) *$MS^G(\Gamma)$ has a G-faithful irreducible representation;*

(iv) *for every finite subset E of I, there exists an element x_E in $MA_E^G(\Gamma) \doteq \bigoplus_{i \in E} B_i$ such that the G-orbit of x_E generates $MA_E^G(\Gamma)$.*

In particular, a countable group Γ has a G-faithful irreducible representation as soon as $MA^G(\Gamma) = \{e\}$, a fortiori as soon as $MS^G(\Gamma) = \{e\}$.

For example, let $\Gamma = \bigoplus_{i \in \mathbb{N}} A_i$ be a countable infinite direct sum of groups A_i indexed by the natural numbers, each of them isomorphic to a given finite cyclic group, and let G be the group of permutations of $\mathbb{N}$, identified in the natural way to a group of automorphisms of Γ. Then Γ is irreducibly underrepresented, but has a G-faithful irreducible character, for example the projection onto A_1 followed by the natural isomorphisms of A_1 with the appropriate group of roots of unity.

Proposition 17 and Theorem 18 can be proved by essentially the same arguments as in Sections 2 and 3.

Lemmas 9 and 10 should be reformulated for a G-invariant subgroup N of Γ; in the new Lemma 9, the G-faithfulness of π implies that σ_ω is G-faithful for almost all $\omega \in \Omega$; in the new Lemma 10, if we assume that σ is G-faithful and that there does not exist any finite G-foot M such that $M \cap N = \{e\}$, then π_ω is G-faithful for

almost all $\omega \in \Omega$. In the new Lemma 13, the groups N, A, S should be G-invariant, but S_i normal (not necessarily G-invariant) and simple (not necessarily G-simple); the conclusion is that N has a G-faithful irreducible representation if and only if A has a G-faithful character. In the new Lemma 14, both A and the A_i should be G-invariant, and χ^{Γ} should be replaced by χ^{G}. The other (minor) modifications, as well as the formulation of one more claim analogous to Claim (v) of Theorem 2, are left to the reader.

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