Comment. Math. Helv. 84 (2009), 135–157 **Commentarii Mathematici Helvetici**

String topology for spheres

Luc Menichi*

With an appendix by Gerald Gaudens and Luc Menichi

Dedicated to Jean-Claude Thomas, on the occasion of his 60th birthday

Abstract. Let M be a compact oriented d-dimensional smooth manifold. Chas and Sullivan have defined a structure of Batalin–Vilkovisky algebra on $\mathbb{H}_*(LM)$. Extending work of Cohen, Jones and Yan, we compute this Batalin–Vilkovisky algebra structure when M is a sphere S^d, $d \geq 1$. In particular, we show that $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and the Hochschild cohomology $HH^*(H^*(S^2);H^*(S^2))$ are surprisingly not isomorphic as Batalin–Vilkovisky algebras, although we prove that, as expected, the underlying Gerstenhaber algebras are isomorphic. The proof requires the knowledge of the Batalin–Vilkovisky algebra $H_*(\Omega^2 S^3;\mathbb{F}_2)$ that we compute in the Appendix.

Mathematics Subject Classification (2000). 55P35, 16E40, 55P62, 57T30, 55U10.

Keywords. String topology, Batalin–Vilkovisky algebra, Gerstenhaber algebra, Hochschild cohomology, free loop space.

1. Introduction

Let M be a compact oriented d-dimensional smooth manifold. Denote by $LM := \operatorname{map}(S^1, M)$ the free loop space on M. In 1999, Chas and Sullivan [2] have shown that the shifted free loop homology $\mathbb{H}_*(LM) := H_{*+d}(LM)$ has a structure of Batalin–Vilkovisky algebra (Definition 5). In particular, they showed that $\mathbb{H}_*(LM)$ is a Gerstenhaber algebra (Definition 8). This Batalin–Vilkovisky algebra has been computed when M is a complex Stiefel manifold [25] and very recently over $\mathbb{Q}$ when M is a $K(\pi,1)$ [28]. In this paper, we compute the Batalin–Vilkovisky algebra $\mathbb{H}_*(LM;\mathbb{k})$ when M is a sphere S^n, $n \geq 1$ over any commutative ring $\mathbb{k}$ (Theorems 10, 16, 17, 24 and 25).

*The author was partially supported by the Mathematics Research Center of Stanford University.

In fact, few calculations of this Batalin–Vilkovisky algebra structure or even of the underlying Gerstenhaber algebra structure have been done because the following conjecture has not yet been proved.

Conjecture 1 (due to [2, "dictionary" p. 5] or [7]?). If M is simply connected then there is an isomorphism of Gerstenhaber algebras $\mathbb{H}_*(LM) \cong HH^*(S^*(M); S^*(M))$ between the free loop space homology and the Hochschild cohomology of the algebra of singular cochains on M.

In [7], [5], Cohen and Jones proved that there is an isomorphism of graded algebras over any field

$$\mathbb{H}_*(LM) \cong HH^*(S^*(M); S^*(M)).$$

Over the reals or over the rationals, two proofs of this isomorphism of graded algebras have been given by Merkulov [23] and Félix, Thomas, Vigué-Poirrier [11]. Motivated by this conjecture, Westerland [30] has computed the Gerstenhaber algebra $HH^*(S^*(M;\mathbb{F}_2); S^*(M;\mathbb{F}_2))$ when M is a sphere or a projective space.

What about the Batalin–Vilkovisky algebra structure?

Suppose that M is formal over a field, then since the Gerstenhaber algebra structure on Hochschild cohomology is preserves by quasi-isomorphism of algebras [10, Theorem 3], we obtain an isomorphism of Gerstenhaber algebras

$$HH^*(S^*(M); S^*(M)) \cong HH^*(H^*(M); H^*(M)). \tag{2}$$

Poincaré duality induces an isomorphism of $H^*(M)$-modules

$$\Theta\colon H^*(M) \to H^*(M)^\vee.$$

Therefore, we obtain the isomorphism

$$HH^*(H^*(M); H^*(M)) \cong HH^*(H^*(M); H^*(M)^\vee)$$

and the Gerstenhaber algebra structure on $HH^*(H^*(M); H^*(M))$ extends to a Batalin–Vilkovisky algebra [26], [22], [19] (See above Proposition 20 for details). This Batalin–Vilkovisky algebra structure is further extended in [27], [9], [20], [21] to a richer algebraic structure. It is natural to conjecture that this Batalin–Vilkovisky algebra on $HH^*(H^*(M); H^*(M))$ is isomorphic to the Batalin–Vilkovisky algebra $\mathbb{H}_*(LM)$. We show (Corollary 30) that this is not the case over $\mathbb{F}_2$ when M is the sphere S^2. See [6, Comments 2, Chapter 1] or the papers of Tradler and Zeinalian [26], [27] for a related conjecture when M is not assumed to be necessarily formal. On the contrary, we prove (Corollary 23) that the above conjecture is satisfied for $M = S^2$ over $\mathbb{F}_2$.

Acknowledgment. We wish to thank Ralph Cohen and Stanford Mathematics Department for providing a friendly atmosphere during my six months of "delegation CNRS". We would like also to thank Yves Félix for a discussion simplifying the proof of Theorem 10.

2. The Batalin–Vilkovisky algebra structure on $\mathbb{H}_*(LM)$

We recall here the definition of the Batalin–Vilkovisky algebra on $\mathbb{H}_*(LM;\mathbb{k})$ given by Chas and Sullivan [2] over any commutative ring $\mathbb{k}$ and deduce that this Batalin–Vilkovisky algebra $\mathbb{H}_*(LM;\mathbb{k})$ behaves well with respect to change of rings.

We first recall the definition of the loop product following Cohen and Jones [7], [6]. Let M be a closed oriented smooth manifold of dimension d. The inclusion $e\colon \mathrm{map}(S^1 \vee S^1, M) \hookrightarrow LM \times LM$ can be viewed as a codimension d embedding between infinite dimension manifolds [24, Proposition 5.3]. Denote by ν its normal bundle. Let $\tau_e\colon LM \times LM \twoheadrightarrow \mathrm{map}(S^1 \vee S^1, M)^\nu$ its Thom–Pontryagin collapse map. Recall that the umkehr (Gysin) map $e_!$ is the composite of τ_e and the Thom isomorphism:

$$H_*(LM \times LM;\mathbb{k}) \xrightarrow{H_*(\tau_e;\mathbb{k})} H_*(\mathrm{map}(S^1 \vee S^1, M)^\nu;\mathbb{k}) \xrightarrow[\cong]{\cap u_{\mathbb{k}}} H_{*-d}(\mathrm{map}(S^1 \vee S^1, M);\mathbb{k}).$$

The Thom isomorphism is given by taking a relative cap product $\cap$ with a Thom class for ν, $u_{\mathbb{k}} \in H^d(\mathrm{map}(S^1 \vee S^1, M)^\nu;\mathbb{k})$. A Thom class with coefficients in $\mathbb{Z}$, $u_{\mathbb{Z}}$, gives rise to a Thom class $u_{\mathbb{k}}$ with coefficients in $\mathbb{k}$, under the morphism

$$H^d(\mathrm{map}(S^1 \vee S^1, M);\mathbb{Z}) \to H^d(\mathrm{map}(S^1 \vee S^1, M);\mathbb{k})$$

induced by the ring homomorphism $\mathbb{Z} \to \mathbb{k}$ [16, p. 441]. So we have the commutative diagram

$$\begin{array}{ccc} H_*(LM \times LM;\mathbb{Z}) & \xrightarrow{e_!} & H_{*-d}(\mathrm{map}(S^1 \vee S^1, M);\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_*(LM \times LM;\mathbb{k}) & \xrightarrow{e_!} & H_{*-d}(\mathrm{map}(S^1 \vee S^1, M);\mathbb{k}). \end{array}$$

Let $\gamma\colon \mathrm{map}(S^1 \vee S^1, M) \to LM$ be the map obtained by composing loops. The loop product is the composite

$$H_*(LM;\mathbb{k}) \otimes H_*(LM;\mathbb{k}) \to H_*(LM \times LM;\mathbb{k}) \xrightarrow{e_!} H_{*-d}(\mathrm{map}(S^1 \vee S^1, M);\mathbb{k}) \xrightarrow{H_{*-d}(\gamma;\mathbb{k})} H_{*-d}(LM;\mathbb{k}).$$

So clearly, we have proved

Lemma 3. *The morphism of abelian groups $\mathbb{H}_*(LM;\mathbb{Z}) \to \mathbb{H}_*(LM;\mathbb{k})$ induced by $\mathbb{Z} \to \mathbb{k}$ is a morphism of graded rings.*

Suppose that the circle S^1 acts on a topological space X. Then we have an action of the algebra $H_*(S^1)$ on $H_*(X)$,

$$H_*(S^1) \otimes H_*(X) \to H_*(X).$$

Denote by $[S^1]$ the fundamental class of the circle. Then we define an operator of degree 1, $\Delta\colon H_*(X;\Bbbk) \to H_{*+1}(X;\Bbbk)$, which sends x to the image of $[S^1] \otimes x$ under the action. Since $[S^1]^2 = 0$, $\Delta \circ \Delta = 0$. The following lemma is obvious.

Lemma 4. *Let X be a S^1-space. We have the commutative diagram*

$$\begin{array}{ccc} H_*(X;\mathbb{Z}) & \xrightarrow{\Delta} & H_{*+1}(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_*(X;\Bbbk) & \xrightarrow{\Delta} & H_{*+1}(X;\Bbbk), \end{array}$$

where the vertical maps are induced by the ring homomorphism $\mathbb{Z} \to \Bbbk$.

The circle S^1 acts on the free loop space on M by rotating the loops. Therefore we have a operator Δ on $\mathbb{H}_*(LM)$. Chas and Sullivan [2] have shown that $\mathbb{H}_*(LM)$ equipped with the loop product and the Δ-operator, is a Batalin–Vilkovisky algebra.

Definition 5. A *Batalin–Vilkovisky algebra* is a commutative graded algebra A equipped with an operator $\Delta\colon A \to A$ of degree 1 such that $\Delta \circ \Delta = 0$ and

$$\begin{aligned} \Delta(abc) = \Delta(ab)c &+ (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) \\ &- (\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c). \end{aligned} \tag{6}$$

Consider the bracket $\{\ ,\ \}$ of degree $+1$ defined by

$$\{a,b\} = (-1)^{|a|}\big(\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b)\big)$$

for any $a, b \in A$. (6) is equivalent to the following relation called the *Poisson relation*:

$$\{a,bc\} = \{a,b\}c + (-1)^{(|a|+1)|b|}b\{a,c\}. \tag{7}$$

Getzler [14, Proposition 1.2] has shown that $\{\,,\}$ is a Lie bracket and therefore that a Batalin–Vilkovisky algebra is a Gerstenhaber algebra.

Definition 8. A *Gerstenhaber algebra* is a commutative graded algebra A equipped with a linear map $\{-,-\}\colon A \otimes AG \to A$ of degree 1 such that:

a) the bracket $\{-,-\}$ gives to A a structure of a graded Lie algebra of degree 1. This means that for each a, b and $c \in A$,

$$\{a,b\} = -(-1)^{(|a|+1)(|b|+1)}\{b,a\},$$

and

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\}.$$

b) The product and the Lie bracket satisfy the Poisson relation (7).

Using Lemma 3 and Lemma 4, we deduce

Proposition 9. *The* $\mathbb{k}$*-linear map*

$$\mathbb{H}_*(LM;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{k} \hookrightarrow \mathbb{H}_*(LM;\mathbb{k})$$

is an inclusion of Batalin–Vilkovisky algebras.

In particular, by the universal coefficient theorem,

$$\mathbb{H}_*(LM;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{H}_*(LM;\mathbb{Q}).$$

More generally, this proposition tells us that if $\mathrm{Tor}^{\mathbb{Z}}(\mathbb{H}_*(LM;\mathbb{Z}),\mathbb{k}) = 0$ then the Batalin–Vilkovisky algebra $\mathbb{H}_*(LM;\mathbb{Z})$ determines the Batalin–Vilkovisky algebra $\mathbb{H}_*(LM;\mathbb{k})$.

3. The circle and an useful lemma

In this section, we compute the structure of the Batalin–Vilkovisky algebra on the homology of the free loop space on the circle S^1 using a lemma which gives information on the image of Δ on elements of lower degree in $H_*(LM)$.

Theorem 10. *As a Batalin–Vilkovisky algebra, the homology of the free loop space on the circle is given by*

$$\mathbb{H}_*(LS^1;\mathbb{k}) \cong \mathbb{k}[\mathbb{Z}] \otimes \Lambda a_{-1}.$$

Denote by x *a generator of* $\mathbb{Z}$*. The operator* Δ *is*

$$\Delta(x^i \otimes a_{-1}) = i(x^i \otimes 1), \quad \Delta(x^i \otimes 1) = 0$$

for all $i \in \mathbb{Z}$*.*

Let X be a pointed topological space. Consider the free loop fibration $\Omega X \overset{j}{\hookrightarrow} LX \overset{\mathrm{ev}}{\twoheadrightarrow} X$. Denote by $\mathrm{hur}_X : \pi_n(X) \to H_n(X)$ the Hurewicz map.

Lemma 11. *Let* $n \in \mathbb{N}$*. Let* $f \in \pi_{n+1}(X)$*. Denote by* $\tilde{f} \in \pi_n(\Omega X)$ *the adjoint of* f*. Then*

$$(H_*(\mathrm{ev}) \circ \Delta \circ H_*(j) \circ \mathrm{hur}_{\Omega X})(\tilde{f}) = \mathrm{hur}_X(f).$$

Proof. Take in homology the image of $[S^1] \otimes [S^n]$ in the following commutative diagram:

$$\begin{array}{ccccc} S^1 \times \Omega X & \xrightarrow{S^1 \times j} & S^1 \times LX & \xrightarrow{\mathrm{act}_{LX}} & LX \\ {\scriptstyle S^1 \times \tilde{f}} \uparrow & & & & \downarrow {\scriptstyle \mathrm{ev}} \\ S^1 \times S^n & \longrightarrow & S^1 \wedge S^n & \xrightarrow[f]{} & X, \end{array}$$

where $\mathrm{act}_{LX}\colon S^1 \times LX \to LX$ is the action of the circle on LX. □

Proof of Theorem 10. More generally, let G be a compact Lie group. Consider the homeomorphism $\Theta_G\colon \Omega G \times G \xrightarrow{\cong} LG$ which sends the couple (w, g) to the free loop $t \mapsto w(t)g$. In fact, Θ_G is an isomorphism of fiberwise monoids. Therefore by [15, Part 2 of Theorem 8.2],

$$\mathbb{H}_*(\Theta_G)\colon H_*(\Omega G) \otimes \mathbb{H}_*(G) \to \mathbb{H}_*(LG)$$

is a morphism of graded algebras. Since $H_*(S^1)$ has no torsion,

$$\mathbb{H}_*(\Theta_{S^1})\colon H_*(\Omega S^1) \otimes \mathbb{H}_*(S^1) \cong \mathbb{H}_*(LS^1)$$

is an isomorphism of algebras. Since Δ preserves path-connected components,

$$\Delta(x^i \otimes a_{-1}) = \alpha(x^i \otimes 1)$$

where $\alpha \in \mathbb{k}$. Denote by $\varepsilon_{\mathbb{k}[\mathbb{Z}]}$ the canonical augmentation of the group ring $\mathbb{k}[\mathbb{Z}]$. Since $H_*(\mathrm{ev} \circ \Theta_{S^1}) = \varepsilon_{\mathbb{k}[\mathbb{Z}]} \otimes H_*(S^1)$,

$$(H_*(\mathrm{ev}) \circ \Delta)(x^i \otimes a_{-1}) = \alpha 1.$$

On the other hand, applying Lemma 11 to the degree i map $S^1 \to S^1$, we obtain that $(H_*(\mathrm{ev}) \circ \Delta \circ H_*(j))(x^i) = i1$. Therefore $\alpha = i$. □

4. Computations using Hochschild homology

In this section, we compute the Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^n)$, $n \geq 2$, using the following elementary technique:

The algebra structure has been computed by Cohen, Jones and Yan using the Serre spectral sequence [8]. On the other hand, the action of $H_*(S^1)$ on $H_*(LS^n)$ can be computed using Hochschild homology. Using the compatibility between the product and Δ, we determine the Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^n)$ up to isomorphism. This elementary technique will fail for $\mathbb{H}_*(LS^2)$.

Let A be an augmented differential graded algebra. Denote by $s\bar{A}$ the suspension of the augmentation ideal $\bar{A}$, $(s\bar{A})_i = \bar{A}_{i-1}$. Let d_1 be the differential on the tensor product of complexes $A \otimes T(s\bar{A})$. The (normalized) Hochschild chain complex, denoted $\mathcal{C}_*(A;A)$, is the complex $(A \otimes T(s\bar{A}), d_1 + d_2)$ where

$$\begin{aligned} d_2 a[sa_1|\dots|sa_k] =& (-1)^{|a|} a a_1 [sa_2|\dots|sa_k] \\ &+ \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} a[sa_1|\dots|sa_i a_{i+1}|\dots|sa_k] \\ &- (-1)^{|sa_k|\varepsilon_{k-1}} a_k a[sa_1|\dots|sa_{k-1}]. \end{aligned}$$

Here $\varepsilon_i = |a| + |sa_1| + \dots + |sa_i|$.

Connes' boundary map B is the map of degree $+1$

$$B\colon A \otimes (s\bar{A})^{\otimes p} \to A \otimes (s\bar{A})^{\otimes p+1}$$

defined by

$$B(a_o[sa_1|\dots|sa_p]) = \sum_{i=0}^{p} (-1)^{|sa_0 \dots sa_{i-1}||sa_i \dots sa_p|} [sa_i|\dots|sa_p|sa_0|\dots|sa_{p-1}].$$

Up to the isomorphism $s^p(A^{\otimes(p+1)}) \to A \otimes (sA)^{\otimes p}$, $s^p(a_0[a_1|\dots|a_p]) \mapsto (-1)^{p|a_0|+(p-1)|a_1|+\dots+|a_{p-1}|} a_0[sa_1|\dots|sa_p]$, our signs coincides with those of [29].

The Hochschild homology of A (with coefficient in A) is the homology of the Hochschild chain complex:

$$HH_*(A;A) := H_*(\mathcal{C}_*(A;A)).$$

The Hochschild cohomology of A (with coefficient in $A^\vee$) is the homology of the dual of the Hochschild chain complex:

$$HH^*(A;A^\vee) := H_*(\mathcal{C}_*(A;A)^\vee).$$

Consider the dual of Connes' boundary map, $B^\vee(\varphi) = (-1)^{|\varphi|}\varphi \circ B$. On $HH^*(A;A^\vee)$, $B^\vee$ defines an action of $H_*(S^1)$.

Example 12. Let $n \geq 2$. Let $\Bbbk$ be any commutative ring. Let $A := H^*(S^n) = \Lambda x_{-n}$ be the exterior algebra on a generator of lower degree $-n$. Denote by $[sx]^k := 1[sx|\dots|sx]$ and $x[sx]^k := x[sx|\dots|sx]$ the elements of $\mathcal{C}_*(A;A)$ where the term sx appears k times. These elements form a basis of $\mathcal{C}_*(A;A)$. Denote by $[sx]^{k\vee}$, $x[sx]^{k\vee}$, $k \geq 0$, the dual basis. The differential $d^\vee$ on $\mathcal{C}_*(A;A)^\vee$ is given by

$d^\vee([sx]^{k\vee}) = 0$ and $d^\vee(x[sx]^{k\vee}) = \pm\left(1-(-1)^{k(n+1)}\right)[sx]^{(k+1)\vee}$. The dual of Connes' boundary map $B^\vee$ is given by

$$B^\vee([sx]^{k\vee}) = \begin{cases} (-1)^{n+1}k\ x[sx]^{(k-1)\vee} & \text{if } (k+1)(n+1) \text{ is even,} \\ 0 & \text{if } (k+1)(n+1) \text{ is odd,} \end{cases}$$

and $B^\vee(x[sx]^{k\vee}) = 0$. We remark that $[sx]^{k\vee}$ is of (lower) degree $k(n-1)$ and $x[sx]^{k\vee}$ of degree $n+k(n-1)$.

Theorem 13 ([17]). *Let X be a simply connected space such that $H_*(X;\mathbb{k})$ is of finite type in each degree. Then there is a natural isomorphism of $H_*(S^1)$-modules between the homology of the free loop space on X and the Hochschild cohomology of the algebra of singular cochain $S^*(X;\mathbb{k})$:*

$$H_*(LX) \cong HH^*(S^*(X;\mathbb{k}); S^*(X;\mathbb{k})^\vee). \tag{14}$$

In this paper, when we will apply this theorem, $H_*(X;\mathbb{k})$ is assumed to be $\mathbb{k}$-free of finite type in each degree and X will be always $\mathbb{k}$-formal: the algebra $S^*(X;\mathbb{k})$ will be linked by quasi-isomorphisms of cochain algebras to $H_*(X;\mathbb{k})$. Therefore

$$HH^*(S^*(X;\mathbb{k}); S^*(X;\mathbb{k})^\vee) \cong HH^*(H^*(X;\mathbb{k}); H^*(X;\mathbb{k})^\vee). \tag{15}$$

Theorem 16. *For $n > 1$ odd, as a Batalin–Vilkovisky algebra,*

$$\mathbb{H}_*(LS^n;\mathbb{k}) = \mathbb{k}[u_{n-1}] \otimes \Lambda a_{-n},$$

$$\Delta(u_{n-1}^i \otimes a_{-n}) = i(u_{n-1}^{i-1} \otimes 1),$$

$$\Delta(u_{n-1}^i \otimes 1) = 0.$$

Proof. For the algebra structure, Cohen, Jones and Yan [8] proved that $\mathbb{H}_*(LS^n;\mathbb{Z}) = \mathbb{k}[u_{n-1}] \otimes \Lambda a_{-n}$ when $\mathbb{k} = \mathbb{Z}$. Their proof works over any $\mathbb{k}$ (alternatively, using Proposition 9, we could assume that $\mathbb{k} = \mathbb{Z}$). Computing Connes' boundary map on $HH^*(H^*(S^n); H_*(S^n))$ (Example 12), we see that Δ on $\mathbb{H}_*(LS^n;\mathbb{k})$ is null in even degree and in degree $-n$, and is an isomorphism in degree -1. Therefore $\Delta(u_{n-1}^i \otimes 1) = 0$, $\Delta(1 \otimes a_{-n}) = 0$ and $\Delta(u_{n-1} \otimes a_{-n}) = \alpha 1$ where α is invertible in $\mathbb{k}$. Replacing a_{-n} by $\frac{1}{\alpha}a_{-n}$ or u_{n-1} by $\frac{1}{\alpha}u_{n-1}$, we can assume up to isomorphisms that $\Delta(u_{n-1} \otimes a_{-n}) = 1$. Therefore $\{u_{n-1}, a_{-n}\} = 1$. Using the Poisson relation (7), $\{u_{n-1}^i, a_{-n}\} = i u_{n-1}^{i-1}$. Therefore $\Delta(u_{n-1}^i \otimes a_{-n}) = i(u_{n-1}^{i-1} \otimes 1)$. □

Theorem 17. *For $n \geq 2$ even, there exists a constant $\varepsilon_0 \in \mathbb{F}_2$ such that as a Batalin–Vilkovisky algebra,*

$$\mathbb{H}_*(LS^n;\mathbb{Z}) = \Lambda b \otimes \frac{\mathbb{Z}[a,v]}{(a^2, ab, 2av)} = \bigoplus_{k=0}^{+\infty} \mathbb{Z}v_{2(n-1)}^k \oplus \bigoplus_{k=0}^{+\infty} \mathbb{Z}b_{-1}v^k \oplus \mathbb{Z}a_{-n} \oplus \bigoplus_{k=1}^{+\infty} \frac{\mathbb{Z}}{2\mathbb{Z}} av^k,$$

for all $k \geq 0$, $\Delta(v^k) = 0$, $\Delta(av^k) = 0$ and

$$\Delta(bv^k) = \begin{cases} (2k+1)v^k + \varepsilon_0 a v^{k+1} & \text{if } n = 2, \\ (2k+1)v^k & \text{if } n \geq 4. \end{cases}$$

Proof. For the algebra structure, Cohen, Jones and Yan [8] proved the equality. Computing Connes' boundary map on $HH^*(H^*(S^n); H_*(S^n))$ (Example 12), we see that Δ on $\mathbb{H}_*(LS^n;\mathbb{k})$ is null in even degree and is injective in odd degree.

Case $n \neq 2$. This case is simple, since all the generators of $\mathbb{H}_*(LS^n)$, v^k, bv^k and av^k, $k \geq 0$, have different degrees. Using Example 12, we also see that for all $k \geq 0$,

$$\Delta\colon \mathbb{H}_{-1+2k(n-1)} = \mathbb{Z}b_{-1}v^k \hookrightarrow \mathbb{H}_{2k(n-1)} = \mathbb{Z}v^k$$

has cokernel isomorphic to $\frac{\mathbb{Z}}{(2k+1)\mathbb{Z}}$. Therefore $\Delta(bv^k) = \pm(2k+1)v^k$. By replacing b_{-1} by $-b_{-1}$, we can assume up to isomorphims that $\Delta(b) = 1$. Let $k \geq 1$. Let $\alpha_k \in \{-2k-1, 2k+1\}$ such that $\Delta(bv^k) = \alpha_k v^k$. Using formula (6), we obtain that $\Delta(bv^k v^k) = (2\alpha_k - 1)v^{2k}$. We know that $\Delta(bv^{2k}) = \pm(4k+1)v^{2k}$. Therefore α_k must be equal to $2k+1$.

Case $n = 2$. This case is complicated, since for $k \geq 0$, v^k and av^{k+1} have the same degree. Using Example 12, we also see that

$$\Delta\colon \mathbb{H}_{-1+2k} = \mathbb{Z}b_{-1}v^k \hookrightarrow \mathbb{H}_{2k} = \mathbb{Z}v^k \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} av^{k+1}$$

has cokernel, denoted CokerΔ, isomorphic to $\frac{\mathbb{Z}}{(2k+1)\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. There exists unique $\alpha_k \in \mathbb{Z}^*$ and $\varepsilon_k \in \frac{\mathbb{Z}}{2\mathbb{Z}}$ such that $\Delta(bv^k) = \alpha_k v^k + \varepsilon_k av^{k+1}$. The injective map Δ

fits into the commutative diagram of short exact sequences (Noether's Lemma)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{H}_{-1+2k} & \xrightarrow{\mathrm{id}} & \mathbb{H}_{-1+2k} & \longrightarrow & 0 & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \times 2} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{H}_{-1+2k} & \xrightarrow{\Delta} & \mathbb{H}_{2k} & \longrightarrow & \mathrm{Coker}\,\Delta & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \frac{\mathbb{Z}}{2\mathbb{Z}} & \xrightarrow{\overline{\Delta}} & \frac{\mathbb{Z}}{2\alpha_k\mathbb{Z}}\oplus\frac{\mathbb{Z}}{2\mathbb{Z}} & \longrightarrow & \mathrm{Coker}\,\overline{\Delta} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

The cokernel of $\overline{\Delta}$, denoted $\mathrm{Coker}\,\overline{\Delta}$ is of cardinal $2|\alpha_k|$. So $|\alpha_k| = 2k+1$. Therefore $\Delta(bv^k) = \pm(2k+1)v^k + \varepsilon_k av^{k+1}$.

By replacing b_{-1} by $-b_{-1}$, we can assume up to isomorphims that $\Delta(b) = 1 + \varepsilon_0 av$. Using formula (6), we obtain that

$$\Delta(bv^k v^l) = (\alpha_k + \alpha_l - 1)v^{k+l} + (\varepsilon_k + \varepsilon_l - \varepsilon_0)av^{k+l+1}.$$

Therefore

$$\Delta(bv^k v^k) = (2\alpha_k - 1)v^{2k} + \varepsilon_0 av^{2k+1} = \pm(4k+1)v^{2k} + \varepsilon_{2k} av^{2k+1}.$$

So $\alpha_k = 2k+1$, $\varepsilon_{2k} = \varepsilon_0$ and $\varepsilon_{2k+1} = \varepsilon_{2k} + \varepsilon_1 - \varepsilon_0 = \varepsilon_1$.

The map $\Theta\colon \mathbb{H}_*(LS^2) \to \mathbb{H}_*(LS^2)$ given by $\Theta(b_{-1}v^k) = b_{-1}v^k$, $\Theta(v^k) = v^k + kav^{k+1}$, $\Theta(av^k) = av^k$, $k \geq 0$ is an involutive isomorphism of algebras. Therefore, by replacing v by $v + av^2$, we can assume that $\varepsilon_1 = \varepsilon_0$. So we have proved

$$\Delta(bv^k) = (2k+1)v^k + \varepsilon_0 av^{k+1}, \quad k \geq 0. \qquad \square$$

These two cases $\varepsilon_0 = 0$ and $\varepsilon_0 = 1$ correspond to two non-isomorphic Batalin–Vilkovisky algebras whose underlying Gerstenhaber algebras are the same. Therefore even if we have not yet computed the Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^2;\mathbb{Z})$, we have computed its underlying Gerstenhaber algebra. Using the definition of the bracket, straightforward computations give the following corollary.

Corollary 18. *For $n \geq 2$ even, as Gerstenhaber algebra*

$$\mathbb{H}_*(LS^n;\mathbb{Z}) = \Lambda b_{-1} \otimes \frac{\mathbb{Z}[a_{-n}, v_{2(n-1)}]}{(a^2, ab, 2av)}$$

with $\{v^k, v^l\} = 0$, $\{bv^k, v^l\} = -2lv^{k+l}$, $\{bv^k, bv^l\} = 2(k-l)bv^{k+l}$, $\{a, v^l\} = 0$, $\{av^k, bv^l\} = -(2l+1)av^{k+l}$ *and* $\{av^k, av^l\} = 0$ *for all* $k, l \geq 0$.

5. When Hochschild cohomology is a Batalin–Vilkovisky algebra

In this section, we recall the structure of Gerstenhaber algebra on the Hochschild cohomology of an algebra whose degrees are bounded. We recall from [26], [22], [27], [19] the Batalin–Vilkovisky algebra on the Hochschild cohomology of the cohomology $H^*(M)$ of a closed oriented manifold M. We compute this Batalin–Vilkovisky algebra $HH^*(H^*(M); H^*(M))$ when M is a sphere.

Throughout in this section we will work over the prime field $\mathbb{F}_2$. Let A be an augmented graded algebra such that the augmentation ideal $\bar{A}$ is concentrated in degree ≤ -2 and bounded below (or concentrated in degree ≥ 0 and bounded above). Then the (normalized) Hochschild cochain complex, denoted $\mathcal{C}^*(A, A)$, is the complex

$$\mathrm{Hom}(Ts\bar{A}, A) \cong \textstyle\bigoplus_{p\geq 0} \mathrm{Hom}((s\bar{A})^{\otimes p}, A)$$

with a differential d_2. For an element f in $\mathrm{Hom}((s\bar{A})^{\otimes p}, A)$, the differential $d_2 f$ in $\mathrm{Hom}((s\bar{A})^{\otimes p+1}, A)$ is given by

$$\begin{aligned}(d_2 f)([sa_1|\dots|sa_{p+1}]) &:= a_1 f([sa_2|\dots|sa_{p+1}]) \\ &\quad + \sum_{i=1}^{p} f([sa_1|\dots|s(a_i a_{i+1})|\dots|sa_{p+1}]) + f([sa_1|\dots|sa_p])a_p.\end{aligned}$$

The Hochschild cohomology of A with coefficient in A is the homology of the Hochschild cochain complex:

$$HH^*(A; A) := H_*(\mathcal{C}^*(A; A)).$$

We remark that $HH^*(A; A)$ is bigraded. Our degree is sometimes called the total degree: sum of the external degree and the internal degree. The Hochschild cochain complex $\mathcal{C}^*(A, A)$ is a differential graded algebra. For $f \in \mathrm{Hom}((s\bar{A})^{\otimes p}, A)$ and $g \in \mathrm{Hom}((s\bar{A})^{\otimes q}, A)$, the (cup) product of f and g, $f \cup g \in \mathrm{Hom}((s\bar{A})^{\otimes p+q}, A)$ is defined by

$$(f \cup g)([sa_1|\dots|sa_{p+q}]) := f([sa_1|\dots|sa_p])g([sa_{p+1}|\dots|sa_{p+q}]).$$

The Hochschild cochain complex $\mathcal{C}^*(A, A)$ has also a Lie bracket of (lower) degree $+1$.

$$\begin{aligned}&(f \mathbin{\bar{\circ}} g)([sa_1|\dots|sa_{p+q-1}]) \\ &\qquad := \sum_{i=1}^{p} f\left([sa_1|\dots|sa_{i-1}|sg([sa_i|\dots|sa_{i+q-1}])|sa_{i+q}|\dots|sa_{p+q-1}]\right).\end{aligned}$$

$\{f, g\} = f \bar{\circ} g - g \bar{\circ} f$. Our formulas are the same as in the non-graded case [13]. We remark that if A is not assumed to be bounded, the formulas are more complicated. Gerstenhaber has shown that $HH^*(A; A)$ equipped with the cup product and the Lie bracket is a Gerstenhaber algebra.

Let M be a closed d-dimensional smooth manifold. Poincaré duality induces an isomorphism of $H^*(M; \mathbb{F}_2)$-modules of (lower) degree d:

$$\Theta\colon H^*(M; \mathbb{F}_2) \xrightarrow{\cap[M]} H_*(M; \mathbb{F}_2) \cong H^*(M; \mathbb{F}_2)^\vee. \tag{19}$$

More generally, let A be a graded algebra equipped with an isomorphism of A-bimodules of degree d, $\Theta\colon A \xrightarrow{\cong} A^\vee$. Then we have the isomorphism

$$HH^*(A, \Theta)\colon HH^*(A, A) \xrightarrow{\cong} HH^*(A, A^\vee).$$

Therefore on $HH^*(A, A)$, we have both a Gerstenhaber algebra structure and an operator Δ given by the dual of Connes' boundary map B. Motivated by the Batalin–Vilkovisky algebra structure of Chas–Sullivan on $\mathbb{H}_*(LM)$, Thomas Tradler [26] proved that $HH^*(A, A)$ is a Batalin–Vilkovisky algebra. See [22, Theorem 1.6] for an explicit proof. In [19] or [27, Corollary 3.4] or [9, Section 1.4] or [20, Theorem B] or [21, Section 11.6], this Batalin–Vilkovisky algebra structure on $HH^*(A, A)$ extends to a structure of algebra on the Hochschild cochain complex $\mathcal{C}^*(A, A)$ over various operads or PROPs: the so-called cyclic Deligne conjecture. Let us compute this Batalin–Vilkovisky algebra structure when M is a sphere.

Proposition 20 ([30] and [31, Corollary 4.2]). *Let $d \geq 2$. As Batalin–Vilkovisky algebra, for the Hochschild cohomology of $H^*(S^d; \mathbb{F}_2) = \Lambda x_{-d}$, we have*

$$HH^*(H^*(S^d; \mathbb{F}_2); H^*(S^d; \mathbb{F}_2)) \cong \Lambda g_{-d} \otimes \mathbb{F}_2[f_{d-1}]$$

with $\Delta(g_{-d} \otimes f_{d-1}^k) = k(1 \otimes f_{d-1}^{k-1})$ and $\Delta(1 \otimes f_{d-1}^k) = 0, k \geq 0$. In particular, the underlying Gerstenhaber algebra is given by $\{f^k, f^l\} = 0$, $\{gf^k, f^l\} = lf^{k+l-1}$ and $\{gf^k, gf^l\} = (k-l)gf^{k+l-1}$ for $k, l \geq 0$.

Proof. Denote by $A := H^*(S^d; \mathbb{F}_2)$. The differential on $\mathcal{C}^*(A; A)$ is null. Let $f \in \mathrm{Hom}(s\bar{A}, A) \subset \mathcal{C}^*(A; A)$ such that $f([sx]) = 1$. Let $g \in \mathrm{Hom}(\mathbb{F}_2, A) = \mathrm{Hom}((s\bar{A})^{\otimes 0}, A) \subset \mathcal{C}^*(A; A)$ such that $g([]) = x$. The k-th power of f is the map $f^k \in \mathrm{Hom}((s\bar{A})^{\otimes k}, A)$ such that $f^k([sx|\dots|sx]) = 1$. The cup product $g \cup f^k \in \mathrm{Hom}((s\bar{A})^{\otimes k}, A)$ sends $[sx|\dots|sx]$ to x. So we have proved that $\mathcal{C}^*(A; A)$ is isomorphic to the tensor product of graded algebras $\Lambda g_{-d} \otimes \mathbb{F}_2[f_{d-1}]$.

The unit 1 and x_{-d} form a linear basis of $H^*(S^d)$. Denote by $1^\vee$ and $x^\vee$ the dual basis of $A^\vee = H^*(S^d)^\vee$. Poincaré duality induces the isomorphism $\Theta\colon H^*(S^d) \xrightarrow{\cong} H^*(S^d)^\vee$, $1 \mapsto x^\vee$ and $x \mapsto 1^\vee$. The two families of elements of the form $1[sx|\dots|sx]$ and of the form $x[sx|\dots|sx]$ form a basis of $\mathcal{C}_*(A;A)$. Denote by $1[sx|\dots|sx]^\vee$ and $x[sx|\dots|sx]^\vee$ the dual basis in $\mathcal{C}_*(A;A)^\vee$. The isomorphism Θ induces an isomorphism of complexes of degree d, $\widehat{\Theta}\colon \mathcal{C}^*(A;A) \xrightarrow[\cong]{\mathcal{C}^*(A;\Theta)} \mathcal{C}^*(A;A^\vee) \xrightarrow{\cong} \mathcal{C}_*(A;A)^\vee$. Explicitly [22, Section 4] this isomorphism sends $f \in \operatorname{Hom}((s\bar{A})^{\otimes p}, A)$ to the linear map $\widehat{\Theta}(f) \in (A \otimes (s\bar{A})^{\otimes p})^\vee \subset \mathcal{C}_*(A;A)^\vee$ defined by

$$\widehat{\Theta}(f)(a_0[sa_1|\dots|sa_p]) = \big((\Theta \circ f)[sa_1|\dots|sa_p]\big)\,(a_0).$$

Here with $A = \Lambda x$, $\widehat{\Theta}(f^k) = x[sx|\dots|sx]^\vee$ and $\widehat{\Theta}(g \cup f^k) = 1[sx|\dots|sx]^\vee$. Computing Connes' boundary map $B^\vee$ on $\mathcal{C}_*(A;A)^\vee$ (Example 12) and using that $\widehat{\Theta} \circ \Delta = B^\vee \circ \widehat{\Theta}$ by definition of Δ, we obtain the desired formula for Δ. □

6. The Gerstenhaber algebra $\mathbb{H}_*(LS^2;\mathbb{F}_2)$

Using the same Hochschild homology technique as in Section 4, we compute, up to an indeterminacy, the Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^2;\mathbb{F}_2)$. Nevertheless, this will give the complete description of the underlying Gerstenhaber algebra on $\mathbb{H}_*(LS^2;\mathbb{F}_2)$.

Lemma 21. *There exists a constant $\varepsilon \in \{0,1\}$ such that as a Batalin–Vilkovisky algebra, the homology of the free space loop on the sphere S^2 is*

$$\mathbb{H}_*(LS^2;\mathbb{F}_2) = \Lambda a_{-2} \otimes \mathbb{F}_2[u_1],$$

$$\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + \varepsilon a_{-2} \otimes u_1^{k+1}) \quad \textit{and} \quad \Delta(1 \otimes u_1^k) = 0, \quad k \geq 0.$$

Proof. In [8], Cohen, Jones and Yan proved that the Serre spectral sequence for the free loop fibration $\Omega M \overset{j}{\hookrightarrow} LM \overset{\mathrm{ev}}{\twoheadrightarrow} M$ is a spectral sequence of algebras converging toward the algebra $\mathbb{H}_*(LM)$. Using Hochschild homology, we see that there is an isomorphism of vector spaces $\mathbb{H}_*(LS^2;\mathbb{F}_2) \cong \mathbb{H}_*(S^2;\mathbb{F}_2) \otimes H_*(\Omega S^2;\mathbb{F}_2)$. Therefore the Serre spectral sequence collapses. Since there is no extension problem, we have the isomorphism of algebras

$$\mathbb{H}_*(LS^2;\mathbb{F}_2) \cong \mathbb{H}_*(S^2;\mathbb{F}_2) \otimes H_*(\Omega S^2;\mathbb{F}_2) = \Lambda(a_{-2}) \otimes \mathbb{F}_2[u_1].$$

Computing Connes' boundary map on $HH^*(H^*(S^2;\mathbb{F}_2); H_*(S^2;\mathbb{F}_2))$ (see Example 12), we see that Δ on $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is null in even degree and that

$$\Delta\colon \mathbb{H}_{2k-1} \to \mathbb{H}_{2k}$$

is a linear map of rank 1, $k \geq 0$. In particular Δ is injective in degree -1.

Applying Lemma 11 to the identity map $\mathrm{id}\colon S^2 \to S^2$, we see that the composite

$$H_1(\Omega S^2;\mathbb{F}_2) \xrightarrow{H_1(j;\mathbb{F}_2)} H_1(LS^2;\mathbb{F}_2) \xrightarrow{\Delta} H_2(LS^2;\mathbb{F}_2) \xrightarrow{H_2(\mathrm{ev};\mathbb{F}_2)} H_2(S^2;\mathbb{F}_2)$$

is non-zero. Since $\mathbb{H}_*(\mathrm{ev})$ is a morphism of algebras, $\mathbb{H}_0(ev)(a_{-2}u_1^2) = 0$. And so $\Delta(a_{-2}u_1) = 1 + \varepsilon a_{-2}u_1^2$ with $\varepsilon \in \mathbb{F}_2$.

We remark that when $b = c$, formula (6) takes the simple form

$$\Delta(ab^2) = \Delta(a)b^2 + a\Delta(b^2). \tag{22}$$

Using this formula, we obtain that

$$\Delta(a_{-2}u_1^{2k+1}) = \Delta((a_{-2}u_1)(u_1^k)^2) = u_1^{2k} + \varepsilon a_{-2}u_1^{2k+2}, \quad k \geq 0.$$

Since $\Delta\colon \mathbb{H}_1 = \mathbb{F}_2 a_{-2}u_1^3 \oplus \mathbb{F}_2 u_1 \to \mathbb{H}_2$ is of rank 1 and $\Delta(a_{-2}u_1^3) \neq 0$, $\Delta(u_1) = \lambda\Delta(a_{-2}u_1^3)$ with $\lambda = 0$ or $\lambda = 1$. Using again formula (22), we have that

$$\Delta(u_1^{2k+1}) = \Delta(u_1(u_1^k)^2) = \lambda\Delta(a_{-2}u_1^3)u_1^{2k} = \lambda\Delta(a_{-2}u_1^{2k+3}), \quad k \geq 0.$$

So finally

$$\Delta(a_{-2}u_1^k) = ku_1^{k-1} + \varepsilon k a_{-2}u_1^{k+1} \text{ and } \Delta(u_1^k) = \lambda\Delta(a_{-2}u_1^{k+2}), \quad k \geq 0.$$

The cases $\lambda = 0$ and $\lambda = 1$ correspond to isomorphic Batalin–Vilkovisky algebras: Let $\Theta\colon \mathbb{H}_*(LS^2;\mathbb{F}_2) \to \mathbb{H}_*(LS^2;\mathbb{F}_2)$ be an automorphism of algebras which is not the identity. Since $\Theta(a_{-2}) \neq 0$, $\Theta(a_{-2}) = a_{-2}$. Since $\Theta(a_{-2})$ and $\Theta(u_1)$ must generate the algebra $\Lambda a_{-2} \otimes \mathbb{F}_2[u_1]$, $\Theta(u_1) \neq a_{-2}u_1^3$. Since $\Theta(u_1) \neq u_1$, $\Theta(u_1) = u_1 + a_{-2}u_1^3$. Therefore there is an unique automorphism of algebras $\Theta\colon \mathbb{H}_*(LS^2;\mathbb{F}_2) \to \mathbb{H}_*(LS^2;\mathbb{F}_2)$ which is not the identity. Explicitly, Θ is given by $\Theta(u_1^k) = u_1^k + ka_{-2}u_1^{k+2}$, $\Theta(a_{-2}u_1^k) = a_{-2}u_1^k$, $k \geq 0$. One can check that Θ is an involutive isomorphism of Batalin–Vilkovisky algebras who transforms the cases $\lambda = 0$ into the cases $\lambda = 1$ without changing ε. Therefore, by replacing u_1 by $u_1 + a_{-2}u_1^3$, we can assume that $\lambda = 0$. □

Consider the Batalin–Vilkovisky algebras $\Lambda a_{-2} \otimes \mathbb{F}_2[u_1]$ with $\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + \varepsilon a_{-2} \otimes u_1^{k+1})$, $\Delta(1 \otimes u_1^k) = \lambda\Delta(a_{-2}u_1^{k+2})$, $k \geq 0$, given by the different values of $\varepsilon, \lambda \in \{0,1\}$. These four Batalin–Vilkovisky algebras have only two underlying Gerstenhaber algebras given by $\{u_1^k, u_1^l\} = 0$, $\{a_{-2}u_1^k, u_1^l\} = lu^{k+l-1} + l(\varepsilon-\lambda)a_{-2}u^{k+l+1}$ and $\{a_{-2}u_1^k, a_{-2}u_1^l\} = (k-l)a_{-2}u^{k+l-1}$ for $k, l \geq 0$. Via the above isomorphism Θ, these two Gerstenhaber algebras are isomorphic.

Corollary 23. *The free loop space modulo* 2 *homology* $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ *is isomorphic as Gerstenhaber algebra to the Hochschild cohomology of* $H^*(S^2;\mathbb{F}_2)$*,*

$$HH^*(H^*(S^2;\mathbb{F}_2); H^*(S^2;\mathbb{F}_2)).$$

7. The Batalin–Vilkovisky algebra $\mathbb{H}_*(LS^2)$

In this section, we complete the calculations of the Batalin–Vilkovisky algebras $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $\mathbb{H}_*(LS^2;\mathbb{Z})$ started respectively in Sections 6 and 4, using a purely homotopic method.

Theorem 24. *As a Batalin–Vilkovisky algebra, the homology of the free loop space on the sphere S^2 with mod 2 coefficients is*

$$\mathbb{H}_*(LS^2;\mathbb{F}_2) = \Lambda a_{-2} \otimes \mathbb{F}_2[u_1],$$

$$\Delta(a_{-2}\otimes u_1^k) = k(1\otimes u_1^{k-1} + a_{-2}\otimes u_1^{k+1}) \quad \textit{and} \quad \Delta(1\otimes u_1^k) = 0, \quad k \geq 0.$$

Theorem 25. *With integer coefficients, as a Batalin–Vilkovisky algebra,*

$$\begin{aligned}\mathbb{H}_*(LS^2;\mathbb{Z}) &= \Lambda b \otimes \frac{\mathbb{Z}[a,v]}{(a^2, ab, 2av)}\\ &= \bigoplus_{k=0}^{+\infty}\mathbb{Z}v_2^k \oplus \bigoplus_{k=0}^{+\infty}\mathbb{Z}b_{-1}v^k \oplus \mathbb{Z}a_{-2} \oplus \bigoplus_{k=1}^{+\infty}\frac{\mathbb{Z}}{2\mathbb{Z}}av^k\end{aligned}$$

for all $k \geq 0$, $\Delta(v^k) = 0$, $\Delta(av^k) = 0$ and $\Delta(bv^k) = (2k+1)v^k + av^{k+1}$.

Denote by $s\colon X \hookrightarrow LX$ the trivial section of the evaluation map $\mathrm{ev}\colon LX \twoheadrightarrow X$.

Lemma 26. *The image of $\Delta\colon H_1(LS^2;\mathbb{F}_2) \to H_2(LS^2;\mathbb{F}_2)$ is not contained in the image of $H_2(s;\mathbb{F}_2)\colon H_2(S^2;\mathbb{F}_2) \hookrightarrow H_2(LS^2;\mathbb{F}_2)$.*

Lemma 27. *The image of $\Delta\colon H_1(LS^2;\mathbb{Z}) \to H_2(LS^2;\mathbb{Z})$ is not contained in the image of $H_2(s;\mathbb{Z})\colon H_2(S^2;\mathbb{Z}) \hookrightarrow H_2(LS^2;\mathbb{Z})$.*

Proof of Lemma 27 assuming Lemma 26. Consider the commutative diagram

$$\begin{array}{ccc} H_1(LS^2;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_2 & \xrightarrow{\cong} & H_1(LS^2;\mathbb{F}_2)\\ \downarrow{\scriptstyle \Delta\otimes_{\mathbb{Z}}\mathbb{F}_2} & & \downarrow{\scriptstyle\Delta}\\ H_2(LS^2;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_2 & \xrightarrow{\cong} & H_2(LS^2;\mathbb{F}_2)\\ \uparrow{\scriptstyle H_2(s;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_2} & & \uparrow{\scriptstyle H_2(s;\mathbb{F}_2)}\\ H_2(S^2;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_2 & \xrightarrow{\cong} & H_2(S^2;\mathbb{F}_2). \end{array}$$

Since $H_1(LS^2;\mathbb{Z}) \cong H_0(LS^2;\mathbb{Z}) \cong \mathbb{Z}$, the horizontal arrows are isomorphisms by the universal coefficient theorem. The top rectangle commutes according to Lemma 4.

Suppose that the image of $\Delta\colon H_1(LS^2;\mathbb{Z}) \to H_2(LS^2;\mathbb{Z})$ is included in the image of $H_2(s;\mathbb{Z})$. Then the image of $\Delta \otimes_{\mathbb{Z}} \mathbb{F}_2$ is included in the image of $H_2(s;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2$. Using the above diagram, the image of $\Delta\colon H_1(LS^2;\mathbb{F}_2) \to H_2(LS^2;\mathbb{F}_2)$ is included in the image of $H_2(s;\mathbb{F}_2)$. This contradicts Lemma 26. □

Proof of Theorem 24 *assuming Lemma* 26. It suffices to show that the constant ε in Lemma 21 is not zero. Suppose that $\varepsilon = 0$. Then by Lemma 21, $\Delta(a_{-2} \otimes u_1) = 1$.

It is well known that $\mathbb{H}_*(s)\colon \mathbb{H}_*(M) \to \mathbb{H}_*(LM)$ is a morphism of algebras. In particular, let $[S^2]$ be the fundamental class of S^2, $H_2(s)([S^2])$ is the unit of $\mathbb{H}_*(LS^2)$. So $\Delta(a_{-2} \otimes u_1) = H_2(s)([S^2])$. This contradicts Lemma 26. □

The proof of Theorem 25 assuming Lemma 27 is the same. To complete the computation of this Batalin–Vilkovisky algebra on the homology of the free loop space of a manifold, we will relate it to another structure of a Batalin–Vilkovisky algebra that arises in algebraic topology: the homology of the double loop space.

Let X be a pointed topological space. The circle S^1 acts on the sphere S^2 by "rotating the earth". Hence the circle also acts on $\Omega^2 X = \operatorname{map}\big((S^2, \text{North pole}), (X,*)\big)$. So we have an induced operator $\Delta\colon H_*(\Omega^2 X) \to H_{*+1}(\Omega^2 X)$. With Theorem 32 and the following proposition, we will able to prove Lemma 26.

Proposition 28. *Let X be a pointed topological space. There is a natural morphism $r\colon L\Omega X \to \operatorname{map}_*(S^2, X)$ of S^1-spaces between the free loop space on the pointed loops of X and the double pointed loop space of X such that:*

- *If we identify S^2 and $S^1 \wedge S^1$, r is a retract up to homotopy of the inclusion $j : \Omega(\Omega X) \hookrightarrow L(\Omega X)$.*
- *The composite $r \circ s : \Omega X \hookrightarrow L(\Omega X) \to \operatorname{map}_*(S^2, X)$ is homotopically trivial.*

Proof. Let $\sigma\colon S^2 \twoheadrightarrow \frac{S^1 \times S^1}{S^1 \times *} = S^1_+ \wedge S^1$ be the quotient map that identifies the North pole and the South pole on the earth S^2. The circle S^1 acts without moving the based point on $S^1_+ \wedge S^1$ by multiplication on the first factor. On the torus $S^1 \times S^1$, the circle can act by multiplication on both factors. But when you pinch a circle to a point in the torus, the circle can act only on one factor. If we make a picture, we easily see that $\sigma : S^2 \twoheadrightarrow S^1_+ \wedge S^1$ is compatible with the actions of S^1. Therefore $r\colon = \operatorname{map}_*(\sigma, X)\colon L\Omega X \to \operatorname{map}_*(S^2, X)$ is a morphism of S^1-spaces.

- Let $\pi : S^1_+ \wedge S^1 \twoheadrightarrow S^1 \wedge S^1 = \frac{S^1_+ \wedge S^1}{* \times S^1}$ be the quotient map. The inclusion map $j : \Omega(\Omega X) \to L(\Omega X)$ is $\operatorname{map}_*(\pi, X)$. The composite $\pi \circ \sigma : S^2 \twoheadrightarrow S^1 \wedge S^1$ is the quotient map obtained by identifying a meridian with a point in the sphere S^2. The composite $\pi \circ \sigma$ can also be viewed as the quotient map from the non-reduced suspension of S^1 to the reduced suspension of S^1. So the composite $\pi \circ \sigma : S^2 \twoheadrightarrow S^1 \wedge S^1$ is a homotopy equivalence. Let $\Theta\colon S^1 \wedge S^1 \xrightarrow{\cong} S^2$ be any given homeomorphism.

The composite $\Theta \circ \pi \circ \sigma \colon S^2 \to S^2$ is of degree ± 1. The reflection through the equatorial plane is a morphism of S^1-spaces. By replacing eventually σ by its composite with the previous reflection, we can suppose that $\Theta \circ \pi \circ \sigma \colon S^2 \to S^2$ is homotopic to the identity map of S^2, i.e. $\sigma \circ \Theta$ is a section of π up to homotopy. Therefore $\mathrm{map}_*(\sigma \circ \Theta, X) = \mathrm{map}_*(\Theta, X) \circ r$ is a retract of j up to homotopy.

• Let $\rho \colon S^1_+ \wedge S^1 = \frac{S^1 \times S^1}{S^1 \times *} \twoheadrightarrow S^1$ be the map induced by the projection on the second factor. Since $\pi_2(S^1) = *$, the composite $\rho \circ \sigma$ is homotopically trivial. Therefore $r \circ s$, the composite of $r = \mathrm{map}_*(\sigma, X)$ and $s = \mathrm{map}_*(\rho, X) \colon \Omega X \to L(\Omega X)$ is also homotopically trivial. □

Proof of Lemma 26. Denote by $\mathrm{ad}_{S^n} \colon S^n \to \Omega S^{n+1}$ the adjoint of the identity map $\mathrm{id} \colon S^{n+1} \to S^{n+1}$. The map $L(\mathrm{ad}_{S^2}) \colon LS^2 \to L\Omega S^3$ is obviously a morphism of S^1-spaces. Therefore using Proposition 28, the composite $r \circ L(\mathrm{ad}_{S^2}) \colon LS^2 \to L\Omega S^3 \to \Omega^2 S^3$ is also a morphism of S^1-spaces. Therefore $H_*(r \circ L(\mathrm{ad}_{S^2}))$ commutes with the corresponding operators Δ in $H_*(LS^2)$ and $H_*(\Omega^2 S^3)$.

Consider the commutative diagram up to homotopy:

$$
\begin{array}{ccccc}
\Omega S^2 & \xrightarrow{j} & LS^2 & \xleftarrow{s} & S^2 \\
\downarrow{\scriptstyle \Omega(\mathrm{ad}_{S^2})} & & \downarrow{\scriptstyle L(\mathrm{ad}_{S^2})} & & \downarrow{\scriptstyle \mathrm{ad}_{S^2}} \\
\Omega^2 S^3 & \xrightarrow{j} & L\Omega S^3 & \xleftarrow{s} & \Omega S^3 \\
& {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle r} & \swarrow{\scriptstyle *} & \\
& & \Omega^2 S^3. & &
\end{array}
\tag{29}
$$

Using the left part of this diagram, we see that $\pi_1(r \circ L(\mathrm{ad}))$ maps the generator of $\pi_1(LS^2) = \mathbb{Z}(j \circ \mathrm{ad}_{S^1})$ to the composite $\Omega(\mathrm{ad}_{S^2}) \circ \mathrm{ad}_{S^1} \colon S^1 \to \Omega S^2 \to \Omega^2 S^3$ which is the generator of $\pi_1(\Omega^2 S^3) \cong \mathbb{Z}$. Therefore $\pi_1(r \circ L(\mathrm{ad}))$ is an isomorphism.

So we have the commutative diagram

$$
\begin{array}{ccccc}
\pi_1(LS^2) \otimes \mathbb{F}_2 & \xrightarrow[\cong]{\mathrm{hur}} & H_1(LS^2; \mathbb{F}_2) & \xrightarrow{\Delta} & H_2(LS^2; \mathbb{F}_2) \\
\downarrow{\scriptstyle \pi_1(r \circ L(\mathrm{ad}_{S^2})) \otimes \mathbb{F}_2}\ \cong & & \downarrow{\scriptstyle H_1(r \circ L(\mathrm{ad}_{S^2}); \mathbb{F}_2)} & & \downarrow{\scriptstyle H_2(r \circ L(\mathrm{ad}_{S^2}); \mathbb{F}_2)} \\
\pi_1(\Omega^2 S^3) \otimes \mathbb{F}_2 & \xrightarrow[\cong]{\mathrm{hur}} & H_1(\Omega^2 S^3; \mathbb{F}_2) & \xrightarrow{\Delta} & H_2(\Omega^2 S^3; \mathbb{F}_2).
\end{array}
$$

By Theorem 32, $\Delta \colon H_1(\Omega^2 S^3; \mathbb{F}_2) \to H_2(\Omega^2 S^3; \mathbb{F}_2)$ is non-zero. Therefore using the above diagram, the composite $H_2(r \circ L(\mathrm{ad}_{S^2})) \circ \Delta$ is also non-zero. On the other hand, using the right part of diagram (29), we have that the composite $H_2(r \circ L(\mathrm{ad}_{S^2})) \circ H_2(s)$ is null. □

Corollary 30. *The free loop space modulo* 2 *homology* $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ *is not isomorphic as Batalin–Vilkovisky algebra to the Hochschild cohomology of* $H^*(S^2;\mathbb{F}_2)$*,* $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$.

This means exactly that there exists no isomorphism between $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$ which at the same time

- is an isomorphism of algebras and
- commutes with the Δ-operators,

although separately

- there exists (Corollary 23) an isomorphism of algebras between $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$ and
- there exists also an isomorphism commuting with the Δ-operators between them.

Proof. By Proposition 20, $HH^*(H^*(S^2);H^*(S^2))$ is the Batalin–Vilkovisky algebra given by $\varepsilon = 0$ in Lemma 21. On the contrary, by Theorem 24, $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is the Batalin–Vilkovisky algebra given by $\varepsilon = 1$. At the end of the proof of Lemma 21, we saw that the two cases $\varepsilon = 0$ and $\varepsilon = 1$ correspond to two non-isomorphic Batalin–Vilkovisky algebras. □

More generally, we believe that for any prime p, the free loop space modulo p of the complex projective space $\mathbb{H}_*(L\mathbb{CP}^{p-1};\mathbb{F}_p)$[1] is not isomorphic as Batalin–Vilkovisky algebra to the Hochschild cohomology

$$HH^*(H^*(\mathbb{CP}^{p-1};\mathbb{F}_p);H^*(\mathbb{CP}^{p-1};\mathbb{F}_p)).$$

Such phenomena for formal manifolds should not appear over a field of characteristic 0.

Recall that by Poincaré duality, we have the isomorphism (cf. Equation (19))

$$\Theta\colon H^*(S^2) \xrightarrow{\cong} H^*(S^2)^\vee.$$

Therefore we have the isomorphism

$$HH^*(H^*(S^2);\Theta)\colon HH^*(H^*(S^2);H^*(S^2)) \xrightarrow{\cong} HH^*(H^*(S^2);H^*(S^2)^\vee).$$

Consider any isomorphism of graded algebras

$$\mathbb{H}_*(LS^2) \cong HH^*(S^*(S^2);S^*(S^2)). \tag{31}$$

[1]Bökstedt and Ottosen [1] have recently announced the computation of the Batalin–Vilkovisky algebra $\mathbb{H}_*(L\mathbb{CP}^n;\mathbb{F}_p)$.

By Corollary 23, such isomorphism exists. Cohen and Jones ([7, Theorem 3] and [5]) proved that such isomorphism exists for any manifold M. Since S^2 is formal, we have the isomorphism of algebras (cf. Equation (2))

$$HH^*(S^*(S^2); S^*(S^2)) \xrightarrow{\cong} HH^*(H^*(S^2); H^*(S^2)).$$

By [17], we have the isomorphisms of $H_*(S^1)$-modules

$$H_*(LS^2) \overset{(14)}{\cong} HH^*(S^*(S^2); S^*(S^2)^\vee) \overset{(15)}{\cong} HH^*(H^*(S^2); H^*(S^2)^\vee).$$

Corollary 30 implies that the following diagram does not commute over $\mathbb{F}_2$:

$$\begin{array}{ccccc} & & HH^*(S^*(S^2); S^*(S^2)^\vee) & \xrightarrow{(15)} & HH^*(H^*(S^2); H^*(S^2)^\vee) \\ & \overset{(14)}{\nearrow} & & & \uparrow {\scriptstyle HH^*(H^*(S^2);\Theta)} \\ H_*(LS^2) & & & & \\ & \underset{(31)}{\searrow} & & & \\ & & HH^*(S^*(S^2); S^*(S^2)) & \xrightarrow{(2)} & HH^*(H^*(S^2); H^*(S^2)). \end{array}$$

This is surprising because as explained by Cohen and Jones [7, p. 792], the composite of the isomorphism (14) given by Jones in [17] and an isomorphism induced by Poincaré duality should give an isomorphism of algebras between $\mathbb{H}_*(LS^2)$ and $HH^*(S^*(S^2); S^*(S^2))$.

8. Appendix by Gerald Gaudens and Luc Menichi

Let X be a pointed topological space. Recall that the circle S^1 acts on the double loop space $\Omega^2 X$. Consider the induced operator $\Delta\colon H_*(\Omega^2 X) \to H_{*+1}(\Omega^2 X)$. Getzler [14] has shown that $H_*(\Omega^2 X)$ equipped with the Pontryagin product and this operator Δ forms a Batalin–Vilkovisky algebra. In [12], Gerald Gaudens and the author have determined this Batalin–Vilkovisky algebra $H_*(\Omega^2 S^3; \mathbb{F}_2)$. The key was the following theorem. In [18, Proposition 7.46], answering to a question of Gerald Gaudens, Sadok Kallel and Paolo Salvatore give another proof of this theorem.

Theorem 32 ([12]). *The operator* $\Delta\colon H_1(\Omega^2 S^3; \mathbb{F}_2) \to H_2(\Omega^2 S^3; \mathbb{F}_2)$ *is non-trivial.*

Both proofs [12] and [18, Proposition 7.46] are unpublished and publicly unavailable yet. So the goal of this section is to give a proof of this theorem which is as simple as possible.

Denote by $*$ the Pontryagin product in $H_*(\Omega^2 X)$ and by $\circ$ the map induced in homology by the composition map $\Omega^2 X \times \Omega^2 S^2 \to \Omega^2 X$. Denote by $\Omega^2_n S^2$, the path-connected component of the degree n maps. Denote by v_1 the generator of $H_1(\Omega^2_0 S^2; \mathbb{F}_2)$ and by [1] the generator of $H_0(\Omega^2_1 S^2; \mathbb{F}_2)$.

Lemma 33. *For $x \in H_*(\Omega^2 X; \mathbb{F}_2)$, $\Delta x = x \circ (v_1 * [1])$.*

Proof. The circle S^1 acts on the sphere S^2. Therefore we have a morphism of topological monoids $\Theta\colon (S^1, 1) \to (\Omega^2_1 S^2, \mathrm{id}_{S^2})$. The action of S^1 on $\Omega^2 X$ is the composite $S^1 \times \Omega^2 X \xrightarrow{\Theta\times\Omega^2 X} \Omega^2_1 S^2 \times \Omega^2 X \xrightarrow{\circ} \Omega^2 X$. Therefore for $x \in H_*(\Omega^2 X; \mathbb{F}_2)$, $\Delta x = x \circ (H_1(\Theta)[S^1])$.

Suppose that $H_1(\Theta)[S^1] = 0$. Then for any topological space X, the operator Δ on $H_*(\Omega^2 X; \mathbb{F}_2)$ is null. Therefore, for any x and $y \in H_*(\Omega^2 X; \mathbb{F}_2)$, $\{x, y\} = \Delta(xy) - (\Delta x)y - x(\Delta y) = 0$. That is the modulo 2 Browder brackets on any double loop space are null. This is obviously false. For example, Cohen in [3] explains that the Gerstenhaber algebra $H_*(\Omega^2\Sigma^2 Y)$ has in general many non-trivial Browder brackets. So the assumption $H_1(\Theta)[S^1] = 0$ is false.

Since the loop multiplication by id_{S^2} in the H-group $\Omega^2 S^2$ is a homotopy equivalence, the Pontryagin product by [1], $*[1]\colon H_*(\Omega^2_0 S^2) \xrightarrow{\cong} H_*(\Omega^2_1 S^2)$ is an isomorphism. Therefore $v_1 * [1]$ is a generator of $H_1(\Omega^2_1 S^2)$, hence $H_1(\Theta)[S^1] = v_1 * [1]$. So finally

$$\Delta x = x \circ (H_1(\Theta)[S^1]) = x \circ (v_1 * [1]). \qquad \square$$

Recall that v_1 denotes the generator of $H_1(\Omega^2_0 S^2; \mathbb{F}_2)$.

Lemma 34. *In the Batalin–Vilkovisky algebra $H_*(\Omega^2 S^2; \mathbb{F}_2)$, $\Delta(v_1) = v_1 * v_1$.*

Proof. Recall that [1] is the generator of $H_0(\Omega^2_1 S^2)$. By Lemma 33,

$$\Delta[1] = [1] \circ (v_1 * [1]) = (v_1 * [1]).$$

Denote by $Q\colon H_q(\Omega^2_n S^2) \to H_{2q+1}(\Omega^2_{2n} S^2)$ the Dyer–Lashof operation. It is well known that $Q[1] = v_1 * [2]$. So by [4, p. 218, Theorem 1.3 (4)]

$$\{v_1 * [2], [1]\} = \{Q[1], [1]\} = \{[1], \{[1], [1]\}\}.$$

By [4, p. 215, Theorem 1.2 (3)], $\{[1], [1]\} = 0$. Therefore on one hand, $\{v_1 * [2], [1]\}$ is null. And on the other hand, using the Poisson relation (7), since $\{[2], [1]\} = \{[1] * [1], [1]\} = 2\{[1], [1]\} * [1] = 0$,

$$\{v_1 * [2], [1]\} = \{v_1, [1]\} * [2] + v_1 * \{[2], [1]\} = \{v_1, [1]\} * [2].$$

Since $*[1]\colon H_*(\Omega^2 S^2) \xrightarrow{\cong} H_*(\Omega^2 S^2)$ is an isomorphism, we obtain that the Browder bracket $\{v_1, [1]\}$ is null. Therefore,

$$\Delta(v_1 * [1]) = (\Delta v_1) * [1] + v_1 * (\Delta[1]) = ((\Delta v_1) - v_1 * v_1) * [1].$$

But $\Delta(v_1 * [1]) = (\Delta \circ \Delta)([1]) = 0$. Therefore (Δv_1) must be equal to $v_1 * v_1$. □

Proof of Theorem 32. We remark that since Δ preserves path-connected components and since the loop multiplication of two homotopically trivial loops is a homotopically trivial loop, $H_*(\Omega_0^2 S^2)$ is a sub Batalin–Vilkovisky algebra of $H_*(\Omega^2 S^2)$.

Let $S^1 \hookrightarrow S^3 \overset{\eta}{\twoheadrightarrow} S^2$ be the Hopf fibration. After double looping, the Hopf fibration gives the fibration $\Omega^2 S^1 \hookrightarrow \Omega^2 S^3 \overset{\Omega^2\eta}{\twoheadrightarrow} \Omega_0^2 S^2$ with contractile fiber $\Omega^2 S^1$ and path-connected base $\Omega_0^2 S^2$. Therefore $\Omega^2\eta\colon \Omega^2 S^3 \xrightarrow{\simeq} \Omega_0^2 S^2$ is a homotopy equivalence. And so $H_*(\Omega^2\eta)\colon H_*(\Omega^2 S^3) \xrightarrow{\simeq} H_*(\Omega_0^2 S^2)$ is an isomorphism of Batalin–Vilkovisky algebras.

Let u_1 be the generator of $H_1(\Omega^2 S^3)$. Lemma 34 implies that $\Delta(u_1) = u_1 * u_1$. Since $u_1 * u_1$ is non-zero in $H_*(\Omega^2 S^3; \mathbb{F}_2)$, $\Delta(u_1)$ is non-trivial. □

References

[1] M. Bökstedt and I. Ottosen, The homology of the free loop space on a projective space. Talk at the *First Copenhagen Topology Conference*, September 1–3, 2006; http://www.math.ku.dk/conf/CTC2006

[2] M. Chas and D. Sullivan, String topology. Preprint 1999, arXiv:math/9911159

[3] F. Cohen, On configuration spaces, their homology, and lie algebras. *J. Pure Appl. Algebra* **100** (1–3) (1995), 19–42. Zbl 0921.57011 MR 1344842

[4] F. Cohen, T. Lada, and J. May, *The homology of iterated loop spaces*. Lecture Notes in Math. 533, Springer-Verlag, Berlin 1976. Zbl 0334.55009 MR 0436146

[5] R. Cohen, Multiplicative properties of Atiyah duality. *Homology Homotopy Appl.* **6** (1) (2004), 269–281. Zbl 1072.55004 MR 2076004

[6] R. Cohen, K. Hess, and A. Voronov, *String topology and cyclic homology*. Lectures from the Summer School held in Almería, 2003 Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel 2006. Zbl 1089.57002 MR 2251006

[7] R. Cohen and J. Jones, A homotopic theoretic realization of string topology. *Math. Ann.* **324** (4) (2002), 773–798. Zbl 1025.55005 MR 1942249

[8] R. Cohen, J. Jones, and J. Yan, The loop homology algebra of spheres and projective spaces. In *Categorical decomposition techniques in algebraic topology* (Isle of Skye 2001), Prog. Math. 215, Birkhäuser, Basel 2004, 77–92. Zbl 1054.55006 MR 2039760

[9] K. Costello, Topological conformal field theories and Calabi-Yau categories. *Adv. Math.* **210** (1) (2007), 165–214. Zbl 05132553 MR 2298823

[10] Y. Félix, L. Menichi, and J.-C. Thomas, Gerstenhaber duality in Hochschild cohomology. *J. Pure Appl. Algebra* **199** (1–3) (2005), 43–59. Zbl 1076.55003 MR 2134291

[11] Y. Félix, J.-C. Thomas, and M. Vigué-Poirrier, Rational string topology. *J. Eur. Math. Soc.* **9** (1) (2005), 123–156. Zbl 05129007 MR 2283106

[12] G. Gaudens and L. Menichi, Batalin-Vilkovisky algebras and the J-homomorphism. *Topology Appl.* **156** (2) (2008), 365–374.

[13] M. Gerstenhaber, The cohomology structure of an associative ring. *Ann. of Math.* **78** (2) (1963), 267–288. Zbl 0131.27302 MR 0161898

[14] E. Getzler, Batalin-Vilkovisky algebras and two-dimensional topological field theories. *Comm. Math. Phys.* **159** (2) (1994), 265–285. Zbl 0131.27302 MR 0161898

[15] K. Gruher and P. Salvatore, Generalized string topology operations. *Proc. Lond. Math. Soc.* (3) **96** (1) (2008), 78–106. Zbl 1143.57012 MR 2392316

[16] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge 2002. Zbl 1044.55001 MR 1867354

[17] J. D. S. Jones, Cyclic homology and equivariant homology. *Invent. Math.* **87** (2) (1987), 403–423. Zbl 1012.20002 MR 0870737

[18] S. Kallel, *Loop spaces, symmetric products and configuration spaces*. Book in preparation; previously available at http://math.univ-lille1.fr/~kallel

[19] R. Kaufmann, A proof of a cyclic version of Deligne's conjecture via Cacti. *Math. Res. Letters* **15** (5) (2008), 901–921. MR 2443991

[20] ——, Moduli space actions on the Hochschild co-chains of a Frobenius algebra I:Cells operads. *J. Noncommut. Geom.* **1** (3) (2007), 333–384. Zbl 1145.55008 MR 2314100

[21] M. Kontsevich and Y. Soibelman, Notes on A-infinity algebras, A-infinity categories and non-commutative geometry. I. Preprint 2006, arXiv:math.RA/0606241

[22] L. Menichi, Batalin-Vilkovisky algebras and cyclic cohomology of Hopf algebras. *K-Theory* **32** (3) (2004), 231–251. Zbl 1101.19003 MR 2114167

[23] S. Merkulov, De Rham model for string topology. Internat. Math. Res. Notices **2004** (55) (2004), 2955–2981. Zbl 1066.55008 MR 2099178

[24] A. Stacey, The differential topology of loop spaces. Preprint 2005, arXiv:math.DG/0510097

[25] H. Tamanoi, Batalin-Vilkovisky Lie algebra structure on the loop homology of complex Stiefel manifolds. *Internat. Math. Res. Notices* **2006** (2006), 1–23. Zbl 1108.55007 MR 2211159

[26] T. Tradler, The BV algebra on Hochschild cohomology induced by infinity inner products. *Ann. Inst. Fourier*, to appear; Preprint 2002, arXiv:math.QA/0210150

[27] T. Tradler and M. Zeinalian, On the cyclic Deligne conjecture. *J. Pure Appl. Algebra* **204** (2) (2006), 280–299. Zbl 02242211 MR 2184812

[28] D. Vaintrob, The string topology BV algebra, Hochschild cohomology and the Goldman bracket on surfaces. Preprint 2007, arXiv:math/0702859

[29] M. Vigué-Poirrier, Decomposition de l'homologie cyclique des algèbres différentielles graduées commutatives. *K-Theory* **4** (5) (1991), 399–410. Zbl 0731.19004 MR 1116926

[30] C. Westerland, Dyer-Lashof operations in the string topology of spheres and projective spaces. *Math. Z.* **250** (3) (2005), 711–727. Zbl 1076.55006 MR 2179618

[31] —–, String homology of spheres and projective spaces. *Algebr. Geom. Topol.* **7** (2007), 309–325. Zbl 05220877 MR 2308947

Received March 15, 2007

Luc Menichi, UMR 6093 associée au CNRS, Université d'Angers, Faculté des Sciences, 2 Boulevard Lavoisier, 49045 Angers, France

E-mail: luc.menichi@univ-angers.fr