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A lower bound on the essential dimension of a connected linear group

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Abstract. Let G be a connected linear algebraic group defined over an algebraically closed field k and H be a finite abelian subgroup of G whose order does not divide $\mathrm{char}(k)$. We show that the essential dimension of G is bounded from below by $\mathrm{rank}(H) - \mathrm{rank}\,C_G(H)^0$, where $\mathrm{rank}\,C_G(H)^0$ denotes the rank of the maximal torus in the centralizer $C_G(H)$. This inequality, conjectured by J.-P. Serre, generalizes previous results of Reichstein–Youssin (where $\mathrm{char}(k)$ is assumed to be 0 and $C_G(H)$ to be finite) and Chernousov–Serre (where H is assumed to be a 2-group).

1. Introduction

Let k be a base field, K/k be a field extension, G/k be a linear algebraic group and $\alpha \in H^1(K, G)$ be a G-torsor over $\mathrm{Spec}(K)$. We will say that α *descends* to a subfield $K_0 \subset K$ if α lies in the image of the natural map $H^1(K_0, G) \to H^1(K, G)$. The essential dimension $\mathrm{ed}_k(\alpha)$ of α, is defined as the minimal value of $\mathrm{trdeg}_k(K_0)$, where α descends to K_0 and $k \subset K_0$. (Throughout this paper we will work over a fixed algebraically closed field k; for this reason we will write ed in place of ed_k.) We also define $\mathrm{ed}(\alpha; l)$ as the minimal value of $\mathrm{ed}(\alpha_L)$, as L/K ranges over all finite field extensions of degree prime to l. Here l is a prime integer. The essential dimension $\mathrm{ed}(G)$ of the group G (respectively, the essential dimension $\mathrm{ed}(G; l)$ of G at l) is defined as the maximal value of $\mathrm{ed}(\alpha)$ (respectively, of $\mathrm{ed}(\alpha; l)$), as K/k ranges over all field extensions and α ranges over $H^1(K, G)$. For details on the notion of essential dimension, its various interpretations and numerous examples, see [Re], [RY$_1$] and [BF].

Many of the best known lower bounds on $\mathrm{ed}(G)$ and $\mathrm{ed}(G; l)$, for specific groups G, are deduced from the following theorem.

1.1 Theorem ([RY$_1$], Theorem 7.8). *Let G be a connected semisimple linear algebraic group defined over an algebraically closed field k of characteristic zero and H be a finite abelian subgroup of G. Assume that the centralizer $C_G(H)$ is finite. Then*

(a) $\mathrm{ed}(G) \geq \mathrm{rank}(H)$ *and*

(b) *if H is an l-group, then* $\mathrm{ed}(G; l) \geq \mathrm{rank}(H)$.

Here by the rank of a finite abelian group H we mean the smallest positive integer r such that H can be written as a direct product of r cyclic groups. Equivalently, $\mathrm{rank}(H)$ is the minimal dimension of a faithful complex linear representation of H.

The purpose of this paper is to prove the following more general inequality conjectured by J.-P. Serre (e-mail, July 25, 2005).

1.2 Theorem. *Let G be a connected reductive linear algebraic group defined over an algebraically closed base field k. Suppose that H is a finite abelian subgroup of G and char(k) does not divide $|H|$. Then*

(a) $\mathrm{ed}(G) \geq \mathrm{rank}\, H - \mathrm{rank}\, C_G(H)^0$.

(b) *Moreover, if H is an l-group, then* $\mathrm{ed}(G; l) \geq \mathrm{rank}\, H - \mathrm{rank}\, C_G(H)^0$.

Here $C_G(H)^0$ denotes the connected component of the centralizer of H in G, and by the rank of this connected group we mean the dimension of its maximal torus. In particular, if $\mathrm{char}(k) = 0$ and the centralizer $C_G(H)$ is finite (i.e., $\mathrm{rank}\, C_G(H)^0 = 0$), then Theorem 1.2 reduces to Theorem 1.1. Note, however, that even in this special case the proof we present here is simpler than the one in [RY$_1$]; in particular, it does not rely on resolution of singularities.

We also remark that our argument shows a bit more, namely that the essential dimension of a particular torsor, which we call a *loop torsor*, is $\geq \mathrm{rank}(H) - \mathrm{rank}\, C_G(H)^0$. Here by a loop torsor we mean the image of a versal H-torsor under the natural map $H^1(*, H) \to H^1(*, G)$. (Such torsors come up in connection with loop algebras; see [GP].)

Chernousov and Serre [CS] used techniques from the theory of quadratic forms to show that, in the case where H is a 2-group, many of the bounds given by Theorem 1.1 (b) remain valid over any algebraically closed field base field k of characteristic $\neq 2$. The "incompressible" quadratic forms they construct are closely related to loop torsors; our arguments may thus be viewed as extending their approach to abelian subgroups H which are not necessarily 2-groups.

In order to clarify the exposition we will give two proofs of Theorem 1.2. The first one, presented in Section 4, is quite short but it relies on resolution of singularities and, in particular, only works in characteristic zero. The second proof, presented in Section 8 requires a bit more work. The advantage of this more elaborate argument is that it is entirely independent of resolution of singularities; in particular, it works in

prime characteristic not dividing $|H|$. Both proofs rely, in a key way, on the existence results for wonderful (and regular) group compactifications from [Br] and [BK] (see Section 2) and on the "reduction of structure" Theorem 1.1 from [CGR]. The case where char(k) divides the order of the Weyl group of G is particularly delicate; here we use a refined version of [CGR, Theorem 1.1], which is proved in [CGR2].

The following symbols will be used for the remainder of the paper.

k, algebraically closed base field of characteristic $p \geq 0$;

G, connected reductive linear group defined over k;

$\overline{G}$, a regular compactification of G;

H, finite abelian group;

ed, essential dimension over k;

$F_n = k((t_1)) \ldots ((t_n))$, iterated Laurent series field in n variables.

2. Regular compactifications

Let G be a connected reductive algebraic group defined over k. Let B and B^- be opposite Borel subgroups of G, containing a maximal torus T. By the Bruhat decomposition, G has finitely many $B \times B^-$ orbits. Hence, by [BK, Proposition 6.2.5], G (viewed as a $G \times G$-variety) has a "regular" compactification in the sense of [BDP]; we will denote this compactification by $\overline{G}$. (Note that the terms "regular" and "smooth" are not interchangeable in this context; a regular compactification is smooth but not the other way around.) In particular, if G is adjoint, then $\overline{G}$ is the wonderful compactification of G constructed in [DP] (and in prime characteristic in [St]). Regular compactifications have many interesting special properties; most of them will not be used in the sequel. The only property of $\overline{G}$ we will need is the following description of the stabilizers of points in $\overline{G}$ from [Br, Proposition A.1].

Every $G \times G$ orbit $\mathcal{O}$ in a regular compactification $\overline{G}$ of G has a unique point $\bar{g}$ such that $(B \times B^-)\bar{g}$ is open in $\mathcal{O}$ and $\bar{g}$ is the limit of some 1-parameter subgroup of T. We shall refer to $\bar{g}$ as a *special point*. For example, the special point in the dense orbit $\mathcal{O} = G$ of $\overline{G}$ is the identity element $\bar{g} = e$.

The stabilizer of a special point $\bar{g}$ in $G \times G$ has the following form. Let P be the projection of $\mathrm{Stab}_{G \times G}(\bar{g})$ to the first factor of $G \times G$ and Q be the projection to the second factor. Then P and Q are opposite parabolic subgroups. Denote the unipotent radicals of P and Q by P_u and Q_u and their common Levi subgroup $P \cap Q$ by L. The stabilizer $\mathrm{Stab}_{G \times G}(\bar{g})$ is then given by
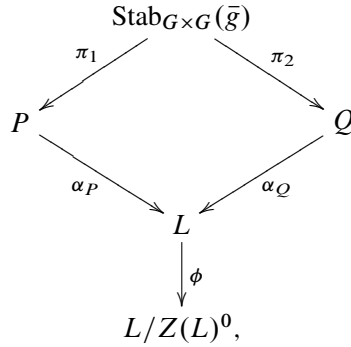
$$\mathrm{Stab}_{G \times G}(\bar{g}) = \{(p_u l z, q_u l) \mid p_u \in P_u, \ q_u \in Q_u, \ l \in L, \ z \in Z\}, \qquad (2.1)$$

where Z is a subgroup of $Z(L)^0$. Of course, the stabilizer of any other point in $\mathcal{O}$ is conjugate to this subgroup.

2.2 Proposition. *Let $\bar{g}$ be a special point in the regular compactification $\bar{G}$ of G and Γ be a finite subgroup of $\mathrm{Stab}_{G \times G}(\bar{g})$ whose order is prime to $\mathrm{char}(k)$. Then $|\mathrm{rank}(\pi_2(\Gamma)) - \mathrm{rank}(\pi_1(\Gamma))| \leq \mathrm{rank}(Z(L)^0)$.*

Here $\pi_1, \pi_2 \colon G \times G \to G$ denote the projection to the first and second factor, respectively, $P = \pi_1 \mathrm{Stab}_{G \times G}(\bar{g})$, $Q = \pi_2 \mathrm{Stab}_{G \times G}(\bar{g})$ and $L = P \cap Q$ as above, and $\mathrm{rank}(\Gamma)$ denotes the maximal value of $\mathrm{rank}(A)$, as A ranges over the abelian subgroups of a finite group Γ.

Proof. The proof is based on tracing Γ through the diagram of natural projections

$$
\begin{array}{ccc}
 & \mathrm{Stab}_{G \times G}(\bar{g}) & \\
{}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\
P & & Q \\
{}^{\alpha_P}\searrow & & \swarrow^{\alpha_Q} \\
 & L & \\
 & \downarrow^{\phi} & \\
 & L/Z(L)^0, &
\end{array}
$$

where $\alpha_P(p_u l) = l$ and $\alpha_Q(q_u l) = l$ for any $p_u \in P_u$, $q_u \in Q_u$ and $l \in L$. Since the kernels P_u and Q_u of α_P and α_Q are unipotent, we see that α_P and α_Q project the finite groups $\pi_1(\Gamma)$ and $\pi_2(\Gamma)$ isomorphically onto subgroups of L, which we will denote by Γ_1 and Γ_2, respectively. By (2.1), Γ_1 and Γ_2 have the same image in $L/Z(L)^0$, which we will denote by Γ_0. Since the natural projection $\phi_{|\Gamma_2} \colon \Gamma_2 \to \Gamma_0$ is surjective with kernel $Z_2 \subset Z(L)^0$, we have

$$\mathrm{rank}(\Gamma_2) \leq \mathrm{rank}(\Gamma_1) + \mathrm{rank}(Z_2) \leq \mathrm{rank}(\Gamma_1) + \mathrm{rank}\, Z(L)^0.$$

By symmetry, we also have the reverse inequality $\mathrm{rank}(\Gamma_1) - \mathrm{rank}(\Gamma_2) \leq \mathrm{rank}\, Z(L)^0$, and the proposition follows. $\square$

3. Compactifications of homogeneous spaces

The following lemma is well known in characteristic zero (see [RY$_2$, Lemma 2.1]).

3.1 Lemma. *Let Γ be a finite group and let X be a normal quasiprojective Γ-variety. Then*

 (a) *X is covered by affine open Γ-invariants subsets.*

Assume moreover that the order of Γ is invertible in k. Then

(b) *there is a geometric quotient map $\pi : X \to X/\Gamma$.*

(c) *Moreover, if X is projective, then so is X/Γ.*

Recall that we are assuming throughout that the base field k is algebraically closed.

Proof. (a) The proof of *loc. cit.* is characteristic free.

(b) Recall first that the group Γ is linearly reductive, see [MFK, §1]. If X is affine, part (b) is proved in [MFK, Theorem 1.1 and Amplification 1.3]. The general case follows from part (a) and the characteristic free glueing assertion in [PV, Theorem 4.14].

(c) See [N, Theorem 3.14].   □

Let G/k be a connected reductive linear algebraic group, $\bar{G}$ be a regular compactification, and Γ be a finite subgroup of G of order prime to $\mathrm{char}(k)$. We shall denote by $\bar{G}/\Gamma$ the geometric quotient of $\bar{G}$ for the action of the finite group Γ (on the right). It may be viewed as a (possibly singular) compactification of the homogeneous space G/Γ. By the properties of geometric quotients

(i) the fibers of the natural projection $\bar{G} \to \bar{G}/\Gamma$ are the Γ-orbits of the right action of Γ on $\bar{G}$, and

(ii) the left action of G on $\bar{G}$ descends to $\bar{G}/\Gamma$.

3.2 Proposition. *Let G be a connected reductive group, $\bar{G}$ be a regular compactification and Γ_1, Γ_2 be finite subgroups of G whose orders are not divisible by $\mathrm{char}(k)$. If $\bar{G}/\Gamma_2$ has an Γ_1-fixed point, then*

$$\mathrm{rank}(\Gamma_1) - \mathrm{rank}(\Gamma_2) \leq \mathrm{rank}\, C_G(\Gamma_1)^0.$$

Proof. Let y be an Γ_1-fixed point of $\bar{G}/\Gamma_2$ and x be a point of $\bar{G}$ lying above y. After translating x by a suitable element $(h_1, h_2) \in G \times G$ and replacing Γ_1, Γ_2 by $h_1 \Gamma_1 h_1^{-1}$, $h_2 \Gamma_2 h_2^{-1}$ respectively, we may assume that x is a special point. Then $P = \pi_1(\mathrm{Stab}_{G \times G}(x))$, $Q = \pi_2(\mathrm{Stab}_{G \times G}(x))$ is a pair of opposite parabolics, as in the previous section. Letting $L = P \cap Q$ be their common Levi subgroup, we see that $\mathrm{Stab}_{G \times G}(x)$ is as in (2.1). (Here, as before, $\pi_i : G \times G \to G$ denotes the projection to the ith factor, where $i = 1$ or 2.)

Let $\Gamma = \mathrm{Stab}_{\Gamma_1 \times \Gamma_2}(x)$. The fact that y is Γ_1-fixed means that for every $g_1 \in \Gamma_1$, there is an $g_2 \in \Gamma_2$ such that $(g_1, g_2) \in \mathrm{Stab}_{G \times G}(x)$. In other words, $\pi_1(\Gamma) = \Gamma_1$ and, in particular, $\Gamma_1 \subset P$. By Proposition 2.2

$$\mathrm{rank}(\Gamma_1) - \mathrm{rank}(\pi_2(\Gamma)) \leq \mathrm{rank}\, Z(L)^0.$$

Since $\operatorname{rank}(\Gamma_2) \geq \operatorname{rank}(\pi_2(\Gamma))$, we have

$$\operatorname{rank}(\Gamma_1) - \operatorname{rank}(\Gamma_2) \leq \operatorname{rank} Z(L)^0.$$

It remains to show that

$$\operatorname{rank} Z(L)^0 \leq \operatorname{rank} C_G(\Gamma_1)^0.$$

By the Levi decomposition, P is isomorphic to a semidirect product $P_u \rtimes L$; see, e.g., [H, 30.2]. Now recall that we are assuming that $\operatorname{char}(k)$ does not divide the order of Γ_1. In particular, Γ_1 is linearly reductive. Thus by [J, Lemma 11.24], Γ_1 is conjugate to a finite subgroup of L. Denote this subgroup by Γ_L. The connected centralizers $C_G(\Gamma_1)^0$ and $C_G(\Gamma_L)^0$ will then also be conjugate in G, and since $Z(L)^0 \subset C_G(\Gamma_L)^0$, we see that

$$\operatorname{rank} C_G(\Gamma_1)^0 = \operatorname{rank} C_G(\Gamma_L)^0 \geq \operatorname{rank} Z(L)^0,$$

as claimed. $\square$

4. Proof of Theorem 1.2 in characteristic zero

The following lemma is well known; we supply a short proof for lack of a direct reference.

4.1 Lemma. *Consider a faithful action of a finite abelian group A on an irreducible algebraic variety X, defined over a field k. Assume $\operatorname{char}(k)$ does not divide $|A|$. If A fixes a smooth k-point in X, then $\dim(X) \geq \operatorname{rank}(A)$.*

Proof. Let $x \in X(k)$ be a smooth A-fixed k-point. Then A acts on the regular local ring $R = \mathcal{O}_x(X)$ and on its maximal ideal $M = \mathcal{M}_x(X)$.

Assume the contrary; $\dim(X) = d$ and $\operatorname{rank}(A) > d$. Then the A-representation on the d-dimensional cotangent space $T_x(X)^* = M/M^2$ cannot be faithful; denote its kernel by $A_0 \neq \{1\}$. Since $|A|$ is prime to $\operatorname{char}(k)$, the map $M \to M/M^2$ of A-representations splits. Thus R has a system of local parameters $t_1, \ldots, t_d \in M$ such that each t_i is fixed by A_0. Then A_0 acts trivially on the completion $\widehat{R} = k[[t_1, \ldots, t_d]]$, hence, on $R \subset \widehat{R}$. Since X is irreducible, A_0 acts trivially on X. This contradicts our assumption that the A-action on X is faithful. $\square$

For the remainder of this section we will assume that k is an algebraically closed field of characteristic zero. Before proceeding with the proof of Theorem 1.2, we recall that every Γ-variety is birationally isomorphic to a smooth projective Γ-variety; cf. [RY$_2$, Proposition 2.2]. Here Γ is an arbitrary linear algebraic group, not necessarily connected. This fact, whose proof relies on equivariant resolution of singularities

and thus requires the assumption that $\text{char}(k) = 0$, will be used repeatedly for the remainder of this section.

Let Γ be a linear algebraic group defined over k and K/k be a finitely generated field extension. Recall that elements of $H^1(K, \Gamma)$ are in a natural 1-1 correspondence with birational isomorphism classes of generically free primitive Γ-varieties Z, with $K = k(Z)^\Gamma$; see, e.g., [Po, Section 1.3]. Here by saying that Z is a *primitive* Γ-variety, we mean that Γ transitively permutes the irreducible components of Y. We will denote the class of the generically free Γ-variety Z in $H^1(K, \Gamma)$ by $[Z]$.

4.2 Lemma. *Every primitive generically free G-variety X is birationally isomorphic to a projective G-variety of the form $(\bar{G} \times Y)/S$, where*

(a) *S is a finite subgroup of G,*

(b) *Y is a smooth irreducible projective S-variety,*

(c) *S acts on $\bar{G} \times Y$ by $s \cdot (h, y) = (hs^{-1}, s \cdot y)$, and $(\bar{G} \times Y)/S$ is the geometric quotient for this action.*

Note that in (c) S is a finite group acting on the smooth projective variety $\bar{G} \times Y$. In this situation a geometric quotient exists and is projective; see, e.g., [PV, Theorem 4.14] or Lemma 3.1. Moreover, the natural (left) G-action on $\bar{G} \times Y$ (via the first factor) commutes with the S-action and thus descends to $(\bar{G} \times Y)/S$. In the statement of the lemma we view $(\bar{G} \times Y)/S$ as a G-variety with respect to this action.

Proof. Let $K = k(X)^G$. By [CGR, Theorem 1.1] there exists a finite subgroup $S \subset G$ and a generically free primitive S-variety Y such that $[X]$ is the image of $[Y] \in H^1(K, S)$ under the natural map $H^1(K, S) \to H^1(K, G)$. In other words, X is birationally isomorphic to $(G \times Y)/S$. Since the S-variety Y is only defined up to birational isomorphism, we may assume without loss of generality that it is smooth and projective. Moreover, $(G \times Y)/S$ is birationally isomorphic to $(\bar{G} \times Y)/S$, as in (c).

In general, the above construction produces only a primitive S-variety Y. However, it can be slightly modified to ensure that Y is irreducible, as follows. Let Y_0 be an irreducible component of Y and S_0 be the subgroup of S consisting of elements that leave Y_0 invariant. Then it is easy to see that $[Y]$ is the image of $[Y_0]$ under the natural map $H^1(K, S_0) \to H^1(K, S)$; cf. e.g., [Re, Example 2.10]. Thus after replacing S by S_0 and Y by Y_0, we may assume that Y is irreducible. $\qquad\square$

4.3 Lemma. *Let G be a reductive group, $H \subset G$ be a finite abelian subgroup, and X be a primitive generically free G-variety. If H fixes a smooth k-point of X, then*

$$\dim k(X)^G \geq \text{rank}(H) - \text{rank}\, C_G(H)^0.$$

Proof. By Lemma 4.2 there exists a birational G-equivariant isomorphism $X \overset{\simeq}{\dashrightarrow} (\bar{G} \times Y)/S$, where Y is a smooth complete S-variety and $S \subset G$ is a finite subgroup, as in Lemma 4.2. Note that $\dim(Y) = \operatorname{trdeg}_k k(X)^G$; we will denote this number by d. By the Going Down Theorem [RY$_1$, Proposition A.2] $(\bar{G} \times Y)/S$ also has an H-fixed k-point; denote it by $x = [\bar{g}, y]$. The fiber of the natural projection $X = (\bar{G} \times Y)/S \to Y/S$ containing x is easily seen to be G-equivariantly isomorphic to $\bar{G}/S_0$, where $S_0 := \operatorname{Stab}_S(y)$. Now observe that since y is a smooth k-point of Y, S_0 can contain no abelian subgroup of rank $\geq d$; see Lemma 4.1. In other words, $\operatorname{rank}(S_0) \leq d$. By Proposition 3.2,

$$\operatorname{rank}(H) - \operatorname{rank}(S_0) \leq \operatorname{rank} C_G(H)^0$$

and thus

$$d \geq \operatorname{rank}(S_0) \geq \operatorname{rank}(H) - \operatorname{rank} C_G(H)^0.$$

This completes the proof of Lemma 4.3.   □

We are now ready to proceed with the proof of Theorem 1.2 (in characteristic zero). Let V be a generically free linear k-representation of G.

(a) The essential dimension $\operatorname{ed}(G)$ is the minimal value of $\operatorname{trdeg}_k k(X)^G$, where the minimum is taken over all dominant rational G-equivariant maps

$$V \dashrightarrow X,$$

such that X is a generically free G-variety; see [Re, Section 3]. Thus our goal is to show that

$$\operatorname{trdeg}_k k(X)^G \geq \operatorname{rank}(H) - \operatorname{rank} C_G(H)^0. \tag{4.4}$$

After replacing X by a birationally equivalent G-variety, we may assume that X is smooth and projective. Since V has a smooth H-fixed k-point (namely the origin), the Going Down Theorem [RY$_1$, Proposition A.2] tells us that X also has an H-fixed k-point (which is smooth, because every k-point of X is smooth). The inequality (4.4) now follows from Lemma 4.3.

(b) It suffices to show that the inequality (4.4) holds if there is a diagram of dominant rational G-equivariant maps of the form

$$
\begin{array}{ccc}
V' & & \\
\big\downarrow & \searrow & \\
V & & X \, ,
\end{array}
$$

where X is a generically free G-variety, $\dim(V') = \dim(V)$ and $[k(V') : k(V)]$ is prime to l. (Note that the G-variety V' is not required to be linear.) Once again, we

may assume without loss of generality that V' and X are smooth and complete. Since H fixes the origin in V, the Going Up Theorem [RY$_1$, Proposition A.4] tells us that V' has an H-fixed k-point. Now by the Going Down Theorem [RY$_1$, Proposition A.2], X has an H-fixed k-point as well, and Lemma 4.3 completes the proof.   □

5. The field of iterated Laurent series

In this section we will describe the structure of the iterated Laurent polynomial field $F_n = k((t_1)) \ldots ((t_n))$. Our proof of Theorem 1.2 in full generality (i.e., without assuming that $\mathrm{char}(k) = 0$) will make use of these results.

We begin by describing the absolute Galois group $\mathrm{Gal}(F_n)$. Since $F_n = F_{n-1}((t_n))$ is a complete valued field with residue field F_{n-1}, we have an exact sequence

$$1 \to I_n \to \mathrm{Gal}(F_n) \xrightarrow{\pi_n} \mathrm{Gal}(F_{n-1}) \to 1$$

where I_n stands for the inertia group; see [GMS, §II.7]. Let $\widehat{\mathbb{Z}}'$ be the prime to p part of $\widehat{\mathbb{Z}}$, i.e. $\widehat{\mathbb{Z}}' = \prod_{q \neq p} \widehat{\mathbb{Z}}_q$. In particular, if $p = 0$, then $\widehat{\mathbb{Z}}' = \widehat{\mathbb{Z}}$.

5.1 Lemma. *There is a split exact sequence*

$$1 \to J_n \to \mathrm{Gal}(F_n) \to (\widehat{\mathbb{Z}}')^n \to 1, \tag{5.2}$$

such that

(1) $J_n = 1$ *if* $p = 0$,

(2) J_n *is a free pro-p-group if* $p > 0$.

Proof. We proceed by induction on n. The group I_n fits in an exact sequence

$$0 \to I_n^{\mathrm{wild}} \to I_n \to \widehat{\mathbb{Z}}' \to 1,$$

where I_n^{wild} is the wild inertia group (it is a pro-p-group). Define

$$F_{n,m} := k((\sqrt[m]{t_1})) \ldots ((\sqrt[m]{t_n})) \quad \text{and} \quad F_{n,\infty} := \varinjlim_{(m,p)=1} F_{n,m}.$$

Since

$$\mathrm{Gal}(F_{n,\infty}/F_{n-1,\infty}((t_n))) = \widehat{\mathbb{Z}}',$$

we have the following commutative diagram of profinite groups:

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I_n^{\mathrm{wild}} & \longrightarrow & J_n & \longrightarrow & J_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & I_n & \longrightarrow & \mathrm{Gal}(F_n) & \longrightarrow & \mathrm{Gal}(F_{n-1}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Gal}(F_{n,\infty}/F_{n-1,\infty}((t_n))) & \longrightarrow & \mathrm{Gal}(F_{n,\infty}/F_n) & \longrightarrow & \mathrm{Gal}(F_{n-1,\infty}/F_{n-1}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1. & &
\end{array}
$$

If $p = 0$, then $I_n^{\mathrm{wild}} = 1$; cf. [GMS, §II.7.1]. Hence, the top row in the above diagram tells us that $J_0 \cong J_1 \cong J_2 \cong \ldots$. On the other hand, since $F_0 = F_{0,\infty} = k$, we see that $J_0 = \{1\}$. We thus conclude that $J_n = 1$ for every n. (This also follows from [GP, Corollary 2.10].)

Similarly if $p > 0$, then I_n^{wild} is a pro-p group (once again, cf. [GMS, §II.7.1]), and we see by induction that J_n is pro-p-group. The group J_n is the absolute Galois group of the field $F_{n,\infty}$, so $\mathrm{cd}_p(J_n) \le 1$ ([Se, II.3.1, Proposition 7]) and J_n is then a free pro-p-group (*ibid*, I.4.2, Corollary 2).

Finally, since $(\widehat{\mathbb{Z}}')^n$ is a prime-to-p group, we conclude that the sequence (5.2) splits; see [Se, I.5.9, Corollary 1].   □

We will now show that every finite field extension of F_n is k-isomorphic to F_n. Recall that the lexicographic order $\prec$ on $\mathbb{Z}^n$ is defined as follows:

$$(m_1, \ldots m_n) \prec (m'_1, \ldots m'_n)$$

if $m_i < m'_i$ for the smallest subscript i with $m_i \ne m'_i$. A valuation

$$v = (v^{(n)}, \ldots, v^{(1)}) \colon E^\times \to \Gamma$$

on a field E is called *n-discrete* if the group Γ is isomorphic to a (lexicographically ordered) subgroup of $\mathbb{Z}^n$; see [F, 1.1.3]. Then E is a 1-discrete valuation with respect to the first component $v^{(n)}$ of v and the residue field E_{n-1} is a $(n-1)$-discrete valuation via $(v^{(n-1)}, \ldots, v^{(1)})$. In this way we obtain a sequence of fields $E = E_n, E_{n-1}, \ldots, E_1$ such that E_i is the residue field of E_{i+1} with respect to a 1-discrete valuation. The residue field E_0 of E_1 then coincides with the residue field $\bar{E}_v$.

The definition of completeness for E is inductive as follows [F, 1.2.1]. The field E is a complete n-discrete field if E_n is complete with respect to $v^{(n)}$ and E_{n-1} is complete. Assume from now on that E is a complete n-discrete field. Then according to [W, 3.1], E is henselian, i.e. its valuation ring is a henselian ring. In particular, given a finite extension E'/E the valuation v extends uniquely to $v' : E'^{\times} \to \frac{1}{[E':E]} \mathbb{Z}^n$, the formula being $v' = \frac{1}{[E':E]} v \circ N_{E'/E}$. Then E' is a n-discrete field which is complete (by induction).

The field $F_n = k((t_1))((t_2)) \ldots ((t_n))$ over iterated Laurent series over a base field k is n-complete; see [W, §3]. Here the valuation v on F_n is defined by

$$v\left(\sum_{i_1} \ldots \sum_{i_n} c_{i_1,\ldots,i_n} t_1^{i_1} \ldots t_n^{i_n} \right) = \text{Min}\big\{ (i_1, \ldots, i_n) \ \mid c_{i_1,\ldots,i_n} \neq 0 \big\}.$$

5.3 Proposition. *Let E be a complete n-discrete k-field. Then*

(1) *E is isomorphic to $\bar{E}_v((t_1))((t_2)) \ldots ((t_n))$;*

(2) *Suppose the valuation v is trivial on a perfect subfield K of E. Then E is K-isomorphic to $\bar{E}_v((t_1))((t_2)) \ldots ((t_n))$.*

Proof. (1) immediately follows from (2) if we take K to be the prime subfield of E.

To prove (2), first assume that $n = 1$. In this case Cohen's structure theorem [Co, Theorem 10] shows that the valuation ring of E is K-isomorphic to $\bar{E}_v[[t_1]]$. Thus E is K-isomorphic to $\bar{E}_v((t_1))$.

Now suppose that $n \geq 2$. Let $E = E_n, E_{n-1}, \ldots, E_1$ be the sequence of fields constructed above. By induction, we may assume that E_{n-1} is K-isomorphic to $\bar{E}_v((t_1))((t_2)) \ldots ((t_{n-1}))$. Since we know part (2) holds for $n = 1$, we conclude that $E = E_n$ is K-isomorphic to $\bar{E}_v((t_1))((t_2)) \ldots ((t_n))$. $\square$

5.4 Corollary. *Assume that k is algebraically closed. Then any finite extension of $F_n = k((t_1)) \ldots ((t_n))$ is k-isomorphic to F_n.*

Proof. A finite extension E of F_n is a complete n-discrete field. Its residue field E_v is a finite extension of k. Since k is algebraically closed, we conclude that $E_v = k$. Proposition 5.3 (2) now shows that E is k-isomorphic to F_n. $\square$

6. Reduction of structure

In this section G/k will denote a linear algebraic group defined over an algebraically closed field k of characteristic ≥ 0, whose identity component G^0 is reductive. F_n will denote the iterated power series field $k((t_1))((t_2)) \ldots ((t_n))$ in variables $t_1, \ldots, t_n$, as in the previous section. As usual, we will say that $\gamma \in H^1(K, G)$

- *descends* to a subfield $K_0 \subset K$ if γ lies in the image of the restriction map $H^1(K_0, G) \to H^1(K, G)$,

- *admits reduction of structure* to a subgroup $A \subset G$ if γ lies in the image of the natural map $H^1(K, A) \to H^1(K, G)$.

6.1 Proposition. *Suppose $\gamma \in H^1(F_n, G)$ descends to a subfield $K \subset F_n$ such that $\mathrm{trdeg}_k(K) = d < \infty$.*

(a) *Assume that $\mathrm{char}(k) = 0$. Then γ admits reduction of structure to a finite abelian subgroup $A \subset G$ of rank $\leq d$.*

(b) *Assume that $\mathrm{char}(k) = p > 0$. Then there exists a finite field extension F'/F_n such that $[F' : F_n]$ is a power of p and $\gamma_{F'}$ admits reduction of structure to a finite abelian subgroup $A \subset G$ of rank $\leq d$, whose order $|A|$ is prime to p.*

Our proof of Proposition 6.1 will make use of the following two simple lemmas.

6.2 Lemma. *Suppose $K \subset E$ is a field extension such that K is algebraically closed in E. Then*

(a) *for every finite Galois field extension K'/K, $K'_E = K' \otimes_K E$ is a field.*

(b) *The absolute Galois group $\mathrm{Gal}(K)$ is a quotient of $\mathrm{Gal}(E)$.*

Proof. (a) By the primitive element theorem we can write K' as $K[x]/(p(x))$ for some irreducible monic polynomial $p(x) \in K[x]$. Then $K'_E = K' \otimes_K E = E[x]/(p(x))$, and we need to show that $p(x)$ remains irreducible over E.

We argue by contradiction. Suppose $p(x) = p_1(x)p_2(x)$ for some non-constant monic polynomials $p_1(x), p_2(x) \in E[x]$. The coefficients of $p_i(x)$ are then polynomials in the roots of $p(x)$. (Here $i = 1$ or 2.) In particular, they are algebraic over K. Since K is algebraically closed in E, we conclude that $p_i(x) \in K[x]$. Thus $p(x)$ is reducible over K, a contradiction.

(b) Let $\bar{K}$ be the algebraic closure of K. Then $\mathrm{Gal}(K) = \mathrm{Gal}(\bar{K}/K) = \mathrm{Gal}(\bar{K}_E/E)$, and $\bar{K}_E = \bar{K} \otimes_K E$ is an E-subfield of $\bar{E}$ by part (a). $\square$

6.3 Lemma. *Let $d \geq 0$ be an integer. Let Γ be a finitely generated abelian profinite group such that $\mathrm{cd}(\Gamma) \leq d$. Then Γ is a direct summand of $\widehat{\mathbb{Z}}^d$.*

Proof. Without loss of generality, we may assume that Γ is a p-profinite group for a prime p. Since $\mathrm{cd}(\Gamma) < \infty$, we see that Γ is torsion free, and since Γ is finitely generated and abelian, we conclude that $\Gamma \cong \mathbb{Z}_p^m$ for some integer m; see [RZ, Theorem 4.3.4. (a)]. Thus $\mathrm{cd}(\Gamma) = m$. Since we are assuming $m \leq d$, this shows that Γ is a direct summand of $\widehat{\mathbb{Z}}^d$. $\square$

Proof of Proposition 6.1. Let W be the Weyl group of G. We recall that there exists a finite subgroup $S \hookrightarrow G$ such that every prime factor of $|S|$ divides $|W|$ and S has the following property.

(i) If $\mathrm{Char}(k)$ does not divide $|W|$, then the map $H^1(K, S) \to H^1(K, G)$ is surjective for every field K/k; see [CGR].

(ii) If $\mathrm{char}(k)$ divides $|W|$, the above map is surjective for every perfect field K/k; see [CGR2, Corollary 1.4].

We fix a finite subgroup S with these properties for the rest of the proof.

Let $\rho: (\widehat{\mathbb{Z}}')^n \to \mathrm{Gal}(F_n)$ be a splitting of the exact sequence (5.2) in Lemma 5.1. Denote the extension of F_n associated by the Galois correspondence to the image of ρ by E_n/F_n and its perfect closure by E_n^{perf}/F_n. Note that if $\mathrm{char}(k) = 0$, then ρ is an isomorphism and $E_n^{\mathrm{perf}} = E_n = F_n$. If $\mathrm{char}(k) = p$, then the degree of any finite subextension of E_n/F_n is a power of p and $\mathrm{Gal}(E_n) = (\widehat{\mathbb{Z}}')^n$. The same is true for the perfect closure E_n^{perf}/F_n.

Since Galois cohomology commutes with direct limits of fields, in order to establish parts (a) and (b) of the proposition, it suffices to show that $\gamma_{E_n^{\mathrm{perf}}}$ admits reduction of structure to some abelian subgroup $A \subset S$ of rank $\leq \mathrm{trdeg}_k(K)$ (where $|A|$ is prime to p, if $p = \mathrm{char}(k) > 0$). After replacing K by its algebraic closure in E_n^{perf}, we may assume that K is algebraically closed in E_n^{perf}. In particular, K is perfect.

By our assumption γ descends to some $\gamma_K \in H^1(K, G)$. On the other hand, by (ii) γ_K is the image of some $\delta_K \in H^1(K, S)$. The class δ_K is represented by a continuous homomorphism $\psi: \mathrm{Gal}(K) \to S$. Clearly δ_K (and hence, γ_K and $\beta_{E_n^{\mathrm{perf}}}$) admit reduction of structure to the subgroup $A = \mathrm{Im}(\psi)$ of S. It remains to show that A is an abelian group of rank $\leq d$ whose order is prime to p.

By Lemma 6.2 we can identify $\mathrm{Gal}(K)$ with a quotient of $\mathrm{Gal}(E_n^{\mathrm{perf}}) = (\widehat{\mathbb{Z}}')^n$. In particular, $\mathrm{Gal}(K)$ is finitely generated, abelian, and the order of every finite quotient of $\mathrm{Gal}(K)$ is prime to p. Moreover, by Tsen's theorem, $\mathrm{cd}(\mathrm{Gal}(K)) \leq d$; cf. [Se, II.4.2]. Thus Lemma 6.3 enables us to conclude that $\mathrm{Gal}(K)$ is a direct summand of $(\widehat{\mathbb{Z}}')^d$. Hence, the finite quotient A of $\mathrm{Gal}(K)$ is an abelian group of rank $\leq d$ whose order is prime to p. $\qquad\square$

6.4 Remark. A minor modification of the above argument (in particular, using (i) instead of (ii)) shows that the assertion of Proposition 6.1 (a) holds whenever $\mathrm{Char}(k)$ does not divide the order of the Weyl group W of G. In other words, in this case we can take F' to be F_n in part (b). Since we will not use this result in the sequel, we leave the details of its proof as an exercise for an interested reader.

7. Fixed points in homogeneous spaces

Let k is an algebraically closed field of characteristic $p \geq 0$, $t_1, \ldots t_n$ are independent variables over k, and $H = (\mathbb{Z}/m\mathbb{Z})^n$. If $p > 0$, we will assume that m is prime to p. We will continue to denote the iterated power series field $k((t_1))((t_2)) \ldots ((t_n))$ by F_n.

The purpose of this section is to establish the following fixed point result which will be used in the proof of Theorem 1.2. For notational convenience, we will consider an arbitrary (not necessarily injective) morphism

$$\phi \colon H = (\mathbb{Z}/m\mathbb{Z})^n \to G \tag{7.1}$$

of algebraic groups. This is slightly more general than considering a finite abelian subgroup of G. We will assume that G, H and ϕ are fixed throughout this section.

7.2 Proposition. *Assume that*

(1) *F'/F_n is a finite field extension of degree prime to $|H|$,*

(2) *$\beta \in H^1(F_n, H)$ is represented by an H-Galois field extension E/F, and*

(3) *$\phi_*(\beta)_{F'} \in H^1(F', G)$ admits reduction of structure to a finite subgroup S of G.*

   Then $\phi(H)$ has a fixed k-point in any G-equivariant compactification Y of G/S.

Here by a G-equivariant compactification of G/S we mean a complete (but not necessarily smooth) G-variety, which contains G/S as a dense open G-subvariety.

Proof. By Corollary 5.4, F' is k-isomorphic to F_n. Thus, after replacing F_n by F' and β by $\beta_{F'}$, we may assume that $F' = F_n$. (Note that $\beta_{F'} \in H^1(F', H)$ is represented by the H-Galois algebra $E_{F'}/F'$, where $E_{F'} = E \otimes_F F'$. Since E is a field and $[F'_n : F_n]$ is prime to $|H| = [E : F]$, $E_{F'}$ is again a field.)

By Lemma 5.1, we may assume that $E = k((s_1)) \ldots ((s_n))$, where $s_i^m = t_i$ and there exists a minimal set of generators $\tau_1, \ldots, \tau_n$ of H such that H acts on $k((s_1)) \ldots ((s_n))$ by

$$\tau_i(s_j) = \begin{cases} \zeta\, s_j & \text{if } i = j, \\ s_j & \text{if } i \neq j, \end{cases} \tag{7.3}$$

where ζ is a primitive mth root of unity (independent of i and j). In the sequel we will denote E by $F_{n,m}$; note that we previously encountered this field in the proof of Lemma 5.1.

Set $\gamma = \phi_*(\beta) \in H^1(F_n, G)$ and consider the twisted F_n-variety $_\gamma Y$ which is a compactification of the twisted variety $_\gamma(G/S)$. By our assumption $_\gamma(G/S)$ has a F_n-point, so a fortiori

$$_\gamma Y(F_{n,m}) = \{ y \in Y(F_{n,m}) \mid \gamma(\sigma).^\sigma y = y \text{ for all } \sigma \in H \} \neq \emptyset.$$

Since Y is complete, this implies $_\gamma Y(F_{n-1,m}[[\sqrt[m]{t_n}]]) \neq \emptyset$. Specializing t_n to 0, we see that

$$\{y \in Y(F_{n-1,m}) \mid \gamma(\sigma).^\sigma y = y \text{ for all } \sigma \in H\} \neq \emptyset,$$

where the Galois action of H on $Y(F_{n-1,m})$ is induced by the canonical projection $H \to \text{Gal}(F_{n,m}/F_n) \to \text{Gal}(F_{n-1,m}/F_{n-1})$. Repeating this process, we finally obtain

$$\{y \in Y(k) \mid \gamma(\sigma).^\sigma y = y \text{ for all } \sigma \in H\} \neq \emptyset.$$

Since k is algebraically closed, we conclude that $\phi(H)$ fixes some k-point of Y. $\qquad\square$

7.4 Corollary. *Let k be an algebraically closed field and G/k be a connected reductive group. Suppose there exists a class $\beta \in H^1(F_n, H)$ such that β is represented by an H-Galois field extension of F_n. If $\phi_*(\beta) \in H^1(F_n, G)$ descends to a k-subfield $K \subset F_n$, then*

$$\text{trdeg}_k(K) \geq \text{rank}\,\phi(H) - \text{rank}\,C_G(\phi(H))^0.$$

Here H and ϕ are as in (7.1).

Proof. Let $\text{trdeg}_k(K) = d$. By Proposition 6.1 there exists a finite extension F'/F_n and a finite abelian subgroup $A \subset G$ of rank $\leq d$ such that

- $|A|$ is prime to $\text{char}(k)$,
- $F' = F_n$ if $\text{char}(k) = 0$, and $[F' : F_n]$ is a power of p if $\text{char}(k) = p$, and
- $\phi_*(\beta)_{F'}$ admits reduction of structure to A.

Let $\bar{G}$ be a regular compactification of G. By Proposition 7.2, $Y = \bar{G}/A$ has a $\phi(H)$-fixed point. Now Proposition 3.2, with $\Gamma_1 = \phi(H)$ and $\Gamma_2 = A$, tells us that

$$\text{rank}\,\phi(H) - \text{rank}\,A \leq \text{rank}\,C_G(\phi(H))^0.$$

Consequently,

$$d \geq \text{rank}\,A \geq \text{rank}\,\phi(H) - \text{rank}\,C_G(\phi(H))^0,$$

as claimed. $\qquad\square$

8. Proof of Theorem 1.2

In the statement of Theorem 1.2, we assume that H is a subgroup of G, where as in the previous section we worked with a homomorphism $\phi \colon H \to G$ instead. For notational consistency, we will restate Theorem 1.2 in the following (clearly equivalent) form.

8.1 Theorem. *Let G be a connected reductive linear algebraic group defined over an algebraically closed base field k, $H \simeq (\mathbb{Z}/m\mathbb{Z})^n$ and $\phi \colon H \to G$ be a (not necessarily injective) homomorphism of algebraic groups. Assume* char(k) *does not divide m. Then*

(a) $\operatorname{ed}(G) \geq \operatorname{rank} \phi(H) - \operatorname{rank} C_G(\phi(H))^0.$

(b) *Moreover, if H is an l-group (i.e., m is a power of a prime integer l) then*
$\operatorname{ed}(G; l) \geq \operatorname{rank} \phi(H) - \operatorname{rank} C_G(\phi(H))^0.$

Let $t_1, \ldots, t_n$ be independent variables over k, $K_n = k(t_1, \ldots, t_n)$ and $K_{n,m} = k(s_1, \ldots, s_n)$, where $s_i^m = t_i$. The H-Galois field extension $K_{n,m}/K_n$ gives rise to a class $\alpha \in H^1(K_n, H)$. We will be interested in the class $\phi_*(\alpha) \in H^1(K_n, G)$, which we will sometimes refer to as a *loop torsor*. (Such torsors naturally come up in connection with loop algebras; see [GP].) We are now ready to proceed with the proof of Theorem 8.1.

(a) By the definition of $\operatorname{ed}(G)$, it suffices to show that

$$\operatorname{ed} \phi_*(\alpha) \geq \operatorname{rank} \phi(H) - \operatorname{rank} C_G(\phi(H))^0.$$

Let $d = \operatorname{ed}(\phi_*(\alpha))$. Let $\beta = \alpha_{F_n} \in H^1(F_n, H)$. Then β is represented by the field extension $F_{n,m}/F_n$. Moreover, $\phi_*(\beta)$ descends to $\phi^*(\alpha) \in H^1(K_n, G)$, which, by our assumption, further descends to a k-subfield of K_n of transcendence degree d. Corollary 7.4 now tells us that

$$d \geq \operatorname{rank} \phi(H) - \operatorname{rank} C_G(\phi(H))^0.$$

This completes the proof of Theorem 8.1 (a).

(b) Once again, we will denote the class of the H-Galois field extension $K_{n,m}/K_n$ in $H^1(K_n, H)$ by α and consider the loop torsor $\phi_*(\alpha) \in H^1(K_n, G)$. By the definition of $\operatorname{ed}(G; l)$ it suffices to show that

$$\operatorname{ed}(\phi_*(\alpha); l) \geq \operatorname{rank} H - \operatorname{rank} C_G(H)^0.$$

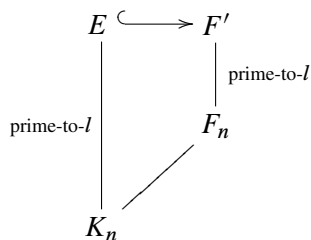Equivalently, we want to show that

$$\operatorname{ed}(\phi_*(\alpha)_E) \geq \operatorname{rank} H - \operatorname{rank} C_G(H)^0$$

for every finite extension E/K_n of degree prime to l. Suppose E/K_n is such an extension and $\operatorname{ed}(\phi_*(\alpha)_E) = d$.

By [KM, Lemma 3.1] there exists a finite field extension F'/F_n of degree prime

to l and a K_n-embedding $E \hookrightarrow F'$ such that the diagram

$$
\begin{array}{ccc}
E & \stackrel{\subset}{\longrightarrow} & F' \\
& & \downarrow \text{\small prime-to-}l \\
\text{\small prime-to-}l \downarrow & & F_n \\
& & \nearrow \\
& K_n &
\end{array}
$$

commutes. We want to conclude that $d \geq \operatorname{rank} H - \operatorname{rank} C_G(H)^0$ by applying Corollary 7.4 to $\beta = \alpha_{F'} \in H^1(F', H)$. Since F' is k-isomorphic to F_n (see Corollary 5.4), Corollary 7.4 can be applied to this β, as long as we can show that

(i) $\phi_*(\beta)$ descends to a k-subfield of F', of transcendence degree d over k, and

(ii) β is represented by an H-Galois field extension of F'.

(i) is clear since $\phi^*(\beta) \in H^1(F', G)$ descends to $\phi^*(\alpha) \in H^1(E, G)$, which, by our assumption, descends to a k-subfield $E_0 \subset E$ with $\operatorname{trdeg}_k(E_0) = d$. To prove (ii), note that α_{F_n} is represented by the field extension $F_{n,m}/F_n$. Thus $\beta = \alpha_{F'}$ is represented by the H-Galois algebra $F_{n,m} \otimes_{F_n} F'$ over F'. Since $[F' : F_n]$ is a finite and prime to l, $F_{n,m} \otimes_{F_n} F'$ is a field. This concludes the proof of (ii) and thus of Theorem 8.1 (b). $\square$

9. Examples

This section contains five examples illustrating Theorem 1.2 (b).

9.1 Example. If $\operatorname{char}(k) \neq 2$, then $\operatorname{ed}(GO_n; 2) \geq n - 1$.

Proof. Let $H \simeq (\mathbb{Z}/2\mathbb{Z})^n$ be the subgroup of diagonal matrices in O_n. Viewing H as a subgroup of GO_n, we easily see that $C^0_{GO_n}(H) = $ the center of GO_n, has rank 1. Applying Theorem 1.2 (b) to this subgroup we obtain the desired bound. $\square$

9.2 Example. If p is a prime and $\operatorname{char}(k) \neq p$, then

(a) $\operatorname{ed}(SL_{p^r}/\mu_{p^s}; p) \geq \begin{cases} 2s + 1 & \text{if } s < r, \\ 2s & \text{if } s = r; \end{cases}$

(b) $\operatorname{ed}(GL_{p^r}/\mu_{p^s}; p) \geq \begin{cases} 2s, & \text{if } s < r, \\ 2s - 1 & \text{if } s = r. \end{cases}$

Proof. (a) The group SL_{p^r}/μ_s has a self-centralizing subgroup

$$H \simeq (\mathbb{Z}/p\mathbb{Z})^r \times \mathbb{Z}/p^{r-s}\mathbb{Z};$$

see [RY$_1$, Lemma 8.12]. Now apply Theorem 1.2 (b) to this group.

(b) We now view H as a subgroup of GL_{p^r}/μ_{p^s}. The centralizer $C^0_{GL_{p^r}/\mu_s}(H)$ is the center of GL_{p^r}/μ_s; it is isomorphic to a 1-dimensional torus. Part (b) now follows from Theorem 1.2 (b). $\qquad\square$

The non-vanishing of the Rost invariant $H^1(*, G) \to H^3(*, \mu_p)$ for a group G and a prime p implies that $ed(G; p) \geq 3$; cf. [Re, Theorem 12.14]. In particular, one can show that $ed(F_4; 3), ed(E_6; 2)$ and $ed(E_7; 3) \geq 3$ in this way. In Examples 9.3–9.6 below we will deduce these inequalities directly from Theorem 1.2 (b) and show that equality holds in each case.

9.3 Example. If $char(k) \neq 3$, then $ed(F_4; 3) = 3$.

Proof. F_4 has a self-centralizing subgroup isomorphic to $(\mu_3)^3$; see [Gr, Theorem 7.4]. Theorem 1.2 now tells us that $ed(F_4; 3) \geq 3$. To prove the opposite inequality, recall that $H^1(K, F_4)$ classifies the exceptional 27-dimensional Jordan algebras J/K. After a quadratic extension K'/K, $J \otimes_K K'$ is given by the first Tits construction. Without loss of generality, we may assume that $J = (A, \nu)$ where A is a central simple K-algebra of degree 3 and ν is a scalar in K. Since A is a symbol algebra $(a, b)_3$, we see that $J \otimes_K K'$ descends to the subfield $k(a, b, \nu)$ of K of transcendence degree ≤ 3. We conclude that $ed(J; 3) \leq 3$ and thus $ed(F_4; 3) \leq 3$, as claimed. $\qquad\square$

9.4 Example. If $char(k) \neq 2$, then $ed(E_6; 2) = 3$.

Here E_6 denotes the simply connected simple group of type E_6. By abuse of notation we will also write E_6 for the Dynkin diagram of E_6.

Proof. By [Gr, Table II], E_6 has a unique (up to conjugation) non-toral subgroup H isomorphic to $(\mathbb{Z}/2\mathbb{Z})^5$. To compute the rank of its centralizer, we make use of its Witt–Tits index $I(H) \subset E_6$ which describes the type of a minimal parabolic subgroup containing H; see [GP, Section 3]. The Dynkin diagram for E_6 is as follows.
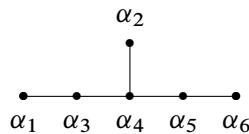
Set $I = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Let P_I be a standard parabolic subgroup and $L_I = Z_G(T_I)$ its standard Levi subgroup. Then $DL_I = \mathrm{Spin}_8$. Since Spin_8 has a maximal non-toral 2-elementary abelian subgroup of rank 5 (see [Gr, Table I]), we may assume that $H \subset \mathrm{Spin}_8$. Moreover, $C_{\mathrm{Spin}_8}(H)$ is finite, so H is irreducible in L_I. It follows that P_I is a minimal parabolic subgroup of E_6 containing H; the Witt–Tits index of H is then I. By [GP, Proposition 3.11], we have

$$\mathrm{rank}\, C_{E_6}(H) = |E_6 \setminus I| = 2.$$

Theorem 1.2 (b) now tells us that $\mathrm{ed}(E_6; 2) \geq 5 - 2 = 3$. (In [GP] the base field is assumed to be of characteristic 0. However, in our situation, [GP, Proposition 3.11] applies as long as $\mathrm{char}(k) \neq 2$.)

To prove the opposite inequality, suppose $\alpha \in H^1(K, E_6)$, where K is a field containing k. Let $L \subset E_6$ be the Levi subgroup of the parabolic $E_6 \setminus \{\alpha_6\}$. We observe that the finite groups $N_L(T)/T$ and $N_{E_6}(T)/T$ have isomorphic 2-Sylow subgroups of order 2^7. By [Gi, Lemme 3.a], it follows that there exists a finite odd degree extension K'/K such that $\alpha_{K'}$ belongs to the image of $H^1(K', L) \to H^1(K', E_6)$. Hence the class $\alpha_{K'}$ is isotropic with respect to the root α_6. By the list of Witt–Tits indices in [T], the class $\alpha_{K'}$ is isotropic with respect to I. So $\alpha_{K'}$ belongs to the image of $H^1(K', L_I) \cong H^1(K', P_I) \to H^1(K', E_6)$.

It thus remains to show that

$$\mathrm{ed}(L_I; 2) \leq 3.$$

To prove this inequality, we need an explicit description of the group L_I. Recall that there is a natural inclusion $\mu_2 \times \mu_2 = C(DL_I) \subset T_I = \mathbb{G}_m \times \mathbb{G}_m$; see [CP, Proof of Proposition 14.a]. Hence we have the following commutative exact diagram:

$$
\begin{array}{ccccccccc}
& & 1 & & 1 & & & & \\
& & \downarrow & & \downarrow & & & & \\
0 & \to & C(DL_I) = \mu_2 \times \mu_2 & \to & DL_I = \mathrm{Spin}_8 & \to & \mathrm{PSO}_8 & \to & 1 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \to & T_I = \mathbb{G}_m \times \mathbb{G}_m & \to & L_I & \to & \mathrm{PSO}_8 & \to & 1 \\
& & {\scriptstyle (2,2)}\downarrow & & \downarrow & & & & \\
& & \mathbb{G}_m \times \mathbb{G}_m & = & \mathbb{G}_m \times \mathbb{G}_m & & & & \\
& & \downarrow & & \downarrow & & & & \\
& & 1 & & 1. & & & &
\end{array}
$$

Taking Galois cohomology of the right square over a field F/k, we obtain the fol-

lowing commutative exact diagram of pointed sets:

$$
\begin{array}{ccc}
H^1(F, \mathrm{Spin}_8) & \longrightarrow & H^1(F, \mathrm{PSO}_8) \\
\downarrow & & \| \\
1 \longrightarrow H^1(F, L_I) & \longrightarrow & H^1(F, \mathrm{PSO}_8) \\
\downarrow & & \\
1. & &
\end{array}
$$

By the usual twisting argument, $H^1(F, L_I)$ injects into $H^1(F, \mathrm{PSO}_8)$, therefore $H^1(F, L_I) = \mathrm{Im}\big(H^1(F, \mathrm{Spin}_8) \to H^1(F, \mathrm{PSO}_8)\big)$. It is well known that the image of $H^1(F, \mathrm{Spin}_8)$ in $H^1(F, \mathrm{PSO}_8)$ classifies the similarity classes of 8-dimensional quadratic F-forms in $I^3(F)$; cf., e.g., [KMRT, pp. 409 and 437]. By the Arason–Pfister Theorem, every 8-dimensional quadratic form $q \in I^3(F)$ is similar to a 3-fold Pfister form $\langle\langle a, b, c \rangle\rangle$. Thus the similarity class of q is defined over $k(a, b, c)$. This shows that $\mathrm{ed}(L_I; 2) \leq 3$, as claimed. □

9.5 Remark. One can show that for every $\alpha \in H^1(K, E_6)$ there is an odd degree field extension L/K such that α_L lies in the image of the natural map $H^1(L, G_2) \to H^1(L, E_6)$; see [GMS, Exercise 22.9]. Since $\mathrm{ed}(G_2) = 3$, this leads to an alternative proof of the inequality $\mathrm{ed}(E_6; 2) \leq 3$.

9.6 Example. If $\mathrm{char}(k) \neq 3$, then $\mathrm{ed}(E_7; 3) = 3$.

Here E_7 denotes the simply connected simple group of type E_7. By abuse of notation we will sometimes also write E_7 for the Dynkin diagram of E_7.

Proof. By [Gr, Table III], E_7 has a unique (up to conjugation) non-toral subgroup H isomorphic to $(\mathbb{Z}/3\mathbb{Z})^5$. To compute the rank of its centralizer, we make use of its Witt–Tits index $I(H) \subset E_7$ [GP, Section 3], where E_7 is the following Dynkin diagram.
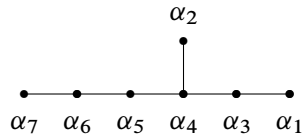
Set $I = E_7 \setminus \{\alpha_7\}$ and let P_I be the standard parabolic subgroup. Denote by $L_I = Z_G(T_I)$ its standard Levi subgroup. Then $DL_I = E_6$, where E_6 denotes a simply connected group of type E_6. Since E_6 has a maximal non-toral 3-elementary abelian subgroup of rank 4 (see [Gr, Table III]), we may assume that $H \subset E_6$.

Moreover, $C_{E_6}(H)$ is finite, so H is irreducible in L_I. It follows that P_I is a minimal parabolic subgroup of E_7 containing H and thus the Witt–Tits index of H is I. By [GP, Proposition 3.19], the group $C_{E_7}(H)$ is of rank 1. Theorem 1.2 (b) now tells us that $\mathrm{ed}(E_7; 3) \geq 4 - 1 = 3$.

To prove the opposite inequality, consider $\alpha \in H^1(K, E_7)$. By [Ga, Example 3.5], the natural map

$$H^1(K, E_6 \rtimes \mu_4) \to H^1(K, E_7)$$

is surjective. Here, once again, E_6 stands for the simply connected group of type E_6. It follows that there exists a quartic extension K'/K such that $\alpha_{K'}$ admits reduction of structure to E_6 (i.e., lies in the image of the map $H^1(K', E_6) \to H^1(K', E_7)$). Thus we may assume without loss of generality that α comes from E_6. Now recall that the natural map $H^1(K, F_4 \rtimes \mu_3) \to H^1(K, E_6)$ is surjective (see [Ga, Example 3.5]); here $\mu_3 = C(E_6)$. Thus there exists a $\beta \in H^1(K, F_4 \rtimes \mu_3)$ mapping to α by the composite map

$$H^1(K, F_4 \rtimes \mu_3) \to H^1(K, E_6) \to H^1(K, E_7).$$

We claim that $\alpha \in H^1(K, E_7)$ admits further reduction of structure to F_4 (i.e., α depends only of the F_4-component of β). If this claim is established the desired inequality $\mathrm{ed}(\alpha; 3) \leq 3$ will immediately follow from Example 9.3.

To prove the claim we again view E_6 inside E_7 as $E_6 = DL_I$. We have $L_I = C_{E_7}(T_I) = DL_I.T_I$, where $T_I = \mathbb{G}_m$ is the standard torus associated to I and $C(L_I) = T_I.C(E_7)$. Since $C(E_7) = \mu_2$, it follows that $\mu_3 = C(E_6) \subset T_I \subset L_I$. We consider the commutative diagram of pairings

$$
\begin{array}{ccc}
H^1(K, \mu_3) \times H^1(K, E_6) & \longrightarrow & H^1(K, E_6) \\
\downarrow & \downarrow & \downarrow \\
H^1(K, T_I) \times H^1(K, L_I) & \longrightarrow & H^1(K, L_I).
\end{array}
$$

From the vanishing of $H^1(K, T_I)$, it follows that the map $H^1(K, E_6) \to H^1(K, L_I)$ is $H^1(K, \mu_3)$-invariant. A fortiori, the image of the map

$$H^1(K, F_4) \to H^1(K, E_6) \to H^1(K, L_I)$$

is $H^1(K, \mu_3)$-invariant. We conclude that the image of $\beta \in H^1(F_4 \times \mu_3)$ in $H^1(K, L_I)$ depends only of its F_4 component, as claimed. $\qquad\square$

9.7 Remark. One can show that if $G \to G'$ is a central isogeny of degree d, then $\mathrm{ed}(G; p) = \mathrm{ed}(G'; p)$ for any prime p not dividing d. In particular, the equalities $\mathrm{ed}(E_6; 2) = 3$ and $\mathrm{ed}(E_7; 3) = 3$ are valid for adjoint E_6 and E_7 as well.

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