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Diameter pinching in almost positive Ricci curvature

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Abstract. In this paper we prove a diameter sphere theorem and its corresponding λ_1 sphere theorem under L^p control of the curvature. They are generalizations of some results due to S. Ilias [8].

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1. Introduction

Let (M^n, g) be a complete manifold with Ricci curvature $\mathrm{Ric} \geq n-1$. Then (M^n, g) satisfies the following classical results (the proofs can be found in [13] for instance):

- $\mathrm{Diam}(M^n, g) \leq \pi$ (S. Myers) with equality iff $(M^n, g) = (\mathbb{S}^n, \mathrm{can})$ (S. Cheng),
- $\lambda_1(M^n, g) \geq n$ (A. Lichnerowicz) with equality iff $(M^n, g) = (\mathbb{S}^n, \mathrm{can})$ (M. Obata),

where Diam is the diameter and λ_1 is the first positive eigenvalue.

Studying the properties of the sphere kept by manifolds with $\mathrm{Ric} \geq n-1$ and almost extremal diameter or λ_1, S. Ilias proved in [8] the following results:

Theorem 1.1 (S. Ilias). *For any $A > 0$, there exists $\epsilon(A, n) > 0$ such that any n-manifold with* $\mathrm{Ric} \geq n-1$, *sectional curvature $\sigma \leq A$ and $\lambda_1 \leq n + \epsilon$ is homeomorphic to $\mathbb{S}^n$.*

Theorem 1.2 (S. Ilias). *For any $A > 0$, there exists $\epsilon(A, n) > 0$ such that any n-manifold with* $\mathrm{Ric} \geq n-1$, *$\sigma \leq A$ and* $\mathrm{Diam}(M) \geq \pi - \epsilon$ *is homeomorphic to $\mathbb{S}^n$.*

Remark 1.3. C. Croke proves in [7] that for n-manifolds with $\mathrm{Ric} \geq n-1$, $\lambda_1(M)$ close to n implies $\mathrm{Diam}(M)$ close to π. The converse is proved in [8] (using a spectral inequality due to S. Cheng [6]).

Remark 1.4. For $n \geq 4$, M. Anderson [1] and Y. Otsu [10] construct sequences of complete metrics g_i with $\mathrm{Ric}(g_i) \geq n-1$, $\lambda_1(g_i) \to n$ and $\mathrm{Diam}(g_i) \to \pi$ on manifolds that are not homotope to $\mathbb{S}^n$ (more precisely, Otsu shows that if $n \geq 5$, these manifolds can have infinitely many different fundamental groups).

Remark 1.5. The two results of S. Ilias have been improved by G. Perelman in [11], where the assumption $\sigma \leq A$ is replaced by $\sigma \geq -A$ (note that under the Ilias's assumptions $\sigma \leq A$ and $\mathrm{Ric} \geq n-1$ we have $|\sigma| \leq (n-2)A$).

Subsequently, we *denote* $\underline{\mathrm{Ric}}(x)$ *the lowest eigenvalue of the Ricci tensor and* $\bar{\sigma}(x)$ *the maximal sectional curvature at* x. In [4], we prove the following generalization of the theorems of Myers and Lichnerowicz:

Theorem 1.6. *For any $p > n/2$, there exists $C(p,n)$ such that if (M^n, g) is a complete manifold with $\int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p < \frac{\mathrm{Vol}\, M}{C(p,n)}$, then M is compact, has finite fundamental group and satisfies*

$$\mathrm{Diam}(M) \leq \pi\Big[1 + C(p,n)\Big(\frac{\rho_p}{\mathrm{Vol}\, M}\Big)^{\frac{1}{10}}\Big],$$

$$\lambda_1(M) \geq n\Big[1 - C(p,n)\Big(\frac{\rho_p}{\mathrm{Vol}\, M}\Big)^{\frac{1}{p}}\Big],$$

where $\rho_p = \int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p$ and $x_- = \max(0, -x)$.

Remark 1.7. It follows from [4] that the constant $C(p,n)$ is computable, that if $\int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p$ is finite (for $p > n/2$), then $\mathrm{Vol}\, M$ is finite, and that we cannot bound the diameter or the first non zero eigenvalue under the assumption $\rho_p \leq \frac{1}{C(p,n)}$ or $\rho_{\frac{n}{2}}$ small (see [4]).

In this paper we prove the following extensions of Ilias's stability results.

Theorem 1.8. *Let $n \geq 2$ be an integer, $A > 0$ and $p > n$ be some reals. There exists a positive constant $C(p,n,A)$ such that any complete n-manifold which satisfies*

$$\int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p < C(p,n,A)\,\mathrm{Vol}\, M, \qquad \int_M \bar{\sigma}_+^p < A\,\mathrm{Vol}\, M$$

and

$$\mathrm{Diam}(M) \geq \pi\big(1 - C(p,n,A)\big)$$

is homeomorphic to $\mathbb{S}^n$ (where $x_+ = \max(0,x)$).

Theorem 1.9. *Let $n \geq 2$ be an integer, $A > 0$ and $p > n$ be some reals. There exists a positive constant $C(p,n,A)$ such that any complete n-manifold which satisfies*

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and

$$\lambda_1(M) \le n\big(1 + C(p,n,A)\big)$$

is homeomorphic to $\mathbb{S}^n$.

Remark 1.10. By the Hölder inequality, the two curvature assumptions of Theorem 1.9 can be replaced by

$$\int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_- < C(p,n,A)\operatorname{Vol} M \quad \text{and} \quad \int_M \sigma^p < A \operatorname{Vol} M,$$

where $\sigma(x)$ is an upper bound for the absolute value of the sectional curvatures at x.

2. Comparison results in almost positive Ricci curvature

Subsequently we denote $B(x,r)$ (resp. $S(x,r)$) the geodesic ball (resp. sphere) with center x and radius r and $L_k(r)$ (resp. $A_k(r)$) the volume of a geodesic sphere (resp. ball) of radius r in $(\mathbb{S}^n, \frac{1}{k}g)$. Besides Theorem 1.6, we will need the following comparison results for manifolds of almost positive Ricci curvature (see [4] for a proof).

Proposition 2.1. *For any $n \ge 2$ and $p > n/2$ ($p \ge 1$ if $n = 2$) there exists a constant $C(p,n)$ such that for any complete Riemannian n-manifold (M^n, g) with $\eta^{10} = \frac{\rho_p}{\operatorname{Vol} M} \le \frac{1}{C(p,n)}$, we have*

$$\Big(\frac{\operatorname{Vol}_{n-1} S(x,R)}{L_{1-\eta}(R)}\Big)^{\frac{1}{2p-1}} - \Big(\frac{\operatorname{Vol}_{n-1} S(x,r)}{L_{1-\eta}(r)}\Big)^{\frac{1}{2p-1}} \le C(p,n)\eta^2 (R-r)^{\frac{2p-n}{2p-1}},$$

$$\frac{\operatorname{Vol} B(x,r)}{\operatorname{Vol} B(x,R)} \ge (1 - C(p,n)\eta)\frac{A_1(r)}{A_1(R)},$$

$$\operatorname{Vol}_{n-1} S(x,R) \le \big(1+\eta^2\big) L_{1-\eta}(R),$$

$$\operatorname{Vol} B(x,R) \le \big(1+\eta\big) A_1(R)$$

for all $x \in M$ and all radii $0 \le r \le R$.

For any $n \ge 2$ and $p > n/2$ there exists a constant $C(p,n)$ such that if (M^n, g) is a complete n-manifold with $\bar\rho_p \le \frac{1}{C(p,n)}$, then $\|u\|_{\frac{2n}{n-2}} \le \mathrm{Diam}(M) C(p,n)\|du\|_2 + \|u\|_2$, for any $u \in H^{1,2}(M)$. In the case $n = 2$, we have $\|u\|_4 \le \mathrm{Diam}(M) C\|du\|_2 + \|u\|_2$ if $\bar\rho_1 \le \frac{1}{C}$.

Similar estimates are proved in [12] under the assumption that M is compact and $\mathrm{Diam}(M)^{2p}\frac{\rho_p}{\operatorname{Vol} M} \le \frac{1}{C(p,n)}$.

3. Theorem 1.9 implies Theorem 1.8

Proposition 3.1. *Let $n \geq 2$ and $p > n/2$. There exists $C(p,n) > 0$ such that if (M^n, g) is a complete n-manifold with $\eta^{10} = \bar{\rho}_p \leq \frac{1}{C(p,n)}$ and $\operatorname{Diam}(M) \geq \pi - \frac{1}{C(p,n)}$, then we have*

$$\lambda_1(M) \leq n + C(p,n)\big[\eta + (\operatorname{Diam}(M) - \pi)_-\big].$$

The main tool to prove this proposition is the following lemma:

Lemma 3.2. *Let $n \geq 2$ and $p > n/2$ ($p \geq 1$ if $n = 2$) and $\bar{x}_0 \in \mathbb{S}^n$. There exists a constant $C(p,n)$ such that if (M^n, g) is a complete n-manifold with $\eta^{10} = \bar{\rho}_p \leq \frac{1}{C(p,n)}$, then there exists $x_0 \in M$ such that for any C^1-function $u\colon [0, 2\pi] \to \mathbb{R}$ we have*

$$\left|\frac{1}{\operatorname{Vol} M}\int_M u \circ d_M(x_0, .)\, dv_g - \frac{1}{\operatorname{Vol} \mathbb{S}^n}\int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x}_0, .)\, dv_{\mathbb{S}^n}\right|$$

$$\leq \|u'\|_\infty C(p,n)\big[\eta + \big(\operatorname{Diam}(M) - \pi\big)_-\big].$$

Proof. Let $(x_0, y_0) \in M^2$ such that $d = \operatorname{Diam}(M) = d(x_0, y_0)$. The functions A, L, A_1 and L_1 are defined in Proposition 2.1 and prolonged by 0 to $\mathbb{R}$ (note that the diameter of M can be greater than π). The function $r \to u(r)A(r)$ is continuous and has right differential on $\mathbb{R}$ equal to $u'A + uL$. We infer the equalities

$$\begin{aligned} u(d)\operatorname{Vol} M &= \int_0^d u(r)L(r)\,dr + \int_0^d u'(r)A(r)\,dr, \\ u(\pi)\operatorname{Vol} \mathbb{S}^n &= \int_0^\pi u(r)L_1(r)\,dr + \int_0^\pi u'(r)A_1(r)\,dr \end{aligned}$$

which imply

$$\begin{aligned} &\left|\frac{1}{\operatorname{Vol} M}\int_M u \circ d_M(x_0, x)\, dv_g - \frac{1}{\operatorname{Vol} \mathbb{S}^n}\int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x}_0, x)\, dv_{\mathbb{S}^n}\right| \\ &\quad = \left|\int_0^d \frac{u(r)L(r)}{\operatorname{Vol} M} dr - \int_0^\pi \frac{u(r)L_1(r)}{\operatorname{Vol} \mathbb{S}^n}\right| \\ &\quad = \left|u(d) - u(\pi) + \int_0^\pi \frac{u'A_1}{\operatorname{Vol} \mathbb{S}^n} - \int_0^d \frac{u'A}{\operatorname{Vol} M}\right| \\ &\quad = \left|\int_0^d u'\left(\frac{A_1}{\operatorname{Vol} \mathbb{S}^n} - \frac{A}{\operatorname{Vol} M}\right) + \int_d^\pi u'\left(\frac{A_1}{\operatorname{Vol} \mathbb{S}^n} - 1\right)\right| \\ &\quad \leq \|u'\|_\infty \left(\int_0^d \left|\frac{A_1}{\operatorname{Vol} \mathbb{S}^n} - \frac{A}{\operatorname{Vol} M}\right| dr + \big|\pi - d\big|\right). \end{aligned}$$

By Proposition 2.1 we have, for all $r \leq d$:

$$\begin{aligned}\big(1 - C(p,n)\eta\big)\frac{A_1(r)}{\operatorname{Vol}\mathbb{S}^n} \leq \frac{A(r)}{\operatorname{Vol} M} &\leq 1 - \frac{\operatorname{Vol} B(y_0, d-r)}{\operatorname{Vol} M}\\ &\leq 1 - \big(1 - C(p,n)\eta\big)\frac{A_1(d-r)}{\operatorname{Vol}\mathbb{S}^n}\\ &\leq \frac{A_1(r+\pi-d)}{\operatorname{Vol}\mathbb{S}^n} + C(p,n)\eta.\end{aligned}$$

Hence $\left|\frac{A(r)}{\operatorname{Vol} M} - \frac{A_1(r)}{\operatorname{Vol}\mathbb{S}^n}\right| \leq C(p,n)\eta + \frac{(A_1(r)-A_1(r+\pi-d))_-}{\operatorname{Vol}\mathbb{S}^n}$. An easy computation gives $\left\|\frac{(A_1(\cdot)-A_1(\cdot+h))_-}{\operatorname{Vol}\mathbb{S}^n}\right\|_\infty \leq C(n)(-h)_-$, and by Proposition 2.1 we get:

$$\begin{aligned}&\left|\frac{1}{\operatorname{Vol} M}\int_M u\circ d_M(x_0,x)\,dv_g - \frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n} u\circ d_{\mathbb{S}^n}(\bar{x}_0,x)\,dv_{\mathbb{S}^n}\right|\\ &\qquad \leq \|u'\|_\infty C(p,n)\big[\eta + (d-\pi)_-\big].\end{aligned}$$ □

We now finish the proof of Proposition 3.1.

Proof. Lemma 3.2 applied to $u = \sin^2$, $u = \cos^2$ and $u = \cos$ gives

$$\left|\int_M \frac{\sin^2 d_M(x_0,.)}{\operatorname{Vol} M} - \int_{\mathbb{S}^n}\frac{\sin^2 d_{\mathbb{S}^n}(\bar{x}_0,.)}{\operatorname{Vol}\mathbb{S}^n}\right| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$

$$\left|\int_M \frac{\cos^2 d_M(x_0,.)}{\operatorname{Vol} M} - \int_{\mathbb{S}^n}\frac{\cos^2 d_{\mathbb{S}^n}(\bar{x}_0,.)}{\operatorname{Vol}\mathbb{S}^n}\right| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$

$$\left|\int_M \frac{\cos d_M(x_0,.)}{\operatorname{Vol} M} - \int_{\mathbb{S}^n}\frac{\cos d_{\mathbb{S}^n}(\bar{x}_0,.)}{\operatorname{Vol}\mathbb{S}^n}\right| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1.$$

Hence, if we set $f = \cos d_M(x_0,.)$, we get

$$\left|\|\nabla f\|_2^2 - \frac{n}{n+1}\right| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$

$$\left|\|f\|_2^2 - \frac{1}{n+1}\right| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$

$$\left|\frac{1}{\operatorname{Vol} M}\int_M f\right| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$

which readily implies that

$$\lambda_1(M) \leq \frac{\|\nabla(f-\bar{f})\|_2^2}{\|f-\bar{f}\|_2} \leq n\big(1 + C(p,n)(\eta + (d-\pi)_-)\big),$$

where we have set $\bar{f} = \frac{1}{\operatorname{Vol} M}\int_M f$. □

Remark 3.3. The same technique as in [5] can be used to prove that manifolds with almost positive Ricci curvature and λ_1 is close to n have a diameter close to π (see [12]).

4. Proof of Theorem 1.9

4.1. Fiber bundle E. Let E be the fiber bundle $TM \oplus \mathbb{R}\, e \to M$ endowed with the following scalar product and linear connection:

$$\langle X + fe, Y + he\rangle_E = g(X,Y) + fh,$$
$$D^E_Z(X + fe) = D^M_Z X + fZ + \big(df(Z) - g(Z,X)\big)\cdot e,$$

where D^M is the Levi-Civita connection of the metric g on M. *We denote by p the orthogonal projection of E on TM,* $\mathrm{Ric}'(S) = \mathrm{Ric}_M\big(p(S)\big) - (n-1)p(S)$ *and* $\triangle_{\mathrm{sph}} = \bar{\triangle}^E + \mathrm{Ric}'$.

The following lemma is proved in [3]:

Lemma 4.1. *If $f: M \to \mathbb{R}$ satisfies $\triangle f = \lambda f$, then $S_f = \nabla f + fe$ satisfies $\triangle_{\mathrm{sph}}(S_f) = (\lambda - n)(\nabla f - fe)$ and $\langle D^E_X S_f, X\rangle = Ddf(X,X) + fg(X,X)$.*

Note also that we have

$$\mathrm{R}^E_{(Z,Y)}(X + fe) = \mathrm{R}^M(Z,Y)X - \big(g(Y,X)Z - g(Z,X)Y\big).$$

4.2. Bound on the Hessian of the first eigenfunction. To prove Theorem 1.9 we need an L^∞ bound on the Hessian of the first eigenfunction. For that purpose, we will modify the proof of Theorem 2.4 in [2] (whose proof would give us only a bound on $\|DS_f\|_{n+\epsilon}/\|S_f\|_\infty$ for a given $\epsilon = \epsilon(p,n)$). In our case we really need to perform a Moser iteration.

Proposition 4.2. *Let $n \geq 2$ and $\infty \geq p > n/2$. There exists a constant $C(p,n)$ such that if (M^n, g) is any manifold with $\bar{\rho}_p \leq \frac{1}{C(p,n)}$ and $\lambda_1 \leq n + \frac{1}{C(p,n)}$, then for $f: M \to \mathrm{R}$ such that $\triangle f = \lambda_1 f$ we have*

$$\frac{\|D^E S_f\|_\infty}{\|S_f\|_\infty} \leq C(p,n)\big(\lambda_1 + \|\mathrm{R}\|_{2p}\big)^\gamma \big(|\lambda_1 - n| + \bar{\rho}_p\big)^{\frac{1}{2(1+\gamma)}},$$

where $S_f = \nabla f + f\cdot e$ and $\gamma = \frac{pn}{2p-n}$.

To prove Proposition 4.2 we need a commutation lemma (see [2]):

Lemma 4.3. *For any section $S \in \Gamma(E)$ we have*

$$\begin{aligned}&\frac{1}{2}\triangle\left(|DS|^2\right) + |D^2S|^2 \\ &\qquad \leq \langle D^*\mathrm{R}^E S, DS\rangle + \underline{\mathrm{Ric}}^-|DS|^2 + \langle D\bar{\triangle}S, DS\rangle + \|\mathrm{R}^E\|\cdot|DS|^2,\end{aligned}$$

where $\|\mathrm{R}^E\|$ is the norm of the linear map $\mathrm{R}^E\colon \bigwedge^2 T_mM \to \bigwedge^2 E_m^*$ defined by $\mathrm{R}^E(u\wedge v)(T,S) = \langle \mathrm{R}^E(u,v)T,S\rangle$.

Remark 4.4. This lemma is valid for any Riemannian fiber bundle $(E, D, \langle\cdot,\cdot\rangle)$.

We now give the proof of Proposition 4.2.

Proof of Proposition 4.2. We set $u = \sqrt{|DS|^2+\epsilon^2}$. We have

$$\begin{aligned}u\triangle u &= \frac{1}{2}\triangle(u^2) + |du|^2 = \frac{1}{2}\triangle\big(u^2\big) + \frac{|\langle D^2S, DS\rangle|^2}{|DS|^2+\epsilon^2} \\ &\leq \frac{1}{2}\triangle\big(|DS|^2\big) + |D^2S|^2.\end{aligned}$$

Hence, by Lemma 4.3

$$\begin{aligned}\int_M |d(u^k)|^2 &\leq \frac{k^2}{2k-1}\int_M\Big(\frac{1}{2}\triangle|DS|^2 + |D^2S|^2\Big)u^{2(k-1)} \\ &\leq \frac{k^2}{2k-1}\bigg(\int_M \underline{\mathrm{Ric}}^- u^{2k} + \int_M \langle D\bar{\triangle}S, DS\rangle u^{2(k-1)} \\ &\qquad + \int_M \langle D^*\mathrm{R}^E S, DS\rangle u^{2(k-1)} + \int_M \|\mathrm{R}^E\| u^{2k}\bigg).\end{aligned}$$

We now apply the divergence theorem to the form $u^{2(k-1)}\langle\bar{\triangle}S, D_\bullet S\rangle$, and get for any $k\geq 1$:

$$\begin{aligned}&\int_M \langle D\bar{\triangle}S, DS\rangle u^{2(k-1)} \\ &\quad = \int_M |\bar{\triangle}S|^2u^{2(k-1)} - 2(k-1)\sum_i\int_M\langle\bar{\triangle}S, DS(i)\rangle du(i)\cdot u^{2k-3} \\ &\quad \leq \int_M |\bar{\triangle}S|^2u^{2(k-1)} + 2(k-1)\int_M|\bar{\triangle}S||du|u^{2(k-1)} \\ &\quad \leq \frac{k-1}{2}\int_M|du|^2u^{2(k-1)} + (2k-1)\int_M|\bar{\triangle}S|^2u^{2(k-1)}.\end{aligned}$$

We do the same with the form $u^{2(k-1)}(tr_{1,3}(\langle \mathrm{R}^E_{(\bullet,\bullet)} S, D_\bullet S\rangle))$ and get

$$\begin{aligned}\int_M \langle D^*\mathrm{R}^E S, DS\rangle u^{2(k-1)} &= \int_M \frac{1}{2}|\mathrm{R}^E S|^2 u^{2(k-1)} + 2(k-1)\sum_{i,j}\int_M \langle \mathrm{R}^E(i,j)S, D_j S\rangle du(i) u^{2k-3}\\ &\le \frac{k-1}{2}\int_M |du|^2 u^{2(k-1)} + (2k-1)\int_M |\mathrm{R}^E S|^2 u^{2(k-1)},\end{aligned}$$

where we have used $\sum_{i,j}\langle \mathrm{R}^E S(i,j), D^2 S(i,j)\rangle = \frac{1}{2}|\mathrm{R}^E S|^2$.

Since $\int_M |du|^2 u^{2(k-1)} = \frac{1}{k^2}\int_M |d(u^k)|^2$, the three last inequalities give, for any $k \ge 1$,

$$\begin{aligned}\|d(u^k)\|_2^2 &\le k\Big(\int_M \underline{\mathrm{Ric}}^- u^{2k} + \int_M \|\mathrm{R}^E\| u^{2k}\Big)\\ &\quad + k(2k-1)\Big(\int_M \|\mathrm{R}^E S\|^2 u^{2k-2} + \int_M \|\bar{\triangle} S\|^2 u^{2k-2}\Big)\\ &\le 4k^2\Big(B_1 \|u\|^{2k}_{\frac{2kp}{p-1}} + B_2\|S\|^2_\infty \|u\|^{2(k-1)}_{\frac{2(k-1)p}{p-1}}\Big),\end{aligned}$$

where we have set

$$\begin{aligned}B_1 &= \|\underline{\mathrm{Ric}}^-\|_p + \|\mathrm{R}^E\|_p \le C(n)\big(\|\mathrm{R}^M\|^2_{2p} + \lambda_1^2\big) = B^2,\\ B_2 &= \frac{\|\bar{\triangle} S\|^2_{2p}}{\|S\|^2_\infty} + \frac{\|\mathrm{R}^E S\|^2_{2p}}{\|S\|^2_\infty} \le \frac{\|\triangle_{\mathrm{sph}} S\|^2_{2p}}{\|S\|^2_\infty} + \|\,\mathrm{Ric}'\,\|^2_{2p} + \|\mathrm{R}^E\|^2_{2p}\\ &\le C(n)\big(\lambda_1^2 + \|\mathrm{R}^M\|^2_{2p}\big) = B^2.\end{aligned}$$

By the Sobolev inequality given by Proposition 2.1 we get

$$\|DS\|^k_{\frac{2kn}{n-2}} \le \|DS\|^k_{2k} + C(p,n)Bk\sqrt{\|DS\|^{2k}_{\frac{2kp}{p-1}} + \|S\|^2_\infty \|DS\|^{2(k-1)}_{\frac{2(k-1)p}{p-1}}},$$

and by $\|DS\|_{2k} \le \|DS\|_{\frac{2kp}{p-1}} \le \|DS\|^{1/k}_\infty \|DS\|^{(1-1/k)}_{\frac{2(k-1)p}{p-1}}$ we have

$$\begin{aligned}&\left(\frac{\|DS\|_{\frac{2kn}{n-2}}}{\|DS\|_\infty}\right)^{\frac{2kn}{n-2}}\\ &\quad\le \left[1 + BkC(p,n)\Big(1 + \frac{\|S\|^2_\infty}{\|DS\|^2_\infty}\Big)\right]^{\frac{2n}{n-2}} \left(\frac{\|DS\|_{\frac{2(k-1)p}{p-1}}}{\|DS\|_\infty}\right)^{\nu\frac{2(k-1)p}{p-1}},\end{aligned}$$

where $\nu = \frac{n(p-1)}{2p(n-2)} > 1$. We set $k = \frac{a_n(p-1)}{2p} + 1$, where $(a_n)_n$ is the sequence defined by $a_0 = \frac{2p}{p-1}$ and $a_{n+1} = \nu a_n + \frac{2n}{n-2}$. Then we get

$$\left(\frac{\|DS\|_{a_{n+1}}}{\|DS\|_\infty}\right)^{\frac{a_{n+1}}{\nu^{n+1}}} \leq \left[1 + a_n C(p,n) B\left(1 + \frac{\|S\|_\infty^2}{\|DS\|_\infty^2}\right)\right]^{\frac{2n}{(n-2)\nu^{n+1}}} \left(\frac{\|DS\|_{a_n}}{\|DS\|_\infty}\right)^{\frac{a_n}{\nu^n}}.$$

Hence

$$1 = \lim_{n\to+\infty}\left(\frac{\|DS\|_{a_n}}{\|DS\|_\infty}\right)^{\frac{a_n}{\nu^n}} \leq \prod_{i=1}^{\infty}\left(1 + C(p,n)a_i B\left(1 + \frac{\|S\|_\infty^2}{\|DS\|_\infty^2}\right)\right)^{\frac{2n}{(n-2)\nu^i}} \left(\frac{\|DS\|_{a_0}}{\|DS\|_\infty}\right)^{a_0}.$$

The Hölder inequality $\|DS\|_{a_0} \leq \|DS\|_2^{1-\frac{1}{p}} \|DS\|_\infty^{\frac{1}{p}}$, gives

$$\|DS\|_\infty \leq \prod_{i=1}^{\infty}\left(1 + C(p,n)a_i B\left(1 + \frac{\|S\|_\infty^2}{\|DS\|_\infty^2}\right)\right)^{\frac{n}{(n-2)\nu^i}} \|DS\|_2. \qquad (*)$$

If $\|DS\|_\infty \geq \|S\|_\infty$, then inequality $(*)$ gives

$$\|DS\|_\infty \leq \prod_{i=1}^{\infty}(1 + C(p,n)a_i B)^{\frac{n}{(n-2)\nu^i}} \|DS\|_2 \leq C(p,n)\big(\lambda_1 + \|\mathrm{R}\|_{2p}\big)^{\frac{pn}{2p-n}} \|DS\|_2.$$

If $\|DS\|_\infty \leq \|S\|_\infty$, then inequality $(*)$ gives

$$\frac{\|DS\|_\infty}{\|DS\|_2} \leq \left(\frac{\|S\|_\infty}{\|DS\|_\infty}\right)^{\frac{2pn}{2p-n}} \prod_{i=1}^{\infty}(1 + C(p,n)a_i B)^{\frac{n}{(n-2)\nu^i}},$$

hence

$$\frac{\|DS\|_\infty}{\|S\|_\infty} \leq C(p,n)(\lambda_1 + \|\mathrm{R}\|_{2p})^{\frac{pn}{2p-n)}} \left(\frac{\|DS\|_2}{\|S\|_\infty}\right)^{\frac{2p-n}{2p-n+2pn}}.$$

At this stage note that, by Lemma 4.1 we have

$$\|DS\|_2^2 = \langle\overline{\triangle}_{\mathrm{sph}}(S), S\rangle_{L^2} - \langle\mathrm{Ric}'(S), S\rangle_{L^2} \leq |\lambda_1 - n|\,\|S\|_2^2 + \int_M \frac{\big(\underline{\mathrm{Ric}} - (n-1)\big)^-}{\mathrm{Vol}\,M}|S|^2 \leq \big(|\lambda_1 - n| + \bar{\rho}_p\big)\|S\|_\infty^2.$$

Since we have $\frac{2p-n}{2p-n+2pn} \leq 1$, we get the result. □

4.3. Critical points of the first eigenfunction. By Proposition 4.2, the section $S_f = \nabla f + fe$ of E satisfies $\|D^E S_f\|_\infty \leq C(p,n,A)\big(|\lambda_1 - n| + \bar{\rho}_p\big)^{\frac{1}{1+\gamma}} \|S_f\|_\infty$. Since we can suppose the pinching on $|\lambda_1 - n|$ and $\bar{\rho}_p$ small enough to have

$$C(p,n,A)\big(|\lambda_1 - n| + \bar{\rho}_p\big)^{\frac{1}{1+\gamma}} \leq 1/4,$$

the previous inequality and Theorem 1.6 give

$$\begin{aligned}\inf |S_f| &\geq \big[1 - C(p,n,A)\big(|\lambda_1 - n| + \bar{\rho}_p\big)^{\frac{1}{1+\gamma}}\big]\|S_f\|_\infty \\ &> C(p,n,A)\big(|\lambda_1 - n| + \bar{\rho}_p\big)^{\frac{1}{1+\gamma}} \|S_f\|_\infty \geq \|D^E S_f\|_\infty.\end{aligned}$$

We infer that if x_0 is a critical point of f, then by Lemma 4.1 we have

$$|Ddf_{x_0}(X,X) + f(x_0)| = |\langle D^E_X S_f, X\rangle_E| \leq \|D^E S_f\|_\infty < |S_f(x_0)| = |f(x_0)|$$

for any unit vector X of $T_{x_0}M$. Hence we have $-|f(x_0)| - f(x_0) < Ddf_{x_0}(X,X) < |f(x_0)| - f(x_0)$ for any critical point x_0 of f. So the only critical points of f are non degenerate global extrema, which implies that M is homeomorphic to $\mathbb{S}^n$ by Reeb's theorem.

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