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Group splittings and integrality of traces

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Abstract. In this paper, we elaborate on Connes' proof of the integrality of the trace conjecture for free groups, in order to show that any action of a group G on a tree leads to a similar integrality assertion concerning the trace on the group algebra $\mathbb{C}G$, which is associated with the set of group elements that stabilize a vertex.

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Introduction

Given a torsion-free group G, the integrality of the trace conjecture is the assertion that the image of the additive map

$$\tau_* \colon K_0(C_r^* G) \longrightarrow \mathbb{C},$$

which is induced by the canonical trace τ on the reduced C^*-algebra $C_r^* G$ of G, is the group $\mathbb{Z}$ of integers. Some evidence for the validity of that conjecture is provided by Zalesskii's theorem [18], which states that for any group G (possibly with torsion) the values of τ_* on K-theory classes that come from the group algebra $\mathbb{C}G$ are rational. By a standard argument, the integrality of the trace conjecture can be shown to imply the triviality of idempotents in $C_r^* G$. In the case where G is a free group, Connes proved in [5, §IV.5] the integrality of the trace conjecture by using a free action of G on a tree and the associated representations of $C_r^* G$ on the Hilbert space $\ell^2 V$, where V is the set of vertices of the tree. We also note that, in the case of a torsion-free abelian group G, the integrality of the trace conjecture is an immediate consequence of the connectedness of the dual group $\widehat{G}$ (cf. [16, Theorem 2]).

In the case where the group G has non-trivial torsion elements, Baum and Connes had conjectured in [3] that the image of τ_* is the subgroup of $\mathbb{Q}$ generated by the inverses of the orders of the finite subgroups of G. This latter conjecture was disproved by Roy [14]. Subsequently, Lück [10] formulated a modified version of that

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conjecture, according to which the image of τ_* is contained in the subring of $\mathbb{Q}$ generated by the inverses of the orders of the finite subgroups of G, and showed that this is indeed the case if the so-called Baum–Connes assembly map is surjective.

In this paper, we are interested in traces defined on the group algebra $\mathbb{C}G$ and examine integrality properties of the induced additive maps on the K-theory group $K_0(\mathbb{C}G)$. If $S \subseteq G$ is a subset closed under conjugation, then the linear map (partial augmentation)

$$\tau_S \colon \mathbb{C}G \longrightarrow \mathbb{C},$$

which is defined by letting $\tau_S\big(\sum_{g\in G} a_g g\big) = \sum_{g\in S} a_g$ for any element $\sum_{g\in G} a_g g \in \mathbb{C}G$, is a trace. As such, it induces an additive map

$$(\tau_S)_* \colon K_0(\mathbb{C}G) \longrightarrow \mathbb{C}.$$

In the special case where $S = G$, the additive map $(\tau_S)_*$ is that induced by the augmentation homomorphism $\mathbb{C}G \to \mathbb{C}$. The map $(\tau_S)_*$ is then referred to as the homological (or naive) rank and its image is the group $\mathbb{Z}$ of integers (cf. [4, Chapter IX, Exercise 2.5]). On the other hand, if $S = \{1\}$, then the map $(\tau_S)_*$ is the Kaplansky rank, whose values are rational in view of Zaleskii's theorem [18]. In fact, if G is torsion-free then a weak version of Bass' trace conjecture [2] asserts that the Kaplansky rank coincides with the homological rank; if this is true, then we must have $\mathrm{im}(\tau_S)_* = \mathbb{Z}$ in this case as well.

We can now state our main result.

Theorem. *Let G be the fundamental group of a connected graph of groups with vertex groups $(G_v)_v$ and edge groups $(G_e)_e$, and consider the subset $S \subseteq G$ which consists of the conjugates of all elements of the set $\bigcup_v G_v$. Then,* $\mathrm{im}(\tau_S)_* = \mathbb{Z}$.

Equivalently, we may state the result above in terms of the universal trace defined by Hattori and Stallings, as follows: If $x \in K_0(\mathbb{C}G)$ is a K-theory class and $r_{[g]}(x) \in \mathbb{C}$ the coefficient of the Hattori–Stallings rank $r_{\mathrm{HS}}(x)$ that corresponds to the conjugacy class $[g]$ of an element $g \in G$, then $\sum_{[g]\in[S]} r_{[g]}(x) \in \mathbb{Z}$, where $[S]$ is the set of conjugacy classes of the elements of S.

We observe that our integrality result would follow immediately if one could show that the additive map

$$\bigoplus_v K_0(\mathbb{C}G_v) \longrightarrow K_0(\mathbb{C}G), \tag{1}$$

which is induced by the inclusion of the vertex groups G_v into G, is surjective. Indeed, for any vertex v the composition

$$K_0(\mathbb{C}G_v) \longrightarrow K_0(\mathbb{C}G) \xrightarrow{(\tau_S)_*} \mathbb{C} \tag{2}$$

is the additive map induced by the restriction of the trace τ_S on $\mathbb{C}G_v$ (cf. Remark 1.1 (ii) below). But S contains G_v and hence the restriction of τ_s on $\mathbb{C}G_v$

is the augmentation homomorphism $\mathbb{C}G_v \to \mathbb{C}$. Therefore, the composition (2) is the homological rank associated with G_v; in particular, its image is the group $\mathbb{Z}$ of integers. In view of the assumed surjectivity of (1), we conclude that the image of $(\tau_S)_*$ is the group $\mathbb{Z}$ as well.

We note that if G is the fundamental group of a graph of groups as above, then any finite subgroup $H \subseteq G$ is contained in a conjugate of G_v for some vertex v of the graph (cf. [15, Chapitre I, Exemple 6.3.1]). Since conjugation by any element of G induces the identity map on $K_0(\mathbb{C}G)$, we conclude that the map (1) is surjective if this is the case for the additive map

$$\bigoplus_H K_0(\mathbb{C}H) \longrightarrow K_0(\mathbb{C}G).$$

Here, the direct sum is over all finite subgroups H of G and the map is induced by the inclusion of the H's into G. In particular, it follows that the map (1) is surjective if the so-called isomorphism conjecture for $K_0(\mathbb{C}G)$ holds (cf. [11, Conjecture 9.40]). On the other hand, we may consider the special case where the graph has one edge e and two distinct vertices v_1 and v_2. In that case, G is the amalgamated free product $G_{v_1} \star_{G_e} G_{v_2}$ and a sufficient condition that guarantees the surjectivity of (1) has been given by Waldhausen (see the discussion following [17, Corollary 11.5]).

From the point of view presented above, there is a formal resemblance between our result and those described in [1] and [12], where it is proved that a certain statement is true for G if it is true for all G_v's.

We also observe that our integrality result would follow if one could show that the group G satisfies (the strong version of) Bass' trace conjecture [2]. Indeed, the latter conjecture asserts that for any $x \in K_0(\mathbb{C}G)$ the coefficient $r_{[g]}(x)$ of the Hattori–Stallings rank $r_{\mathrm{HS}}(x)$ vanishes if $g \in G$ is an element of infinite order. Since S contains all group elements of finite order (as we have already noted above), this would imply that $(\tau_S)_*(x)$ is the homological rank of x; in particular, it would follow that $(\tau_S)_*(x) \in \mathbb{Z}$.

The contents of the paper are as follows: In Section 1, we recall certain well-known facts concerning traces on an algebra and the induced additive maps on the K_0-group. In the following section, we consider a group G acting on a tree and examine certain representations of the group algebra $\mathbb{C}G$. In Section 3, we prove our main result and explicit the integrality assertions in the special cases where G is an amalgamated free product or an HNN extension. Finally, in the last section, we examine whether our integrality result can be extended from the group algebra $\mathbb{C}G$ to the reduced group C^*-algebra C_r^*G of G.

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1. Traces and the K_0-group

Let R be a unital ring, V an abelian group and $\tau\colon R \to V$ a trace, i.e. an additive map which vanishes on the commutators $xy - yx$ for all $x, y \in R$. Then, for any positive integer n the map

$$\tau_n : \mathrm{M}_n(R) \longrightarrow V,$$

which is defined by letting $\tau_n(A) = \sum_{i=1}^{n} \tau(a_{ii})$ for any matrix $A = (a_{ij})_{i,j} \in \mathrm{M}_n(R)$, is a trace as well. These traces induce an additive map

$$\tau_*\colon K_0(R) \longrightarrow V,$$

by mapping the K-theory class of any idempotent matrix $E \in \mathrm{M}_n(R)$ onto $\tau_n(E)$.

Remarks 1.1. (i) Let R be a ring and $f : V \to V'$ an abelian group homomorphism. We consider a trace $\tau\colon R \to V$ and the V'-valued trace $f \circ \tau$ on R. Then, the induced additive map $(f \circ \tau)_*\colon K_0(R) \to V'$ is the composition

$$K_0(R) \xrightarrow{\tau_*} V \xrightarrow{f} V',$$

where $\tau_*\colon K_0(R) \to V$ is the additive map induced by the trace τ.

(ii) Let $\varphi\colon R \to S$ be a ring homomorphism and V an abelian group. We consider a trace $\tau\colon S \to V$ and the V-valued trace $\tau \circ \varphi$ on R. Then, the induced additive map $(\tau \circ \varphi)_*\colon K_0(R) \to V$ is the composition

$$K_0(R) \xrightarrow{K_0(\varphi)} K_0(S) \xrightarrow{\tau_*} V,$$

where $\tau_*\colon K_0(S) \to V$ is the additive map induced by the trace τ.

(iii) Let R be a ring and $[R, R]$ the additive subgroup of it generated by the commutators $xy - yx$, $x, y \in R$. Then, the quotient map $p\colon R \to R/[R, R]$ is the universal trace defined on R and induces the Hattori–Stallings rank map

$$r_{\mathrm{HS}}\colon K_0(R) \longrightarrow R/[R, R];$$

see also [4, Chapter IX, §2].

(iv) In the special case where $R = \mathbb{C}G$ is the group algebra of a group G, the quotient group $R/[R, R] = \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G]$ is a complex vector space with basis the set $\mathfrak{C}(G)$ of conjugacy classes of elements of G. If $[g] \in \mathfrak{C}(G)$ is the conjugacy class of an element $g \in G$, then the linear functional (partial augmentation) $\sum_{h\in G} a_h h \mapsto \sum_{h\in[g]} a_h$, $\sum_{h\in G} a_h h \in \mathbb{C}G$, is a trace and hence induces an additive map

$$r_{[g]}\colon K_0(\mathbb{C}G) \longrightarrow \mathbb{C}.$$

These maps determine the Hattori–Stallings rank of any K-theory class $x \in K_0(\mathbb{C}G)$, since we have $r_{\mathrm{HS}}(x) = \sum_{[g]\in\mathfrak{C}(G)} r_{[g]}(x)[g] \in \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G]$.

We now consider a non-unital ring I and let I^+ be the associated unital ring. Here, $I^+ = I \oplus \mathbb{Z}$ as an abelian group, whereas the product of any two elements $(x,n),(y,m) \in I^+$ is equal to $(xy + ny + mx, nm) \in I^+$. The K_0-group of I is defined by means of the split extension

$$0 \longrightarrow I \longrightarrow I^+ \xrightarrow{\pi} \mathbb{Z} \longrightarrow 0,$$

where π is the projection $(x,n) \mapsto n$, $(x,n) \in I^+$. More precisely, $K_0(I)$ is the kernel of the induced additive map $K_0(\pi)\colon K_0(I^+) \to K_0(\mathbb{Z})$. Let $\tau\colon I \to V$ be a V-valued trace on I; by this, we mean that τ is an additive map which vanishes on the commutators $xy - yx$ for all $x, y \in I$. Then, τ extends to an additive map τ^+ on the associated unital ring I^+, by letting $\tau^+(0,1) = 0$; in fact, τ^+ is a trace. The induced additive map $\tau_*\colon K_0(I) \to V$ is defined as the restriction of $(\tau^+)_*\colon K_0(I^+) \to V$ to the subgroup $K_0(I) \subseteq K_0(I^+)$.

Example 1.2. Let U be a complex vector space with basis $(\xi_i)_i$ and $L(U)$ the algebra of linear endomorphisms of U. We consider the ideal $\mathfrak{F} \subseteq L(U)$ consisting of those endomorphisms of U that have finite rank. Then, for any $a \in \mathfrak{F}$ the family of complex numbers $([a(\xi_i), \xi_i^*])_i$ has finite support. Here, we denote for all i by ξ_i^* the linear functional on U which maps ξ_i onto 1 and vanishes on ξ_j for $j \neq i$, whereas $[\ ,\ ]$ denotes the standard pairing between U and its dual. Moreover, the map

$$\mathrm{Tr}\colon \mathfrak{F} \longrightarrow \mathbb{C},$$

which is defined by letting $\mathrm{Tr}(a) = \sum_i [a(\xi_i), \xi_i^*]$ for all $a \in \mathfrak{F}$, does not depend upon the choice of the basis $(\xi_i)_i$ and vanishes on the elements of the form $ab - ba$, $a \in \mathfrak{F}$, $b \in L(U)$. In particular, Tr is a trace on $\mathfrak{F}$. In view of the Morita invariance and the continuity of the functor K_0 (cf. [13, Chapter 1, §2]), the induced additive map

$$\mathrm{Tr}_*\colon K_0(\mathfrak{F}) \longrightarrow \mathbb{C}$$

identifies $K_0(\mathfrak{F})$ with the subgroup $\mathbb{Z} \subseteq \mathbb{C}$. More generally, let R be a complex algebra and consider a linear trace τ on R with values in a complex vector space V. Then, the linear map

$$\mathrm{Tr} \otimes \tau\colon \mathfrak{F} \otimes R \longrightarrow \mathbb{C} \otimes V \simeq V,$$

which is defined by letting $a \otimes x \mapsto \mathrm{Tr}(a)\tau(x)$ for all elementary tensors $a \otimes x \in \mathfrak{F} \otimes R$, is also a trace. Moreover, the induced additive map

$$(\mathrm{Tr} \otimes \tau)_*\colon K_0(\mathfrak{F} \otimes R) \longrightarrow V$$

is identified with the additive map

$$\tau_*\colon K_0(R) \longrightarrow V,$$

which is induced by τ, in view of the Morita isomorphism $K_0(\mathfrak{F} \otimes R) \simeq K_0(R)$.

A proof of the following result may be found in [9, Proposition 1.44].

Proposition 1.3. *Let $\varphi, \psi\colon A \to B$ be two homomorphisms of non-unital rings and $I \subseteq B$ an ideal such that $\psi(a) - \varphi(a) \in I$ for all $a \in A$. We consider an abelian group V and an additive map $\tau\colon I \to V$ that vanishes on elements of the form $xy - yx$ for all $x \in I$ and $y \in B$; in particular, τ is a trace on I. Let $t\colon A \to V$ be the additive map which is defined by letting $t(a) = \tau(\psi(a) - \varphi(a))$ for all $a \in A$. Then:*

(i) *The map t is a trace on A.*

(ii) *The image of the additive map $t_*\colon K_0(A) \to V$ is contained in the image of the additive map $\tau_*\colon K_0(I) \to V$.* □

2. Trees and group actions

Let X be a graph and denote by V, E^{or} the set of vertices and oriented edges of it respectively. A path on X is a finite sequence $(e_1, \dots, e_n)$ of oriented edges such that the terminus v_i of e_i is the origin of e_{i+1} for all $i = 1, \dots, n-1$. We say that a path as above has origin v_0 equal to the origin of e_1, terminus v_n equal to the terminus of e_n and passes through the vertices $v_1, \dots, v_{n-1}$. The path is reduced if there is no i such that e_{i+1} is equal to the reverse edge of e_i. The graph X is a tree if for any two vertices $v, v' \in V$ with $v \neq v'$ there is a unique reduced path with origin v and terminus v'; this path, denoted by $[v, v']$, is called the geodesic joining v and v'.

Let X be a tree and denote by E the corresponding set of un-oriented edges. It is well known that the number of vertices exceeds the number of un-oriented edges by one. More precisely, having fixed a vertex $v_0 \in V$, we consider for any $v \in V \setminus \{v_0\}$ the geodesic $[v_0, v] = (e_1, \dots, e_n)$ and define the map

$$\lambda\colon V \setminus \{v_0\} \longrightarrow E,$$

by letting $\lambda(v)$ be the un-oriented edge associated with the oriented edge e_n. The proof of the next result is straightforward.

Lemma 2.1. *Let X be a tree and fix a vertex $v_0 \in V$.*

(i) *The map λ defined above is bijective.*

(ii) *For another vertex $v_0' \in V$ consider the corresponding map $\lambda'\colon V \setminus \{v_0'\} \to E$. If the geodesic $[v_0, v_0']$ passes through the vertices $v_1, \dots, v_{n-1}$, then we have $\lambda(v) = \lambda'(v)$ for all vertices $v \in V \setminus \{v_0, v_1, \dots, v_{n-1}, v_0'\}$.* □

Let α be an automorphism of the tree X and denote by α_V (resp. α_E) the corresponding bijection of the set of vertices (resp. edges) of X. We fix a vertex $v_1 \in V$

and consider the associated bijection $\lambda_1 \colon V \setminus \{v_1\} \to E$. We also consider the vertex $v_2 = \alpha_V(v_1) \in V$ and the associated bijection $\lambda_2 \colon V \setminus \{v_2\} \to E$. Then, it is easily seen that

$$\alpha_E \circ \lambda_1 = \lambda_2 \circ \alpha'_V, \tag{3}$$

where α'_V denotes the restriction of α_V to the subset $V \setminus \{v_1\} \subseteq V$. The automorphism α is said to have no inversions if there is no edge $e \in E^{\mathrm{or}}$ such that $\alpha(e)$ is the reverse edge of e.

Proposition 2.2. *Let X be a tree, $v_0 \in V$ a vertex and $\lambda \colon V \setminus \{v_0\} \to E$ the associated bijection. We consider a group G acting on X and fix an element $g \in G$.*

(i) *If $g \cdot v_0 = v_0$, then we have $g \cdot \lambda(v) = \lambda(g \cdot v)$ for all $v \in V \setminus \{v_0\}$.*

(ii) *If $g \cdot v_0 \neq v_0$ and the geodesic $[v_0, g^{-1} \cdot v_0]$ passes through the vertices $v_1, \dots, v_{n-1}$, then $g \cdot \lambda(v) = \lambda(g \cdot v)$ for all $v \in V \setminus \{v_0, v_1, \dots, v_{n-1}, g^{-1} \cdot v_0\}$.*

Proof. We consider the vertex $g^{-1} \cdot v_0$ and let

$$\lambda' \colon V \setminus \{g^{-1} \cdot v_0\} \longrightarrow E$$

be the associated bijection. The element $g \in G$ induces an automorphism of the tree X, which maps the vertex $g^{-1} \cdot v_0$ onto v_0, and hence Equation (3) above implies that $g \cdot \lambda'(v) = \lambda(g \cdot v)$ for all $v \in V \setminus \{g^{-1} \cdot v_0\}$. This completes the proof in the case where $g \cdot v_0 = v_0$, since we then have $\lambda = \lambda'$. If $g \cdot v_0 \neq v_0$, the proof is finished by invoking Lemma 2.1 (ii), which implies that $\lambda(v) = \lambda'(v)$ for all vertices $v \in V \setminus \{v_0, v_1, \dots, v_{n-1}, g^{-1} \cdot v_0\}$. □

Let X be a graph and consider the sets V, E of vertices and (un-oriented) edges of X respectively and the complex vector spaces $\mathbb{C}^{(V)} = \bigoplus_{v \in V} \mathbb{C} \cdot \xi_v$ and $\mathbb{C}^{(E)} = \bigoplus_{e \in E} \mathbb{C} \cdot \xi_e$. If G is a group acting on X, then for any element $g \in G$ we denote by

$$\varrho_V(g) \colon \mathbb{C}^{(V)} \longrightarrow \mathbb{C}^{(V)} \quad \text{and} \quad \varrho_E(g) \colon \mathbb{C}^{(E)} \longrightarrow \mathbb{C}^{(E)}$$

the linear maps which are defined by letting $\xi_v \mapsto \xi_{g \cdot v}$ for all $v \in V$ and $\xi_e \mapsto \xi_{g \cdot e}$ for all $e \in E$. These linear maps induce algebra homomorphisms

$$\varrho_V \colon \mathbb{C}G \longrightarrow L\big(\mathbb{C}^{(V)}\big) \quad \text{and} \quad \varrho_E \colon \mathbb{C}G \longrightarrow L\big(\mathbb{C}^{(E)}\big).$$

We now assume that X is a tree and fix a vertex $v_0 \in V$. Then, using the associated bijection $\lambda \colon V \setminus \{v_0\} \to E$, we may define the linear maps

$$p \colon \mathbb{C}^{(V)} \longrightarrow \mathbb{C}^{(E)} \quad \text{and} \quad q \colon \mathbb{C}^{(E)} \longrightarrow \mathbb{C}^{(V)},$$

by letting $p(\xi_v) = \xi_{\lambda(v)}$ for all $v \in V \setminus \{v_0\}$, $p(\xi_{v_0}) = 0$ and $q(\xi_e) = \xi_{\lambda^{-1}(e)}$ for all $e \in E$. It is easily seen that $p \circ q = 1 \in L(\mathbb{C}^{(E)})$ and $q \circ p = 1 - p_0 \in L(\mathbb{C}^{(V)})$, where $p_0 \in L(\mathbb{C}^{(V)})$ is the projection onto the 1-dimensional subspace $\mathbb{C}\cdot\xi_{v_0}$, which vanishes on ξ_v for all $v \in V \setminus \{v_0\}$. In particular, the linear map

$$\widetilde{\varrho_E}\colon \mathbb{C}G \longrightarrow L\big(\mathbb{C}^{(V)}\big),$$

which is defined by letting $\widetilde{\varrho_E}(g) = q \circ \varrho_E(g) \circ p$ for all $g \in G$, is a homomorphism of non-unital algebras.

We say that the group G acts without inversions on the tree X if the automorphism induced by the element $g \in G$ on X has no inversions for all $g \in G$.

Proposition 2.3. *Let G be a group acting on a tree X without inversions, fix a vertex $v_0 \in V$ and consider the algebra homomorphisms ϱ_V and $\widetilde{\varrho_E}$ defined above. Then, for any element $g \in G$ the operator $\varrho_V(g) - \widetilde{\varrho_E}(g) \in L(\mathbb{C}^{(V)})$ is of finite rank, whereas its trace* (*cf. Example* 1.2) *is equal to* 1 *if g stabilizes some vertex of the tree and vanishes otherwise.*

Proof. First of all, we note that $\varrho_V(g)(\xi_v) = \xi_{g\cdot v}$ for all $v \in V$ and $\widetilde{\varrho_E}(g)(\xi_v) = \xi_{v'}$, where $v' = \lambda^{-1}(g\cdot\lambda(v))$ for all $v \in V \setminus \{v_0\}$. Moreover, we have

$$\lambda^{-1}(g\cdot\lambda(v)) = g\cdot v \iff g\cdot\lambda(v) = \lambda(g\cdot v)$$

for all $v \in V \setminus \{v_0, g^{-1}\cdot v_0\}$. Hence, Proposition 2.2 (i) shows that if $g\cdot v_0 = v_0$ then the operator $\varrho_V(g) - \widetilde{\varrho_E}(g)$ is equal to the projection p_0 onto the 1-dimensional subspace $\mathbb{C}\cdot\xi_{v_0}$, which vanishes on ξ_v for all $v \in V \setminus \{v_0\}$. It follows that $\varrho_V(g) - \widetilde{\varrho_E}(g)$ is of finite rank and $\operatorname{Tr}[\varrho_V(g) - \widetilde{\varrho_E}(g)] = 1$. We now assume that $g\cdot v_0 \neq v_0$. In that case, we let $(e_1,\dots,e_n)$ be the geodesic $[v_0, g^{-1}\cdot v_0]$ and consider for all $i = 1,\dots,n$ the terminal vertex $v_i \in V$ of e_i; in particular, $v_n = g^{-1}\cdot v_0$. Then, Proposition 2.2 (ii) implies that the operator $\varrho_V(g) - \widetilde{\varrho_E}(g)$ vanishes on ξ_v for all $v \in V \setminus \{v_0, v_1,\dots,v_{n-1}, v_n\}$; in particular, $\varrho_V(g) - \widetilde{\varrho_E}(g)$ is of finite rank. On the other hand, it is easily seen that

$$(\varrho_V(g) - \widetilde{\varrho_E}(g))\,(\xi_{v_i}) = \begin{cases} \xi_{g\cdot v_0} & \text{if } i = 0,\\ \xi_{g\cdot v_i} - \xi_{g\cdot v_{i-1}} & \text{if } i = 1,\dots,n. \end{cases}$$

Therefore, the final assertion in the statement of the proposition to be proved follows readily from the next lemma. □

Lemma 2.4. *Let X be a tree and α an automorphism of X. We consider a vertex $v_0 \in V$ such that $v' = \alpha(v_0) \neq v_0$, and the geodesic $[v_0, v'] = (e_1,\dots,e_n)$. We denote by v_i the terminal vertex of e_i for all $i = 1,\dots,n$; in particular, $v_n = v'$. Then:*

(i) *If α has no inversions, then $\alpha(v_i) \neq v_{i-1}$ for all $i = 1,\dots,n$.*

(ii) *If α fixes some vertex $v \in V$, then there is a unique $i \in \{1,\dots,n-1\}$ such that $\alpha(v_i) = v_i$.* □

3. Actions on trees and integrality of traces

We assume that G is a group acting on a tree X without inversions. We let V be the set of vertices of X and consider for any element $g \in G$ the fixed point set $V^g = \{v \in V : g \cdot v = v\}$. We also consider the subset $S \subseteq G$, consisting of those elements $g \in G$ for which the fixed point set V^g is non-empty. In other words, S consists of those group elements that stabilize some vertex of the tree, i.e. $S = \bigcup_{v \in V} \mathrm{Stab}_v$. Since the set S is closed under conjugation, the linear map (partial augmentation)

$$\tau_S \colon \mathbb{C}G \longrightarrow \mathbb{C},$$

which maps any element $\sum_{g \in G} a_g g \in \mathbb{C}G$ onto the complex number $\sum_{g \in S} a_g$, is easily seen to be a trace. The trace τ_S maps a group element $g \in G$ onto 1 (resp. onto 0) if g stabilizes a vertex (resp. if g does not stabilize any vertex). We consider the subset $[S] \subseteq \mathfrak{C}(G)$ which consists of the conjugacy classes of the elements of S, i.e. we let

$$[S] = \{[g] \in \mathfrak{C}(G) : g \in S\} = \{[g] \in \mathfrak{C}(G) : V^g \neq \emptyset\}.$$

Then, the trace τ_S factors through the quotient $\mathbb{C}G/[\mathbb{C}G, \mathbb{C}G]$ as the composition

$$\mathbb{C}G \xrightarrow{p} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \xrightarrow{\overline{\tau_S}} \mathbb{C}.$$

Here, p is the quotient map, whereas $\overline{\tau_S}$ maps any element $\sum_{[g] \in \mathfrak{C}(G)} a_{[g]}[g] \in \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G]$ onto the complex number $\sum_{[g] \in [S]} a_{[g]}$. In view of Remark 1.1 (i), we conclude that the additive map

$$(\tau_S)_* \colon K_0(\mathbb{C}G) \longrightarrow \mathbb{C},$$

which is induced by the trace τ_S, coincides with the composition

$$K_0(\mathbb{C}G) \xrightarrow{r_{\mathrm{HS}}} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \xrightarrow{\overline{\tau_S}} \mathbb{C}.$$

Therefore, $(\tau_S)_*$ maps any element $x \in K_0(\mathbb{C}G)$ with Hattori–Stallings rank $\sum_{[g] \in \mathfrak{C}(G)} r_{[g]}(x)[g]$ onto the complex number $\sum_{[g] \in [S]} r_{[g]}(x)$. Since the subset $S \subseteq G$ is obviously closed under n-th powers for all $n \geq 1$, it follows from [8, Proposition 3.2] that $\sum_{[g] \in [S]} r_{[g]}(x) \in \mathbb{Q}$. The following result strengthens that assertion, as it states that the above rational number is, in fact, an integer.

Theorem 3.1. *Let G be a group acting on a tree X without inversions and consider the subset $S \subseteq G$ and the additive map*

$$(\tau_S)_* \colon K_0(\mathbb{C}G) \longrightarrow \mathbb{C}$$

defined above. Then, $\mathrm{im}(\tau_S)_* = \mathbb{Z} \subseteq \mathbb{C}$.

Proof. Since $\tau_S(1) = 1$, it follows that $\mathbb{Z} \subseteq \operatorname{im}(\tau_S)_*$. In order to prove the reverse inclusion, we shall use the following result.

Theorem 3.2. *Let G be a group acting on a tree X without inversions and consider the subset $S \subseteq G$ defined above. Then, for any $x \in K_0(\mathbb{C}G)$ there exists a suitable element $y \in K_0(\mathbb{C}G)$ such that $r_{[g]}(y) = r_{[g]}(x)$ if $g \in S$ and $r_{[g]}(y) = 0$ if $g \notin S$.*

Proof. We fix a vertex $v_0 \in V$ and consider the representations

$$\varrho_V : \mathbb{C}G \longrightarrow L\big(\mathbb{C}^{(V)}\big) \quad \text{and} \quad \widetilde{\varrho_E} : \mathbb{C}G \longrightarrow L\big(\mathbb{C}^{(V)}\big)$$

which were defined before the statement of Proposition 2.3. Using the Hopf algebra structure of $\mathbb{C}G$, we now define the algebra homomorphisms

$$\sigma_V : \mathbb{C}G \longrightarrow L\big(\mathbb{C}^{(V)}\big) \otimes \mathbb{C}G \quad \text{and} \quad \widetilde{\sigma_E} : \mathbb{C}G \longrightarrow L\big(\mathbb{C}^{(V)}\big) \otimes \mathbb{C}G,$$

by letting $\sigma_V(g) = \varrho_V(g) \otimes g$ and $\widetilde{\sigma_E}(g) = \widetilde{\varrho_E}(g) \otimes g$ for all $g \in G$. Then, for any $g \in G$ we have $\sigma_V(g) - \widetilde{\sigma_E}(g) = [\varrho_V(g) - \widetilde{\varrho_E}(g)] \otimes g$ and hence Proposition 2.3 implies that $\sigma_V(a) - \widetilde{\sigma_E}(a) \in \mathfrak{F} \otimes \mathbb{C}G$ for all $a \in \mathbb{C}G$, where $\mathfrak{F} \subseteq L(\mathbb{C}^{(V)})$ is the ideal of finite rank operators on $\mathbb{C}^{(V)}$.

We also consider the trace

$$\operatorname{Tr} \otimes p : \mathfrak{F} \otimes \mathbb{C}G \longrightarrow \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G],$$

where Tr is the standard trace on $\mathfrak{F}$ and p the universal trace on $\mathbb{C}G$ (cf. Remark 1.1 (iii) and Example 1.2), and define the map t as the composition

$$\mathbb{C}G \xrightarrow{\sigma_V - \widetilde{\sigma_E}} \mathfrak{F} \otimes \mathbb{C}G \xrightarrow{\operatorname{Tr}\otimes p} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G].$$

Then, t is a trace as well, in view of Proposition 1.3 (i). Moreover, Proposition 1.3 (ii) implies that the image of the induced additive map

$$t_* : K_0(\mathbb{C}G) \longrightarrow \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G]$$

is contained in the image of the additive map

$$(\operatorname{Tr} \otimes p)_* : K_0(\mathfrak{F} \otimes \mathbb{C}G) \longrightarrow \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G].$$

Hence, in view of the identification of the latter map with the Hattori–Stallings rank map r_{HS} on $K_0(\mathbb{C}G)$ (cf. Remark 1.1 (iii) and Example 1.2), we conclude that $\operatorname{im} t_* \subseteq \operatorname{im} r_{\mathrm{HS}}$.

On the other hand, Proposition 2.3 implies that the trace t maps any group element $g \in G$ onto $[g]$ (resp. onto 0) if $g \in S$ (resp. if $g \notin S$). It follows that t factors as the composition

$$\mathbb{C}G \xrightarrow{p} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \xrightarrow{\bar{t}} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G],$$

where p is the quotient map and $\bar{t}$ maps any element $\sum_{[g]\in\mathfrak{C}(G)} a_{[g]}[g]$ onto the partial sum $\sum_{[g]\in[S]} a_{[g]}[g]$. Hence, invoking Remark 1.1 (i), we conclude that the additive map t_* coincides with the composition

$$K_0(\mathbb{C}G) \xrightarrow{r_{\mathrm{HS}}} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G] \xrightarrow{\bar{t}} \mathbb{C}G/[\mathbb{C}G, \mathbb{C}G].$$

It follows that t_* maps any element $x \in K_0(\mathbb{C}G)$ with Hattori–Stallings rank $\sum_{[g]\in\mathfrak{C}(G)} r_{[g]}(x)[g]$ onto $\sum_{[g]\in[S]} r_{[g]}(x)[g]$. It follows that the assertion in the statement of Theorem 3.2 is equivalent to the inclusion $\mathrm{im} t_* \subseteq \mathrm{im} r_{\mathrm{HS}}$, that we have already established. □

Proof of Theorem 3.1 (*continued*). We fix a K-theory class $x \in K_0(\mathbb{C}G)$ and choose $y \in K_0(\mathbb{C}G)$ according to in the statement of Theorem 3.2. Then,

$$(\tau_S)_*(x) = \sum_{[g]\in[S]} r_{[g]}(x) = \sum_{[g]\in\mathfrak{C}(G)} r_{[g]}(y)$$

is the homological rank of y; in particular, we have $(\tau_S)_*(x) \in \mathbb{Z}$. □

At this point, we recall that there is a close relationship between group actions on trees on one hand and group splittings on the other. Using the notion of a graph of groups (cf. [6], [15]), this relationship can be described by the Bass–Serre theory, as follows:

(i) If G is a group acting without inversions on a tree X, then there is a structure of a graph of groups on the quotient graph $Y = X/G$ such that the corresponding fundamental group is isomorphic to G.

(ii) Conversely, for any graph of groups on a connected graph Y with fundamental group G there is a tree X, the so-called universal tree of the graph, on which G acts without inversions, in such a way that $X/G \simeq Y$ and the stabilizer of any vertex (resp. edge) of X is a conjugate in G of a vertex group (resp. edge group) of the graph of groups.

Hence, we may rephrase Theorem 3.1 as follows: Let G be the fundamental group of a connected graph of groups with vertex groups $(G_v)_v$. For any vertex v of the graph we regard the group G_v as a subgroup of G and define

$$[G_v] = \{[g] \in \mathfrak{C}(G) : g \in G_v\}.$$

Then, for any element $x \in K_0(\mathbb{C}G)$ with Hattori–Stallings rank $\sum_{[g]\in\mathfrak{C}(G)} r_{[g]}(x)[g]$ the complex number $\sum\{r_{[g]}(x) : [g] \in \bigcup_v [G_v]\}$ is, in fact, an integer.

In particular, we obtain the following two results concerning amalgamated free products and HNN extensions:

Corollary 3.3. *Let $G = A \star_H B$ be the amalgamated free product of two groups A and B along a common subgroup H of theirs and consider an element $x \in K_0(\mathbb{C}G)$ with Hattori–Stallings rank $\sum_{[g]\in\mathfrak{C}(G)} r_{[g]}(x)[g]$. We view A and B as subgroups of G and define*

$$[A] = \{[g] \in \mathfrak{C}(G) : g \in A\} \quad \textit{and} \quad [B] = \{[g] \in \mathfrak{C}(G) : g \in B\}.$$

Then, the complex number $\sum_{[g]\in[A]\cup[B]} r_{[g]}(x)$ is, in fact, an integer.

Proof. Let Y be the graph consisting of an edge e and two distinct vertices $v = o(e)$ and $v' = t(e)$. Then, the result follows from Theorem 3.1, in view of the discussion above, by considering the graph of groups on Y which is given by letting $G_e = H$, $G_v = A$ and $G_{v'} = B$ with homomorphisms $G_e \to G_{o(v)}$ and $G_e \to G_{t(e)}$ the inclusion maps of H into A and B respectively. □

Corollary 3.4. *Let A be a group, $H \subseteq A$ a subgroup and $\varphi\colon H \to A$ a monomorphism. We consider the corresponding HNN extension $G = A\star_\varphi$ and let $x \in K_0(\mathbb{C}G)$ be an element with Hattori–Stallings rank $\sum_{[g]\in\mathfrak{C}(G)} r_{[g]}(x)[g]$. We view A as a subgroup of G and define*

$$[A] = \{[g] \in \mathfrak{C}(G) : g \in A\}.$$

Then, the complex number $\sum_{[g]\in[A]} r_{[g]}(x)$ is, in fact, an integer.

Proof. Let Y be the graph consisting of an edge e and a vertex $v = o(e) = t(e)$. Then, the result follows from Theorem 3.1, in view of the discussion above, by considering the graph of groups on Y which is given by letting $G_e = H$, $G_v = A$ with homomorphisms $G_e \to G_{o(e)}$ and $G_e \to G_{t(e)}$ the inclusion map of H into A and $\varphi\colon H \to A$ respectively. □

Remark 3.5. The result of Corollary 3.3 admits an alternative homological proof, if the group H therein is trivial. Indeed, let $G = A \star B$ be the free product of two groups A, B and consider an element $g \in G$ which is not conjugate to any element of A nor B, i.e. an element $g \in G$ for which $[g] \notin [A] \cup [B]$. Then, the centralizer C_g of g in G is easily seen to be infinite cyclic; this can be proved, for example, by invoking the Bass–Serre theory of groups acting on trees. In particular, the quotient group $N_g = C_g/\langle g\rangle$ is finite and hence one may use the Connes–Karoubi character map from $K_0(\mathbb{C}G)$ to the second cyclic homology group of the group algebra $\mathbb{C}G$, in order to show that the coefficient $r_{[g]}(x)$ of the Hattori–Stallings rank $r_{\mathrm{HS}}(x)$ of any element $x \in K_0(\mathbb{C}G)$ vanishes (cf. [7]). In particular, for any $x \in K_0(\mathbb{C}G)$ we have $\sum_{[g]\in[A]\cup[B]} r_{[g]}(x) = \sum_{[g]\in\mathfrak{C}(G)} r_{[g]}(x)$. Since the right-hand side of the latter equality is the homological rank of x, we conclude that $\sum_{[g]\in[A]\cup[B]} r_{[g]}(x) \in \mathbb{Z}$.

On the other hand, if $G = A \star B$ then the additive map

$$K_0(\mathbb{C}A) \oplus K_0(\mathbb{C}B) \longrightarrow K_0(\mathbb{C}G),$$

which is induced by the inclusions of A and B into G, is surjective; this follows from the discussion following [17, Corollary 11.5]. As explained in the Introduction, the surjectivity of the above map provides yet another proof of Corollary 3.3 (in the case where $H = 1$).

4. Group actions on trees with finite S

Our goal in this final section is to examine the extent to which Theorem 3.1 can be generalized to an integrality result concerning a trace defined on the reduced C^*-algebra of a group. Unfortunately, it will turn out that our approach does not lead to any really new results in that direction.

First of all, we recall that the group G acts on the Hilbert space $\ell^2 G$ by left translations and denote by

$$L\colon \mathbb{C}G \longrightarrow \mathfrak{B}(\ell^2 G)$$

the induced algebra homomorphism. Then, L is injective, its image $L(\mathbb{C}G)$ is a self-adjoint subalgebra of $\mathfrak{B}(\ell^2 G)$ and the reduced C^*-algebra C_r^*G of G is the operator norm closure of $L(\mathbb{C}G)$ in $\mathfrak{B}(\ell^2 G)$. The linear functional

$$\tau\colon C_r^*G \longrightarrow \mathbb{C},$$

which is defined by letting $\tau(a) = \langle a(\delta_1), \delta_1\rangle$ for all $a \in C_r^*G$, is a continuous positive faithful and normalized trace, which is referred to as the canonical trace on C_r^*G. (Here, we denote by $(\delta_g)_g$ the standard orthonormal basis of $\ell^2 G$.) For later use, we note that for any element $g \in G$ the linear map $a \mapsto \tau(L(g)^* a)$, $a \in C_r^*G$, restricts to the subspace $\mathbb{C}G \simeq L(\mathbb{C}G)$ to the linear map $\sum_{h\in G} a_h h \mapsto a_g$, $\sum_{h\in G} a_h h \in \mathbb{C}G$.

In order to extend the trace τ_S on the group algebra $\mathbb{C}G$, which was defined in the beginning of §3, to a trace on C_r^*G, we shall make the following assumption: *The group G acts without inversions on a tree X in such a way that the subset $S = \bigcup_{v\in V} \mathrm{Stab}_v$ of G, which consists of those group elements that stabilize a vertex, is finite.* We note that, under this assumption, the trace τ_S on $\mathbb{C}G \simeq L(\mathbb{C}G)$ extends to a continuous trace

$$\tau_S\colon C_r^*G \longrightarrow \mathbb{C},$$

by letting $\tau_S(a) = \sum_{g\in S} \tau(L(g)^* a)$ for all $a \in C_r^*G$. Indeed, the set S being finite, τ_S is a continuous linear functional on C_r^*G. In view of the remark made above, that linear functional restricts to the subspace $\mathbb{C}G \simeq L(\mathbb{C}G)$ to the trace τ_S on $\mathbb{C}G$. It follows by continuity that τ_S satisfies the trace property on C_r^*G as well. Since the set S is obviously closed under inverses, we also have $\tau_S(a) = \sum_{g \in S} \tau(L(g)a)$ for all $a \in C_r^*G$.

It turns out that the finiteness assumption on S places some severe restrictions on the group G. In fact, we shall prove that S must be a normal subgroup of G such that the quotient G/S is free. Then, the integrality of the trace τ_S on C_r^*G will be an immediate consequence of Connes' result [5, §IV.5] that free groups satisfy the integrality of the trace conjecture.

Let us consider the subset (normal subgroup) $G_f \subseteq G$ consisting of those elements that have only finitely many conjugates; in other words, we let

$$G_f = \{g \in G : \text{the conjugacy class } [g] \text{ is finite}\}.$$

We recall that a group is 2-ended if and only if it has an infinite cyclic subgroup of finite index (cf. [6, Chapter IV, Theorem 6.12]).

Proposition 4.1. *Let G be a group acting without inversions on a tree X, in such a way that the subset $S = \bigcup_{v \in V} \mathrm{Stab}_v$ of G is finite. Then:*

(i) *The stabilizer subgroup Stab_v is a finite subgroup of G_f for all $v \in V$.*

(ii) $S = \{g \in G$: *the order of g is finite*$\} \subseteq G_f$.

(iii) *The group G has a free subgroup of finite index.*

(iv) *If G is not 2-ended, then $S = G_f$ and the quotient group G/G_f is free.*

(v) *If G is 2-ended, then S is a normal subgroup of G and the quotient group G/S is infinite cyclic.*

Proof. (i) Let us fix a vertex $v \in V$. Then, the finiteness of Stab_v is clear, since $\mathrm{Stab}_v \subseteq S$. On the other hand, S is closed under conjugation and hence for any $g \in \mathrm{Stab}_v$ the conjugacy class $[g]$ is contained in S; in particular, $[g]$ is a finite set, i.e. $g \in G_f$.

(ii) Since $S = \bigcup_{v \in V} \mathrm{Stab}_v$ is a union of finite subgroups of G_f (in view of (i) above), it is contained itself in G_f and consists of elements of finite order. On the other hand, any torsion element $g \in G$ acts on the tree X by fixing some vertex (cf. [15, Chapitre I, Exemple 6.3.1]); hence, $g \in S$. We conclude that $S = \{g \in G :$ the order of g is finite$\}$.

(iii) Since the orders of the stabilizer subgroups Stab_v, $v \in V$, are obviously bounded by some integer, the result follows from [6, Chapter IV, Theorem 1.6].

(iv) We fix a free normal subgroup $N \subseteq G$ of finite index; such a subgroup exists, in view of (iii) above. Since the group G is not 2-ended, the free group N is not infinite cyclic. Hence, all non-identity elements of N have infinitely many conjugates in N

and, *a fortiori*, in G; in particular, $N \cap G_f = 1$. It follows that G_f embeds in G/N and hence G_f is a finite group. As such, G_f is contained in the subset of torsion elements of G and hence $G_f = S$, in view of (ii) above. Since the free group N embeds as a subgroup of finite index in G/G_f, we may invoke [6, Chapter IV, Theorem 1.6] once again, in order to conclude that there is a tree T on which G/G_f acts without inversions, in such a way that the vertex stabilizer subgroups are finite (and have orders bounded by some integer). On the other hand, since G_f coincides with the subset of torsion elements of G, the group G/G_f is easily seen to be torsion-free. It follows that the action of G/G_f on the tree T must be free. Hence, invoking [15, Chapitre I, §3.3], we conclude that the group G/G_f is free.

(v) It is well known that a 2-ended group G admits a surjective homomorphism with finite kernel onto the infinite cyclic group $\mathbb{Z}$ or else onto the infinite dihedral group D_∞. The latter case cannot occur, since D_∞ has infinitely many elements of finite order, whereas the corresponding set for G is finite (in view of (ii) above). Therefore, there is a finite normal subgroup H of G such that $G/H \simeq \mathbb{Z}$. It is now clear that H coincides with the set of elements of finite order in G and hence the proof is finished. □

Let us now consider the group G which acts without inversions on a tree X, in such a way that the subset $S \subseteq G$ consisting of those group elements that stabilize a vertex is finite. Then, it follows from Proposition 4.1 that S is a finite normal subgroup of G, whereas the quotient group $\overline{G} = G/S$ is free. In view of the finiteness of S, the quotient map $G \to \overline{G}$ induces an algebra homomorphism

$$\pi_0 \colon \mathbb{C}G \longrightarrow \mathbb{C}\overline{G},$$

which can be extended to a $*$-algebra homomorphism

$$\pi \colon C_r^* G \longrightarrow C_r^* \overline{G}.$$

We note that the trace τ_S on $C_r^* G$, which was defined in the beginning of this section, coincides with the composition

$$C_r^* G \xrightarrow{\pi} C_r^* \overline{G} \xrightarrow{\bar{\tau}} \mathbb{C},$$

where $\bar{\tau}$ is the canonical trace on $C_r^* \overline{G}$. In order to verify this latter assertion, it suffices (by continuity) to show that the trace τ_S on $\mathbb{C}G$, which was defined in the beginning of §3, coincides with the composition

$$\mathbb{C}G \xrightarrow{\pi_0} \mathbb{C}\overline{G} \xrightarrow{\bar{\tau}} \mathbb{C},$$

where $\bar{\tau}$ is the linear trace on $\mathbb{C}\overline{G}$, which maps $\bar{1} \in \overline{G}$ onto 1 and any element $\bar{g} \in \overline{G} \setminus \{\bar{1}\}$ onto 0. But this is clear, in view of the definitions. Invoking now

Remark 1.1 (ii), we conclude that the additive map

$$(\tau_S)_*\colon K_0(C_r^*G) \longrightarrow \mathbb{C}, \tag{4}$$

which is induced by the trace τ_S on C_r^*G, coincides with the composition

$$K_0(C_r^*G) \xrightarrow{K_0(\pi)} K_0\left(C_r^*\overline{G}\right) \xrightarrow{\bar{\tau}_*} \mathbb{C},$$

where $\bar{\tau}_*$ is the additive map induced by the canonical trace $\bar{\tau}$ on $C_r^*\overline{G}$. Therefore, the group $\overline{G}$ being free, we may invoke Connes' result [5, §IV.5] that free groups satisfy the integrality of the trace conjecture, in order to conclude that the image of the additive map (4) is the group $\mathbb{Z}$ of integers.

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