

The geometry of genus-one helicoids

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Abstract. We prove: a properly embedded, genus-one, minimal surface that is asymptotic to a helicoid and that contains two straight lines must intersect that helicoid precisely in those two lines. In particular, the two lines divide the surface into two connected components that lie on either side of the helicoid. We prove an analogous result for periodic helicoid-like surfaces. We also give a simple condition guaranteeing that an immersed minimal surface with finite genus and bounded curvature is asymptotic to a helicoid at infinity.

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1. Introduction and statement of results

In this paper we consider properly immersed minimal surfaces $\mathcal{S} \subset \mathbb{R}^3$ that have one end asymptotic to the helicoid, and genus equal to one. We will call such a surface a nonperiodic genus-one helicoid. We are interested in embedded, nonperiodic genus-one helicoids. Without loss of generality, we may assume that \mathcal{S} is asymptotic to a vertical helicoid whose axis is the z -axis, Z . If \mathcal{S} contains Z and one horizontal line we will refer to \mathcal{S} as a *symmetric*, nonperiodic genus-one helicoid. Schwarz reflection¹ about the lines on the surface provides the symmetries. Without loss of generality we may assume that this line is the x -axis, X , and that \mathcal{S} is asymptotic to the standard helicoid H , half of which is parametrized by

$$(r, \theta) \rightarrow (r \cos \theta, r \sin \theta, \theta), \quad (1)$$

$r \in [0, \infty)$, $\theta \in \mathbb{R}$. (The other half is obtained by Schwarz reflection about Z .) Note that $X \cup Z \subset H$.

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¹A minimal surface that contains a line in its interior is symmetric under reflection in that line; this symmetry is called Schwarz reflection and is a consequence of the Schwarz Reflection Principle in complex analysis. By the same token, a minimal surface that contains a straight line in its boundary can be continued analytically across that line by reflection. See [HK97] or [Oss86].

Hoffman, Weber and Wolf [WHW06] proved the existence of a symmetric, embedded, nonperiodic genus-one helicoid. In [HW08] we gave a variational construction for such surfaces. The examples we constructed in that paper have the following property:

$$\begin{aligned} \mathcal{S} \cap H &= X \cup Z, \text{ and} \\ \mathcal{S} \setminus H &\text{ consists of two congruent simply connected components.} \end{aligned} \tag{2}$$

In Theorem 2.5 of Section 2.4, we prove that *every embedded, symmetric, nonperiodic genus-one helicoid satisfies (2)*.

We also establish a parallel result for embedded *periodic* genus-one helicoids, by which we mean properly embedded minimal surfaces $S \subset \mathbb{R}^3$ that are invariant under a screw motion

$$\sigma_{2h}(r \cos \theta, r \sin \theta, z) = (r \cos(\theta + 2h), r \sin(\theta + 2h), z + 2h) \tag{3}$$

for some $h > 0$, and for which S/σ_{2h} has genus one and is asymptotic to H/σ_{2h} at infinity. Let $S^* = S \cap \{z : -h < z \leq h\}$, and note that S^* is a fundamental domain for S . If S contains Z and if S^* contains *two* horizontal lines, then we will refer to S as a *symmetric, periodic genus-one helicoid*. Without loss of generality, we may assume that $X \subset S^*$, which implies that the other horizontal line in S^* is $\sigma_h(X)$. For $h > \pi/2$, such surfaces were proved to exist in [HKW99], [WHW06], and by variational means in [HW08]. Without loss of generality, we may assume that $\{z = h\} \cap S = \sigma_h(X)$, which implies that S^* is bounded by two lines. This follows, for example, from Lemma 1(vii) of [WHW06] together with an application of the maximum principle. Define $X^* = X \cup \sigma_h(X)$. The construction in [HW08] produces periodic surfaces satisfying an analog of (2) above:

$$\begin{aligned} S^* \cap H &= X^* \cup Z^*, \text{ and} \\ S^* \setminus H &\text{ consists of two symmetric, simply connected components,} \end{aligned} \tag{4}$$

where $Z^* = \{(0, 0, t) : -h < t \leq h\}$. In this paper, we prove that *every embedded, symmetric, periodic genus-one helicoid satisfies (4)*. This is Theorem 2.7.

We prove in Theorems 2.5 and 2.7 that embedded symmetric genus-one helicoids have simple intersections with all rotations of H :

Let \hat{H} be the result of rotating H about the z -axis, Z through an angle in $(0, \pi)$. If \mathcal{S} is nonperiodic, then $\mathcal{S} \cap \hat{H}$ consists of Z together with a smooth embedded closed curve that intersects Z twice, once above and once below the xy -plane. If S is periodic, $S^ \cap \hat{H}$ consists of Z^* together with a smooth embedded closed curve that intersects Z^* twice, once above and once below the xy -plane.*

Section 2 concludes with a uniqueness result for half-helicoids, Theorem 2.11: *Suppose M is a connected minimal surface that lies in the closure of a component of $\mathbb{R}^3 \setminus H$, with ∂M lying in the closure of a component, Σ , of $H \setminus Z$. If M is bounded*

or if M is asymptotic to Σ , then $M \subset \Sigma$. The proof of this result uses the fact that $\mathbb{R}^3 \setminus Z$ is foliated by half-helicoids. Our approach is close to that taken by Hardt and Rosenberg in [HR90].

As mentioned above, [WHW06] and [HW08] proved existence of σ_{2h} -invariant, symmetric genus-one helicoids for every $h > \pi/2$. In Theorem 3.4 of Section 3.3, we prove that the condition $h > \pi/2$ is necessary: for $h \leq \pi/2$, there are no embedded, symmetric, periodic genus- g helicoids (with $g \geq 1$) invariant under the screw motion σ_{2h} . To our knowledge, this was first observed by Bill Meeks for $h < \pi/2$. Our result requires only the presence of two horizontal lines in S/σ_{2h} (no assumption that S contains the axis Z). This proof uses Proposition 5.1, which gives estimates of the radial decay of the vertical distance between the end of a symmetric, periodic genus-one helicoid and the end of a helicoid. We use the same estimates to prove (Theorem 5.4) that the σ_{2h} -invariant, helicoid-like surfaces constructed in [HW08] are asymptotic to helicoids and thus are in fact periodic genus-one helicoids.

In Section 4, we investigate the geometry of properly immersed minimal surfaces with finite genus and one end. With a few additional assumptions, we prove that such a surface is asymptotic to a helicoid:

Let $\mathcal{S} \subset \mathbb{R}^3$ be a properly immersed minimal surface with finite genus, one end and bounded Gauss curvature. Suppose that \mathcal{S} contains $X \cup Z$, and that one level set $\{x_3 = c\} \cap \mathcal{S}$ has precisely one divergent component and a finite number of singular points. Then \mathcal{S} is conformally a compact Riemann surface punctured in one point corresponding to the end, and that end is asymptotic to a helicoid.

This is Theorem 4.1. This result gives another proof that the genus-one surfaces constructed in [HW08] are asymptotic to the helicoid. The method of proof here is a slight generalization of the method used in that paper. (See Theorem 6.1 in [HW08].)

2. Structural properties of symmetric genus-one helicoids

An embedded, nonperiodic genus-one helicoid $\mathcal{S} \subset \mathbb{R}^3$ is a properly embedded minimal surface in \mathbb{R}^3 that is asymptotic to the helicoid at infinity. Without loss of generality, we will assume that \mathcal{S} is asymptotic to the helicoid H defined in the first paragraph of the Introduction. The surface H is a right-handed helicoid that contains Z and X . We say that \mathcal{S} is *symmetric* if it contains Z and X . Similarly, an embedded, *periodic* genus-one helicoid is a properly embedded minimal surface $S \subset \mathbb{R}^3$ invariant under a screw motion (3), such that S/σ_{2h} has genus one and two helicoidal ends. We say that S is *symmetric* if $Z \subset S$, and the fundamental domain $S^* = S \cap \{-h < z \leq h\}$ contains $X^* = X \cup \sigma_h(X)$.

In this section we will prove that embedded, symmetric genus-one helicoids are cut by H precisely along $X \cup Z$ into two congruent simply connected domains.

We also prove the analogous result for periodic genus-one helicoids. The technique involves the study of minimal surfaces with boundary lying in a half-helicoid.

2.1. Removal of the axes results in two congruent, simply connected domains.

We begin by showing that removal of $X \cup Z$ from a properly embedded, nonperiodic, symmetric genus-one helicoid produces two congruent, simply connected domains. Similarly, removal of $X^* \cup Z^*$ from a fundamental domain of a properly embedded, *periodic*, symmetric genus-one helicoid also produces two congruent simply connected domains.

Lemma 2.2. *Suppose \mathcal{S} is a properly embedded, nonperiodic symmetric genus-one helicoid. Then $\mathcal{S} \setminus Z$ and $\mathcal{S} \setminus X$ are annuli, and $\mathcal{S} \setminus (X \cup Z)$ is a pair of congruent, simply connected domains.*

Lemma 2.3. *Suppose S is a properly embedded, periodic, symmetric genus-one helicoid invariant under σ_{2h} . Let $S^* = \{-h < z \leq h\}$ be a fundamental domain of S , and $Z^* = Z \cap \{-h < z \leq h\}$. Then $S^* \setminus Z^*$ and $S^* \setminus X^*$ are annuli, and $S^* \setminus (Z^* \cup X^*)$ is a pair of congruent, simply connected domains.*

Proof of Lemma 2.2. The surface \mathcal{S} is topologically a once-punctured torus. (In fact by Theorem 4.1, \mathcal{S} is conformally a once-punctured torus, but we will not use that here.) Thus the one-point compactification $T = \mathcal{S} \cup \{\infty\}$ is a torus and $Z \cup \{\infty\}$ is a simple closed curve in T . Removing a simple closed curve from a torus either separates it into a disk and a once-punctured torus, or else results in a single annulus.

The rotation ρ_Z is an isometry of \mathcal{S} that leaves Z invariant. Therefore, Z cannot divide T into a disk and punctured torus, because these pieces would have to be homeomorphic (by the involution ρ_Z). Thus the result is a single annulus A .

For similar reasons, X , viewed as a curve in T , is a simple closed curve. The same argument shows that removal of X from T produces an annulus.

Note that Z and X cross at the origin and at the point at infinity. These two points are represented as points on the boundary of the annulus $A = T \setminus Z$. The positive ray of X is a simple curve in A going from one boundary point to another. If it went from one boundary component of A to the same boundary component, it would divide A into two components, one a disk, the other an annulus. But if there were two components, they would be homeomorphic (by ρ_X). Thus, the positive ray of X goes from one boundary component of A to the other, and removing it results in a disk. Now removing the negative ray of X divides that disk into two disks. The two disks are congruent since they are related by the isometry ρ_X \square

Proof of Lemma 2.3. Note that the one-point compactification of $S \cap \{-h < z < h\}$ is a torus T . The proof of Lemma 2.3 is exactly the same as the proof of Lemma 2.2, except that one replaces \mathcal{S} and Z by $S \cap \{-h < z < h\}$ and $Z \cap \{-h < z < h\}$. \square

2.4. The decomposition theorem for nonperiodic symmetric genus-one helicoids.

The helicoid H divides \mathbb{R}^3 into two simply connected regions. Let H^+ be the region that contains Y^+ , the positive ray of the y -axis, and let H^- be the other region. The axis Z is contained in the helicoid H , and $H \setminus Z$ consists of two simply connected components, each of which we will refer to as a *half-helicoid*. We will denote by Σ_0 the half-helicoid that contains the X^+ , the positive x -axis.

More generally, let us extend the definition of a half-helicoid to include any surface obtained by rotating one of the components of $H \setminus Z$ through some angle about Z . Thus the half-helicoids form a foliation of $\mathbb{R}^3 \setminus Z$. In particular, rotating Σ_0 through angles in $(0, \pi)$ produces a foliation of H^+ .

Theorem 2.5. *Let \mathcal{S} be an embedded, nonperiodic, symmetric genus-one helicoid. Then*

1. $\mathcal{S} \setminus (X \cup Z)$ consists of two simply connected, congruent components D and D' .
2. D and D' lie in $\mathbb{R}^3 \setminus H$, one in H^+ , the other in H^- .
3. Let \hat{H} be a helicoid obtained by rotating H about Z through an angle in $(0, \pi)$. Then $D \cap \hat{H}$ (resp. $D' \cap \hat{H}$) is a smooth embedded curve with one endpoint in Z^+ and the other endpoint in Z^- .

Proof. Statement 1 is Lemma 2.2. It remains to prove statements 2 and 3. We begin by observing that by assumption \mathcal{S} contains the axes X and Z and $\partial D = \partial D' = X \cup Z$. Let ρ_X and ρ_Z denote rotations by π about X and Z , respectively. These symmetries are orientation-reversing on \mathcal{S} , and it is easy to see that they interchange D and D' . It follows that their composition, $\rho_Y = \rho_X \circ \rho_Z$, rotation by π about the axis Y , leaves D and D' invariant and preserves orientation on \mathcal{S} .

If $p \in \mathbb{R}^3 \setminus Z$, let $\Sigma(p)$ be the half-helicoid that contains p . If $p \in Z \setminus \{0\}$, let $\Sigma(p)$ be the half-helicoid with the property that $\Sigma(p)$ and D have the same tangent half-plane at p . Note that

$$\text{If } p \in Z \setminus \{0\}, \text{ then } D \cap \Sigma(p) \text{ contains a smooth curve, one of whose endpoints is } p. \tag{5}$$

Since $\bar{D} \setminus Z$ is simply connected, there is a smooth function

$$\theta: \bar{D} \setminus Z \rightarrow \mathbb{R}$$

such that

$$q = (x, y, z) = (r \cos \theta, r \sin \theta, z) \tag{6}$$

for $q \in \bar{D} \setminus Z$, where $r = r(x, y, z) = \sqrt{x^2 + y^2}$. We may normalize θ so that

$$\theta = 0 \quad \text{on } X^+, \tag{7}$$

where X^+ is the positive x -axis. It follows that for some integer k ,

$$\theta = (2k + 1)\pi \quad \text{on } X^-, \tag{8}$$

where $X^- = \rho_Y(X^+)$, the negative x -axis. Since $\bar{D} \setminus \{0\}$ is a smooth manifold with boundary $X \cup Z \setminus \{0\}$, the function θ extends smoothly to $Z \setminus \{0\}$. Thus θ is a smooth function on $\bar{D} \setminus \{0\}$.

Since $\theta = 0$ on X^+ and $\theta = (2k + 1)\pi$ on X^- , extending θ to 0 is somewhat problematic. We get around that by using the geodesic completion D^* of $\bar{D} \setminus \{0\}$. Note that D^* is $\bar{D} \setminus \{0\}$ together with two points, 0^+ and 0^- , which are the limits of $(x, 0, 0)$ as $x \downarrow 0$ and $x \uparrow 0$, respectively. We let

$$\theta(0^+) = 0 \quad \text{and} \quad \theta(0^-) = (2k + 1)\pi, \tag{9}$$

which makes θ continuous on all of D^* .

We now use the ρ_Y symmetry of D . Since $\rho_Y(x, y, z) = (-x, y, -z)$,

$$\cos(\theta \circ \rho_Y) = -\cos \theta \quad \text{and} \quad \sin(\theta \circ \rho_Y) = \sin \theta,$$

from which it follows that $\theta \circ \rho_Y$ and $-\theta$ differ by an odd multiple of π . In fact, since $\rho_Y(X^+) = X^-$, it follows from (7) and (8) that

$$\theta \circ \rho_Y = -\theta + (2k + 1)\pi. \tag{10}$$

Since D is a disk and ρ_Y is an orientation-preserving isometric involution of D , ρ_Y has a unique fixed point. That is, $D \cap Y$ is a single point. Without loss of generality, we can assume that this fixed point is in H^+ . (Since $D' = \rho_Z(D)$, we can simply relabel the disks.) In particular,

$$D \cap Y^- = \emptyset, \tag{11}$$

where Y^- is the negative y -axis. Let

$$F : D \rightarrow \mathbb{R}, \quad (x, y, z) \mapsto \theta(x, y, z) - z. \tag{12}$$

The function F is connected to the geometry of our situation by the following elementary observation:

$$F \text{ is constant on every half-helicoid.} \tag{13}$$

Claim 1. *F has no local maxima or local minima on D.*

Proof of Claim 1. Suppose F has a local maximum or minimum at $p \in D$. Let \mathbb{B} be a ball centered at p , small enough so that \mathbb{B} is disjoint from Z and that $D \cap \mathbb{B}$ is connected. Then there is a unique continuous extension of θ to $D \cup \mathbb{B}$ so that (6) still

holds. Note that if we use this extended θ together with (12) to define F on \mathbb{B} , then it follows from (13) that

$$\{q \in \mathbb{B} : F(q) = F(p)\} = \Sigma(p) \cap \mathbb{B}.$$

Thus $D \cap \mathbb{B}$ lies in the closure of one of the connected components of $\mathbb{B} \setminus \Sigma(p)$, and $D \cap \mathbb{B}$ and $\Sigma(p) \cap \mathbb{B}$ are tangent at p . By the maximum principle, $D \cap \mathbb{B}$ and $\Sigma(p) \cap \mathbb{B}$ coincide. By analyticity, all of D is contained in a helicoid, a contradiction. \square

Claim 2. F has no local maxima or local minima on $Z \setminus \{0\}$.

Proof of Claim 2. Suppose $p \in Z \setminus \{0\}$. Since $\Sigma(p)$ and $\bar{D} \setminus \{0\}$ are minimal surfaces with boundary and since they are tangent at p , $D \cap \Sigma(p)$ contains a smooth curve C with p as one of its endpoints. Note that F is constant along C by (13). By Claim 1, none of the points of C is a local maximum or local minimum of F . Thus p is neither a local maximum nor a local minimum of F . \square

Claim 3. Suppose α is not an integral multiple of π . Then either $F^{-1}(\alpha)$ is empty, or it is a single smooth curve with one endpoint on Z^+ and the other endpoint on Z^- .

Proof of Claim 3. Suppose $p \in C = F^{-1}(\alpha)$. Then $C \subset \Sigma(p)$. Note that D is asymptotic to H at infinity and $\Sigma(p)$ is not (because α is not an integral multiple of π), so C lies in a bounded region of \mathbb{R}^3 . Now C cannot contain a closed curve, because then that curve would bound a region in D , and F would have an interior maximum or minimum in that region, violating Claim 1. Thus each connected component T of C has the structure of a tree whose endpoints are on $X \cup Z$. Since $F = 0$ on X^+ and $F = (2k + 1)\pi$ on X^- , the endpoints must be on Z . Since 0 and $(2k + 1)\pi$ are the only subsequential limits of $F(p)$ as $p \rightarrow 0$, in fact the endpoints of T must be on $Z \setminus \{0\}$.

Now T cannot have two endpoints on Z^+ . For if it did, T would contain a curve Γ joining those endpoints, and that curve together with the interval $I \subset Z$ joining the endpoints would bound a region U in D . Since F is constant on Γ , F would have a local maximum or a local minimum at some point $p \in U \cup I$. But that is impossible by Claims 1 and 2.

Thus T has at most one endpoint on Z^+ and (by the same reasoning) at most one endpoint on Z^- . It follows that T is a smooth curve joining a point on Z^+ to a point on Z^- .

We have shown: $C = F^{-1}(\alpha)$ is a union of disjoint curves, each of which joins a point in Z^+ to a point in Z^- . Furthermore, there cannot be more than one such curve. For if there were two such curves T_1 and T_2 , then those curves together with a pair of intervals I^+ and I^- (in Z^+ and Z^- , respectively) would bound a region U . Since F is not constant on \bar{U} , its maximum on \bar{U} is greater than α or its minimum on

\bar{U} is less than α . Thus F has a local maximum or a local minimum on $U \cup I^+ \cup I^-$. But that is impossible by Claims 1 and 2. \square

Claim 4. *If $p \in Y^-$, then $D \cap \Sigma(p) = \emptyset$.*

Proof of Claim 4. Note that $D \cap \Sigma(p)$ is the union of $F^{-1}(\alpha)$ over all α that are congruent to $3\pi/2 \pmod{2\pi}$. Thus if $D \cap \Sigma(p)$ were nonempty, by Claim 3 it would contain a curve joining Z^+ to Z^- . But any such curve in $\Sigma(p)$ must cross Y^- , and D does not contain any points in Y^- . Thus $D \cap \Sigma(p) = \emptyset$. \square

By Claim 4, the set $F(D)$ does not contain any values equal to $3\pi/2 \pmod{2\pi}$. In particular, it contains neither $-\pi/2$ nor $3\pi/2$. Since D is connected, $F(D)$ is an interval. Since $F|_D$ has no local maxima or local minima, $F(D)$ is an open interval (a, b) . From (9), $[a, b]$ contains 0 and $(2k + 1)\pi$. Thus

$$F(D) = (a, b), \quad \text{where } -\pi/2 \leq a \leq 0 \text{ and } \pi \leq b \leq 3\pi/2, \tag{14}$$

and $k = 0$. In particular, from (10) we have

$$\theta \circ \rho_Y = -\theta + \pi. \tag{15}$$

Claim 5. *$a = 0$ and $b = \pi$.*

Proof of Claim 5. If $b \neq \pi$, then by (14), $\pi < b \leq 3\pi/2$. Let $b_i \in (\pi/2, b)$, $b_i \neq \pi$, with $b_i \rightarrow b$. Then $F^{-1}(b_i)$ is nonempty, and by Claim 3, it must contain a point p_i in the xy -plane. Note that p_i has the form

$$p_i = (r_i \cos b_i, r_i \sin b_i, 0),$$

where $r_i > 0$. By passing to a subsequence, we may assume that the r_i converge to a limit $r \in [0, \infty]$. Now r cannot be 0 since $\theta(0^+) = 0$ and $\theta(0^-) = \pi$. Also, r cannot be a finite nonzero number since otherwise F would attain its maximum, contradicting Claim 1.

Finally, if $r_i \rightarrow \infty$, then $\text{dist}(p_i, H) \rightarrow b - \pi > 0$, which is impossible since D is asymptotic to H at infinity. The contradiction proves that $b = \pi$. The proof that $a = 0$ is essentially the same. (It also follows from the ρ_Y symmetry of D .) \square

Since $F(D) = (0, \pi)$, it follows that D intersects only those half-helicoids produced by rotating Σ_0 through an angle in $(0, \pi)$. As observed in the paragraph just before the statement of the theorem, those half-helicoids foliate H^+ , so D lies in H^+ . It follows that $D' = \rho_Y D$ lies in $H^- = \rho_Y H^+$. This completes the proof of statement (2).

Statement (3) of the theorem follows from Claim 3, together with the fact that $F(D) = (0, \pi)$. \square

2.6. The decomposition theorem for periodic symmetric genus-one helicoids.

There is an analogous result to Theorem 2.5 for periodic, symmetric genus-one helicoids.

Theorem 2.7. *Let S be a periodic, embedded, symmetric genus-one helicoid. Then*

1. $S^* \setminus (X^* \cup Z^*)$ consists of two simply connected components D and D' .
2. D and D' lie in $\mathbb{R}^3 \setminus H$, one in H^+ the other in H^- .
3. Let \hat{H} be a helicoid obtained by rotating H about Z through an angle in $(0, \pi)$. Then $D \cap H'$ (resp. $D' \cap \hat{H}$) is a smooth embedded curve with one endpoint in Z^+ and the other endpoint in Z^- .

The sets S^* and X^* are defined in the Introduction, just before equation (4). The proof of Theorem 2.6 is a straightforward adaptation of the proof of Theorem 2.5. (Statement 1 was already proved in Lemma 2.2.)

2.8. A half-helicoid uniqueness theorem. We end this section with a uniqueness theorem for minimal surfaces that lie in the closure of a component of $\mathbb{R}^3 \setminus H$, and that are either compact or asymptotic to H at infinity.

Definition 2.9. Let S be an unbounded, oriented, embedded surface in \mathbb{R}^3 . We say that another surface M is *asymptotic to S at infinity* provided there is a domain $\Omega \subset S$ and a function $u: \Omega \rightarrow \mathbb{R}$ such that

$$\lim_{|p| \rightarrow \infty} (|u(p)| + |Du(p)|) = 0 \quad (16)$$

and such that outside of a compact subset of \mathbb{R}^3 , the surface M coincides with the graph

$$\{p + u(p)v(p) : p \in \Omega\},$$

where $v(p)$ is the unit normal to S at p .

Remark 2.10. If M and S have compact boundaries and bounded principal curvatures, then the C^1 condition (16) follows (by an Arzela–Ascoli type argument) from the analogous C^0 condition $\lim_{|p| \rightarrow \infty} |u(p)| = 0$.

Theorem 2.11. *Let Σ be one of the components of $H \setminus Z$. Suppose M is a connected minimal surface in $\overline{H^+} \setminus Z$ such that $\partial M \subset \overline{\Sigma}$ and such that M is either bounded or asymptotic to Σ at infinity.*

Then M is a subset of Σ .

Proof. We may assume without loss of generality that Σ is the component of $H \setminus Z$ containing the positive x -axis. We may also assume that M is bounded: If M is not bounded then by hypothesis M is asymptotic to Σ at infinity. A sufficiently small rotation of M about Z , say through an angle $-\epsilon$, will produce a surface with nonempty intersection with H^+ . Since M is asymptotic to $\Sigma \subset H$ at infinity, the intersection of the rotated M with H^+ is bounded. Denote this intersection by $M(\epsilon)$. If M were a counterexample to the theorem, then the surface $M(\epsilon)$ would also be a counterexample, a bounded one.

Thus from now on we assume that M is bounded.

Let C be a closed, solid circular cylinder of finite height that contains M and whose axis of symmetry is Z . Let Γ be the boundary of $C \cap \Sigma$. Minimize area among disks in $\overline{H^+}$ with boundary Γ and with M as an obstacle (i.e., among disks Δ such that M is contained in the closed region bounded by $\Delta \cup (C \cap \Sigma)$.) Call the resulting disk D .

Let Σ' be the half-helicoid $H \setminus (Z \cup \Sigma)$. Rotate Σ' in H^+ until it touches D at an interior point or until it becomes tangent to D at some point of the interval $I := \Gamma \cap Z$. Call the resulting half-helicoid Σ^* .

Note that one of the following must occur:

1. Σ^* touches D at an interior point of D .
2. Σ^* is tangent to D at an interior point of I .
3. Σ^* is tangent to D at an endpoint of I .

In case 1, D is contained in Σ^* by the maximum principle. In case 2, D is contained in Σ^* by the boundary maximum principle. In case 3, $\Sigma^* = \Sigma$ (since D and Σ are tangent at the endpoints of I) and thus D is contained in Σ^* .

In all three cases, we have shown that D is contained in Σ^* . Since $\partial D \subset \Sigma$, this implies that $\Sigma^* = \Sigma$. Since M lies between Σ and Σ^* , in fact M is contained in Σ . \square

3. Nonexistence of embedded, periodic, higher genus helicoids with small twist angles

In this section we study properly embedded, periodic minimal surfaces invariant under a screw motion σ_{2h} and asymptotic to the helicoid. We will show that if $h \leq \pi/2$ and if the intersection of the surface with some horizontal plane is a line, then the surface must be the helicoid.

3.1. The total curvature of almost-helicoidal curves. The curvature of a space curve $\theta \mapsto c(\theta)$ is given by $\frac{|c'(\theta) \times c''(\theta)|}{|c'(\theta)|^3}$, and therefore the total curvature from

$\theta = 0$ to $\theta = A$ is

$$\int_c k ds = \int_0^A \frac{|c'(\theta) \times c''(\theta)|}{|c'(\theta)|^2} d\theta. \quad (17)$$

Now suppose that

$$c(\theta) = (r \cos \theta, r \sin \theta, \theta + f(r, \theta))$$

for some function $f(r, \theta)$. In the special case that f is constant, the curve c is a standard helix. More generally, suppose that $\frac{\partial f}{\partial \theta}$ and $\frac{\partial^2 f}{\partial \theta^2}$ tend to zero uniformly as $r \rightarrow \infty$. Since $|c'(\theta)|^2 = r^2 + (1 + \frac{\partial f}{\partial \theta})^2$ and $|c''(\theta)|^2 = r^2 + (\frac{\partial^2 f}{\partial \theta^2})^2$, it follows that

$$|c''(\theta)| < |c'(\theta)|$$

for r sufficiently large. Therefore

$$\frac{|c'(\theta) \times c''(\theta)|}{|c'(\theta)|^2} \leq \frac{|c'(\theta)| |c''(\theta)|}{|c'(\theta)|^2} < \frac{|c'(\theta)|^2}{|c'(\theta)|^2} = 1$$

for r sufficiently large. It follows that the total curvature (17) is strictly less than A . We state this observation as a lemma.

Lemma 3.2. *Let $f(r, \theta)$ be a function defined for $r \geq R_0$ and $\theta \in [0, A]$. Suppose that $\frac{\partial f}{\partial \theta}$ and $\frac{\partial^2 f}{\partial \theta^2}$ tend to zero uniformly as $r \rightarrow \infty$. Then for every sufficiently large r , the curve*

$$\theta \in [0, A] \mapsto (r \cos \theta, r \sin \theta, \theta + f(r, \theta))$$

has total curvature strictly less than A .

3.3. Nonexistence of examples with $h \leq \pi/2$. We will use Lemma 3.2 to prove

Theorem 3.4. *Let S be a properly immersed minimal surface that lies in the slab $\{-h \leq z \leq h\}$ and that is bounded by the two lines $H \cap \{z = h\}$ and $H \cap \{z = -h\}$, where H is the standard helicoid (1). Suppose that S is asymptotic² to*

$$H \cap \{-h \leq z \leq h\}.$$

If $h \leq \pi/2$, then $S = H \cap \{-h \leq z \leq h\}$.

As a corollary we have

Theorem 3.5. *Suppose that S is a properly immersed minimal surface invariant under a screw motion σ_{2h} with $h \leq \pi/2$, and that S is asymptotic to H as $r = \sqrt{x^2 + y^2}$ tends to infinity. If the intersection of S with some horizontal plane is a line, then $S = H$.*

²See Definition 2.9.

Proof. Without loss of generality we may assume that $S \cap \{z = -h\}$ is a line. By the σ_{2h} invariance, $S \cap \{z = h\}$ must also be a line. Since S is asymptotic to H away from Z , these lines must be the lines $H \cap \{z = -h\}$ and $H \cap \{z = h\}$. Thus $S \cap \{-h \leq z \leq h\}$ satisfies the hypotheses of Theorem 3.4, from which we conclude that $S = H$. □

Proof of Theorem 3.4. By assumption, outside of any sufficiently large cylinder $C_R = \{x^2 + y^2 \leq R^2, |z| \leq h\}$, one end of S is a graph of the form

$$\{(r \cos \theta, r \sin \theta, \theta + f(r, \theta)) : -h \leq \theta \leq h, r > R\},$$

with $f(-h, r) = f(h, r) = 0$. (A similar discussion applies to the other end. Indeed, after rotation by ρ_Z , the other end has the same form.) By Proposition 5.1 (20),

$$\left| \frac{\partial w}{\partial \theta} \right| = o(r^{-\beta}), \quad \text{and} \quad \left| \frac{\partial^2 w}{\partial \theta^2} \right| = o(r^{-\beta})$$

for every $\beta < \pi/2h$. Since $h \leq \pi/2$, these estimates hold for every $\beta < 1$.

Let $S_R = S \cap C_R$ be the portion of S inside the cylinder C_R . Note that ∂S_R is an extremal curve: it lies on the boundary of the convex set $C_R \cap \{|z| \leq h\}$.

Claim. For R sufficiently large, the total curvature of ∂S_R is strictly less than 4π .

Proof of Claim. The curve ∂S_R consists of two line segments—one on the top and one on the bottom disk of ∂C_R —and two nearly helical curves on ∂C_R . There are four corners where the curves and the line segments meet orthogonally.

By Lemma 3.2, each of the two nearly helical arcs has total curvature strictly less than $2h$, provided R is sufficiently large. Thus ∂S_R has total curvature strictly less than

$$2(2h) + 4(\pi/2) = 4h + 2\pi \leq 4\pi$$

since $h \leq \pi/2$ and since each corner contributes $\pi/2$ to the total curvature. □

We now apply the following uniqueness result :

Theorem 3.6. *A smooth, extremal Jordan curve with total curvature at most 4π bounds precisely one minimal surface and that minimal surface is an embedded disk.*

Meeks and Yau [MY82] prove that a smooth, extremal Jordan curve either bounds two distinct embedded minimal disks or it bounds a unique minimal disk and no other minimal surface of any genus. Together with the result of Nitsche [Nit73] that a smooth Jordan curve with total curvature not greater than 4π bounds a unique minimal disk, they arrive at Theorem 3.6.

The curve ∂S_R we are dealing with has four corners that can be smoothed in the surface with an arbitrarily small increase in total curvature, so that the smoothed

curve will also have total curvature strictly less than 4π . Hence by Theorem 3.6 (applied to the smoothed curve), S_R is simply connected for all sufficiently large R , and therefore S is simply connected.

Thus the surface obtained from S by repeated Schwarz reflection about the boundary lines is a nonplanar, singly periodic, embedded and simply connected minimal surface. By a theorem of Meeks and Rosenberg [MR93], the only such surface is the helicoid. \square

Remark 3.7. For $h < \pi/2$, it is also possible to prove Theorem 3.4, without recourse to Theorem 3.6, as follows. The decay estimates of Proposition 5.1 can be used to show that (for R sufficiently large) the curve ∂S_R projects monotonically to the boundary of a convex region Ω in the plane $\{x = 0\}$. (One shows that the curvature at each point of the projections of the perturbed helical arcs is strictly positive.) By a theorem of Rado (see for example Sections 398 and 400 in Nitsche [Nit89]), S_R must be a graph over Ω . In particular, S_R must be simply connected.

If $h = \pi/2$, the curve ∂S_R still projects monotonically to the boundary of a region Ω in the plane $\{x = 0\}$. However, convexity of Ω at the projections of the corner points of ∂S_R seems to be delicate. In particular, the convexity does not seem to follow from the decay estimates in Proposition 5.1.

4. Asymptotic behavior of symmetric, properly immersed minimal surfaces with one end and finite topology

The goal of this section is to prove the following theorem:

Theorem 4.1. *Let $\mathcal{S} \subset \mathbb{R}^3$ be a properly immersed, nonplanar minimal surface with finite genus, one end, and bounded curvature. Suppose*

1. \mathcal{S} contains $X \cup Z$, and
2. for some value of c , $\{x_3 = c\} \cap \mathcal{S}$ has precisely one divergent component.

Then \mathcal{S} is conformally a once-punctured Riemann surface, and \mathcal{S} is asymptotic to a helicoid.

Note that any level set $M \cap \{z = c\}$ of a properly embedded minimal surface $M \subset \mathbb{R}^3$ can be decomposed uniquely as a union of connected C^1 , properly immersed curves, all intersections and self-intersections of which are transverse. The intersection points are precisely the points of tangency of M and the plane $\{z = c\}$. Thus hypothesis 2 of Theorem 4.1 is equivalent to: *there are only finitely many points of tangency of $\{z = c\}$ and M , and $M \cap \{z = c\}$ can be written as the union of finitely many connected C^1 immersed curves, exactly one of which is not closed.*

Remark 4.2. Our proof of Theorem 4.1 is similar to the proof in [HW, §6.1] that the surfaces constructed in that paper are conformally punctured tori and are asymptotic to a helicoid at infinity.

Proof of Theorem 4.1. The surface \mathcal{S} satisfies the hypotheses of the following theorem of Rodriguez and Rosenberg [RR98]:

Suppose $\mathcal{S} \subset \mathbb{R}^3$ is a properly immersed minimal surface with one end and with bounded curvature. Suppose also that at some level $x_3 = c$, the intersection $\{x_3 = c\} \cap \mathcal{S}$ consists of finitely many curves with finitely many intersections. Then \mathcal{S} is of finite type, i.e., \mathcal{S} is conformally a once-punctured Riemann surface, the puncture corresponding to the end, and the one-forms dg/g and dh are meromorphic on the compact surface.

Here, g is the stereographic projection of the Gauss map from the north pole, and $dh = dx_3 + i dx_3^*$ is a holomorphic one form on M . (The function x_3^* is a harmonic conjugate of x_3 ; it is locally well-defined up to an additive constant.) Note that dh is closed but is not, in general, exact.

Claim. The one form dh has a double pole at the puncture and no residue. The one form dg/g also has a double pole at the puncture.

Assuming the claim, we can complete the proof of the theorem by using the following result of Hoffman and McCuan [HM03]: *Let $E \subset \mathbb{R}^3$ be a properly immersed, minimal annular end that is conformally a punctured disk. Suppose that dg/g and dh both have double poles at the puncture and that dh has no residue at the puncture. If E contains a vertical ray and a horizontal ray, then E is asymptotic to a helicoid at infinity.*

We apply this theorem to an end E of \mathcal{S} corresponding to a neighborhood of the puncture. That E contains the requisite rays follows from assumption 1 of the theorem.

Proof of Claim. By assumption 2, in a neighborhood of the puncture, the level curve $\{x_3 = c\} \cap \mathcal{S}$ consists of two smooth curves emanating from the puncture. Hence dh has a pole of order two at that point. (That dh must have a pole at the puncture follows from the maximum principle and the fact that \mathcal{S} has one end.) Since dh is holomorphic on S , it follows from Stokes' Theorem that dh has no residue at the puncture. In a (possibly smaller) neighborhood of the puncture, dh can be assumed to have no zeros. Since dh has a double pole at the puncture, the level curves $\{x_3 = a\} \cap \mathcal{S}$, for any value of a are embedded in this neighborhood. In particular, \mathcal{S} has an embedded end.

We claim that dg/g must have a pole at the puncture. For suppose it does not. Then g has a well-defined value at the puncture, and g is meromorphic on the one-point compactification of \mathcal{S} . Thus \mathcal{S} has finite total curvature ([Oss63], [Oss86], Chapter 9, [HK97], Section 2.3) and, as observed above, an embedded end. Such

ends are asymptotic to a plane or to an end of the catenoid ([Sch83], [LM83], [HK97], Section 2.3). On a catenoid C , dh has a simple pole at an end (observe that the level curves $\{x_3 = a\} \cap C$ are circles that do not pass through the point corresponding to the end.) Therefore, \mathcal{S} is asymptotic to a plane, and since it has one end, the maximum principle implies that it is equal to a plane. Since we are assuming that \mathcal{S} is nonplanar, the contradiction shows that dg/g has a pole at the puncture.

We now determine the order of the pole of dg/g at the puncture. First of all we will show that the order of the pole is even. Note that on the compact Riemann surface $\mathcal{S} \cup \{\infty\}$, the number of zeros minus the number of poles (counting multiplicities) is even. (It is $2(1 - m)$ where m is the genus of \mathcal{S} .) Hence to show that the pole at the puncture has even order, it suffices to show that

- (i) The number of poles of dg/g on \mathcal{S} is even.
- (ii) The number of zeros of dg/g on $\mathcal{S} \setminus \{0\}$ is even.
- (iii) The origin (if it is a zero of dg/g) is a zero of even order.

On \mathcal{S} , dg/g has poles precisely at the zeros and poles of g . Along Z , g is unitary so no poles of dg/g occur there. By the 180° rotation ρ_Z about Z , the zeros and poles of g are paired, counting multiplicity. Hence, dg/g has an even number of poles on \mathcal{S} . This establishes (i). The zeros of dg/g occur at branch points of g (zeros of the Gauss curvature), and the multiplicity of the zero of dg/g is equal to the branching order of g . Except for the origin, these zeros also occur in pairs (the point $p \in \mathcal{S} \setminus Z$ being paired with $\rho_Z(p)$ and $q \in Z \setminus \{0\}$ being paired with $-q$.) This proves (ii). The tangent plane at the origin is the plane P given by $x_2 = 0$. The 180° rotations ρ_X and ρ_Z about X and Z are symmetries of \mathcal{S} and therefore also of $P \cap \mathcal{S}$. Thus $P \cap \mathcal{S}$ must have an even number, say $2k$, of curves passing through the origin. The order of branching of g at the origin is then $2(k - 1)$. This establishes (iii), completing the proof that dg/g has a pole at infinity of even order.

The principal curvature function of a minimal surface is given by the expression ([HK97], page 15)

$$k = \sqrt{-K} = \frac{4|dg/g|/|dh|}{(|g| + 1/|g|)^2}.$$

Along Z , the tangent plane to \mathcal{S} is vertical, so g is unitary and

$$k = \frac{|dg/g|}{|dh|}. \tag{18}$$

Since the curvature of \mathcal{S} is bounded by assumption and since (as we have already shown) dh has a pole of order two at the end, we see from (18) that dg/g has a pole at infinity of order at most two. Since the order is even, it must be exactly two. \square

5. Asymptotic behavior of symmetric, periodic, properly embedded minimal surfaces with finite topology and one end

In this section we give estimates for the rate that a minimal graph with certain helicoidal qualities actually converges to a helicoid. We will use the estimates to prove that the periodic examples constructed in [HW08] are asymptotic to the helicoid. The estimates of Proposition 5.1 were also used in Section 3, to prove that periodic examples with small twist angles do not exist.

Proposition 5.1. *Let $v(r, \theta) = \theta$, a function whose multigraph S over $\mathbb{R}^2 \setminus \{0\}$ is a half-helicoid of H . Suppose S' is another minimal multigraph of a function u over a region of the form*

$$W_A = \{(r, \theta) : r \geq A, |\theta| < h\}$$

with the property that for $r \geq A$,

$$u(r, \pm h) = v(r, \pm h) = \pm h.$$

Suppose further that S' has asymptotically vertical normals as $R \rightarrow \infty$. If $w = u - v$ is bounded, then

$$\begin{aligned} |w| &= o(r^{-\beta}), \\ |Dw| &= o(r^{-(1+\beta)}), \\ |D^2w| &= o(r^{-(2+\beta)}) \end{aligned} \tag{19}$$

for any $\beta < \pi/2h$, where $D = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, and $x_1 = r \cos \theta$, $x_2 = r \sin \theta$. In particular,

$$\left| \frac{\partial w}{\partial \theta} \right| = o(r^{-\beta}), \quad \text{and} \quad \left| \frac{\partial^2 w}{\partial \theta^2} \right| = o(r^{-\beta}) \tag{20}$$

for any $\beta < \pi/2h$.

The proof of Proposition 5.1 involves a Phragmén–Lindelöf-type argument. To apply it, we first show that w is the solution of a linear elliptic equation.

Lemma 5.2. *The vertical distance function between two minimal graphs satisfies a linear elliptic equation*

This is a special case of a well known result for quasilinear, elliptic partial differential equations. (See Gilbarg–Trudinger, [GT98], Chapter 10.) For the reader's convenience we include a proof of the lemma.

Proof of Lemma 5.2. A function u whose graph is a minimal surface satisfies

$$Q(u) = (1 + u_2^2)u_{11} + (1 + u_1^2)u_{22} - (2u_1u_2)u_{12} = 0.$$

If v is another function whose graph is a minimal surface, then writing $w = u - v$,

$$\begin{aligned}
 0 &= Qu - Qv \\
 &= (1 + u_2^2)w_{11} + (1 + u_1^2)w_{22} - (2u_1u_2)w_{12} \\
 &\quad + v_{11}(u_2^2 - v_2^2) + v_{22}(u_1^2 - v_1^2) - 2v_{12}(u_1u_2 - v_1v_2) \\
 &= a_{ij}w_{ij} + [v_{11}(u_2 + v_2)]w_2 + [v_{22}(u_1 + v_1)]w_1 \\
 &\quad - v_{12}((u_2 + v_2)w_1 + (v_1 + u_1)w_2) \\
 &= a_{ij}w_{ij} + [v_{22}(u_1 + v_1) - v_{12}(u_2 + v_2)]w_1 \\
 &\quad + [v_{11}(u_2 + v_2) - v_{12}(u_1 + v_1)]w_2 \\
 &= a_{ij}w_{ij} + b_k w_k,
 \end{aligned}$$

where

$$\begin{aligned}
 a_{11} &= 1 + u_2^2, \quad a_{22} = 1 + u_1^2, \quad a_{12} = a_{21} = u_1u_2, \\
 b_1 &= v_{22}(u_1 + v_1) - v_{12}(u_2 + v_2), \\
 b_2 &= v_{11}(u_2 + v_2) - v_{12}(u_1 + v_1).
 \end{aligned} \tag{21}$$

The operator L , defined by

$$Lw := a_{ij}w_{ij} + b_k w_k = 0, \tag{22}$$

is elliptic and linear (its coefficients do not depend on w or its derivatives), and $Lw = 0$. \square

Suppose u_n and v_n are sequences of solutions to the minimal surface equation on a domain Ω . Then $w_n = u_n - v_n$ satisfies $L_n w_n = 0$, where $L_n = L(u_n, v_n)$ is the linear elliptic operator defined in Lemma 5.2. We will have need of the following result in the proof of Proposition 5.1.

Corollary 5.3. *If u_n and v_n converge uniformly to zero, then L_n converges smoothly to the Laplacian on compact subsets of Ω .*

Proof. Let $f_i : \Omega \rightarrow \mathbb{R}$ be a sequence of solutions to the minimal surface equation that converge uniformly to zero, and let K be a compact subset of Ω . Fix a positive integer k and let

$$\Lambda = \limsup_i \left(\max_{x \in K} |D^k f_i(x)| \right).$$

By passing to a subsequence if necessary we may assume that the \limsup is a limit. By [GT98], Corollary 16.7, $\|f_i\|_{C^{k+1}}$ is uniformly bounded on compact subsets of Ω . Thus, by passing to a further subsequence if necessary, we may assume that the f_i converge, on compact subsets in the C^k norm, to a limit function f . But since the f_i converge uniformly to zero, f is the zero function. Hence $\Lambda = 0$.

We now apply the conclusion of the previous paragraph to the sequences u_n and v_n . From (21), it follows that L_n converges smoothly on compact subsets to the Laplacian. \square

Proof of Proposition 5.1. Let $w = u - v$ be the difference between the two functions that define the minimal multigraphs on W_A . The function w is zero on the rays in ∂W_A and bounded above on the circular arc of radius A in ∂W_A . By Lemma 5.2, w satisfies $Lw = 0$ for the linear elliptic operator L defined in (22).

For any $0 < \beta < \alpha < \frac{\pi}{2h}$, the functions

$$\begin{aligned} f(r, \theta) &= r^{-\beta} \cos(\alpha\theta), \\ g(r, \theta) &= r^{\beta} \cos(\alpha\theta) \end{aligned}$$

are both positive on \overline{W}_A , and they satisfy

$$\begin{aligned} \Delta f &= (\beta^2 - \alpha^2)r^{-2}f < 0, \\ \Delta g &= (\beta^2 - \alpha^2)r^{-2}g < 0 \end{aligned} \tag{23}$$

on W_A .

Claim. There exists an $A' \geq A$, such that

$$Lf < 0 \quad \text{and} \quad Lg < 0 \quad \text{on } W_{A'} = \{(r, \theta) : r \geq A', |\theta| < h\}.$$

Proof of Claim. On S , the half helicoid that is the multigraph of $v(r, \theta) = \theta$, the normal is asymptotically vertical as $r \rightarrow \infty$, and by assumption the same is true for S' , the graph of $u(r, \theta)$. Therefore, $|Du| \rightarrow 0$ and $|Dv| \rightarrow 0$ as $r \rightarrow \infty$. Moreover it is elementary to calculate that $|Dv| = O(r^{-1})$ and $|D^2v| = O(r^{-2})$.

From (21), (23), and the preceding paragraph, we may compute

$$\begin{aligned} Lf &= \Delta f + u_2^2 f_{11} + u_1^2 f_{22} - 2u_1 u_2 f_{12} + b_1 f_1 + b_2 f_2 \\ &= Cr^{-2}f + o(1)(f_{11} + f_{22} - 2f_{12}) + O(r^{-2})(f_1 + f_2), \end{aligned} \tag{24}$$

where $C = (\beta^2 - \alpha^2) < 0$, and the b_k are defined in (21). It is straightforward to compute that $|Df| = O(r^{-1})f$ and $|D^2f| = O(r^{-2})f$, from which it follows that

$$\begin{aligned} Lf &= Cr^{-2}f + o(1)O(r^{-2})f + O(r^{-3})f \\ &= f[Cr^{-2} + o(1)O(r^{-2}) + O(r^{-3})]. \end{aligned} \tag{25}$$

Since $f > 0$ on W_A and $C < 0$, it follows from (25) that $Lf < 0$ for r sufficiently large.

An almost identical proof establishes that $Lg < 0$ for r sufficiently large. \square

Let A' be the constant whose existence is established by Claim 1. Since w is bounded on $\overline{W_{A'}}$ and f is strictly positive on $\overline{W_{A'}}$, there exists a $\lambda > 0$ large enough so that $\lambda f > w$ on the circular arc in $\partial W_{A'}$ of radius A' . On the rays in $\partial W_{A'}$, w is identically zero and both f and g are strictly positive. Therefore, for any $\epsilon > 0$,

$$\tilde{w}_\epsilon := w - \lambda f - \epsilon g < 0 \quad \text{on } \partial W_{A'}.$$

Also, for any $\epsilon > 0$, $\tilde{w}_\epsilon(r, \theta)$ is negative for r sufficiently large. (This is because w is bounded, $f \rightarrow 0$ and $g \rightarrow \infty$ as $r \rightarrow \infty$.) From Claim 1 we have

$$L\tilde{w}_\epsilon > 0 \quad \text{on } W_{A'}.$$

By the maximum principle, we may conclude that $\tilde{w}_\epsilon < 0$ on $W_{A'}$. Since this is valid for any positive ϵ , it follows that $w \leq \lambda f$ on $W_{A'}$, which implies that for any $\beta < \frac{\pi}{2h}$,

$$w \leq \lambda r^{-\beta}.$$

We can repeat the same argument for $-w$ and conclude that $-w \leq \lambda r^{-\beta}$. We have proved the first equation of (19) for any $\beta < \frac{\pi}{2h}$. In particular, S is asymptotic to S' .

It remains to establish the asymptotic decay rates, stated in (19), of Dw and D^2w . If a surface is minimal, then its image under rescaling of \mathbb{R}^3 is also minimal. Therefore

$$u_R(p) := \frac{u(Rp)}{R} \quad \text{and} \quad v_R(p) := \frac{v(Rp)}{R} \tag{26}$$

are solutions of minimal surface equation defined for $r > A/R$ and $|\theta| \leq h$. By Schwartz reflection in the boundary rays, we may extend u_R and v_R to the region defined by $r > A/R$ and $|\theta| \leq 2h$. Moreover, $u_R \rightarrow 0$ and $v_R \rightarrow 0$ as $R \rightarrow \infty$. By Lemma 5.2 and Corollary 5.3, $w_R = u_R - v_R$ satisfies a linear elliptic equation, $L_R w_R = 0$, and as $R \rightarrow \infty$, L_R converges smoothly to the Laplacian on compact subsets of the region where $r > 0$ and $|\theta| < 2h$.

Let Γ be the arc given by $r = 1$ and $|\theta| \leq h$. Let Ω be the open set defined by $1/2 < r < 2$ and $|\theta| \leq \frac{3}{2}h$. We will apply the Schauder interior estimates ([GT98], Theorem 6.2) for the operator L_R on Ω . Since L_R is converging smoothly on Ω , there exist positive constants C and R^* such that for $R > R^*$:

$$\begin{aligned} \sup_{p \in \Gamma} |Dw_R(p)| &\leq C \sup_{p \in \Omega} |w_R(p)|, \\ \sup_{p \in \Gamma} |D^2w_R(p)| &\leq C \sup_{p \in \Omega} |w_R(p)|. \end{aligned} \tag{27}$$

Since $w_R(p) = \frac{w(Rp)}{R}$, the first and second derivatives of w_R satisfy

$$\begin{aligned} Dw_R(p) &= Dw(Rp), \\ D^2w_R(p) &= RD^2w(Rp). \end{aligned} \tag{28}$$

Let $R\Gamma := \{Rp : p \in \Gamma\}$ and $R\Omega := \{Rp : p \in \Omega\}$. It now follows from (27) and (28) that for $R > R^*$:

$$\begin{aligned} \sup_{q \in R\Omega} |Dw(q)| &\leq R^{-1}C \sup_{q \in R\Omega} |w(q)|, \\ R \sup_{q \in R\Omega} |D^2w(q)| &\leq R^{-1}C \sup_{q \in R\Omega} |w(q)|. \end{aligned} \tag{29}$$

Since we have already established that $w = o(r^{-\beta})$, the last two estimates of (19) follow immediately from (29). The bounds (20) follow directly from (19). \square

We will use Proposition 5.1 to prove the main result of this section, which concerns the asymptotic behavior of ends of embedded screw-motion-invariant minimal surfaces with some of the properties of the surfaces constructed in [HW08].

Proposition 5.4. *Let $S \subset \mathbb{R}^3$ be a properly embedded minimal surface that is invariant under a screw motion*

$$\sigma : (r \cos \theta, r \sin \theta, \theta) \mapsto (r \cos(\theta + \beta), r \sin(\theta + \beta), \theta + t) \tag{30}$$

with $t \neq 0$. Suppose that the slope of the tangent plane tends to 0 as $r = \sqrt{x^2 + y^2}$ tends to infinity, that the intersection of S with some horizontal plane coincides with a line L outside of a compact set, and that the two ends of the line L correspond to different ends of S/σ .

Then S is asymptotic to a helicoid H' with axis Z . Indeed, the vertical distance between S and H' decays faster than $r^{-\beta}$ for every

$$\beta < \left| \frac{p}{t} \right| \pi,$$

where p is the pitch of the helicoid H' .

The pitch of H' may be defined as the (constant) value that $\partial z / \partial \theta$ takes on $H' \setminus Z$. Note that if $\beta = t = 2h$, then σ is the screw motion σ_{2h} used elsewhere in this paper (3).

Remark 5.5. The theorem is also true for surfaces S with boundary, provided the boundary lies within a bounded distance of Z .

Proof. By translation and rotation, we may assume that the line L is X . We will allow S to have boundary as indicated in the remark. Let C be the interior of a closed solid cylinder about Z that contains ∂S . Choose the radius R of the cylinder large enough that the tangent planes to $S \setminus C$ are all nearly horizontal, and so that $S \cap \{z = 0\}$ coincides with X outside of C . We may assume that $S \subset \mathbb{R}^3 \setminus C$ and that $\partial S \subset \partial C$; otherwise replace S by $S \setminus C$.

Let X^+ and X^- denote the positive and negative portions of $X \setminus C$:

$$\begin{aligned} X^+ &= \{(x, 0, 0) : x \geq R\}, \\ X^- &= \{(x, 0, 0) : x \leq -R\}. \end{aligned}$$

Thus we have

$$S \cap \{z = 0\} = X^+ \cup X^-. \quad (31)$$

Note that each component V of S is a covering space of $\mathbb{R}^2 \setminus \mathbb{B}(0, R)$. Since V is embedded, it must be either a single-sheeted covering or an infinite covering. Thus V can be parametrized as

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta, f(r, \theta)) \quad (r \geq R, \theta \in \mathbb{R}) \quad (32)$$

where this map is either periodic (with period 2π) or else one-to-one, according to whether V is a single-sheeted or not. Note that if f is not periodic, then by properness f is not bounded above or below.

Now let V be the component of S containing X^+ . By the hypothesis about the ends of L , V cannot be the component of S that contains X^- , nor can it be the component that contains any of the rays identified with X^- in S/σ . Thus

$$V \cap \sigma^n X^- = \emptyset \quad \text{for } n \in \mathbb{Z}. \quad (33)$$

Note that we can parametrize V as in (32) with a function f satisfying

$$f(r, 0) \equiv 0.$$

By Schwartz reflection, $f(r, -\theta) \equiv -f(r, \theta)$. In particular, $f(r, -\pi) = -f(r, \pi)$. Now $f(r, \pi) \neq 0$ since V is disjoint from X^- . Thus $f(r, \pi) \neq f(r, -\pi)$, so $f(r, \theta)$ does not have period 2π and therefore is not periodic.

It follows (from (31)) that $f(r, \theta) = 0$ if and only if $\theta = 0$.

By reflecting in the plane $\{z = 0\}$, if necessary, we may assume that $f(r, \theta) > 0$ for $\theta > 0$. We may also assume that $t > 0$. (Otherwise replace σ by σ^{-1} .) Since f is not bounded above, it must intersect the plane $\{z = t\}$. By (31) and by the σ -invariance,

$$S \cap \{z = t\} = \sigma X^+ \cup \sigma X^-.$$

Thus $V \cap \{z = t\}$ must be one or both of the rays σX^\pm . By (33), it must be the ray σX^+ . Let Θ be the value of the parameter θ corresponding to this ray. Then we have:

$$\begin{aligned} 0 &< f(r, \theta) < t \quad \text{for } 0 < \theta < \Theta, \\ f(r, 0) &\equiv 0, \quad \text{and} \quad f(r, \Theta) \equiv t. \end{aligned}$$

If we dilate the surface S by $\lambda > 0$, then R , t , and $f(r, \theta)$ get replaced by λR , λt , and $f(r/\lambda, \theta)$, but Θ does not change.

Thus by scaling by $\lambda = \Theta/t$, we can assume that $t = \Theta$. Hence $\sigma = \sigma_{2h}$ (see (3)), where $h = t/2$. Now V (or more precisely $\sigma_{-h}V$) satisfies the hypotheses of Proposition 5.1. Hence V is asymptotic to H with the asserted decay rate.

Let W be the component of S containing X^- . Exactly the same argument shows that W is also asymptotic (with the asserted decay rate) to some helicoid H' with axis Z and containing X . Note that $H' = H$ since otherwise V and W would intersect.

Finally, S can have no component other than V and W , because any such component, being trapped between V and W , would have to intersect the plane $\{z = 0\}$, and by (31) the only possible intersections are X^+ and X^- . \square

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