Zariski k-plets via dessins d'enfants

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Abstract. We construct exponentially large collections of pairwise distinct equisingular deformation families of irreducible plane curves sharing the same sets of singularities. The fundamental groups of all curves constructed are abelian.

1. Introduction

1.1. Motivation and principal results. Throughout this paper, the *type* of a singular point is its embedded *piecewise linear* type, and equisingular deformations of curves in surfaces are understood in the *piecewise linear* sense, *i.e.*, the *PL*-type of each singular point should be preserved during the deformation. This convention is essential as some of the curves considered have non-simple singularities.

Recall that a *Zariski k-plet* is a collection $C_1, \dots, C_k$ of plane curves, all of the same degree m, such that

(1) all curves have the same *combinatorial data* (see [5] for the definition; for irreducible curves, this means the set of types of singular points), and

(2) the curves are pairwise not equisingular deformation equivalent.

Note that Condition (2) in the definition differs from paper to paper, the most common being the requirement that the pairs $(\mathbb{P}^2, C_i)$ (or complements $\mathbb{P}^2 \smallsetminus C_i$) should not be homeomorphic. In this paper, we choose equisingular deformation equivalence, *i.e.*, being in the same component of the moduli space, as it is the strongest topologically meaningful 'global' equivalence relation. In any case, the construction of topologically distinguishable Zariski k-plets usually starts with finding curves satisfying (2) above.

Historically, the first example of Zariski pairs was found by O. Zariski [33], [34], who constructed a pair of irreducible sextics C_1, C_2, with six cusps each, which differ by the fundamental groups $\pi_1(\mathbb{P}^2 \smallsetminus C_i)$. Since then, a great number

of other examples has been found. Citing recent results only, one can mention a
large series of papers by E. Artal Bartolo, J. Carmona Ruber, J. I. Cogolludo Agustín,
and H. Tokunaga (see [5], [6] and more recent papers [2]–[4] for further references),
A. Degtyarev [11], [13], [14] (paper [11] deals with a direct generalization of Zariski's
example: pairs of sextics distinguished by their Alexander polynomial), C. Eyral and
M. Oka [16], [17], [25], G.-M. Greuel, C. Lossen, and E. Shustin [19] (Zariski pairs
with abelian fundamental groups), Vik. S. Kulikov [23], A. Özgüner [28] (a complete
list of Zariski pairs of irreducible sextics that are distinguished by their Alexander
polynomial), I. Shimada [29]–[31] (a complete list of Zariski pairs of sextics with the
maximal total Milnor number $\mu = 19$, as well as a list of arithmetic Zariski pairs of
sextics), and A. M. Uludağ [32]. The amount of literature on the subject definitely
calls for a comprehensive survey!

With very few exceptions, the examples found in the literature are those of Zariski
pairs or *triples*. To my knowledge, the largest known Zariski k-plets are those con-
structed in Artal Bartolo, Tokunaga [6]: for each integer $m \geqslant 6$, there is a collection
of $([m/2] - 1)$ reducible curves of degree m sharing the same combinatorial data.
The principal result of this paper is the following Theorem 1.1.1, which states that
the size of Zariski k-plets can grow exponentially with the degree. (Theorem 1.1.3
below gives a slightly better count for reducible curves.)

1.1.1 Theorem. *For each integer $m \geqslant 8$, there is a set of singularities shared by*

$$Z(m) = \frac{1}{k} \binom{2k-2}{k-1} \binom{k}{[k/2]} \binom{[k/2]}{\epsilon}$$

*pairwise distinct equisingular deformation families of irreducible plane curves C_i of
degree m, where $k = [(m-2)/2]$ and $\epsilon = m - 2k - 2 \in \{0, 1\}$. The fundamental
groups of all curves C_i are abelian: one has $\pi_1(\mathbb{P}^2 \smallsetminus C_i) = \mathbb{Z}_m$.*

Recall that a *real structure* on a complex surface X is an anti-holomorphic in-
volution conj: $X \to X$. A curve $C \subset X$ is called *real* (with respect to conj) if
conj$(C) = C$, and a deformation C_t, $|t| \leqslant 1$, is called *real* if $C_{\bar{t}} = $ conj C_t. Up to
projective equivalence, there is a unique real structure on $\mathbb{P}^2$; in appropriate coordi-
nates it is given by $(z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$.

For completeness, we enumerate the families containing real curves.

1.1.2 Theorem. *If $m = 8t + 2$ for some $t \in \mathbb{Z}$, then $Z(4t + 2)$ of the families given
by Theorem* 1.1.1 *contain real curves* (*with respect to some real structure in $\mathbb{P}^2$*). *All
other curves* (*and all curves for other values of m*) *split into pairs of disjoint complex
conjugate equisingular deformation families.*

1.1.3 Theorem. *For each integer $m \geqslant 8$, there is a set of combinatorial data shared by*

$$R(m) = \frac{1}{m-5}\binom{2m-12}{m-6}$$

pairwise distinct equisingular deformation families of plane curves C_i of degree m (each curve splitting into an irreducible component of degree $(m-1)$ and a line). The fundamental groups of all curves C_i are abelian: one has $\pi_1(\mathbb{P}^2 \smallsetminus C_i) = \mathbb{Z}$.

If $m = 2t + 1$ is odd, then $R(t+3)$ of the families above contain real curves (with respect to some real structure in $\mathbb{P}^2$). All other curves (and all curves for m even) split into pairs of disjoint complex conjugate equisingular deformation families.

Theorems 1.1.1–1.1.3 are proved in Sections 7.3–7.5, respectively.

It is easy to see that the counts $Z(m)$ and $R(m)$ given by the theorems grow faster than $a^{3m/2}$ and a^{2m}, respectively, for any $a < 2$. A few values of Z and R are listed in the table below.

m	8	9	10	11	12	13	14	...	20	40	80
$Z(m)$	6	6	30	60	140	280	840	...	$2 \cdot 10^5$	$4 \cdot 10^{13}$	$1 \cdot 10^{31}$
$R(m)$	2	5	14	42	132	429	1430	...	$3 \cdot 10^6$	$8 \cdot 10^{17}$	$3 \cdot 10^{41}$

Note that we are not trying to set a record here; probably, there are much larger collections of curves constituting Zariski k-plets. The principal emphasis of this paper is the fact that Zariski k-plets can be exponentially large.

1.2. Other results and tools. The curves given by Theorems 1.1.1 and 1.1.3 are plane curves of degree m with a singular point of multiplicity $(m-3)$. (In a sense, this is the first nontrivial case, as curves with a singular point of multiplicity $(m-2)$ or $(m-1)$ do not produce Zariski pairs, see [10].) When the singular point is blown up, the proper transform of the curve becomes a (generalized) trigonal curve in a rational ruled surface. We explain this relation in Section 2, and the bulk of the paper deals with trigonal curves, whose theory is rather parallel to Kodaira's theory of Jacobian elliptic fibrations.

A trigonal curve can be characterized by its functional j-invariant, which is a rational function $j \colon \mathbb{P}^1 \to \mathbb{P}^1$, so that the singular fibers of the curve are encoded in terms of the pull-back $j^{-1}\{0, 1, \infty\}$ (see Table 1). To study the j-invariants, we follow S. Orevkov's approach [26], [27] (see also [15]) and use a modified version of Grothendieck's *dessins d'enfants*, see Section 4, reducing the classification of trigonal curves with prescribed combinatorial type of singular fibers to a graph theoretical problem. The resulting problem is rather difficult, as the graphs are allowed to undergo a number of modifications (see 4.4) caused by the fact that j may have critical values other than 0, 1, or ∞. To avoid this difficulty, we concentrate on a special case

of the so called *maximal* curves, see 4.4.4, which can be characterized as trigonal curves not admitting any further degeneration (Proposition 4.4.8); the classification of maximal curves reduces to the enumeration of connected planar maps with vertices of valency $\leqslant 3$, see Theorem 4.5.1. We exploit this relation and use oriented rooted binary trees to produce large Zariski k-plets of trigonal curves, see Proposition 7.1.1 and a slight modification in Proposition 8.1.1.

It is worth mentioning that the curves given by Propositions 7.1.1 and 8.1.1 are defined over algebraic number fields (like all maximal curves), and in Theorem 8.1.2 we use this fact to construct a slightly smaller, but still exponentially large, Zariski k-plet of plain curves with discrete moduli space. All these examples seem to be good candidates for exponentially large *arithmetic Zariski k-plets* (in rational ruled surfaces and in the plane) in the sense of Shimada [29], [30].

An important question that remains open is whether the curves constituting various Zariski k-plets constructed in the paper can be distinguished topologically. As a first step in this direction, we calculate the braid monodromy of the trigonal curves, see 7.2. (For the relation between the braid monodromy and the topology of the curve, see Orevkov [27], Vik. S. Kulikov and M. Teicher [24], or Carmona Ruber [9].) In 6.5, we give a general description of the braid monodromy of a trigonal curve in terms of its dessin; it covers all maximal curves with the exception of four explicitly described series. As a simple application, we obtain a criterion of reducibility of a maximal trigonal curve in terms of its skeleton, see Corollary 6.6.1.

As another direct application of the construction, we produce exponentially large Zariski k-plets of Jacobian elliptic surfaces, see 8.2. (Here, by a Zariski k-plet we mean a collection of not fiberwise deformation equivalent surfaces sharing the same combinatorial type of singular fibers.) The series given by Theorem 8.2.2 are related to positive definite lattices of large rank; this gives one hope to distinguish the surfaces, and hence their branch loci, topologically.

1.3. Contents of the paper. In Section 2, we introduce trigonal curves in rational ruled surfaces and discuss their relation to plane curves with a singular point of multiplicity degree $- 3$. Section 3 reminds the basic properties of the j-invariant of a trigonal curve, and Section 4 introduces the dessin of a trigonal curve and the skeleton of a maximal curve. In Section 5, we prove a few technical statements on the fundamental group of a generalized trigonal curve. Section 6 deals with the braid monodromy. The principal results of the paper, Theorems 1.1.1–1.1.3, are proved in Section 7. Finally, in Section 8, we discuss a few modifications of the construction and state a few open problems.

2. Trigonal models

2.1. Hirzebruch surfaces. Recall that the *Hirzebruch surface* Σ_k, $k \geqslant 0$, is a rational geometrically ruled surface with a section E of self-intersection $-k$. If $k > 0$, the ruling is unique and there is a unique section E of self-intersection $-k$; it is called the *exceptional section*. In the exceptional case $k = 0$, the surface $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ admits two rulings, and we choose and fix one of them; any fiber of the other ruling can be chosen for the exceptional section. The fibers of the ruling are referred to as the fibers of Σ_k. The semigroup of classes of effective divisors on Σ_k is generated by the classes of the exceptional section E and a fiber F; one has $E^2 = -k$, $F^2 = 0$, and $E \cdot F = 1$.

An *elementary transformation* of a Hirzebruch surface Σ_k is the birational transformation consisting in blowing up a point $O \in \Sigma_k$ and blowing down the proper transform of the fiber through O. If the blow-up center O does (respectively, does not) belong to the exceptional section $E \subset \Sigma_k$, the result of the elementary transformation is the Hirzebruch surface Σ_{k+1} (respectively, Σ_{k-1}).

2.2. Trigonal curves. A *generalized trigonal curve* on a Hirzebruch surface Σ_k is a reduced curve not containing the exceptional section E and intersecting each generic fiber at three points. Note that a generalized trigonal curve $B \subset \Sigma_k$ may contain fibers of Σ_k as components; we will call them the *linear components* of B.

A *singular fiber* of a generalized trigonal curve $B \subset \Sigma_k$ is a fiber F of Σ_k that is not transversal to the union $B \cup E$. Thus, F is either a linear component of B, or the fiber through a point of intersection of B and E, or the fiber over a critical value of the restriction to B of the projection $\Sigma_k \to \mathbb{P}^1$.

A *trigonal curve* is a generalized trigonal curve disjoint from the exceptional section. (In particular, trigonal curves have no linear components.) For a trigonal curve $B \subset \Sigma_k$, one has $|B| = |3E + 3kF|$; conversely, any curve $B \in |3E + 3kF|$ not containing E as a component is a trigonal curve.

Let F be a singular fiber of a trigonal curve B. If B has at most simple singular points on F and F is not a component of B, then locally $B \cup E$ is the branch locus of a Jacobian elliptic surface X, and the pull-back of F is a singular fiber of X. In this case, we use the standard notation for singular elliptic fibers (referring to the extended Dynkin diagrams) to describe the type of F. Otherwise, B has a singular point of type $\mathbf{J}_{k,p}$ or $\mathbf{E}_{6k+\epsilon}$, see [1] for the notation, and we use the notation $\check{\mathbf{J}}_{k,p}$ and $\widetilde{\mathbf{E}}_{6k+\epsilon}$, respectively, to describe the type of F.

2.2.1 Remark. We will not attempt to give a formal definition of the type of a singular fiber F of a trigonal curve B. One can understand it as the topological type of the boundary singularity (B, F), see [1] for details. As a result of the classification, one can conclude that this type is determined by whether F is a component of B and

(the conjugacy class of) the braid monodromy about F, see Section 6 below for the definition. Alternatively, if F is not a component of B and B has at worst simple singularities on F (which is always the case in this paper), then the type of F is determined by Kodaira's type of the singular fiber of the Jacobian elliptic surface ramified at $B \cup F$, see above.

Any generalized trigonal curve B without linear components can be converted to a trigonal curve by a sequence of elementary transformations, at each step blowing up a point of intersection of B and the exceptional section and blowing down the corresponding fiber.

2.3. Simplified models. Let Σ' be a Hirzebruch surface, and let Σ'' be obtained from Σ' by an elementary transformation. Denote by $O' \in \Sigma'$ and $O'' \in \Sigma''$ the blow-up centers of the transformation and its inverse, respectively, and let $F' \subset \Sigma'$ and $F'' \subset \Sigma''$ be the fibers through O' and O'', respectively. The *transform* $B'' \subset \Sigma''$ of a generalized trigonal curve $B' \subset \Sigma'$ is defined as follows: if B' does not (respectively, does) contain F' as a linear component, then B'' is the proper transform of B' (respectively, the union of the proper transform and fiber F''). In the above notation, there is an obvious diffeomorphism

$$\Sigma' \smallsetminus (B' \cup E' \cup F') \cong \Sigma'' \smallsetminus (B'' \cup E'' \cup F''), \qquad (2.1)$$

where $E' \subset \Sigma'$ and $E'' \subset \Sigma''$ are the exceptional sections.

A trigonal curve $B \subset \Sigma_k$ is called *simplified* if all its singular points are double, *i.e.*, those of type $\mathbf{A}_p$. Clearly, each trigonal curve has a unique *simplified model* $\bar{B} \subset \Sigma_l$, which is obtained from B by a series of elementary transformations: one blows up a triple point of the curve and blows down the corresponding fiber, repeating this process until there are no triple points left.

2.4. Deformations. Let $B \subset \Sigma_k$ be a generalized trigonal curve and $E \subset \Sigma_k$ the exceptional section. We define a *fiberwise deformation* of B as an equisingular deformation (path in the space of curves) preserving the topological types of all singular fibers. Alternatively, a fiberwise deformation can be defined as an equisingular deformation of the curve $B \cup E \cup$ (all singular fibers of B).

A *degeneration* of a generalized trigonal curve B is a family B_t, $|t| \leqslant 1$, of generalized trigonal curves such that $B = B_1$ and the restriction of B_t to the annulus $0 < |t| \leqslant 1$ is a fiberwise deformation. A degeneration is called *nontrivial* if B_0 is not fiberwise deformation equivalent to B.

Let $B_k \subset \Sigma_k$ and $B_{k+1} \subset \Sigma_{k+1}$ be two generalized trigonal curves related by an elementary transformation, and let $E_i \subset \Sigma_i$, $i = k, k+1$, be the respective exceptional sections. In general, it is *not* true that an equisingular deformation of B_k or $B_k \cup E_k$ is necessarily followed by an equisingular deformation of B_{k+1} (respectively,

$B_{k+1} \cup E_{k+1}$) or *vice versa*: it may happen that a singular fiber splits into two and this operation affects the topology of one of the curves without affecting the topology of the other. However, it obviously *is* true that the fiberwise deformations of B_k are in a natural one-to-one correspondence with the fiberwise deformations of B_{k+1}. A precise statement relating deformations of B_k and B_{k+1} would require simple but tedious analysis of a number of types of singular fibers. Instead of attempting to study this problem in full generality (which becomes even more involved if the two curves are related by a series of elementary transformations), we just make sure that, in the examples considered in this paper (see 7.3, 7.5, and 8.1.2), generic equisingular deformations of each curve $B \cup E$ are fiberwise. (In 7.5, a linear component is added to the curve for this purpose.) In more details this issue is addressed in 7.6.

2.5. The trigonal model of a plane curve. Let $C \subset \mathbb{P}^2$ be a reduced curve, $\deg C = m$, and let O be a distinguished singular point of C of multiplicity $(m-3)$. (Such a point is unique whenever $m > 7$ or $m > 6$ and C is irreducible.) By a *linear component* of C we mean a component of degree 1 passing through O.

Blow O up and denote the result by Y_1; it is a Hirzebruch surface Σ_1, and the proper transform $\widetilde{C} = B_1 \subset Y_1$ of C is a generalized trigonal curve. Clearly, the combinatorial type of C determines and is determined by that of $B_1 \cup E_1$, the type of O itself being recovered from the singularities of $B_1 \cup E_1$ located in the exceptional section E_1. Furthermore, equisingular deformations of the pair (C, O) are in a one-to-one correspondence with equisingular deformations of $B_1 \cup E_1$.

Let B_1' be the curve obtained from B_1 by removing its linear components. As in 2.2, one can apply a sequence of elementary transformations to get a sequence of curves $B_i, B_i' \subset Y_i \cong \Sigma_i$, $i = 1, \ldots, k$, so that B_k' is a true trigonal curve. (Here, B_{i+1} is the transform of B_i, and B_{i+1}' is obtained from B_{i+1} by removing its linear components. In other words, we pass to the trigonal model of B_1' while keeping track of the linear components of C.) The curve B_k' is called the *trigonal model* of C. Finally, passing from B_k' to its simplified model $B' \subset Y \cong \Sigma_l$, one obtains the *simplified trigonal model B'* of C.

3. The j-invariant

The contents of this section is a translation to the language of trigonal curves of certain well known notions and facts about elliptic surfaces; for more details we refer to the excellent founding paper by K. Kodaira [22] or to more recent monographs [18] and [7]. In the theory of elliptic surfaces, trigonal curves (in the sense of this paper) arise as the branch loci of the Weierstraß models of Jacobian elliptic surfaces over a rational base. These curves have at most simple singularities and belong to even Hirzebruch surfaces Σ_{2s}. However, most notions and statements extend, more or less directly, to trigonal curves in odd Hirzebruch surfaces Σ_{2s+1}.

3.1. Weierstraß equation. Let $\Sigma_k \to \mathbb{P}^1$ be a Hirzebruch surface. Any trigonal curve $B \subset \Sigma_k$ can be given by a *Weierstraß equation*; in appropriate affine charts it has the form

$$x^3 + g_2 x + g_3 = 0,$$

where g_2 and g_3 are certain sections of $\mathcal{O}_{\mathbb{P}^1}(2k)$ and $\mathcal{O}_{\mathbb{P}^1}(3k)$, respectively, and x is a coordinate such that $x = 0$ is the zero section and $x = \infty$ is the exceptional section $E \subset \Sigma_k$. The sections g_2, g_3 are determined by the curve uniquely up to the transformation

$$(g_2, g_3) \mapsto (t^2 g_2, t^3 g_3), \quad t \in \mathbb{C}^*. \tag{3.1}$$

The following statement is straightforward.

3.1.1 Proposition. *A trigonal curve B as in 3.1 is simplified if and only if there is no point $z \in \mathbb{P}^1$ which is a root of g_2 of multiplicity $\geqslant 2$ and a root of g_3 of multiplicity $\geqslant 3$.* □

3.2. The (*functional*) *j-invariant* of a trigonal curve $B \subset \Sigma_k$ is the meromorphic function $j = j_B : \mathbb{P}^1 \to \mathbb{P}^1$ given by

$$j = \frac{4g_2^3}{\Delta}, \quad \Delta = 4g_2^3 + 27g_3^2,$$

where g_2 and g_3 are the coefficients of the Weierstraß equation of B, see 3.1. Here, the domain of j is the base of the ruling $\Sigma_k \to \mathbb{P}^1$, whereas its range is the standard projective line $\mathbb{P}^1 = \mathbb{C}^1 \cup \{\infty\}$. If the fiber F_z over $z \in \mathbb{P}^1$ is nonsingular, then the value $j(z)$ is the usual j-invariant (divided by the magic number $1728 = 12^3$) of the quadruple of points cut on F_z by the union $B \cup E$ (or, in more conventional terms, the j-invariant of the elliptic curve that is the double of $F_z \cong \mathbb{P}^1$ ramified at the four points above). The values of j at the finitely many remaining points corresponding to the singular fibers of B are obtained by analytic continuation.

Since j_B is defined *via* affine charts and analytic continuation, it is obviously invariant under elementary transformations. In particular, the notion of j-invariant can be extended to generalized trigonal curves (by ignoring the linear components and passing to a trigonal model), and the j-invariant of a trigonal curve B is the same as that of the simplified model of B.

3.3. The j-invariant $j_B : \mathbb{P}^1 \to \mathbb{P}^1$ has three 'special' values: 0, 1, and ∞. The correspondence between the type of a fiber F_z, see remark in Section 2.2, and the value $j(z)$ (and the ramification index $\mathrm{ind}_z j$ of j at z) is shown in Table 1. (We confine ourselves to the curves with at worst simple singular points. In fact, in view of the invariance of the j-invariant under elementary transformations, it would suffice to consider type $\tilde{\mathbf{A}}$ singular fibers only. For the reader's convenience, we

also cite Kodaira's notation for the types of singular elliptic fibers, *cf.* Section 2.2.) If B is a curve in Σ_k, the maximal degree of j_B is $6k$. However, $\deg j_B$ drops if B has triple singular points or type $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^*$, or $\tilde{\mathbf{A}}_2^*$ singular fibers, see $\Delta \deg j$ in Table 1. It is worth mentioning that the j-invariant of a generic trigonal curve is highly *non*-generic, as it takes values 0 and 1 with multiplicities 3 and 2 respectively (see Comments to Table 1); conversely, a generic function $j : \mathbb{P}^1 \to \mathbb{P}^1$ would arise as the j-invariant of a trigonal curve with a large number of type $\tilde{\mathbf{A}}_0^{**}$ and $\tilde{\mathbf{A}}_1^*$ singular fibers.

Table 1. The values $j(z)$ at singular fibers F_z.

Type of F_z		$j(z)$	$\mathrm{ind}_z\, j$	$\Delta \deg j$	$\mathrm{mult}\, F_z$
$\tilde{\mathbf{A}}_p\ (\tilde{\mathbf{D}}_{p+5})$, $p \geqslant 1$	$\mathrm{I}_{p+1}\ (\mathrm{I}^*_{p+1})$	∞	$p+1$	$0\ (-6)$	$p+1\ (p+7)$
$\tilde{\mathbf{A}}_0^*\ (\tilde{\mathbf{D}}_5)$	$\mathrm{I}_1\ (\mathrm{I}^*_1)$	∞	1	$0\ (-6)$	$1\ (7)$
$\tilde{\mathbf{A}}_0^{**}\ (\tilde{\mathbf{E}}_6)$	$\mathrm{II}\ (\mathrm{II}^*)$	0	$1 \bmod 3$	$-2\ (-8)$	$2\ (8)$
$\tilde{\mathbf{A}}_1^*\ (\tilde{\mathbf{E}}_7)$	$\mathrm{III}\ (\mathrm{III}^*)$	1	$1 \bmod 2$	$-3\ (-9)$	$3\ (9)$
$\tilde{\mathbf{A}}_2^*\ (\tilde{\mathbf{E}}_8)$	$\mathrm{IV}\ (\mathrm{IV}^*)$	0	$2 \bmod 3$	$-4\ (-10)$	$4\ (10)$

Comments. Fibers of type $\tilde{\mathbf{A}}_0$ (Kodaira's I_0) are not singular. For a nonsingular fiber F_z with complex multiplication of order 2 (respectively, 3) one has $j(z) = 1$ and $\mathrm{ind}_z\, j = 0 \bmod 2$ (respectively, $j(z) = 0$ and $\mathrm{ind}_z\, j = 0 \bmod 3$). Singular fibers of type $\tilde{\mathbf{D}}_4$ (Kodaira's I_0^*) are not detected by the j-invariant, except that each such fiber decreases the degree of j by 6. The multiplicity $\mathrm{mult}\, F_z$ is the number of simplest (*i.e.*, type $\tilde{\mathbf{A}}_0^*$) singular fibers resulting from a generic perturbation of F_z.

3.4. Isotrivial curves. A trigonal curve $B \subset \Sigma_k$ is called *isotrivial* if $j_B = \mathrm{const}$. All simplified isotrivial curves can easily be classified.

(1) If $j_B \equiv 0$, then $g_2 \equiv 0$ and g_3 is a section of $\mathcal{O}_{\mathbb{P}^1}(3k)$ whose all roots are simple or double, see Proposition 3.1.1. The singular fibers of B are of type $\tilde{\mathbf{A}}_0^{**}$ (over the simple roots of g_3) or $\tilde{\mathbf{A}}_2^*$ (over the double roots of g_3).

(2) If $j_B \equiv 1$, then $g_3 \equiv 0$ and g_2 is a section of $\mathcal{O}_{\mathbb{P}^1}(2k)$ with simple roots only, see Proposition 3.1.1. All singular fibers of B are of type $\tilde{\mathbf{A}}_1^*$ (over the roots of g_2).

(3) If $j_B = \mathrm{const} \neq 0, 1$, then $g_2^3 \equiv \lambda g_2^2$ for some $\lambda \in \mathbb{C}^*$; in view of Proposition 3.1.1, this implies that $k = 0$ and $g_2, g_3 = \mathrm{const}$, *i.e.*, B is a union of disjoint sections of Σ_0. (In particular, B has no singular fibers.)

Note that an isotrivial trigonal curve cannot be fiberwise deformation equivalent to a non-isotrivial one, as a non-constant j-invariant j_B would take value ∞ and hence the curve would have a singular fiber of type $\tilde{\mathbf{A}}_0^*$ or $\tilde{\mathbf{A}}_p$, $p > 0$, see Table 1.

3.4.1 Proposition. *Any non-constant meromorphic function $j : \mathbb{P}^1 \to \mathbb{P}^1$ is the j-invariant of a certain simplified trigonal curve $B \subset \Sigma_k$; the latter is unique up to the change of coordinates given by* (3.1).

Proof. For simplicity, restrict all functions/sections to an affine portion $\mathbb{C}^1 \subset \mathbb{P}^1$, which we assume to contain all pull-backs $j^{-1}(0)$ and $j^{-1}(1)$. Represent the function $l = j/(1 - j)$ by an irreducible fraction p/q. Since $l(\infty) \neq 0, \infty$, one has deg $p =$ deg q. For each root a of p of multiplicity 1 mod 3 (respectively, 2 mod 3), multiply both p and q by $(z-a)^2$ (respectively, $(z-a)^4$), and for each root b of q of multiplicity 1 mod 2, multiply both p and q by $(z-b)^3$. In the resulting representation $l = \bar{p}/\bar{q}$, the multiplicity of each root of $\bar{p}$ (respectively, $\bar{q}$) is divisible by 3 (respectively, 2), and $\bar{p}$ and $\bar{q}$ have no common roots of multiplicity $\geqslant 6$. Hence, one has $\bar{p} = 4g_2^3$ and $\bar{q} = 27g_3^2$ for some polynomials g_2, g_3 satisfying the condition in Proposition 3.1.1, and the function $j = l/(l + 1) = \bar{p}/(\bar{p} + \bar{q})$ is the j-invariant of the simplified trigonal curve $B \subset \Sigma_k$ given by the Weierstraß equation with coefficients g_2, g_3, where $k = \frac{1}{6}$ deg $\bar{p} = \frac{1}{6}$ deg $\bar{q}$. Clearly, the polynomials g_2, g_3 as above are defined by l uniquely up to the transformation given by (3.1). $\square$

3.4.2 Proposition. *A fiberwise deformation of a non-isotrivial trigonal curve B results in a deformation of its j-invariant $j = j_B : \mathbb{P}^1 \to \mathbb{P}^1$ with the following properties*:

(1) *the degree of the map $j : \mathbb{P}^1 \to \mathbb{P}^1$ remains constant*;

(2) *distinct poles of j remain distinct, and their multiplicities remain constant*;

(3) *the multiplicity of each root of j remains constant* mod 3;

(4) *the multiplicity of each root of $j - 1$ remains constant* mod 2.

Conversely, any deformation of nonconstant meromorphic functions $j : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying conditions (1)–(4) *above results in a fiberwise deformation of the corresponding* (via *Proposition* 3.4.1) *simplified trigonal curves.*

3.4.3 Remark. Condition 3.4.2 (3) means that a root of j of multiplicity divisible by 3 may join another root and, conversely, a root of large multiplicity may break into several roots, all but one having multiplicities divisible by 3. Condition 3.4.2 (4) should be interpreted similarly.

3.4.4 Remark. Note that just an equisingular (not necessarily fiberwise) deformation of trigonal curves does not always result in a deformation of their j-invariants. In the case of simplified curves, the degree of j_B drops whenever a type $\tilde{\mathbf{A}}_0^*$ singular fiber of B joins another singular fiber, of type $\tilde{\mathbf{A}}_0^*, \tilde{\mathbf{A}}_1$, or $\tilde{\mathbf{A}}_2$, to form a fiber of type $\tilde{\mathbf{A}}_0^{**}$, $\tilde{\mathbf{A}}_1^*$, or $\tilde{\mathbf{A}}_2^*$, respectively, see Table 1.

Proof. The direct statement follows essentially from Table 1. Indeed, the multiplicities of the poles of j_B, (mod 3)-multiplicities of its roots, and (mod 2)-multiplicities of the roots of $j_B - 1$ are encoded in the singular fibers of B, and the degree $\deg j_B$ can be found as the sum of the multiplicities of all poles of j_B. Since the expression for j_B depends 'continuously' on the coefficients of the Weierstrass equation and $\deg j_B$ remains constant, there is no extra cancellation during the deformation and the map $j_B \colon \mathbb{P}^1 \to \mathbb{P}^1$ changes continuously.

The converse statement follows from the construction of the simplified trigonal curve B from a given j-invariant j, see the proof of Proposition 3.4.1. Since the degree $\deg l = \deg j$ remains constant, the polynomials p and q in the irreducible representation $l = p/q$ change continuously during the deformation. Crucial is the fact that the passage from p/q to $\bar{p}/\bar{q}$ depends only on the roots of p and q whose multiplicity is not divisible by 3 and 2, respectively. Hence, due to Conditions 3.4.2 (3) and (4), the degree $\deg \bar{p} = \deg \bar{q}$ will remain constant, the polynomials $\bar{p}$ and $\bar{q}$ will change continuously, and so will the coefficients g_2, g_3 of the Weierstrass equation. The fact that the resulting deformation of the trigonal curves is fiberwise follows again from Table 1. $\square$

4. Dessins d'enfants and skeletons

According to Propositions 3.4.1 and 3.4.2, the study of simplified trigonal curves in Hirzebruch surfaces is reduced to the study of meromorphic functions $j : \mathbb{P}^1 \to \mathbb{P}^1$ with three 'essential' critical values 0, 1, and ∞ and, possibly, a few other critical values. Following S. Orevkov [26], [27], we employ a modified version of Grothendieck's *dessins d'enfants*. Below, we outline briefly the basic concepts and principal results; for more details and proofs we refer to [15], Sections 5.1 and 5.2. Note that [15] deals with a real version of the theory, where functions (graphs) are supplied with an anti-holomorphic (respectively, orientation reversing) involution; however, all proofs apply to the settings of this paper literally, with the real structure ignored.

Since, in this paper, we deal with rational ruled surfaces only, we restrict the further exhibition to the case of graphs in the sphere $S^2 \cong \mathbb{P}^1$.

4.1. Trichotomic graphs. Given a graph $\Gamma \subset S^2$, we denote by S^2_Γ the closed cut of S^2 along Γ. The connected components of S^2_Γ are called the *regions* of Γ. (Unless specified otherwise, in the topological part of this section we are working in the *PL*-category.)

A *trichotomic graph* is an embedded oriented graph $\Gamma \subset S^2$ decorated with the following additional structures (referred to as *colorings* of the edges and vertices of Γ, respectively):

– each edge of Γ is of one of the three kinds: solid, bold, or dotted;

– each vertex of Γ is of one of the four kinds: $\bullet$, $\circ$, $\times$, or monochrome (the vertices of the first three kinds being called *essential*)

and satisfying the following conditions:

(1) the valency of each essential vertex of Γ is at least 2, and the valency of each monochrome vertex of Γ is at least 3;

(2) the orientations of the edges of Γ form an orientation of the boundary ∂S_Γ^2; this orientation extends to an orientation of S_Γ^2;

(3) all edges incident to a monochrome vertex are of the same kind;

(4) $\times$-vertices are incident to incoming dotted edges and outgoing solid edges;

(5) $\bullet$-vertices are incident to incoming solid edges and outgoing bold edges;

(6) $\circ$-vertices are incident to incoming bold edges and outgoing dotted edges.

In (4)–(6) the lists are complete, *i.e.*, vertices cannot be incident to edges of other kinds or with different orientation.

Condition (2) implies that the orientations of the edges incident to a vertex alternate. In particular, all vertices of Γ have even valencies.

4.2. Dessins. In view of 4.1 (3), the monochrome vertices of a trichotomic graph Γ can further be subdivided into solid, bold, and dotted, according to their incident edges. A path in Γ is called *monochrome* if all its vertices are monochrome. (Then, all vertices of the path are of the same kind, and all its edges are of the same kind as its vertices.) Given two monochrome vertices $u, v \in \Gamma$, we say that $u \prec v$ if there is an oriented monochrome path from u to v. (Clearly, only vertices of the same kind can be compatible.) The graph is called *admissible* if $\prec$ is a partial order. Since $\prec$ is obviously transitive, this condition is equivalent to the requirement that Γ should have no oriented monochrome cycles.

In this paper, an admissible trichotomic graph is called a *dessin*.

4.2.1 Remark. Note that the orientation of Γ is almost superfluous. Indeed, Γ may have at most two orientations satisfying 4.1 (2), and if Γ has at least one essential vertex, its orientation is uniquely determined by 4.1 (4)–(6). Note also that each connected component of an admissible graph does have essential vertices (of all three kinds), as otherwise any component of ∂S_Γ^2 would be an oriented monochrome cycle.

4.2.2 Remark. In fact, all three decorations of a dessin Γ (orientation and the two colorings) can be recovered from any of the colorings. However, for clarity we retain both colorings in the diagrams.

4.3. The dessin of a trigonal curve. Any orientation preserving ramified covering $j: S^2 \to \mathbb{P}^1$ defines a trichotomic graph $\Gamma(j) \subset S^2$. As a set, $\Gamma(j)$ is the pull-back $j^{-1}(\mathbb{P}^1_\mathbb{R})$. (Here, $\mathbb{P}^1_\mathbb{R} \subset \mathbb{P}^1$ is the fixed point set of the standard real structure $z \mapsto \bar{z}$.) The trichotomic graph structure on $\Gamma(j)$ is introduced as follows: the •-, ∘-, and ×-vertices are the pull-backs of 0, 1, and ∞, respectively (monochrome vertices being the ramification points with other real critical values), the edges are solid, bold, or dotted provided that their images belong to $[\infty, 0]$, $[0, 1]$, or $[1, \infty]$, respectively, and the orientation of $\Gamma(j)$ is that induced from the positive orientation of $\mathbb{P}^1_\mathbb{R}$ (*i.e.*, order of $\mathbb{R}$).

As shown in [15], a trichotomic graph $\Gamma \subset S^2$ is a dessin if and only if it has the form $\Gamma(j)$ for some orientation preserving ramified covering $j: S^2 \to \mathbb{P}^1$; the latter is determined by Γ uniquely up to homotopy in the class of ramified coverings having a fixed trichotomic graph.

We define the *dessin* $\Gamma(B)$ of a trigonal curve B as the dessin $\Gamma(j)$ of its j-invariant $j: \mathbb{P}^1 \to \mathbb{P}^1$. The correspondence between the singular fibers of a simplified trigonal curve B and the vertices of its dessin $\Gamma(B)$ is given by Table 1 (see $j(z)$), the valency of a vertex z being twice the ramification index $\text{ind}_z j$. The •- (respectively, ∘-) vertices of $\Gamma(B)$ of valency 0 mod 6 (respectively, 0 mod 4) correspond to the nonsingular fibers of B with complex multiplication of order 3 (respectively, 2); such vertices are called *nonsingular*, whereas all other essential vertices of Γ are called *singular*.

4.4. Equivalence of dessins. Let $\Gamma \subset S^2$ be a trichotomic graph, and let v be a vertex of Γ. Pick a regular neighborhood $U \subset S^2$ of v and replace the intersection $\Gamma \cap U$ with another decorated graph, so that the result Γ' is again a trichotomic graph. If $\Gamma' \cap U$ contains essential vertices of at most one kind, then Γ' is called a *perturbation* of Γ (at v), and the original graph Γ is called a *degeneration* of Γ'.

A perturbation Γ' of a dessin is also a dessin if and only if the intersection $\Gamma' \cap U$ contains no oriented monochrome cycles. There are no simple local criteria for the admissibility of a degeneration.

4.4.1 Remark. Assume that the perturbation Γ' is a dessin. Since the intersection $\Gamma' \cap \partial U$ is fixed, the assumption on $\Gamma' \cap U$ implies that $\Gamma' \cap U$ either is monochrome (if v is monochrome) or consists of monochrome vertices, essential vertices of the same kind as v, and edges of the two kinds incident to v.

A perturbation Γ' of a dessin Γ at a vertex v (and the inverse degeneration of Γ' to Γ) is called *equisingular* if v is not a ×-vertex and the intersection $\Gamma' \cap U$ contains at most one singular •- or ∘-vertex. Two dessins $\Gamma', \Gamma'' \subset S^2$ are said to be *equivalent* if they can be connected by a chain $\Gamma' = \Gamma_0, \Gamma_1, \ldots, \Gamma_n = \Gamma''$ of dessins, where each Γ_i, $1 \leqslant i \leqslant n$, either is isotopic to Γ_{i-1} or is an equisingular perturbation or degeneration of Γ_{i-1}. Clearly, equivalence of dessins is an equivalence relation.

4.4.2 Remark. By an *isotopy* between two dessins Γ' and Γ'' we mean a *PL*-family ϕ_t of *PL*-autohomeomorphisms of S^2 such that $\phi_0 = \mathrm{id}$ and $\phi_1(\Gamma') = \Gamma''$, the latter map taking vertices to vertices and edges to edges and preserving both colorings of the dessins. Note that, since the mapping class group of S^2 is trivial, one can just require that Γ' is taken to Γ'' by an *orientation preserving PL*-autohomeomorphism of S^2 respecting the graph structure and the colorings.

The following statement, essentially based on the Riemann existence theorem, is an immediate consequence of Propositions 3.4.1 and 3.4.2 and the results of [15] (particularly, Corollaries 5.1.8 and 5.2.3, with the real structure ignored).

4.4.3 Theorem. *The map $B \mapsto \Gamma(B)$ sending a trigonal curve B to its dessin establishes a one-to-one correspondence between the set of fiberwise deformation classes of simplified trigonal curves in Hirzebruch surfaces and the set of equivalence classes of dessins.* $\qquad\square$

4.4.4 Definition (Maximal curves and dessins). A dessin $\Gamma \subset S^2$ is called *maximal* if it satisfies the following conditions:

 (1) all vertices of Γ are essential;

 (2) all ●- (respectively, ○-) vertices of Γ have valency $\leqslant 6$ (respectively, $\leqslant 4$);

 (3) all regions of Γ are triangles.

A simplified trigonal curve B is called *maximal* if its dessin $\Gamma(B)$ is maximal.

4.4.5 Remark. Conditions 4.4.4 (1) and (3) in the definition of a maximal dessin can be restated as the requirement that the function $j : S^2 \to \mathbb{P}^1$ constructed from Γ, see 4.3, should have no critical values other than 0, 1, and ∞.

4.4.6 Remark. Any maximal trigonal curve is defined over an algebraic number field. Indeed, as any function with three critical values, the rational function $j : \mathbb{P}^1 \to \mathbb{P}^1$ has finitely many Galois conjugates and hence is defined over an algebraic number field. Then, the construction in the proof of Proposition 3.4.1 shows that the coefficients g_2, g_3 of the Weierstraß equation are defined over the splitting field of j. (One may need to add to the field some roots and poles of j.)

4.4.7 Proposition. *A trigonal curve B' is fiberwise deformation equivalent to a maximal trigonal curve B if and only if the dessins $\Gamma(B')$ and $\Gamma(B)$ are isotopic. Furthermore, a permutation of the singular fibers of a maximal trigonal curve B is realized by a fiberwise self-deformation if and only if the corresponding permutation of the vertices of $\Gamma(B)$ is induced by an isotopy of $\Gamma(B)$.*

Proof. A maximal dessin does not admit nontrivial equisingular perturbations (due to Conditions (1) and (2), as an equisingular perturbation requires a vertex of high valency) or degenerations (due to Condition (3), as a perturbation produces more than triangle regions). Hence, any equivalence to a maximal dessin is an isotopy.   □

4.4.8 Proposition. *A trigonal curve B is maximal if and only if it does not admit a nontrivial degeneration, see* 2.4, *to a non-isotrivial curve.*

Proof. Let $\Gamma = \Gamma(B)$. After a small deformation, we can assume that Γ satisfies the general position assumptions 4.4.4 (1) and (2). Then, if B is not maximal, Γ has a region R whose boundary contains at least two ×-vertices, and these two vertices can be brought together within R. This degeneration of Γ results in a nontrivial degeneration of the curve.

Conversely, assume that B has a nontrivial degeneration to a curve B_0, which is necessarily trigonal. Then, up to isotopy, Γ is obtained from $\Gamma(B_0)$ by removing disjoint regular neighborhoods of some of its vertices and replacing them with new decorated graphs. (Since deg j may change, it is no longer required that each of the new graphs should contain essential vertices of at most one kind. Note that we do not discuss the realizability of any such modification by an actual degeneration of curves.) If this procedure is nontrivial, it results in a graph Γ with at least one non-triangular region.   □

4.5. Skeletons. An *abstract skeleton* is a connected planar map Sk $\subset S^2$ whose vertices have valencies at most three; we allow the possibility of *hanging edges*, *i.e.*, edges with only one end attached to a vertex. An *isomorphism* between two abstract skeletons Sk$'$ and Sk$''$ is an orientation preserving *PL*-autohomeomorphism of S^2 taking Sk$'$ to Sk$''$.

The *skeleton* of a maximal trigonal curve B is the skeleton Sk$(B) \subset S^2$ obtained from the dessin $\Gamma(B)$ by removing all ×-vertices and incident edges (*i.e.*, all solid and dotted edges) and disregarding the ∘-vertices. (Note that the resulting graph is indeed connected due to Condition 4.4.4 (3).) Clearly, a maximal dessin Γ is uniquely (up to homotopy) recovered from its skeleton Sk: one should place a ∘-vertex at the middle of each edge (at the free end of each hanging edge), place a ×-vertex v_R at the center of each region R of Sk, and connect this vertex v_R to the •- and ∘-vertices in the boundary ∂R by appropriate (respectively, solid and dotted) edges. The last operation is unambiguous as, due to the connectedness of Sk, each open region R is a topological disk.

4.5.1 Theorem. *The map $B \mapsto \mathrm{Sk}(B)$ sending a maximal trigonal curve B to its skeleton establishes a one-to-one correspondence between the set of fiberwise deformation classes of maximal trigonal curves in Hirzebruch surfaces and the set of isomorphism classes of abstract skeletons in S^2.*

Proof. The statement follows from the correspondence between maximal dessins and skeletons described above, Theorem 4.4.3, and Proposition 4.4.7.   □

4.5.2 Remark. Removing from a dessin $\Gamma(j)$ all ×-vertices and incident edges results in a classical *dessin d'enfants* in the sense of Grothendieck, *i.e.*, the bipartite graph obtained as the pull-back $j^{-1}([0, 1])$. The passage to the skeletons is a further simplification due to the fact that, under the assumptions on maximal dessins, all ◦-vertices have valency at most two.

4.5.3 Remark. Theorem 4.5.1 suggests that, in general, the classification of maximal trigonal curves with a prescribed combinatorial type of singular fibers is a wild problem: one would have to enumerate all planar maps with prescribed valencies of vertices and numbers of edges of regions. The only general result in this direction that I am aware of is the Hurwitz formula [20] (see also [8]), which establishes a relation between a certain weighed count of planar maps (more precisely, ramified coverings of $\mathbb{P}^1$, not necessarily connected) and characters of symmetric groups.

4.6. Vertex count. We conclude this section with a few simple counts. For a dessin Γ, denote by $\#_* = \#_*(\Gamma)$ the total number of ∗-vertices (where ∗ is either •, or ◦, or ×), and by $\#_*(i)$, $i \in \mathbb{N}$, the number of ∗-vertices of valency $2i$. (Recall that valencies of all vertices of a dessin are even.) Consider a trigonal curve $B \subset \Sigma_k$, its j-invariant $j \colon \mathbb{P}^1 \to \mathbb{P}^1$, and its dessin $\Gamma = \Gamma(B)$. Counting the number of points in one of the three special fibers of j, one obtains

$$\deg j = \sum_{i>0} i \#_\bullet(i) = \sum_{i>0} i \#_\circ(i) = \sum_{i>0} i \#_\times(i). \tag{4.1}$$

Since B can be perturbed to a generic trigonal curve in the same surface Σ_k, and a generic curve has $\deg \Delta = 6k$ simplest singular fibers, Table 1 yields

$$6k = \sum_{i>0} i \#_\times(i) + 2 \sum_{i=1(3)} \#_\bullet(i) + 3 \sum_{i=1(2)} \#_\circ(i) + 4 \sum_{i=2(3)} \#_\bullet(i). \tag{4.2}$$

(Alternatively, one can notice that the first term in (4.2) equals $\deg j$, see (4.1), and the remaining part of the sum is $6k - \deg j$, see $\Delta \deg j$ in Table 1.) Finally, the Riemann–Hurwitz formula applied to j results in the inequality

$$\#_\bullet + \#_\circ + \#_\times \geqslant \deg j + 2, \tag{4.3}$$

which turns into an equality if and only if j has no critical values other than 0, 1, and ∞, *i.e.*, Conditions 4.4.4 (1) and (3) are satisfied.

5. The fundamental group

5.1. The braid group. Recall that the *braid group* $\mathbb{B}_3$ can be defined as the group of automorphisms of the free group $G = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ sending each generator to a conjugate of another generator and leaving the product $\alpha_1 \alpha_2 \alpha_3$ fixed. We assume that the action of $\mathbb{B}_3$ on G is *from the left*. One has $\mathbb{B}_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle$, where

$$\sigma_1 \colon (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1 \alpha_2 \alpha_1^{-1}, \alpha_1, \alpha_3), \quad \sigma_2 \colon (\alpha_1, \alpha_2, \alpha_3) \mapsto (\alpha_1, \alpha_2 \alpha_3 \alpha_2^{-1}, \alpha_2).$$

We will also consider the elements $\sigma_3 = \sigma_1^{-1} \sigma_2 \sigma_1$ and $\tau = \sigma_2 \sigma_1 = \sigma_3 \sigma_2 = \sigma_1 \sigma_3$. The center of $\mathbb{B}_3$ is the infinite cyclic group generated by τ^3.

Note that the maps $(\sigma_1, \sigma_2) \mapsto (\sigma_2, \sigma_3) \mapsto (\sigma_3, \sigma_1)$ define automorphisms of $\mathbb{B}_3$; in particular, the pairs (σ_2, σ_3) and (σ_3, σ_1) are subject to all relations that hold for (σ_1, σ_2). In what follows, we use the convention $\sigma_{3l+i} = \sigma_i$, $i = 1, 2, 3$, $l \in \mathbb{Z}$.

The *degree* $\deg \beta$ of a braid $\beta \in \mathbb{B}_3$ is defined as its image under the abelianization homomorphism $\mathbb{B}_3 \to \mathbb{Z}$, $\sigma_1, \sigma_2 \mapsto 1$. A braid is uniquely recovered from its degree and its image in the quotient $\mathbb{B}_3 / \tau^3$.

5.2. Van Kampen's method. Let $B \subset \Sigma = \Sigma_k$ be a generalized trigonal curve, and let $E \subset \Sigma$ be the exceptional section. The fundamental group $\pi_1(\Sigma \smallsetminus (B \cup E))$ can be found using an analogue of van Kampen's method [21] applied to the ruling of Σ. Pick a fiber F_∞ (singular or not) over a point $\infty \in \mathbb{P}^1$ and trivialize the ruling over $\mathbb{P}^1 \smallsetminus \infty$. Let $F_1, \ldots, F_r$ be the singular fibers of B other than F_∞. Pick a nonsingular fiber F distinct from F_∞ and a generic section S disjoint from E and intersecting all fibers $F, F_1, \ldots, F_r, F_\infty$ outside of B.

Clearly, $F \smallsetminus (B \cup E)$ is the plane $\mathbb{C}^1 = F \smallsetminus E$ with three punctures. Consider the group $G = \pi_1(F \smallsetminus (B \cup E), F \cap S)$, and let α_1, α_2, α_3 be a standard set of generators of G. Let, further, $\gamma_1, \ldots \gamma_r$ be a standard set of generators of the fundamental group $\pi_1(S \smallsetminus (F_\infty \cup \bigcup_{j=1}^r F_r), S \cap F)$, so that γ_j is a loop around F_j, $j = 1, \ldots, r$. For each $j = 1, \ldots, r$, dragging the fiber F along γ_j and keeping the base point in S results in a certain automorphism $m_j \colon G \to G$, called the *braid monodromy* along γ_j. Strictly speaking, m_j is not necessarily a braid (unless B is disjoint from E); however, it still has the property that the image $m_j(\alpha_i)$ of each standard generator α_i, $i = 1, 2, 3$, is a conjugate of another generator $\alpha_{i'}$.

According to van Kampen, the group $\pi_1(\Sigma \smallsetminus (B \cup E \cup F_\infty \cup \bigcup_{j=1}^r F_r), S \cap F)$ is given by the representation

$$\langle \alpha_1, \alpha_2, \alpha_3, \gamma_1, \ldots, \gamma_r \mid \gamma_j^{-1} \alpha_i \gamma_j = m_j(\alpha_i), i = 1, 2, 3, j = 1, \ldots, r \rangle,$$

and patching back a fiber F_j, $j = 1, \ldots, r$, results in an additional relation $\gamma_j = 1$. Thus, if B has no linear components, the resulting representation for the group

$\pi_1(\Sigma \smallsetminus (B \cup E \cup F_\infty), S \cap F)$ is

$$\langle \alpha_1, \alpha_2, \alpha_3 \mid \alpha_i = m_j(\alpha_i), i = 1, 2, 3, j = 1, \dots, r \rangle.$$

Patching back the remaining fiber F_∞ gives one more relation $\gamma = 1$, where γ is the class of a small loop in S around $S \cap F_\infty$; an expression of γ in terms of $\alpha_1, \alpha_2, \alpha_3$ in the special case of trigonal curves is found below, see Remark 6.2.1.

5.2.1 Remark. Van Kampen's approach applies as well in the case when the curve has linear components: for each such component, one should keep the corresponding generator γ_j and keep the relation $\gamma_j^{-1} \alpha_i \gamma_j = m_j(\alpha_i)$ instead of $\alpha_i = m_j(\alpha_i)$.

5.2.2 Proposition. *Let $B_k \subset \Sigma_k$ and $B_{k+1} \subset \Sigma_{k+1}$ be two generalized trigonal curves, so that B_k is obtained from B_{k+1} by an elementary transformation whose blow-up center O does not belong to B_{k+1}. Then there is a natural isomorphism*

$$\pi_1(\Sigma_k \smallsetminus (B_k \cup E_k)) = \pi_1(\Sigma_{k+1} \smallsetminus (B_{k+1} \cup E_{k+1})),$$

where $E_i \subset \Sigma_i$, $i = k, k+1$, are the exceptional sections.

Proof. Let $F_{k+1} \subset \Sigma_{k+1}$ be the fiber through O, and let $F_k \subset \Sigma_k$ be the fiber contracted by the inverse elementary transformation. The diffeomorphism (2.1) induces an isomorphism

$$\pi_1(\Sigma_k \smallsetminus (B_k \cup E_k \cup F_k)) = \pi_1(\Sigma_{k+1} \smallsetminus (B_{k+1} \cup E_{k+1} \cup F_{k+1})).$$

The group $\pi_1(\Sigma_k \smallsetminus (B_k \cup E_k))$ is obtained from $\pi_1(\Sigma_k \smallsetminus (B_k \cup E_k \cup F_k))$ by adding the relation $[\partial\Gamma_k] = 1$, where $\Gamma_k \subset \Sigma_k$ is a small analytic disk transversal to F_k and disjoint from all other curves involved. Similarly, patching the fiber F_{k+1} results in an additional relation $[\partial\Gamma_{k+1}] = 1$, where $\Gamma_{k+1} \subset \Sigma_{k+1}$ is a small analytic disk transversal to F_{k+1} and disjoint from the other curves in Σ_{k+1}. Under the assumptions, one can choose Γ_{k+1} passing through the blow-up center O; then its proper transform can be taken for Γ_k. Hence, one has $[\partial\Gamma_k] = [\partial\Gamma_{k+1}]$, and the two quotient groups are isomorphic. $\square$

5.2.3 Proposition. *Let $C \subset \mathbb{P}^2$ be an algebraic curve of degree m with a distinguished singular point O of multiplicity $(m-3)$ and without linear components. Assume that C has a branch b at O of type $\mathbf{E}_{12}$. Then C is irreducible and the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C) = \mathbb{Z}_m$ is abelian.*

Proof. Blow O up and consider the proper transform $B_1 \subset \Sigma_1$ of C, see 2.5. The transform of b is a type $\mathbf{E}_6$ singular point of B_1, and the elementary transformation centered at this point converts B_1 to a generalized trigonal curve $B_2 \subset \Sigma_2$ with a type $\tilde{\mathbf{A}}_0^{**}$ singular fiber. In particular, the curve is irreducible.

The inverse transformation is as in Proposition 5.2.2, *i.e.*, its blow-up center does not belong to the curve B_2 or the exceptional section E_2. Hence, one has

$$\pi_1(\mathbb{P}^2 \smallsetminus C) = \pi_1(\Sigma_1 \smallsetminus (B_1 \cup E_1)) = \pi_1(\Sigma_2 \smallsetminus (B_2 \cup E_2)).$$

(The first isomorphism is obvious; the second one is given by Proposition 5.2.2.) The last group can be found using van Kampen's method, see 5.2. Under an appropriate choice of the generators $\alpha_1, \alpha_2, \alpha_3$, the braid monodromy m about a type $\tilde{\mathbf{A}}_0^{**}$ singular fiber is $\tau \in \mathbb{B}_3$, and the relations $m(\alpha_i) = \alpha_i$, $i = 1, 2, 3$, yield $\alpha_1 = \alpha_2 = \alpha_3$. Hence, the group is abelian. □

5.2.4 Proposition. *Let C be the union of an irreducible curve as in Proposition 5.2.3 and $r \geqslant 1$ linear components none of which is tangent to the branch b of type $\mathbf{E}_{12}$. Then one has $\pi_1(\mathbb{P}^2 \smallsetminus C) = \mathbb{Z} \times \langle \gamma_1, \dots \gamma_{r-1} \rangle$. In particular, if $r \leqslant 2$, the group is still abelian.*

Proof. As in the proof of Proposition 5.2.3, there is a relation $\alpha_1 = \alpha_2 = \alpha_3$, and due to the properties of the braid monodromy (each generator is taken to a conjugate of a generator) the relations $\gamma_j^{-1} \alpha_i \gamma_j = m_j(\alpha_i)$ turn into $[\gamma_j, \alpha_i] = 1$. □

6. The braid monodromy

In this section, we describe the braid monodromy of a simplified trigonal curve. We fix such a curve $B \subset \Sigma = \Sigma_k$ and let $\Gamma = \Gamma(B)$. Further, we denote by F_z the fiber over a point $z \in \mathbb{P}^1$, and let $B_z = B \cap F_z$ and $E_z = E \cap F_z$, where $E \subset \Sigma$ is the exceptional section. Note that $F_z \smallsetminus E_z$ is an affine space over $\mathbb{C}^1$; in particular, one can speak about its orientation, lines, circles, angles, and length ratios. We use the notation F_z° for the punctured plane $F_z \smallsetminus (B_z \cup E_z)$.

6.1. Geometry of the fibers. The definition of the j-invariant gives an easy way to recover the topology of B from its dessin Γ. The set B_z consists of a single triple point if z is a singular •- or ∘-vertex. If z is a ×-vertex, B_z consists of two points, one simple and one double. In all other cases, B_z consists of three simple points, whose position in $F_z \smallsetminus E_z$ can be characterized as follows.

(1) If z is an inner point of a region of Γ, the three points of B_z form a triangle with all three edges distinct. Hence, the restriction of the projection $B \to \mathbb{P}^1$ to the interior of each region of Γ is a trivial covering.

(2) If z belongs to a dotted edge of Γ, the three points of B_z are collinear. The ratio (smallest distance)/(largest distance) is in $(0, \frac{1}{2})$; it tends to 0 (respectively, $\frac{1}{2}$) when z approaches a ×- (respectively, ∘-) vertex.

(3) If z belongs to a solid (bold) edge of Γ, the three points of B_z form an isosceles triangle with the angle at the vertex less than (respectively, greater than) $\pi/3$. The angle tends to 0, $\pi/3$, or π when z approaches, respectively, a ×-, •-, or ∘-vertex.

Furthermore, a simple model example proves the following statement.

(4) For a point z as in (1), arrange the vertices of B_z in ascending order based on the length of the opposite edge. The resulting orientation of B_z is counterclockwise if and only if $\Im j_B(z) > 0$.

6.2. Proper sections. To define the braid monodromy, we need to fix a 'fiber at infinity' F_∞, see 5.2, and a generic section S that would provide the base points $S_z = S \cap F_z \in F_z^\circ$. We take for F_∞ the fiber over a fixed point $\infty \notin \Gamma$, and construct S as a small perturbation of $E + kF'$, where F' is the fiber over a point z' in the same open region of Γ as ∞. If the perturbation is sufficiently small, the section S has the following property: *there is a closed neighborhood $K \ni \infty$ disjoint from Γ and such that, for each point $z \in \mathbb{P}^1 \smallsetminus K$, the base point $S_z \in F_z$ is outside a disk $U_z \subset F_z$ containing B_z and centered at its barycenter* (cf. *Figure* 1, *right, below*). In what follows, a section S satisfying this property is called *proper* and, when speaking about the fundamental group $\pi_1(F_z^\circ, S_z)$, we always assume that the point z is outside the above closed neighborhood K.

Note that, together with the exceptional section E and the zero section given by $z \mapsto$ (the barycenter of B_z), a proper section S gives a trivialization of the ruling over $\mathbb{P}^1 \smallsetminus K$, which is necessary to define the braid monodromy.

6.2.1 Remark. From the construction of a proper section S it follows that the class γ of a small loop in S surrounding $F_\infty \cap S$ (see 5.2) is, up to conjugation, given by $\gamma = (\alpha_1\alpha_2\alpha_3)^k\gamma_1 \ldots \gamma_r$. Hence, in this case, the final relation in van Kampen's method is $(\alpha_1\alpha_2\alpha_3)^k = 1$.

6.3. Markings and canonical bases. Let $z \in \Gamma$ be a nonsingular •-vertex. According to 6.1 (3), the three points of the set B_z form an equilateral triangle. There is a natural one-to-one correspondence between the bold edges incident to z and the points of B_z: an edge e corresponds to the point $p \in B_z$ that turns into the vertex of the isosceles triangle when z slides from its original position along e. In fact, the same point p turns into the vertex of the isosceles triangle when z slides along the solid edge e' opposite to e, so that the two other points are brought together over the ×-vertex ending e'.

In what follows, we always assume that the three bold edges e_1, e_2, e_3 incident to z are oriented in the *counterclockwise* direction, as in Figure 1, left. Such an ordering is called a *marking* at z, and an edge e_i incident to z is said to have *index i* at z. A marking at z is uniquely determined by assigning an index to one of the three bold

edges incident to z. Alternatively, a marking is determined by assigning an index to one of the three points constituting B_z.

A *marking* of a dessin Γ is defined as a collection of markings at each nonsingular •-vertex of Γ. The notion of marking and index of edges extends to skeletons in the obvious way.

Using 6.1 (1)–(3), from 6.1 (4) it follows that *if e_1, e_2, e_3 is a marking at a nonsingular •-vertex z, the corresponding points $p_1, p_2, p_3 \in B_z$ form the clockwise orientation of the triangle B_z (Figure 1, right).*
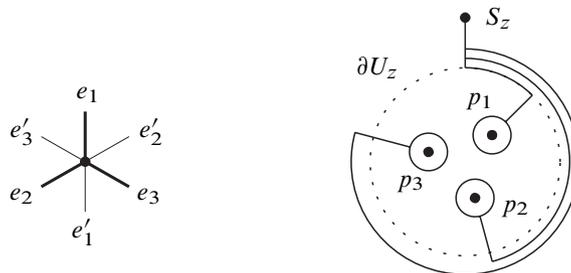
Figure 1. A canonical basis for G_z.

Pick a proper section S, see 6.2, and consider the group $G_z = \pi_1(F_z^\circ, S_z)$. A *canonical basis* for G_z is a basis $\alpha_1, \alpha_2, \alpha_3$ shown in Figure 1, right, where the space F_z° is regarded as the affine line $F_z \smallsetminus E_z$ punctured at $p_1, p_2, p_3 \in B_z$. More precisely, each element α_i is the class of the loop formed by a small counterclockwise circle about p_i, $i = 1, 2, 3$, which is connected to S_z by a radial segment, an arc of a circle ∂U_z separating S_z from B_z (*cf.* 6.2), and another radial segment, common for all three loops. It is required that each consecutive arc is $2\pi/3$ longer than the previous one; however, we do not make any assumption about the length of the first arc: it is defined up to a multiple of 2π. As a result, *a canonical basis $\alpha_1, \alpha_2, \alpha_3$ is determined by a marking at z uniquely up to conjugation by $\alpha_1 \alpha_2 \alpha_3$, i.e., up to the central element $\tau^3 \in \mathbb{B}_3$.*

A canonical basis defines an isomorphism $\rho_z \colon G_z \to G$ to the 'standard' free group $G = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$. This isomorphism is determined by a marking at z up to τ^3. Below, *all braids involving ρ_z are considered up to a power of τ^3.* The isomorphism ρ_z' defined by the cyclic permutation e_2, e_3, e_1 of the bold edges is given by $\rho_z' = \tau \circ \rho_z$.

6.3.1 Remark. In a similar way, one can define a canonical basis and isomorphism $\rho_z \colon G_z \to G$ for a nonsingular ∘-vertex z. The basis and the isomorphism are determined up to a power of τ^3 by an ordering of the two bold edges incident to z. Our choice of •-vertices is motivated by the fact that we will apply the results to skeletons.

6.4. Assumptions and settings. For the rest of this section, we make the following assumptions about Γ:

(1) Γ has no monochrome vertices, all its •-vertices have valency $\leqslant 6$, and all its ∘-vertices have valency $\leqslant 4$;

(2) the union of all •- and ∘-vertices of Γ and its bold edges is connected;

(3) Γ has at least one nonsingular •-vertex.

Note that Condition (1) means that the curve is generic within its fiberwise deformation class, and (2) can be satisfied after a sequence of equisingular perturbations and degenerations, *cf.* [15]. Thus, the only true restriction is (3). In particular, any maximal dessin satisfies (1) and (2), and the remaining Condition (3) rules out four series of maximal curves: those whose skeleton is a simple cycle (one curve in Σ_k for each $k \geqslant 1$) or a linear tree (two curves in Σ_1 and three curves in Σ_k for $k \geqslant 2$; a curve is determined by the number of hanging edges in the skeleton). All these curves are irreducible.

Chose and fix the 'fiber at infinity' F_∞ over a point $\infty \notin \Gamma$ and a proper section S, see 6.2. Denote by $S^\circ \subset \mathbb{P}^1 \cong S^2$ the affine plane $\mathbb{P}^1 \smallsetminus \infty$ punctured at the singular fibers of B. (Since S is a section, S° can as well be regarded as a subset of S.)

As above, let $G = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ be the free group on three generators. Fix a marking of Γ and consider the corresponding isomorphisms $\rho_z : G_z \to G$, see 6.3. Given a path γ in S° connecting two nonsingular •-vertices z' and z'', consider the monodromy $\tilde{m}_\gamma : G_{z'} \to G_{z''}$ and define the automorphism $m_\gamma = \rho_{z''} \circ \tilde{m}_\gamma \circ \rho_{z'}^{-1}$ of G. It is a braid (due to the fact that S is proper). We consider m_γ as an element of the reduced group $\mathbb{B}_3 / \tau^3$, thus removing the ambiguity in the definition of ρ. In the special case $z' = z''$, *i.e.*, when γ is a loop, m_γ is a well defined element of $\mathbb{B}_3$. It can be recovered from its image in $\mathbb{B}_3 / \tau^3$ using the following obvious statement.

6.4.1 Proposition. *The degree of the monodromy $\tilde{m}_\gamma : G_z \to G_z$ defined by a simple loop γ in S° is equal to the total multiplicity $\sum \mathrm{mult}\, F_i$ (see Table 1) of the singular fibers of B encompassed by γ, i.e., separated by γ from ∞.* ☐

6.5. The monodromy. To uniformize the formulas below, we use the convention $e_{3l+i} = e_i$, $i = 1, 2, 3$, $l \in \mathbb{Z}$, for the ordered edges incident to a given nonsingular •-vertex (*cf.* similar convention for the braid group in 5.1).

Let z', z'' be two nonsingular •-vertices, connected by the path γ in Γ formed by two bold edges incident to the same ∘-vertex. Denote $m_\gamma = m_{i,j} \in \mathbb{B}_3 / \tau^3$, where i, j are the indices of the edges constituting γ at z' and z'', respectively. Then

$$m_{i,i+1} = \sigma_i, \quad m_{i+1,i} = \sigma_i^{-1}, \quad \text{and} \quad m_{i,i} = \sigma_i \sigma_{i-1} \sigma_i. \tag{6.1}$$

More generally, let $s \geqslant 0$ be an integer, and let γ be a simple path from z' to z'' composed of $2s$ bold edges, $(s+1)$ ∘-vertices, and s •-vertices of valency 4. Perturb γ

so that each singular •-vertex is circumvented in the counterclockwise direction, and denote by $m_{i,j}(s) \in \mathbb{B}_3/\tau^3$ the resulting monodromy. Then, for all integers $s, t \geqslant 0$, there is a reciprocity relation

$$m_{j+1,i}(t) \cdot m_{i+1,j}(s) \cdot \sigma_i^{s+t+2} = 1, \tag{6.2}$$

which can be used to find $m_{*,*}(s)$ in terms of $m_{*,*}(0) = m_{*,*}$. One has

$$m_{i,i+1}(s) = \sigma_{i+1}^{-s}\sigma_i, \quad m_{i+1,i}(s) = \sigma_i^{-s-1}, \quad \text{and} \quad m_{i,i}(s) = \sigma_i^{-s-2}\sigma_{i+1}^{-1}.$$

Now, let γ be the loop composed of a small counterclockwise circle around a ×-vertex of valency $2d$ connected along a solid edge e_i' (see Figure 1, left) to a nonsingular •-vertex z. The resulting monodromy $c_i(d) = m_\gamma \in \mathbb{B}_3$ is given by

$$c_i(d) = \sigma_{i+1}^d. \tag{6.3}$$

Finally, consider a chain of distinct bold edges starting from an edge e_i at a non-singular •-vertex z and ending at a singular vertex. Let γ be a simple loop at z encompassing all vertices of the chain (except z itself) and oriented in the counterclockwise direction, and let $l_i(d) = m_\gamma \in \mathbb{B}_3$ be the monodromy, where $d = \deg l_i(d)$. (If the chain contains s •-vertices of valency 4, then d can take the values $4s$, $4s + 2$, or $4s + 3$, depending on whether the chain ends at a •-vertex of valency 4, ○-vertex, or •-vertex of valency 2.) One has

$$l_i(4s) = \sigma_i^{-s}\sigma_{i-1}^{-s}\tau^{3s} \quad \text{and} \quad l_i(4s + \epsilon) = \sigma_i^{-2s-5+\epsilon}\sigma_{i+1}^{-1}\tau^{3s+3}, \tag{6.4}$$

where $\epsilon = 2$ or 3.

6.6. Proofs. But for the choice of the trivialization of the ruling, which is also accountable for the τ^3-ambiguity, the monodromy m_γ is local with respect to γ, and it can be found using the description of the geometry of the fibers given in 6.1. We do use this straightforward approach to establish relations (6.1) and (6.3). The expression for $l_i(4s)$ in (6.4) follows from Proposition 6.4.1 and the obvious relation

$$l_i(4s) = m_{j,i}(s) \cdot m_{i,j}(s), \quad j \in \mathbb{Z},$$

in $\mathbb{B}_3/\tau^3$, which is due to our convention that the paths are perturbed so as to circumvent all singular vertices in the counterclockwise direction.

For the rest, we observe that the monodromy related to a fragment of Γ can be found in any other dessin containing this fragment. The reciprocity relation (6.2) is obtained assuming that the two paths resulting in the two $m_{*,*}$ monodromies form the boundary (oriented in the clockwise direction) of a single region R of the skeleton of the dessin, so that R contains a single ×-vertex. (The factor σ_i^{s+t+2} in the relation is, in fact, $c_{i-1}(s + t + 2)$.) The expressions for l_i are obtained in a similar way: we

close the unused bold edges e_{i-1}, e_{i+1} at z 'around' the chain of edges in question and place a single ×-vertex at the center of the resulting region R. Computing the monodromy around ∂R gives the relations

$$c_{i-1}(2s + 5 - \epsilon) \cdot l_i(4s + \epsilon) = l_i(4s + \epsilon) \cdot c_{i+1}(2s + 5 - \epsilon) = m_{i-1,i+1}$$

in $\mathbb{B}_3/\tau^3$ (where $\epsilon = 2$ or 3), which can be used to find l_i.

6.6.1 Corollary. *A maximal trigonal curve B is reducible if and only if all vertices of its skeleton* Sk *are nonsingular* (i.e., *have valency 3) and* Sk *admits a marking with the following properties*:

(1) *each hanging edge has index 1 at the (only) vertex incident to it;*

(2) *any other edge has indices* (1, 1)*,* (2, 3)*, or* (3, 2) *at its two endpoints.*

6.6.2 Remark. Clearly, a marking at any vertex of Sk extends to at most one marking satisfying Condition 6.6.1 (2). If Sk has a hanging edge, it admits at most one marking satisfying 6.6.1 (1) and (2).

6.6.3 Remark. Corollary 6.6.1 still makes sense for a trigonal curve B, not necessarily maximal, whose dessin Γ satisfies Conditions 6.4 (1) and (2). In this case, the existence of a marking as in Corollary 6.6.1 is necessary for B to be reducible; in general, it is not sufficient.

Proof. Let B° be the portion of the curve over S°, and let $\mathrm{pr}\colon B^\circ \to S^\circ$ be the restriction of the projection $\Sigma_k \to \mathbb{P}^1$. It is a triple covering whose monodromy is obtained by downgrading the braid monodromy to the symmetric group $\mathbb{S}_3$. From (6.4) it follows that the monodromy about a singular •-vertex acts transitively on the decks of pr, and hence any curve with such a vertex is irreducible. (As this argument is local, it applies as well to the four exceptional series mentioned in 6.4, proving that they are all irreducible.)

Assume that B is reducible. Then it contains as a component a section of the ruling. Any such section $B_1 \subset B$ defines a marking of Sk: one assigns index 1 to the point $B_1 \cap F_z \in B_z$, see 6.3, and Conditions 6.6.1 (1) and (2) merely list all monodromies $l_i(3)$ and $m_{i,j}$ preserving p_1. Conversely, for any marking as in the statement, the points $p_1 \in B_z$ over all •-vertices $z \in \Gamma(B)$ belong to a deck of pr which is preserved by the monodromy. (One needs to take into account the obvious fact that, for a maximal curve without singular •-vertices, the inclusion homomorphism $\pi_1(\mathrm{Sk}) \to \pi_1(S^\circ)$ is an isomorphism.) Hence, the curve contains a section of the ruling as a component. $\qquad\square$

7. The construction

7.1. Proofs of Theorems 1.1.1 and 1.1.3 are based on the existence of large Zariski k-plets of maximal trigonal curves in Hirzebruch surfaces.

7.1.1 Proposition. *For each integer $k \geqslant 2$, there exists a collection of*

$$C(k-1) = \frac{1}{k}\binom{2k-2}{k-1}$$

pairwise distinct fiberwise deformation families of irreducible maximal trigonal curves $B \subset \Sigma_k$ with the following properties:

(1) *each curve has one fiber of type $\tilde{\mathbf{A}}_0^{**}$, one fiber of type $\tilde{\mathbf{A}}_{5k-3}$, and k fibers of type $\tilde{\mathbf{A}}_0^*$ (and no other singular fibers)*;

(2) *none of the curves admits a fiberwise self-deformation inducing a non-trivial permutation of the singular fibers of the curve.*

Proof. Denote by $\mathcal{T}_s$, $s \geqslant 1$, the set of all binary rooted trees on s vertices. Recall that the cardinality of $\mathcal{T}_s$ is given by the Catalan number $C(s)$,

$$\#\mathcal{T}_s = C(s) = \frac{1}{s+1}\binom{2s}{s}.$$

Each tree $T \in \mathcal{T}_s$ admits a standard 'monotonous' geometric realization $|T| \subset \mathbb{R}^2$, see Figure 2, left. For example, one can map the level l, $l \geqslant 0$, vertices of T to the points $v_{l,i} = (-1 + (2i+1)/2^l, l)$, $i = 0, \ldots, 2^l - 1$, so that the left (respectively, right) edge originating at $v_{l,i}$ connects $v_{l,i}$ to $v_{l+1,2i}$ (respectively, $v_{l+1,2i+1}$).
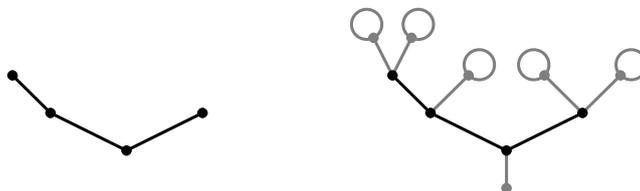

Figure 2. Extending a binary tree T to a skeleton $\mathrm{Sk}(T)$.

Pick a tree $T \in \mathcal{T}_{k-1}$ and extend its geometric realization $|T| \subset \mathbb{R}^2 \subset \mathbb{P}^1$ to a skeleton $\mathrm{Sk}(T)$ as follows: mark the root of T by adding a monovalent vertex at $(0, -1)$ and connecting it to $v_{0,0}$ by an edge, and complete the valency of each vertex of $|T|$ to three by replacing the missing branches with 'leaves', each leaf consisting of a vertex (at an appropriate point $v_{l,i}$, $l > 0$), a loop at this vertex, and a stem

connecting the vertex to the point $v_{l-1,[i/2]}$. (See Figure 2, right, where the trunk and the k leaves added to $|T|$ are shown in grey.)

The resulting skeleton $\mathrm{Sk}(T)$ has one monovalent and $(2k-1)$ trivalent vertices; its faces are k monogons (the interiors of the leaves) and one $(5k-2)$-gon (the outer region). Furthermore, one can easily observe that none of $\mathrm{Sk}(T)$ has a nontrivial automorphism and that two skeletons $\mathrm{Sk}(T_1)$, $\mathrm{Sk}(T_2)$ are isomorphic if and only if $T_1 = T_2$ in $\mathcal{T}_{k-1}$. Here, the key observation is the fact that the root of the original tree T is 'marked' by the only monovalent vertex of the skeleton $\mathrm{Sk}(T)$. Hence, any isomorphism of the skeletons would induce an isomorphism of oriented *rooted* trees (as it also preserves the orientation of S^2). In particular, essentially by its very definition, an oriented *rooted* tree never admits an orientation preserving automorphism.

Applying Theorem 4.5.1, one obtains $\#\mathcal{T}_{k-1} = C(k-1)$ deformation families of maximal trigonal curves with the desired properties. (Each curve is irreducible since it has a type $\tilde{\mathbf{A}}_0^{**}$ singular fiber.) $\square$

7.1.2 Proposition. *If $k = 2s$ is even, then $C(s-1)$ of the trigonal curves given by Proposition 7.1.1 are real (with respect to some real structure on Σ_k). All other curves (and all curves for k odd) split into pairs of complex conjugate curves.*

Proof. A maximal trigonal curve is real if and only if its skeleton is symmetric with respect to some *orientation reversing* involution of the base S^2 (cf. [15], §5 and especially Corollary 5.1.8, where real dessins of real curves are considered: clearly, a symmetric skeleton can be completed to a symmetric dessin, due to the results of [15] cited above, a symmetric dessin gives rise to a real j-invariant, and the further passage from the j-invariant to a trigonal curve is equivariant, cf. the proof of Proposition 3.4.1). A binary rooted tree can be symmetric only if its number of vertices is odd, and all symmetric trees in $\mathcal{T}_{2s-1}$ can be parametrized by their 'left halves', *i.e.*, by $\mathcal{T}_{s-1}$. $\square$

7.2. The braid monodromy. In this section, we apply the results of 6.5 to describe the braid monodromy of the curves given by Proposition 7.1.1.

Fix a curve B corresponding to a tree $T \in \mathcal{T}_{k-1}$ and let $\Gamma = \Gamma(B)$, $\mathrm{Sk} = \mathrm{Sk}(B)$. Let $v_{0,0}$ be the root of the original tree T. Denote by $\Gamma_\times$ the set of ×-vertices of Γ of valency 2 (equivalently, the set of type $\tilde{\mathbf{A}}_0^*$ singular fibers of B). Each vertex $u \in \Gamma_\times$ can be encoded by a word w_u in the alphabet $\{r, l\}$ as follows: let $\bar{u}$ be the •-vertex in the leaf encompassing u, and let ξ_u be the simple path in Sk from $v_{0,0}$ to $\bar{u}$; starting from $v_{0,0}$ and the empty word, walk along ξ_u and, at each vertex, add to the word r or l if the right (respectively, left) branch is chosen at this vertex. (For example, in Figure 2, the ×-vertices encompassed by the five leaves are encoded, from right to left, by the words rr, rl, lr, llr, and lll.) Order $\Gamma_\times$ lexicographically, with $r < l$. (This is the right to left order in the standard geometric realization of the graph, cf. Figure 2.)

As in 6.4, pick a point $\infty \in \mathbb{P}^1 \smallsetminus \Gamma$ and denote by S° the plane $\mathbb{P}^1 \smallsetminus \infty$ punctured at the singular vertices of Γ. Take $v_{0,0}$ for the base point, and consider the basis γ_u, $u \in \Gamma_\times$, $\delta_\times$, $\delta_\bullet$ for $\pi_1(S^\circ, v_{0,0})$ defined as follows:

(1) γ_u, $u \in \Gamma_\times$, is the loop in Sk formed by the circumference of the leaf surrounding u connected to $v_{0,0}$ by the simple path ξ_u;

(2) $\delta_\times$ is a small circle surrounding the $\times$-vertex of valency $10k-4$, connected to $v_{0,0}$ by the left solid edge at $v_{0,0}$;

(3) $\delta_\bullet$ is a small circle surrounding the singular $\bullet$-vertex, connected to $v_{0,0}$ by the bold edge.

(All loops are oriented in the counterclockwise direction.) Then, the braid monodromy $\pi_1(S^\circ, v_{0,0}) \to \mathbb{B}_3$ is given by the following relations:

$$\gamma_u \mapsto \bar{w}_u \sigma_3 \bar{w}_u^{-1}, \quad \delta_\times \mapsto \sigma_1^{5k-2}, \quad \delta_\bullet \mapsto \sigma_1 \sigma_2,$$

where $\bar{w}_u$ is the braid obtained from w_u by replacing each instance of r and l with σ_2 and σ_1^{-1}, respectively.

Proof. We choose a marking at each nonsingular $\bullet$-vertex of Sk so that e_2 is the edge pointing downwards (and hence e_1 and e_3 are, respectively, the left and right branches of the tree). Then $\delta_\times \mapsto c_3(5k-2)$, see (6.3), $\delta_\bullet \mapsto l_2(2)$, see (6.4), and the image of each element γ_u is found by composing appropriate monodromies $m_{i,j}$, see (6.1). $\square$

7.3. Proof of Theorem 1.1.1.

Let k and ϵ be as in the statement. Note that $k \geqslant 3$ and $\epsilon \leqslant [k/2]$. Pick one of the trigonal curves $B \subset \Sigma_k$ given by Proposition 7.1.1. In order to convert B to a plane curve, we need to perform $(k-1)$ elementary transformations. We choose the transformations so as to contract the type $\tilde{\mathbf{A}}_0^{**}$ fiber of B, $[k/2]$ of its k type $\tilde{\mathbf{A}}_0^*$ fibers, and $[(k-3)/2]$ nonsingular fibers. In the type $\tilde{\mathbf{A}}_0^{**}$ fiber and ϵ type $\tilde{\mathbf{A}}_0^*$ fibers the blow-up centers are chosen outside of the curve and the exceptional section; in each other fiber the blow-up center is taken on a branch of B transversal to the fiber. The total number of deformation families thus obtained is

$$Z(m) = C(k-1) \cdot \binom{k}{[k/2]} \cdot \binom{[k/2]}{\epsilon},$$

the three factors standing, respectively, for the choice of B, the choice of $[k/2]$ of its k type $\tilde{\mathbf{A}}_0^*$ fibers to be contracted, and the choice of ϵ of the $[k/2]$ fibers where the blow-up center is not on B. In each case, the transform is an irreducible curve $\tilde{C} = B_1 \subset \Sigma_1$ with the set of singularities

$$\mathbf{A}_{5k-3} + \mathbf{E}_6 + \epsilon \mathbf{D}_5 + \left(\left[\frac{k}{2}\right] - \epsilon\right)\mathbf{A}_2 + \left[\frac{k-3}{2}\right]\mathbf{A}_1, \tag{7.1}$$

so that all points except the first $\mathbf{A}_{5k-3}$ are in the exceptional section $E_1 \subset \Sigma_1$ and the local intersection index of $\widetilde{C}$ and E_1 at each singular point is minimal possible (*i.e.*, 2 at a double point and 3 at a triple point). Blowing E_1 down, one obtains an irreducible plane curve C of degree $2k + 2 + \epsilon = m$. Since the combinatorial data of C are determined by the those of $\widetilde{C} + E_1$, all curves thus obtained share the same set of singularities.

The fundamental groups $\pi_1(\mathbb{P}^2 \smallsetminus C)$ are all abelian due to Proposition 5.2.3. $\square$

7.4. Proof of Theorem 1.1.2. A real curve C is obtained from a real trigonal curve $B \subset \Sigma_k$; hence, $k = 2s$ is even and the number of real trigonal curves is given by Proposition 7.1.2. Next, one should choose a real (*i.e.*, invariant under the complex conjugation) collection of blow-up centers for the elementary transformations converting B to $\widetilde{C}$, see 7.3. Since the k type $\tilde{\mathbf{A}}_0^*$ singular fibers of B split into $k/2 = s$ conjugate pairs and $k/2 = s$ blow-up centers should be chosen in these fibers, s must also be even, $s = 2t$, and the number of choices is $\binom{s}{t}$: one chooses t of the s conjugate pairs. Finally, $\epsilon = 0$ as one cannot choose only one special fiber with the blow-up center not on the curve: for the transformation to be real, the conjugate fiber would have to have the same property. $\square$

7.4.1 Remark. It is worth mentioning that, in the settings of Theorem 1.1.2, each deformation class containing a real curve splits into at least t^2 equisingular real deformation classes. Indeed, let $\mathbb{P}^1_{\mathbb{R}} \subset \mathbb{P}^1$ be the real part of the base of the ruling. It contains the singular •-vertex of Γ, the root of the original tree, and the ×-vertex of Γ of valency $10k - 4$. Thus, the singular fibers of B divide $\mathbb{P}^1_{\mathbb{R}}$ into two distinguishable intervals, and each of the $2t - 2$ nonsingular fibers containing blow-up centers can be chosen either in a conjugate pair or over one of the two intervals. The number of choices is the number of ordered pairs $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ such that $a, b \geqslant 0$, $a + b \leqslant 2t - 2$, and $a + b$ is even. It is t^2.

7.5. Proof of Theorem 1.1.3. The proof is similar to that of Theorem 1.1.1. Let $k = m - 5$, and pick one of the trigonal curves $B \subset \Sigma_k$ given by Proposition 7.1.1. Blow up the only singular point of B and blow down the corresponding fiber. Repeat this procedure $(k - 2)$ times. The result is an irreducible curve $B_2 \subset \Sigma_2$ which intersects the exceptional section at a nonsingular point P with multiplicity $(k-2)$ and has a type $\mathbf{A}_{3k+1}$ singular point Q in the fiber F_P through P. Now, add the fiber F_P as a component, perform an elementary transformation to contract the type $\tilde{\mathbf{A}}_0^{**}$ fiber of B_2 to a type $\mathbf{E}_6$ singular point in the exceptional section, and blow down the exceptional section. (The fiber F_P is added to the curve to make sure that, during the deformations, the intersection point P and the singular point Q remain in the same fiber.) The result is a plane curve C of degree m. Clearly, all $C(k - 1)$ curves obtained in this way share the same combinatorial data. The fundamental groups $\pi_1(\mathbb{P}^2 \smallsetminus C)$ are all abelian due to Proposition 5.2.4.

The count for the number of real curves is based on Proposition 7.1.2: from the construction it follows that a family contains a real curve if and only if the original trigonal curve B is real.   □

7.6. A remark on deformations. From the construction (creating a branch of type $\mathbf{E}_{12}$ and, in 7.5, adding a linear component) it follows that any equisingular deformation of the plane curve C must preserve the type $\tilde{\mathbf{A}}_0^{**}$ and type $\tilde{\mathbf{A}}_{5k-3}$ singular fibers of the original trigonal curve B. Since all other singular fibers of B are of type $\tilde{\mathbf{A}}_0^*$ and B is maximal, the resulting deformation of B is fiberwise, see Proposition 4.4.8. The blow-up centers chosen in the branches of B transversal to its type $\tilde{\mathbf{A}}_0^*$ singular fibers (see 7.3) should stay fixed, as otherwise the type of singularity of C at O would change. (This observation is also crucial in the proof of Theorem 8.1.2 below.) In 7.3, a blow-up center in a nonsingular fiber of B may move to a type $\tilde{\mathbf{A}}_0^*$ singular fiber (not containing another blow-up center), to the branch of B *tangent* to the fiber. This degeneration corresponds to one of the branches of one of the type $\mathbf{A}_1$ points of $\widetilde{C}$, see (7.1), becoming tangent to the fiber; it is equisingular for C. Clearly, these modifications do not affect the number of deformation families.

8. Further applications

8.1. In this section, we present a slight modification of the construction used in Proposition 7.1.1 and discuss a few further applications.

8.1.1 Proposition. *For each integer $k \geqslant 2$, there exists a collection of $C(k-1)$ pairwise distinct fiberwise deformation families of pairs (B, F), where $B \subset \Sigma_k$ is an irreducible maximal trigonal curve with one fiber of type $\tilde{\mathbf{A}}_{5k-2}$ and $(k+1)$ fibers of type $\tilde{\mathbf{A}}_0^*$ (and no other singular fibers) and F is a distinguished type $\tilde{\mathbf{A}}_0^*$ fiber of B. None of the curves admits a fiberwise self-deformation inducing a non-trivial permutation of its singular fibers preserving F.*

Proof. Modify the construction of Proposition 7.1.1 by replacing the monovalent vertex with an extra leaf attached to the root of the original tree T and selecting the corresponding type $\tilde{\mathbf{A}}_0^*$ fiber for F. All curves obtained are irreducible due to Corollary 6.6.1: to show that a marking as in the corollary does not exist, it suffices to consider the two leaves attached to any maximal (in the partial order defined by level) vertex of T.   □

8.1.2 Theorem (Rigid plane curves). *For each odd integer $m = 2k + 1 \geqslant 5$, there is a set of singularities shared by*

$$Z_{\mathrm{ar}}(m) \geqslant \frac{1}{2}\binom{2k-2}{k-1}$$

pairwise distinct equisingular deformation families of irreducible plane curves C_i of degree m. Within each family, all curves are projectively equivalent and defined over an algebraic number field.

8.1.3 Remark. The set of singularities constructed in the proof has a point of type $\mathbf{A}_{5k-2}$ and a point of transversal intersection of $(k-1)$ branches of type $\mathbf{A}_4$. One has $Z_{\mathrm{ar}}(5) = 1$, and the only curve of degree 5 given by the theorem is the well known quintic with the set of singularities $\mathbf{A}_8 + \mathbf{A}_4$, see [10]; it is defined over $\mathbb{Q}$. (Note that in this case the fundamental group $\pi_1(\mathbb{P}^2 \smallsetminus C)$ is abelian, see [12].) For large values of m, the count $Z_{\mathrm{ar}}(m)$ grows faster than a^m for any $a < 2$.

8.1.4 Remark. The curves given by Theorem 8.1.2 seem to be good candidates for examples of exponentially large *arithmetic Zariski k-plets* in the sense of Shimada, see [29], [30]. At present, I do not know whether all/some of the curves C_i are indeed Galois conjugate over an algebraic number field (except the trivial case of pairs of complex conjugate curves). Whether the pairs $(\mathbb{P}^2, C_i)$ or complements $\mathbb{P}^2 \smallsetminus C_i$ are homeomorphic is also an open question.

Proof. Similar to 7.3, we start with a trigonal curve $B \subset \Sigma_k$ as in Proposition 8.1.1, perform $(k-1)$ elementary transformations to convert Σ_k to Σ_1, and blow down the exceptional section of Σ_1 to get a plane curve. The $(k-1)$ blow-up centers are taken in type $\tilde{\mathbf{A}}_0^*$ singular fibers of B, on the branch of B transversal to the fiber. (This choice makes the construction rigid, so that the resulting plane curves have 0-dimensional moduli spaces and are defined over algebraic number fields. Indeed, since B itself is defined over a certain algebraic number field $\Bbbk$, see remark after 4.4.4, all its singular fibers F_j are defined over a finite extension of $\Bbbk$, and so are the intersection points $B \cap F_j$. Hence, each curve C_i is also defined over a finite extension of $\Bbbk$.) The total number of choices is $C(k-1)$ (for the pair (B, F)) times $k(k+1)/2$ (for the choice of $(k-1)$ singular fibers containing the blow-up centers). Since, in each skeleton, the distinguished leaf can be chosen in $(k+1)$ ways, we divide the resulting count by $(k+1)$. $\square$

8.2. Elliptic surfaces. Below, an *elliptic surface* is a compact complex surface X with a distinguished rational pencil of elliptic curves, *i.e.*, elliptic fibration over a rational base. We assume that the pencil has no multiple fibers; then it is unique unless the topological Euler characteristic of X is 24, *i.e.*, X is a $K3$-surface. By a *fiberwise deformation* of elliptic surfaces we mean a deformation preserving the elliptic pencil and the types of its singular fibers. All surfaces mentioned in Theorems 8.2.1 and 8.2.2 are defined over algebraic number fields.

8.2.1 Theorem. *For each integer $s \geqslant 1$, there are $C(2s-1)$ distinct fiberwise deformation families of Jacobian relatively minimal elliptic surfaces of topological*

Euler characteristic $\chi = 12s$ *and having one fiber of type* $\tilde{\mathbf{A}}_0^{**}$, *one fiber of type* $\tilde{\mathbf{A}}_{10s-3}$, *and* $2s$ *fibers of type* $\tilde{\mathbf{A}}_0^*$ (*and no other singular fibers*).

8.2.2 Theorem. *For each integer* $s \geqslant 1$, *there are at least* $C(2s-1)/(2s+1)$ *distinct fiberwise deformation families of Jacobian relatively minimal elliptic surfaces of topological Euler characteristic* $\chi = 12s$ *and having one fiber of type* $\tilde{\mathbf{A}}_{10s-2}$ *and* $(2s+1)$ *fibers of type* $\tilde{\mathbf{A}}_0^*$ (*and no other singular fibers*).

Proof of Theorems 8.2.1 *and* 8.2.2. The statements follow from Propositions 7.1.1 and 8.1.1 applied to $k = 2s$. Each surface is obtained as the minimal resolution of singularities of the double covering of Σ_k branched over the exceptional section E and a trigonal curve B given by the appropriate proposition. $\qquad\square$

8.2.3 Remark. Let X be one of the surfaces given by Theorem 8.2.1 or 8.2.2, and let $L = H_2(X)$ be its intersection lattice. Consider the sublattice $S \subset L$ spanned by the components of the pull-back of $B \cup E$. Over $\mathbb{Q}$, it is spanned by the section of X, its generic fiber, and the exceptional divisors over the only singular point of B. Hence, S is nondegenerate. The advantage of Theorem 8.2.2 is the fact that, in this case, the orthogonal complement $S^\perp$ is an even *positive definite* lattice of rank $2s-2$. Given that positive definite lattices tend to have many isomorphism classes within the same genus, one can hope to use Shimada's invariant [29] to distinguish the surfaces topologically.

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