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Topological rigidity and Gromov simplicial volume

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Abstract. A natural problem in the theory of 3-manifolds is the question of whether two 3-manifolds are homeomorphic or not. The aim of this paper is to study this problem for the class of closed Haken manifolds using degree one maps.

To this purpose we introduce an invariant $\tau(N) = (\mathrm{Vol}(N), \|N\|)$, where $\|N\|$ denotes the Gromov simplicial volume of N and $\mathrm{Vol}(N)$ is a 2-dimensional simplicial volume which measures the volume of the base 2-orbifolds of the Seifert pieces of N.

After studying the behavior of $\tau(N)$ under the action of non-zero degree maps, we prove that if M and N are closed Haken manifolds such that $\|M\| = |\deg(f)|\|N\|$ and $\mathrm{Vol}(M) = \mathrm{Vol}(N)$ then any non-zero degree map $f : M \to N$ is homotopic to a covering map. As a corollary we prove that if M and N are closed Haken manifolds such that $\tau(N)$ is sufficiently close to $\tau(M)$ then any degree one map $f : M \to N$ is homotopic to a homeomorphism.

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Contents

1. Introduction

1.1. Simplicial volume of a manifold. Let N^n be an n-dimensional manifold. The simplicial volume of N is a homotopy invariant of N defined by M. Gromov in [G] using the l^1-pseudo norm on singular homology as follows: for an element

$h \in H_*(N, \partial N; \mathbb{R})$, the Gromov norm is given by

$$\|h\| = \inf \left\{ \textstyle\sum_{i=1}^{i=r} |a_i|, \text{ when } \sum_{i=1}^{i=r} a_i \sigma_i \text{ represents } h \right\}.$$

The Gromov simplicial volume of N, denoted by $\|N\|$, is the Gromov norm of the image of a generator of $H_n(N, \partial N; \mathbb{Z})$ under the canonical homomorphism $H_n(N, \partial N; \mathbb{Z}) \to H_n(N, \partial N; \mathbb{R}) \simeq H_n(N, \partial N; \mathbb{Z}) \otimes \mathbb{R}$.

1.2. Simplicial volume of a Haken manifold. Let N be a closed Haken manifold. Given a submanifold K of N we denote by $W(K)$ a regular neighborhood of K in N. Denote by $\mathcal{T}_N$ the JSJ-family of N, by $\mathcal{S}(N)$, resp. $\mathcal{H}(N)$, the Seifert, resp. hyperbolic, components of $N^* = N \setminus W(\mathcal{T}_N)$ and by $\Sigma(N) = (\Sigma(N), \emptyset)$ the characteristic Seifert pair of N (see [JS] and [J]). The Cutting-off Theorem of Gromov ([G]) combined with the fact that manifolds admitting a fixed point free $\mathbb{S}^1$-action have zero Gromov simplicial volume (by the Mapping Theorem of Gromov) implies that

$$\|N\| = \sum_{H \in \mathcal{H}(N)} \|H\|.$$

In particular this means that the Gromov simplicial volume of a Haken manifold only depends on its hyperbolic pieces. In the following it will be convenient to decompose $\mathcal{S}(N)$ into two parts depending on the geometry of the components of $\mathcal{S}(N)$. We denote by $\mathcal{S}_{\mathrm{h}}(N)$, resp. by $\mathcal{S}_{\mathrm{e}}(N)$, the components of $\mathcal{S}(N)$ admitting a Seifert fibration with hyperbolic, resp. Euclidean, base 2-orbifold.

1.3. Extending the simplicial volume. To get a rigidity theorem for Haken manifolds we need to add another invariant of N which does not vanish on $\mathcal{S}(N)$ when $\mathcal{S}(N)$ is "non-trivial" (i.e. when $\mathcal{S}_{\mathrm{h}}(N) \neq \emptyset$). To this purpose we define a kind of 2-dimensional simplicial volume for N. More precisely, let S be a component of $\mathcal{S}(N)$. Fix a Seifert fibration for S and denote by $\mathcal{O}_S$ the base 2-orbifold of S with respect to the fixed Seifert fibration. Then we set $\mathrm{Vol}(S) = |\chi(\mathcal{O}_S)|$, where $\chi(\mathcal{O}_S)$ denotes the (rational) Euler characteristic of $\mathcal{O}_S$. We then define the 2-dimensional volume of N by setting

$$\mathrm{Vol}(N) = \sum_{S \in \mathcal{S}(N)} \mathrm{Vol}(S).$$

Lemma 1.1. *If N is a closed Haken manifold, the 2-dimensional volume* $\mathrm{Vol}(N)$*, and thus the pair $\tau(N) = (\mathrm{Vol}(N), \|N\|)$, is an invariant of N. Moreover $\tau(N) = 0$ iff N is a virtual torus bundle.*

It will be convenient to use the following convention: we say that $(a, b) \geq (c, d)$ if and only if $a \geq c$ and $b \geq d$, where (a, b) and (c, d) are in $\mathbb{R}^2$.

1.4. Nonzero degree maps decrease the volume. It follows from the definition of the Gromov simplicial volume that non-zero degree maps “decrease the simplicial volume” in the following sense. Let $f: M \to N$ be a proper non-zero degree map between orientable n-dimensional manifolds. Then $\|M\| \geq |\deg(f)|\|N\|$. This inequality does not hold for $\tau(N)$. In particular the relation $\mathrm{Vol}(M) \geq |\deg(f)|\mathrm{Vol}(N)$ is not true. However we have the following comparison result.

Theorem 1.2. *Let $f: M \to N$ be a non-zero degree map between closed Haken manifolds. If $\|M\| = |\deg(f)|\|N\|$ then $\mathrm{Vol}(M) \geq \mathrm{Vol}(N)$. Moreover, if there exists a canonical torus T of M such that $f|T: T \to N$ is not π_1-injective, then $\mathrm{Vol}(M) > \mathrm{Vol}(N)$.*

Note that the condition on the Gromov simplicial volume is necessary in Theorem 1.2. Indeed by a construction of [BW] using null-homotopic hyperbolic knots, we know that for any aspherical Seifert fibered space Σ there always exist a hyperbolic 3-manifold M and a degree one map $f: M \to \Sigma$. In this case $\mathrm{Vol}(M) = 0$ and Σ can be chosen so that $\mathrm{Vol}(\Sigma) > 0$.

In view of Theorem 1.2 the following question is natural: If $\|M\| = |\deg(f)|\|N\|$, what happens when $\mathrm{Vol}(M) = \mathrm{Vol}(N)$? The answer is given in the following section.

1.5. Volume and topological rigidity. The purpose of this paper is to characterize those degree one (resp. non-zero degree) maps between closed Haken manifolds which are homotopic to a homeomorphism (resp. covering). Then our main result can be stated as follows.

Theorem 1.3. *Let $f: M \to N$ be a non-zero degree map between closed Haken manifolds such that $\|M\| = |\deg(f)|\|N\|$. If $\mathrm{Vol}(M) = \mathrm{Vol}(N)$ then f is homotopic to a $\deg(f)$-fold covering.*

Remark 1.4. In Theorem 1.3 we can obviously decompose f into two covering maps which preserve the JSJ-decomposition. This means that after a homotopy, f induces two covering maps $f|\mathcal{H}(M): \mathcal{H}(M) \to \mathcal{H}(N)$ and $f|\mathcal{S}(M): \mathcal{S}(M) \to \mathcal{S}(N)$. Since a Seifert fibered space can be seen as a *generalized* $\mathbb{S}^1$-bundle over a 2-dimensional orbifold, it could be convenient to make precise the behavior of the covering map $f|\mathcal{S}(M)$ with respect to this anisotropic structure. Actually, when the fibration of a Seifert manifold S is unique (up to isotopy), the action of $f|S$ can be unambiguously decomposed into two transversal actions: a *vertical action* (i.e. an action along the $\mathbb{S}^1$-fibers of S) and a *horizontal action* (i.e. an action along the 2-orbifold of S). Then in the proof of Theorem 1.3 we will see that the hypothesis $\mathrm{Vol}(M) = \mathrm{Vol}(N)$ implies that $f|\mathcal{S}_{\mathrm{h}}(M)$ acts only vertically and that the horizontal action is trivial.

Remark 1.5. Note that in [W1], S. Wang proved that a proper map of non-zero degree $f: M \to M$ from a Haken manifold M to itself necessarily induces an injective homomorphism at the fundamental group level. Then Theorem 1.3 gives an extension of this result since when $M = N$ the conditions on the volume are satisfied.

If we consider only degree one maps then one can relax the hypothesis concerning the volumes. More precisely, combining Theorem 1.3 and Theorem 1.2 in [D] we get the following result.

Theorem 1.6. *For any closed Haken manifold M there exists a constant $\eta_M \in (0,1)$ depending only on M such that any degree one map $f: M \to N$ onto a closed Haken manifold is homotopic to a homeomorphism iff $\tau(N) \geq \tau(M)(1-\eta_M)$.*

1.6. Some known results on topological rigidity

1.6.1. Rigidity of surface bundles. The above problem has been studied by S. Wang and M. Boileau in [W] and [BW] for non-zero degree maps, when the domain M is a surface bundle over the circle and when the target N is irreducible. In particular, Wang proved in [W] that if M is a virtual torus bundle over the circle then f is homotopic to a covering map. When M is a bundle over $\mathbb{S}^1$ with a fiber of negative Euler characteristic, denote by α the cohomology class corresponding to the fibration of M. Then in [BW], Boileau and Wang proved that if there is a rational cohomology class β in N with $f^*(\beta) = \alpha$ and such that $\|\alpha\|_{\mathrm{Th}} = |\deg(f)|\|\beta\|_{\mathrm{Th}}$ then f is homotopic to a covering map. Here $\|\cdot\|_{\mathrm{Th}}$ denotes the Thurston norm.

Remark 1.7. Notice that the constant $\mathrm{Vol}(M)$ in Theorem 1.3 can be seen as the analogous of the Thurston norm of α in the result of Boileau and Wang in [BW, Theorem 2.1].

1.6.2. Rigidity of hyperbolic manifolds. The rigidity problem is completely solved for hyperbolic manifolds by a result of Gromov and Thurston which reads as follows.

Theorem 1.8 (M. Gromov, W. Thurston). *Let M and N be two complete finite volume hyperbolic 3-manifolds. Then a proper non-zero degree map $f: M \to N$ is homotopic to a $\deg(f)$-fold covering iff $\|M\| = |\deg(f)|\|N\|$.*

Recall that T. Soma gave a generalization (see [S2]) of this result for degree one maps by proving the following result.

Theorem 1.9 (T. Soma). *For any $\varepsilon > 0$ there is a constant $\eta_\varepsilon > 0$ which depends only on ε such that any degree one map $f: M \to N$ between closed hyperbolic*

3-manifolds satisfying $\|M\| \leq \varepsilon$ and $\|N\| \geq \|M\|(1-\eta_\varepsilon)$ is homotopic to an isometry.

Notice that $\lim_{\varepsilon\to+\infty} \eta_\varepsilon = 0$ (see also [S2]). Note also that this kind of result cannot be extended to Haken manifolds even if the target is a closed hyperbolic manifold. This results from the Thurston hyperbolic surgery theorem.

Indeed, let Y be a complete finite volume orientable hyperbolic 3-manifold with $\partial Y \simeq \mathbb{S}^1 \times \mathbb{S}^1$ and let X denote an orientable graph manifold with $\partial X \simeq \mathbb{S}^1 \times \mathbb{S}^1$ in such a way that there exists a simple closed curve l in ∂X such that the pair (X, l) is *pinchable*. This means that there exists a proper degree one map $\pi\colon (X, \partial X) \to (V, \partial V)$, where V is a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ such that $\pi\colon \partial X \to \partial V$ is a homeomorphism which sends l to the meridian $m = \partial\mathbb{D}^2 \times \{*\}$ in ∂V. To perform this operation it is sufficient to choose X so that l is nul-homologous in $H_1(X;\mathbb{Z})$ (for instance $X = F \times \mathbb{S}^1$, where F is an orientable surface with connected boundary and $l = \partial F$).

Let $\{l_n, n \in \mathbb{N}\}$ be a sequence of simple closed curves in ∂Y such that $\{\text{lenght}(l_n), n \in \mathbb{N}\}$ defines a strictly increasing sequence with $\lim_{n\to\infty} \text{lenght}(l_n) = +\infty$, where lenght denotes the length for the Euclidean metric on ∂Y induced by the hyperbolic metric of $\text{int}(Y)$. Denote by M_n the closed Haken manifold obtained by gluing X and Y along ∂X and ∂Y in such a way that l is identified with l_n and denote by N_n the 3-manifold obtained from Y after performing a Dehn filling along the curve l_n. Thus the map π can be extended by the identity to construct a degree one map $f_n\colon M_n \to N_n$. Then $\|M_n\| = \|Y\| > 0$. By the Thurston hyperbolic surgery theorem, one sees that the N_n's are closed hyperbolic manifolds for n sufficiently large and $\{\|N_n\|, n \in \mathbb{N}\}$ is a strictly increasing sequence such that $\lim_{n\to\infty} \|N_n\| = \|Y\|$. Moreover the maps f_n are neither homotopic to a homeomorphism.

1.7. Organization of the paper. This paper is organized as follows.

In Section 2 we recall some terminology and we state some technical results concerning the following points: finite coverings of Haken manifolds, standard form of non-zero degree maps, and a *thick–thin* decomposition of M with respect to a *non-degenerate*, non-zero degree map $f\colon M \to N$.

Sections 3 and 4 are devoted to the study of non-degenerate proper maps $f\colon M \to N$ of non-zero degree from a Haken graph manifold with toral boundary to a circle bundle N. The aim of these sections is to give a construction allowing us to compare the volume of the thick part of M with $\text{Vol}(N)$ using *efficient surfaces* and *minimal connection graphs* (see Propositions 3.1 and 4.1). These sections are essential for the proof of Theorem 1.2.

Section 5 is devoted to the proof of Theorems 1.2, 1.3 and 1.6. Note that in this paper all the 3-manifolds are orientable.

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2. Preliminaries

Let Σ be an orientable Seifert fibered space. Then Σ is an $\mathbb{S}^1$-bundle over its base 2-orbifold $\mathcal{O}_\Sigma$ and the $\mathbb{S}^1$-action is globally well defined since Σ is orientable. Recall that if $\overline{\mathcal{O}}_\Sigma$ denotes the underlying space of $\mathcal{O}_\Sigma$ and if $c_1, \dots, c_r$ denote the exceptional points of $\mathcal{O}_\Sigma$ with index $\mu_1, \dots, \mu_r$ respectively then

$$\chi(\mathcal{O}_\Sigma) = \chi(\overline{\mathcal{O}}_\Sigma) - \sum_{i=1}^{i=r} \left(1 - \frac{1}{\mu_i}\right).$$

The geometry of $\mathcal{O}_\Sigma$ is hyperbolic, Euclidean or spherical when $\chi(\mathcal{O}_\Sigma)$ is < 0, $= 0$ or > 0, respectively. Hence the geometry of Σ depends of the geometry of $\mathcal{O}_\Sigma$ combined with the rational Euler number $\mathbf{e}(\Sigma)$ of the fibration. More precisely, when $\mathbf{e}(\Sigma) = 0$ then we get respectively an $\mathbb{H}^2 \times \mathbb{R}$, Euclidean, $\mathbb{S}^2 \times \mathbb{R}$-structure and when $\mathbf{e}(\Sigma) \neq 0$ we get respectively a $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, Nil, spherical structure. Note that if N is a Sol-manifold then we consider it as a Haken manifold with non-empty JSJ-decomposition so that the Seifert pieces of N are Euclidean manifolds.

2.1. Two-dimensional simplicial volume. In this paragraph we prove Lemma 1.1. Since the JSJ-decomposition of closed Haken manifolds is unique up to isotopy, we only have to check that the volume $\mathrm{Vol}(N)$ does not depend on the chosen Seifert fibration on the components of $\mathcal{S}(N)$. Let Σ be a Seifert piece of N. Since N is a closed Haken manifold, Σ admits one of the following geometries: $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}}_2(\mathbb{R})$, Nil or Euclidean geometry. The only aspherical Seifert fibered spaces which admit more than one non-isotopic Seifert fibration are Euclidean manifolds. But in this case the Euler characteristic of the base orbifold of Σ is always zero. Hence the invariance is immediate.

It remains to check the second assertion of the lemma. Assume that N admits a finite covering $\pi\colon \widetilde{N} \to N$ which is a torus bundle over the circle. Then $\widetilde{N}$ is a geometric manifold and the structure depends on the monodromy of the bundle. Then N admits a Euclidean, a Nil, or a Sol geometry. In the case of Euclidean or Nil geometry N is a Seifert fibered space and the base 2-orbifold $\mathcal{O}_N$ is Euclidean and thus $\tau(N) = 0$. If N is a Sol-manifold then each component of $N \setminus \mathcal{T}_N$ is a Euclidean manifold and hence $\tau(N) = 0$. Assume that $\tau(N) = 0$. If $\mathcal{T}_N = \emptyset$ then N has Euclidean or Nil-geometry. In any case N is a virtual torus bundle. If $\mathcal{T}_N \neq \emptyset$ then $\mathcal{H}(N) = \emptyset$ and each Seifert piece of N is a Euclidean manifold with non-empty boundary. Then by minimality of the JSJ-decomposition either

(i) N is made of two twisted I-bundles over the Klein bottle glued along their boundary, or

(ii) N is $\mathbb{S}^1 \times \mathbb{S}^1 \times I/\langle\varphi\rangle$, where $\varphi\colon \mathbb{S}^1 \times \mathbb{S}^1 \times \{0\} \to \mathbb{S}^1 \times \mathbb{S}^1 \times \{1\}$ is an Anosov diffeomorphism.

In case (ii) N is a torus bundle over the circle (actually a Sol-manifold) and in case (i) N admits a 2-fold covering that is a torus bundle over the circle. This completes the proof of Lemma 1.1.

2.2. Dehn fillings. We define *Seifert Dehn fillings*. Suppose Σ is an orientable Seifert fibered space with $\partial\Sigma \neq \emptyset$ and let T be a component of $\partial\Sigma$. Since Σ is orientable, $T \simeq \mathbb{S}^1 \times \mathbb{S}^1$. Let α be a simple closed curve in T. Performing a Dehn filling on T along α means that we glue a solid torus $V = \mathbb{D}^2 \times \mathbb{S}^1$ identifying $\partial\mathbb{D}^2 \times \mathbb{S}^1$ with T so that α is glued with the meridian $\partial\mathbb{D}^2 \times \{*\}$ of V. Denote by $\widehat{\Sigma} = \Sigma(\alpha)$ the resulting manifold. When α is not isotopic to a generic fiber of Σ then the fixed Seifert fibration of Σ extends to a Seifert fibration of $\widehat{\Sigma}$ and we say that we have performed a Seifert Dehn filling.

2.3. Morphisms. Let $f\colon \Sigma \to \Sigma'$ be a map between orientable Seifert fibered spaces. We say that f is a *bundle homomorphism* is there exists a Seifert fibration of Σ and Σ' so that f is a homomorphism for the $\mathbb{S}^1$-bundle structures on Σ and Σ'. According to [Ro], for bundle homomorphisms we define the following degrees:

The *fiber degree* of f is the integer $|n|$ given by $f_*(h) = t^n$, where h, resp. t, denotes the generic fiber of Σ, resp. of Σ', and we denote it by $G_h(f)$.

The *orbifold degree* $G_{\mathrm{ob}}(f)$ is the minimum number of regular fibers in $g^{-1}(t)$, where g runs over all bundle homomorphisms properly homotopic to f and transverse to t.

For a bundle homomorphism $f\colon \Sigma \to \Sigma'$ we have

$$|\deg(f)| \leq G_h(f)G_{\mathrm{ob}}(f).$$

We say that a bundle homomorphism is *allowable* if $|\deg(f)| = G_h(f)G_{\mathrm{ob}}(f)$. In particular, a bundle homomorphism $f\colon (\Sigma, \partial\Sigma) \to (\Sigma', \partial\Sigma')$ between orientable Seifert fibered spaces with non-empty boundary which is proper (i.e. $f^{-1}(\partial\Sigma') = \partial\Sigma$) is *allowable*.

2.4. Non-degenerate maps. Let $f\colon S \to N$ be a map from a Seifert manifold to a Haken manifold. We say that f is *non-degenerate* if $f_*(\pi_1 S)$ is not cyclic and if $f_*([\gamma]) \neq \{1\}$ for any fiber of any Seifert fibration on S. A map $f\colon M \to N$ from a Haken manifold with toral boundary M is non-degenerate if $f|S$ is non-degenerate for any Seifert piece of M. A non-degenerate map $f\colon M \to N$ is $\mathcal{T}$-injective if for any component T of $\mathcal{T}_M \cup \partial M$ the map $f|T\colon T \to N$ is π_1-injective.

2.5. Finite coverings of a map. Let $f\colon X \to Y$ be a continuous map between topological spaces. Let $p\colon \widetilde{Y} \to Y$ be a covering map and denote by $q\colon \widetilde{X} \to X$ the covering of X corresponding to the subgroup $f_*^{-1}(p_*(\pi_1\widetilde{Y}))$. A finite covering of f

associated to $p\colon \widetilde{Y} \to Y$ is a lifting $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ of $f \circ q$. In particular, if p is a finite covering then $1 \le \deg(q) \le \deg(p)$ and if f is π_1-surjective then $\deg(p) = \deg(q)$.

2.6. Finite coverings of Seifert and Haken manifolds

Lemma 2.1 ([JS, Lemma II.6.1]). *Let Σ be a Seifert fibered space. Then any finite covering $\pi\colon \widetilde{\Sigma} \to \Sigma$ admits a Seifert fibration so that π is an allowable bundle homomorphism. Moreover the Euler characteristic of the base orbifolds satisfy*

$$\chi(\mathcal{O}_{\widetilde{\Sigma}}) = G_{\mathrm{ob}}(f)\chi(\mathcal{O}_{\Sigma}).$$

Proof. The proof can be found in [JS]. □

Let $\mathcal{T}$ be a union of tori and let m be a positive integer. Call a covering $p\colon \widetilde{\mathcal{T}} \to \mathcal{T}$ *m-characteristic* if for each component T of $\mathcal{T}$ and for each component $\widetilde{T}$ of $\widetilde{\mathcal{T}}$ over T, the restriction $p|\colon \widetilde{T} \to T$ is the covering map associated to the characteristic subgroup of index $m \times m$ in $\pi_1 T$. Call a covering $\widetilde{N} \to N$ of a Haken manifold N m-characteristic if its restriction to $\mathcal{T}_{\widetilde{N}} \to \mathcal{T}_N$ is m-characteristic.

Lemma 2.2. (i) *Any orientable Seifert manifold endowed with hyperbolic base 2-orbifold admits a fiber degree one finite covering which is homeomorphic to an orientable $\mathbb{S}^1$-bundle over an orientable hyperbolic surface.*

(ii) *Any closed Haken manifold admits a 1-characteristic 2-fold covering space which contains no embedded Klein bottle.*

Proof. Point (i) follows from Selberg's lemma ([Al]) and point (ii) is immediate using orientation coverings. □

Lemma 2.3. *Let $f\colon M \to N$ be a non-degenerate map from an orientable aspherical Seifert manifold to an orientable circle bundle over an orientable hyperbolic surface F. Then each Seifert fibration of M has an orientable base 2-orbifold (in particular M is not homeomorphic to the twisted I-bundle over the Klein bottle).*

Proof. Assume first that $f_*(\pi_1 M)$ is abelian. If M admits a fibration over a non-orientable 2-orbifold, then there exists $g \in \pi_1 M$ such that $ghg^{-1} = h^{-1}$, where h denotes the homotopy class of the generic fiber of the fixed Seifert fibration on M. Since $\pi_1 N$ is torsion free this implies that $f_*(h) = 1$. This is a contradiction since f is a non-degenerate map.

Suppose that $f_*(\pi_1 M)$ is non-abelian. If M admits a fibration over a non-orientable 2-orbifold $\mathcal{O}_M$ then consider the double covering $p\colon \widetilde{M} \to M$ corresponding to the orientation covering of $\mathcal{O}_M$.

If $f_*(\pi_1 \widetilde{M})$ is abelian then $f_*(\pi_1 M)$ contains an index 2 free abelian group H. Now $\mathrm{Rank}(H) \le 2$ since N is a circle bundle over an orientable hyperbolic surface, and since f is non-degenerate, $\mathrm{Rank}(H) \geq 2$. This implies that $f_*(\pi_1 M)$ is the fundamental group of a Klein bottle. This is impossible since N is an orientable circle bundle over an orientable hyperbolic surface.

If $f_*(\pi_1 \widetilde{M})$ is non-abelian then $f_*(\tilde{h})$ has a non-abelian centralizer, where $\tilde{h}$ denotes the homotopy class of the generic fiber of $\widetilde{M}$. Then, since N contains no embedded Klein bottle, we know by [JS] that $f_*(\tilde{h}) \in \langle t \rangle$ and thus $f_*(h) \in \langle t \rangle$, where t denotes the homotopy class of the fiber of N. Denote by a the non-zero integer such that $f_*(h) = t^a$. Since $\mathcal{O}_M$ is non-orientable, there exists $g \in \pi_1 M$ such that $ghg^{-1} = h^{-1}$. Since t is central in $\pi_1 N$, this gives a contradiction. This completes the proof of the lemma. □

Lemma 2.4. *Let N be a Haken manifold and let $p\colon \widetilde{N} \to N$ be a finite covering of N. Then* $\mathrm{Vol}(N) \leq \mathrm{Vol}(\widetilde{N}) \leq |\deg(p)|\mathrm{Vol}(N)$. *Moreover, if p has fiber degree one over each component of $\mathcal{S}(N)$ then* $\mathrm{Vol}(\widetilde{N}) = |\deg(p)|\mathrm{Vol}(N)$.

Proof. Let S be a component of $\mathcal{S}(N)$. Choose a component $\widetilde{S}$ of $p^{-1}(S)$ in $\widetilde{N}$ and choose a Seifert fibration on $\widetilde{S}$ so that $p|\widetilde{S}$ is a bundle homomorphism. Denote by $\mathcal{O}_S$ and by $\mathcal{O}_{\widetilde{S}}$ the base 2-orbifold of S and $\widetilde{S}$. Denote by n the integer such that $p_*(\tilde{h}) = h^n$, where h and $\tilde{h}$ are the homotopy classes of the generic fiber of S and $\widetilde{S}$, respectively. Then by Lemma 2.1 we know that

$$|\deg(p|\widetilde{S})| = |n|.G_{\mathrm{ob}}(p|\widetilde{S}) \quad \text{and} \quad \mathrm{Vol}(\widetilde{S}) = G_{\mathrm{ob}}(p|\widetilde{S})\mathrm{Vol}(S) \geq \mathrm{Vol}(S).$$

On the other hand, notice that

$$|\deg(p)| = \sum_{\widetilde{S} \in p^{-1}(S)} |\deg(p|\widetilde{S})|.$$

Hence $|\deg(p)|\mathrm{Vol}(N) \geq \mathrm{Vol}(\widetilde{N}) \geq \mathrm{Vol}(N)$. It remains to prove the second part of the lemma. Let Σ be a component of $\mathcal{S}(N)$ and denote by $\Sigma_1, \dots, \Sigma_k$ the components of $\widetilde{\Sigma} = p^{-1}(\Sigma)$.

Denote by $h, h_1, \dots, h_k$ the homotopy class of the generic fiber of $\Sigma, \Sigma_1, \dots, \Sigma_k$. Since by hypothesis $p_*(h_i) = h^{\pm 1}$ for $i = 1, \dots, k$, each covering $p_i = p|\Sigma_i$ satisfies $|\deg(p_i)| = G_{\mathrm{ob}}(p_i)$ and thus $\mathrm{Vol}(\Sigma_i) = |\deg(p_i)|\mathrm{Vol}(\Sigma)$ for any $i = 1, \dots, k$. Finally, since $|\deg(p)| = |\deg(p|\widetilde{\Sigma})| = |\deg(p|\Sigma_1)| + \dots + |\deg(p|\Sigma_k)|$, we have $\mathrm{Vol}(\widetilde{\Sigma}) = \mathrm{Vol}(\Sigma_1) + \dots + \mathrm{Vol}(\Sigma_k) = |\deg(p)|\mathrm{Vol}(\Sigma)$. This ends the proof of the lemma. □

Lemma 2.5. *Let $f : M \to N$ be a π_1-surjective non-zero degree map between Haken manifolds and assume that there exists a finite covering $\tilde{f} : \widetilde{M} \to \widetilde{N}$ of $f : M \to N$ such that* $\mathrm{Vol}(\widetilde{M}) > \mathrm{Vol}(\widetilde{N})$. *If the covering $\widetilde{N} \to N$ has fiber degree one over the Seifert pieces of N then* $\mathrm{Vol}(M) > \mathrm{Vol}(N)$.

Proof. We keep the same notations as above. Assume that $\mathrm{Vol}(\widetilde{N}) < \mathrm{Vol}(\widetilde{M})$. Since $q\colon \widetilde{N} \to N$ induces the trivial covering over the fibers, it follows from Lemma 2.4 that $\mathrm{Vol}(\widetilde{N}) = \deg(q)\mathrm{Vol}(N)$. On the other hand, $\mathrm{Vol}(\widetilde{M}) \leq \deg(p)\mathrm{Vol}(M)$. Since $f\colon M \to N$ is π_1-surjective, $\deg(p) = \deg(q)$ and thus $\mathrm{Vol}(M) > \mathrm{Vol}(N)$. This completes the proof of the lemma. □

Lemma 2.6. *Let N denote an orientable $\mathbb{S}^1$-bundle over an orientable hyperbolic surface F with bundle projection $\xi\colon N \to F$, and let $\mathcal{U} = \{u_1, \dots, u_q\}$ denote a family of homotopically non-trivial simple closed curves in F. Then there exists a fiber degree one finite covering $p\colon (\widetilde{N}, \widetilde{F}, \tilde{\xi}) \to (N, F, \xi)$ such that each component of $\widetilde{\mathcal{U}} = (p|\widetilde{F})^{-1}(\mathcal{U})$ is of infinite order in $H_1(\widetilde{F}; \mathbb{Z})$.*

Proof. If u_i is not of infinite order in $H_1(F;\mathbb{Z})$, since the group $H_1(F;\mathbb{Z})$ is torsion free, then u_i is a separating curve in F. Denote by A and B the components of $F \setminus u_i$.

Case 1: Assume first that both $H_1(A, u_i;\mathbb{Z})$ and $H_1(B, u_i;\mathbb{Z})$ are non-zero. Then one can construct epimorphisms $\rho_A\colon H_1(A;\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ and $\rho_B\colon H_1(B;\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ such that $\ker \rho_A \supset \langle [u_i] \rangle$ and $\ker \rho_B \supset \langle [u_i] \rangle$. Using the exact sequence

$$H_1([u_i];\mathbb{Z}) \to H_1(A;\mathbb{Z}) \oplus H_1(B;\mathbb{Z}) \to H_1(F;\mathbb{Z}) \to \{0\}$$

we get an epimorphism $\rho\colon H_1(F;\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, well defined by the formula $\rho(x) = \rho_A(a) + \rho_B(b)$, where (a,b) represents x in $H_1(A;\mathbb{Z}) \oplus H_1(B;\mathbb{Z})$. Then denote by $p_i\colon (N_i, F_i, \xi_i) \to (N, F, \xi)$ the 2-fold covering corresponding to the homomorphism

$$\pi_1 N \xrightarrow{\xi_*} \pi_1 F \to H_1(F;\mathbb{Z}) \xrightarrow{\rho} \mathbb{Z}/2\mathbb{Z}.$$

Then $(p_i|F_i)^{-1}(u_i)$ consists of two simple closed curves of infinite order in $H_1(F_i;\mathbb{Z})$ and $\langle t \rangle < (p_i)_*(\pi_1 N_i)$, where t denotes the homotopy class of the fiber of N.

Case 2: Assume that $H_1(A, u_i;\mathbb{Z}) = \{0\}$, say. This means that $H_1(u_i;\mathbb{Z}) \to H_1(A;\mathbb{Z})$ is an epimorphism and thus $H_1(A;\mathbb{Z})$ is $\{0\}$ or $\mathbb{Z}$. In the first case A is a disk which is impossible since u_i is homotopically non-trivial and in the second case A is an annulus. This means that u_i is ∂-parallel in F. Moreover, since u_i is nul-homologous, $[u_i] = [\partial F]$ and in particular F has connected boundary. Since F is hyperbolic, $H_1(F;\mathbb{Z}) \neq \{0\}$. Then there exists a non-trivial finite abelian group L and an epimorphism $\rho\colon H_1(F;\mathbb{Z}) \to L$. We denote by $p_i\colon (N_i, F_i, \xi_i) \to (N, F, \xi)$ a finite abelian covering corresponding to the homomorphism

$$\pi_1 N \xrightarrow{\xi_*} \pi_1 F \to H_1(F;\mathbb{Z}) \xrightarrow{\rho} L.$$

Then $(p_i|F_i)^{-1}(u_i)$ consists of $\mathrm{Card}(L)$ simple closed curves of infinite order in $H_1(F_i;\mathbb{Z})$ and $\langle t \rangle < (p_i)_*(\pi_1 N_i)$.

Hence the covering of N corresponding to the subgroup $(p_1)_*(\pi_1 N_1) \cap \cdots \cap (p_q)_*(\pi_1 N_q)$ of $\pi_1 N$ satisfies the conclusion of the lemma. □

2.7. Separability of fundamental groups. The following result is a direct consequence of a separability result of Allman and Hamilton combined with the residual q-nilpotence of free groups, for any prime q, proved by Gruenberg.

Lemma 2.7 ([AH], [Gr]). *Let F be an orientable hyperbolic surface and let $u \in \pi_1 F$ be a non-trivial element. Then for any prime q there exists a finite group H_q and an epimorphism $\tau\colon \pi_1 F \to H_q$ such that $\tau(u) \neq 1$ and q divides the order of $\tau(u)$.*

Proof. Consider $\pi_1 F$ as a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

Assume first that u is a hyperbolic isometry (i.e. u has exactly two fixed points both in $\partial_\infty \mathbb{H}^{2,+}$). Then the proof of the lemma follows directly from Proposition 1 of [AH] in this case. Indeed the eigenvalues of the matrix representing u in $\mathrm{SL}_2(\mathbb{C})$ are not roots of unity.

Assume now that u is a parabolic isometry (i.e. u has exactly one fixed point and it lies in $\partial_\infty \mathbb{H}^{2,+}$). In this case, necessarily $\partial F \neq \emptyset$ and thus $\pi_1 F$ is a free group. Then it follows from [Gr] that $\pi_1 F$ is residually q-nilpotent for any prime q. This means that there exists a finite q-group H_q and an epimorphism $\tau\colon \pi_1 F \to H_q$ such that $\tau(u) \neq 1$. This completes the proof of the lemma. □

We end this section with the following result which follows from the residual finiteness of surface groups.

Lemma 2.8. *Let $f : S \to \Sigma$ be a $\mathcal{T}$-injective map from an orientable aspherical Seifert fibered space to an orientable $\mathbb{S}^1$-bundle over an orientable hyperbolic surface F such that $f_*(\pi_1 S)$ is non-abelian. Then for any $n \in \mathbb{N}$ there exists a finite covering $\tilde{f}_n\colon \tilde{S}_n \to \tilde{\Sigma}_n$ satisfying the following properties:*

(i) *the covering $\tilde{\Sigma}_n \to \Sigma$ has fiber degree one,*

(ii) *each component of $\tilde{S}_n$ has a base 2-orbifold of genus at least n.*

Proof. By Lemma 2.3, S is based on an orientable orbifold. Let $T_1, \dots, T_p$ be the components of ∂S. Denote by t the homotopy class of the fiber of Σ and by $\xi\colon \Sigma \to F$ the bundle projection. Denote by $d_1, \dots, d_p$ the chosen sections of ∂S with respect to the fixed Seifert fibration of S and let $c_1, \dots, c_r$ denote the homotopy classes of the exceptional fibers of S with index $\mu_1, \dots, \mu_r$ respectively.

Since $f_*(\pi_1 S)$ is non-abelian, it follows from [JS] that $f_*(v) \in \langle t \rangle$ for any fiber v of S. Denote by $\mathcal{O}_S$ the base 2-orbifold of S and by $\overline{\mathcal{O}}_S$ the underlying space and set $g_S = \mathrm{genus}(\overline{\mathcal{O}}_S)$.

Let $q\colon \tilde{\Sigma} \to \Sigma$ be a finite regular covering. Consider the corresponding epimorphism $\varphi\colon \pi_1 \Sigma \to K$, where K is a finite group, and denote by $p\colon \tilde{S} \to S$ the finite covering corresponding to the homomorphism $\varphi \circ f_*$. This covering induces a branched covering of degree σ between the underlying spaces of the base 2-orbifolds

$\mathcal{O}_{\widetilde{S}}$ of $\widetilde{S}$ and $\mathcal{O}_S$ of S. Let β_j denote the order of $\varphi f_*(c_j)$ and for each $i = 1, \dots, p$ denote by r_i the number of components of $\partial\widetilde{S}$ over T_i and set $n_i = \sigma/r_i$. Then the Riemann–Hurwitz formula allows us to compute the genus of $\mathcal{O}_{\widetilde{S}}$ using the data of $\mathcal{O}_S$ and those of $p\colon \widetilde{S} \to S$:

$$2g_{\widetilde{S}} = 2 + \sigma\Big(p + 2g_S + r - 2 - \sum_{i=1}^{i=p} \frac{1}{n_i} - \sum_{i=1}^{i=r} \frac{1}{(\mu_i, \beta_i)}\Big).$$

Case 1: Assume $g_S \geq 2$. First note that since $f_*(\pi_1 S)$ is non-cyclic there exists an element $a \in \pi_1 S$ such that $\xi_* \circ f_*(a) \neq 1$ in $\pi_1 F$. Since surface groups are residually finite, there exists a finite group K and an epimorphism $\varepsilon\colon \pi_1 F \to K$ such that $\varepsilon(\xi_* \circ f_*(a)) \neq 1$. Consider the homomorphism $\varphi = \varepsilon \circ \xi_*$. Note that since the regular fiber h of S is sent via f to the fiber of Σ, we necessarily have $\sigma \geq 2$. Then the Riemann–Hurwitz formula gives

$$2g_{\widetilde{S}} \geq 2 + \sigma\,(2g_S - 2)\,.$$

Thus, since $g_S \geq 2$ and $\sigma \geq 2$, we get $2 + \sigma\,(2g_S - 2) > 2g_S$. This proves that $g_{\widetilde{S}} > g_S$ and completes the proof of the lemma in this case.

Case 2: Assume $g_S = 1$. Then we claim that $p \geq 1$. Suppose the contrary. Let a and b be the standard generators of $\pi_1\overline{\mathcal{O}}_S$ and denote by $q_1, \dots, q_r$ the sections corresponding to the exceptional fibers $c_1, \dots, c_r$. Since $f_*(h) \in \langle t\rangle$ and since $\pi_1 F$ is torsion free, it follows that $f_*(q_i) \in \langle t\rangle$ for $i = 1, \dots, r$. Hence $f_*([a, b]) \in \langle t\rangle$ because $[a, b]q_1 \dots q_r = h^b$ and thus $[\xi_* f_*(a), \xi_* f_*(b)] = 1$ in $\pi_1 F$. Since F is a hyperbolic surface, there exists $u \in \pi_1 F$ such that $\langle u\rangle = \langle \xi_* f_*(a), \xi_* f_*(b)\rangle$. Let $g \in \pi_1\Sigma$ such that $\xi_*(g) = u$. Then $f_*(\pi_1 S) \subset \langle g, t\rangle \simeq \mathbb{Z} \times \mathbb{Z}$. A contradiction. Now $\xi_* f_*(d_i) \neq 1$ in $\pi_1 F$ because $f|\partial S\colon \partial S \to \Sigma$ is π_1-injective. Thus there exists an epimorphism $\varepsilon\colon \pi_1 F \to K$ into a finite group K such that $\varepsilon\xi_* f_*(d_i) \neq 1$ for $i = 1, \dots, p$. Consider the homomorphisms $\varphi = \varepsilon\xi_*$ and $\varphi \circ f_*$ and the associated coverings $\widetilde{\Sigma}$ and $\widetilde{S}$. Then it follows from our construction that $n_i \geq 2$ for $i = 1, \dots, p$ and $\sigma \geq 2$. Then the Riemann–Hurwitz formula gives

$$2g_{\widetilde{S}} \geq 2 + \sigma\frac{p}{2} > 2.$$

Thus $g_{\widetilde{S}} \geq 2$ and we have a reduction to the first case.

Case 3: Assume $g_S = 0$. In this case the fundamental group of S admits a presentation

$$\big\langle d_1, \dots, d_p, q_1, \dots, q_r, h : [h, d_i] = [h, q_j] = 1, q_i^{\mu_i} = h^{\gamma_i}, d_1 \dots d_p q_1 \dots q_r = h^b\big\rangle.$$

Note that when $p > 0$, i.e., when $\partial S \neq \emptyset$, then one can choose $b = 0$. Since $f_*(h) \in \langle t\rangle$ and since $\pi_1 F$ is torsion free, it holds that $f_*(q_i)$ is in $\langle t\rangle$ for $i = 1, \dots, r$.

Then we first check (using the presentation above and the fact that $f_*(\pi_1 S)$ is non-abelian) that $p \geq 3$. If $p \leq 1$ then we get $f_*(\pi_1 S) \subset \langle t \rangle$. A contradiction. Assume that $p = 2$. Using the presentation of $\pi_1 S$ we get $f_*(\pi_1 S) \subset \langle f_*(d_1), t\rangle \simeq \mathbb{Z} \times \mathbb{Z}$, a contradiction again. From now on may we assume that $p \geq 3$.

Note that since $f|\partial S \colon \partial S \to \Sigma$ is π_1-injective, it follows that $\xi_* f_*(d_i) \neq 1$ in $\pi_1 F$. Thus there exists an epimorphism $\varepsilon \colon \pi_1 F \to K$ into a finite group K such that $\varepsilon\xi_* f_*(d_i) \neq 1$ for $i = 1, \dots, p$. Consider the homomorphisms $\varphi = \varepsilon\xi_*$ and φf_* and the associated coverings $\widetilde{\Sigma}$ and $\widetilde{S}$. Then it follows from our construction that $n_i \geq 2$ for $i = 1, \dots, p$ and $\sigma \geq 2$. Then the Riemann–Hurwitz formula gives

$$2g_{\widetilde{S}} \geq 2 + \sigma\left(\frac{p}{2} - 2\right).$$

Subcase 1: Assume $g_S = 0$ and $p \geq 4$. This implies that $g_{\widetilde{S}} \geq 1$ and we have a reduction to the second case.

Subcase 2: Assume $g_S = 0$ and $p = 3$. If the number of connected components $\widetilde{p}$ of $\widetilde{S}$ is ≥ 4 then we have a reduction to the subcase 1. Hence assume that $\widetilde{p} = 3$. The Riemann–Hurwitz formula gives

$$2g_{\widetilde{S}} = 2 - \widetilde{p} + \sigma\left(p + r - 2 - \sum_{i=1}^{i=r} \frac{1}{(\mu_i, \beta_i)}\right) \geq \sigma - 1 \geq 1.$$

Then we get $g_{\widetilde{S}} \geq 1$. This completes the proof of the lemma. □

2.8. Characteristic maps between Haken manifolds. First recall that a codimension 0 submanifold L of a closed Haken manifold M is termed a *characteristic submanifold* if L is a component of $M \setminus \mathcal{T}$, where $\mathcal{T}$ is a subfamily of $\mathcal{T}_M$.

Next we define *characteristic maps*. Let $f : M \to N$ be a map between closed Haken manifolds. We say that f is *standard* if $f(\mathcal{H}(M)) \subset \operatorname{int}\mathcal{H}(N)$ and $f(\mathcal{S}(M)) \subset \operatorname{int}\Sigma(N)$, where $\Sigma(N)$ denotes the characteristic pair defined in Paragraph 1.2. We say that a standard map f is *characteristic* if for any component $T \in \mathcal{T}_N$ the space $f^{-1}(T)$ is the disjoint union of components of $\mathcal{T}_M$.

Lemma 2.9. *Let $f : M \to N$ be a map between closed Haken manifolds and assume that N is not a virtual torus bundle. If f is standard then it is homotopic to a characteristic map.*

Proof. The proof follows from cut and paste arguments of [Wa]. □

Lemma 2.10. *Let $f : M \to N$ be a non-degenerate, non-zero degree map between closed Haken manifolds. Then if $\|M\| = |\deg(f)|\|N\|$ then f is homotopic to a standard map.*

Proof. This follows from the Mapping Theorem of [JS] and from the Rigidity Theorem of [S1]. □

More generally we have

Lemma 2.11. *Let $f: M \to N$ be a non-zero degree map between closed Haken manifolds with $\|M\| = \deg(f)\|N\|$. Then there exists a connected characteristic submanifold $M_1 \subset M$ which contains $\mathcal{H}(M)$ in its interior (in particular if $\partial Q \cap \partial M_1 \neq \emptyset$ for $Q \in M^*$ then Q is Seifert), a closed Haken manifold $\widehat{M}_1$ obtained from M_1 after Seifert Dehn fillings along ∂M_1 and a $\mathcal{T}$-injective non-zero degree extension $\hat{f}_1: \widehat{M}_1 \to N$ of $f_1 = f|M_1: M_1 \to N$ such that $\|\widehat{M}_1\| = |\deg(\hat{f}_1)|\|N\|$.*

Proof. The proof of Lemma 2.11 follows from the arguments used in [Ro1] without any essential change. □

2.9. A thick–thin decomposition of Haken manifolds with respect to standard maps. Let M and N be two Haken manifolds with toral boundary (if non-empty) and let $f: M \to N$ be a standard map. For each Seifert piece S of M denote by Σ_S the component of $\Sigma(N)$ such that $f(S) \subset \mathrm{int}(\Sigma_S)$. Denote by M_+ the disjoint union of the Seifert pieces S of M^* such that there exists a Seifert fibration on S with fiber h and a Seifert fibration on Σ_S with fiber t such that $f_*([h]) \in \langle [t] \rangle$ (at the π_1-level) and set $M_- = \overline{M \setminus (M_+ \cup \mathcal{H}(M))}$.

Lemma 2.12. *Let $f: M \to N$ be a non-degenerate standard map between Haken manifolds and let S be a Seifert piece of M. If Σ_S is an $\mathbb{S}^1$-bundle over an orientable hyperbolic surface then we have $f_*(\pi_1 S) \simeq \mathbb{Z} \times \mathbb{Z}$ for $S \in M_-$.*

Proof. Let S be a Seifert piece of M_-. By Lemma 2.3 each Seifert fibration on S has an orientable basis. Due to the fact that Σ_S contains no Klein bottles, it follows from [JS, Addendum to Theorem VI.I.6] that $f_*(\pi_1 S)$ is abelian. Furthermore $f_*(\pi_1 S) \simeq \mathbb{Z}^r$ as $\pi_1 N$ is torsion free. Since $f|S$ is a non-degenerate map, we have $r \geq 2$, and since N is a three-dimensional manifold, we have $r \leq 3$ because the subgroup $f_*(\pi_1 S)$ must have cohomological dimension at most 3. Finally, the fundamental group of N cannot contain a group isomorphic to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ since Σ_S is an $\mathbb{S}^1$-bundle over an orientable hyperbolic surface. Then necessarily $f_*(\pi_1 S) \simeq \mathbb{Z}\times\mathbb{Z}$. □

Lemma 2.13. *Let $f: (M, \partial M) \to (N, \partial N)$ be a non-degenerate, proper non-zero degree map from a Haken graph manifold with toral boundary to an orientable circle bundle over an orientable hyperbolic surface.*

Let $\tilde{f}: \widetilde{M} \to \widetilde{N}$ be a finite covering of f and let $p: \widetilde{M} \to M$ denote the corresponding finite covering of M. Then $p^{-1}(M_-) = \widetilde{M}_-$ and $\widetilde{M}_+ = p^{-1}(M_+)$.

Proof. If M is a Seifert manifold, then $M = M_+$ since f is a non-zero degree map and thus $\widetilde{M}_+ = \widetilde{M}$ and the result is obvious.

Assume that M is not a Seifert manifold and let S be a Seifert piece of M. Since N is an orientable circle bundle over an orientable hyperbolic surface, by Lemma 2.3 S admits an $\mathbb{H}^2 \times \mathbb{R}$ structure. In particular S admits a unique Seifert fibration. The proof of the lemma follows. □

Proposition 2.14. *Let $f: G \to \Sigma$ be a $\mathcal{T}$-injective proper non-zero degree map from a Haken graph manifold with toral boundary to an orientable $\mathbb{S}^1$-bundle over an orientable hyperbolic surface F. Let L be a characteristic submanifold of G such that each Seifert piece S of L satisfies $f_*(\pi_1 S) \simeq \mathbb{Z} \times \mathbb{Z}$. There exists a finite covering $\tilde{f}: \widetilde{G} \to \widetilde{\Sigma}$ of f and a finite family of vertical tori $\mathcal{T}_L$ in $\widetilde{\Sigma}$ satisfying the following properties:*

(i) *$\widetilde{\Sigma} \to \Sigma$ has fiber degree one.*

(ii) *After a homotopy, for each component $\widetilde{L}$ of $p^{-1}(L)$, $\tilde{f}(\widetilde{L})$ is contained in a component of $\mathcal{T}_L$, where $p: \widetilde{G} \to G$ denotes the finite covering corresponding to $\tilde{f}$.*

To prove this result, we first need some preliminary lemmas.

Lemma 2.15. *Let F denote an orientable hyperbolic surface and let $f: \mathbb{S}^1 \to F$ be a geodesic loop and assume that $\langle [f] \rangle$ is a maximal abelian subgroup of $\pi_1 F$. Then there exists a finite regular covering $q: \widehat{F} \to F$ such that any lifting $\hat{f}: \mathbb{S}^1 \to \widehat{F}$ of $f \circ p$ is an embedding, where $p: \mathbb{S}^1 \to \mathbb{S}^1$ denotes the finite covering corresponding to the subgroup $f_*^{-1}(q_*(\pi_1 \widehat{F}))$.*

Proof. Denote by $p: \mathbb{H}^2 \to F$ the universal covering of F and denote by τ the isometry of $\mathbb{H}^2$ corresponding to $[f]$. Note that if τ is a parabolic isometry then the lemma is obvious. Thus let us assume that τ is a hyperbolic isometry. Let l be the unique τ-invariant geodesic line in $\mathbb{H}^2$. Denote by D a (compact) fundamental domain in l for the action of τ. Denote by $\{g_1, \dots, g_n\}$ the finite subset of $\pi_1 F$ defined by $\{g \in \pi_1 F \setminus \langle \tau \rangle \mid g(D) \cap D \neq \emptyset\}$. Moreover, there exists a finite group K and an epimorphism $\varphi: \pi_1 F \to K$ such that $\varphi(g_i) \notin \varphi(\langle \tau \rangle)$, for $i = 1, \dots, n$. Then the finite regular covering $\mathbb{H}^2 / \ker(\varphi) \to F$ satisfies the conclusion of the lemma. □

Lemma 2.16. *Let Σ denote an orientable circle bundle over an orientable hyperbolic surface F and let $f: \mathbb{S}^1 \times \mathbb{S}^1 \to \Sigma$ be a π_1-injective map. Then there exists a finite, fiber degree one, regular covering $q: \widehat{\Sigma} \to \Sigma$ such that for each lifting $\hat{f}: \mathbb{S}^1 \times \mathbb{S}^1 \to \widehat{\Sigma}$ of $f \circ p$ there is a vertical torus $\widehat{T}$ in $\widehat{\Sigma}$ such that $\hat{f}(\mathbb{S}^1 \times \mathbb{S}^1) \subset \widehat{T}$, where $p: \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ denotes the finite covering corresponding to the subgroup $f_*^{-1}(q_*(\pi_1 \widehat{\Sigma}))$.*

Proof. Denote by G a maximal abelian subgroup of $\pi_1\Sigma$ of rank 2 containing $f_*(\pi_1(\mathbb{S}^1 \times \mathbb{S}^1))$. Then G can be written as $\langle t, b\rangle$, where b generates a maximal abelian subgroup of $\pi_1 F$ and t denotes the homotopy class of the fiber of Σ. It is well known that b is freely homotopic to an element of $\pi_1 F$ which can be represented by a closed geodesic loop $g\colon \mathbb{S}^1 \to F$. Then changing f by a homotopy, we may assume that $b = [g]$. Denote by $q\colon \widehat{F} \to F$ the finite regular covering of F satisfying the conclusion of Lemma 2.15 with the map g. Then the finite, fiber degree one, regular covering $q\colon \widehat{\Sigma} \to \Sigma$ obtained as a pullback of q via the bundle projection $\xi\colon \Sigma \to F$ satisfies the conclusion of the lemma. □

In the sequel we will need the following definition: let T denote a vertical torus in a Seifert manifold Σ endowed with a base point x. We say that T is a *maximal vertical torus* if there are no rank 2 free abelian subgroups of $\pi_1(\Sigma, x)$ strictly containing $\pi_1(T, x)$.

Lemma 2.17. *Let $f: (V, \partial_0 V) \to (\Sigma, \boldsymbol{T})$ be a $\mathcal{T}$-injective map from a connected Haken graph manifold V with toral boundary to an orientable $\mathbb{S}^1$-bundle Σ over an orientable hyperbolic surface F with bundle projection $\xi\colon \Sigma \to F$. Assume that $\partial_0 V$ is a non-empty subset of ∂V, $\boldsymbol{T}$ is a maximal vertical torus in Σ and that $f_*(\pi_1 S) \simeq \mathbb{Z} \times \mathbb{Z}$ for each Seifert piece S of V. Then f is homotopic, rel. $\partial_0 V$, to a map g such that $g(V) \subset \boldsymbol{T}$.*

Note that $\pi_1\boldsymbol{T}$ can be presented as $\langle t, b\rangle$, where $\xi(b)$ is represented by an embedded geodesic curve which generates a maximal cyclic subgroup of $\pi_1 F$ and t is the fiber of Σ. We first check the following

Claim 2.18. *For each Seifert piece S of V, the map $f|S$ is homotopic, rel. to $f|\partial_0 V$, to a map g such that $g(S) \subset \boldsymbol{T}$.*

Proof of Claim. Let S be a Seifert piece such that $\partial S \cap \partial_0 V \neq \emptyset$. Let T_0 be a component of $\partial S \cap \partial_0 V$. Let $x \in T_0$ be a base point and let $y = f(x) \in \boldsymbol{T}$. Let H be a free abelian subgroup of $\pi_1(\Sigma, y)$ such that $K_0 = H \cap \pi_1(\boldsymbol{T}, y)$ is a free abelian group of rank 2. Thus K_0 is a finite index subgroup of H. Hence, since $K_0 \subset \pi_1(\boldsymbol{T}, y)$ for any $g \in H$, there exists an integer $n_g \in \mathbb{Z}$ such that $g^{n_g} \in \pi_1(\boldsymbol{T}, y) = \langle t, b\rangle$. On the other hand, there exists an integer $\beta \neq 0$ and an element $\alpha \in \pi_1\Sigma$ such that $\xi_*(\alpha) \in \pi_1 F \setminus \{1\}$ and such that $H = \langle t^\beta, \alpha\rangle$. Then, in particular, there exist two non-zero integers $n, m \in \mathbb{Z}$ such that $\xi_*(\alpha)^n = \xi_*(b)^m$. It is easy to check that $\langle \xi_*(\alpha), \xi_*(b)\rangle$ is an infinite cyclic subgroup of $\pi_1 F$, using the classification of isometries of $\mathbb{H}^{2,+}$. Therefore, since $\boldsymbol{T}$ is a maximal torus, $\xi_*(\alpha) \in \langle \xi_*(b)\rangle$, and thus $H \subset \pi_1\boldsymbol{T}$.

Using the above construction with $H = f_*(\pi_1 S)$ which contains $H \cap \pi_1(\boldsymbol{T}, y) \supset K_0 = f_*(\pi_1(T_0, x))$ we deduce that $f_*(\pi_1 S) \subset \pi_1\boldsymbol{T}$. Hence one can change $f|S$

by a homotopy, rel. to $f|\partial_0 V$, such that $f(S) \subset \boldsymbol{T}$. Since V is connected, this completes the proof of the claim by repeating the argument for each Seifert piece of V. □

Proof of Lemma 2.17. We argue by induction on the complexity of the dual graph Γ_V of V. Fix a Seifert piece S_0 such that $\partial S_0 \cap \partial_0 V \neq \emptyset$. By the claim above we may assume that $f(S_0) \subset \boldsymbol{T}$.

Case 1: Assume that Γ_V is a tree. Let S be a Seifert piece of V adjacent to S_0. Note that $\partial S_0 \cap \partial S$ is a connected canonical torus T since Γ_V is a tree. Fix a base point $x \in T$ and $y = f(x) \in \boldsymbol{T}$. It follows from the claim above that $f_*(\pi_1(S, x))$ is a subgroup of $\pi_1(\boldsymbol{T}, y)$. Since $S_0 \cap S$ is connected, it follows from the van Kampen Theorem that $f_*(\pi_1(S \cup_T S_0))$ is a subgroup of $\pi_1(\boldsymbol{T}, y)$. Hence, after a homotopy rel. to $\partial_0 V$, we may assume that $f(S \cup_T S_0) \subset \boldsymbol{T}$. This completes the proof of the lemma when Γ_V is a tree by repeating this process.

Case 2: If Γ_V is not a tree then $\mathrm{Rank}(H_1(\Gamma_V;\mathbb{R})) \geq 1$. Choose a characteristic non-separating torus T in V. By Claim 2.18, one can change f by a homotopy rel. to $\partial_0 V$ such that $f(T) \subset \boldsymbol{T}$. Next, consider the space $\widehat{V}$ obtained by cutting V along T. Then $\mathrm{Rank}(H_1(\Gamma_{\widehat{V}};\mathbb{R})) < \mathrm{Rank}(H_1(\Gamma_V;\mathbb{R}))$. Denote by U_1, U_2 the components of $\partial\widehat{V}$ over T and write $\partial_0\widehat{V} = \partial_0 V \cup U_1 \cup U_2$. Consider the map $f_1 = f|\widehat{V} \colon (\widehat{V}, \partial_0\widehat{V}) \to (\Sigma, \boldsymbol{T})$. We know from the induction hypothesis that there exists a map g_1 homotopic to f_1 rel. to $\partial_0\widehat{V}$ such that $g_1(\widehat{V}) \subset \boldsymbol{T}$. Thus it follows from our construction that g_1 factors through V. This completes the proof of the lemma. □

Proof of Proposition 2.14. Let S be a Seifert piece of L. Then, by Lemma 2.16 there exists a fiber degree one finite covering $\widetilde{\Sigma}_S \to \Sigma$ satisfying the following property. Consider the covering $\tilde{f}_S \colon \widetilde{G}_S \to \widetilde{\Sigma}_S$ of f corresponding to $\widetilde{\Sigma}_S \to \Sigma$. Then for each component $\widetilde{S}$ over S in $\widetilde{G}_S$ there exists a maximal vertical torus in $\widetilde{\Sigma}_S$ containing $\tilde{f}_S(\widetilde{S})$. Consider the covering $\widetilde{\Sigma}$ of Σ corresponding to the finite index subgroup

$$\bigcap_{S \in L^*} \pi_1 \widetilde{\Sigma}_S$$

of $\pi_1\Sigma$. Denote by $\tilde{f} \colon \widetilde{G} \to \widetilde{\Sigma}$ the covering of f corresponding to $\widetilde{\Sigma} \to \Sigma$ and by $p \colon \widetilde{G} \to G$ the corresponding covering of G. Hence for each Seifert piece $\widetilde{S}$ of $p^{-1}(L)$, there exists a maximal vertical torus $T_{\widetilde{S}}$ in $\widetilde{\Sigma}$ containing $\tilde{f}(\widetilde{S})$. It remains to prove that the same property remains true by replacing $\widetilde{S}$ by the connected component $\widetilde{L}$ of $p^{-1}(L)$ which contains $\widetilde{S}$. This last point follow directly from Lemma 2.17. This completes the proof of the proposition. □

We end this section with the following result.

Lemma 2.19. *Let $f:(M,\partial M)\to(N,\partial N)$ be a $\mathcal{T}$-injective, proper, non-zero degree map from a Haken graph manifold with toral boundary to an orientable circle bundle over an orientable hyperbolic surface. Then*

(i) *there exists at least one Seifert piece S of M such that $f_*(\pi_1 S)$ is non-abelian,*

(ii) *if $f_*(\pi_1 S)$ is non-abelian then S is a component of G_+, and*

(iii) *if there exists a component T of $\mathcal{T}_M$ shared by two Seifert pieces S_1 and S_2 of M then S_1 or S_2 is in G_-.*

Proof. If $f_*(\pi_1 S)$ is abelian for any Seifert piece S of M then by Proposition 2.14, there exists a finite covering $\tilde f:\widetilde M\to\widetilde N$ such that $\tilde f_*(\pi_1\widetilde M)\simeq\mathbb{Z}\times\mathbb{Z}$. Since $\deg(f)\neq 0$ also $\deg(\tilde f)\neq 0$. This implies that $\pi_1\widetilde N$ contains a finite index abelian subgroup. This is impossible since N is a circle bundle over a hyperbolic surface.

Assume that $f_*(\pi_1 S)$ is non-abelian. By Lemma 2.3 we know that S admits a Seifert fibration over an orientable basis. Moreover, the map $f|S$ is homotopic to a fiber preserving map since N contains no embedded Klein bottle.

Let T be a component of $\mathcal{T}_M$ shared by two Seifert pieces S_1 and S_2 of M such that the maps $f|S_i$ are homotopic to fiber preserving maps. Fix a base point x in T and denote by h_i, $i=1,2$, the homotopy class of the regular fiber in S_i represented in T. Since the $f|S_i$ are fiber preserving, the map $f|T$ cannot be π_1-injective by the minimality of the JSJ decomposition. This is a contradiction and completes the proof of the lemma. □

3. Comparing the volume of the base 2-orbifolds

Let $f:(G,\partial G)\to(\Sigma,\partial\Sigma)$ be a $\mathcal{T}$-injective, proper, non-zero degree map from a Haken graph manifold G with toral boundary to an orientable $\mathbb{S}^1$-bundle (Σ,ξ,F) over an orientable hyperbolic surface F with bundle projection $\xi:\Sigma\to F$. Then one can associate to f a thick–thin decomposition of G into $G_+\cup G_-$. In this section we give a general formula which allows us to compare $\mathrm{Vol}(G_+)$ and $\mathrm{Vol}(\Sigma)$. To this purpose we need to have a relation as precise as possible between $\mathrm{Rank}(H_1(\overline{\mathcal{O}}_+;\mathbb{Z}))$ and $\mathrm{Rank}(H_1(F;\mathbb{Z}))$, where $\overline{\mathcal{O}}_+$ denotes the disjoint union of the base surfaces of the Seifert pieces of G_+ (see Proposition 3.1).

3.1. Efficient surfaces. Let $\mathcal{F}$ be a connected, embedded, orientable surface in G. We say that $\mathcal{F}$ is an *efficient surface* if it satisfies the following properties:

(1) $\mathcal{F}$ is transversal to $\mathcal{T}_G$ in the sense that $\partial\mathcal{F}\cap\mathcal{T}_G=\emptyset$ and that each component of $\mathcal{F}\cap\mathcal{T}_G$ is an essential simple closed curve in $\mathcal{T}_G$.

(2) The torus-decomposition of G gives a "circle" decomposition of $\mathcal{F}$ into $\mathcal{F}^*=\mathcal{F}\setminus(\mathcal{T}_G\cap\mathcal{F})$.

(3) The thick–thin decomposition of G induces a thick–thin decomposition of $\mathcal{F}$ such that $\mathcal{F}_- = \mathcal{F} \cap G_-$ is incompressible and well-embedded in G_- and the thick part $\mathcal{F}_+ = \mathcal{F} \cap G_+$ of $\mathcal{F}$ is horizontal (this means that each component F of $\mathcal{F}_+$ is transversal to the fibers of the Seifert piece of G_+ containing F).

We denote by G^o the characteristic submanifold of G which consists of the Seifert pieces of G which meet $\mathcal{F}$. Denote by G^o_+ and G^o_- the thick–thin decomposition of G^o and by $\bar{\mathcal{O}}^o_+$ the disjoint union of the base surfaces of the Seifert pieces of G^o_+.

We associate to an efficient surface $\mathcal{F}$ a graph Γ in the following way. First consider the dual graph Γ_0 with respect to the circle decomposition of $\mathcal{F}$ and denote by V_0, resp. E_0, the vertex space, resp. the edge space of Γ_0. For each edge $e \in E_0$, $e \cap \mathcal{T}_G$ then consists of a single point $v_e(T)$, where T denotes the component of $\mathcal{T}_G$ such that $e \cap T \neq \emptyset$. The set $\{v_e(T), e \in E_0, T \in \mathcal{T}_G\} = \Gamma_0 \cap \mathcal{T}_G$ will be termed the *middle space* of Γ_0 and we denote it by M_0. Then consider the graph $\Gamma = \Gamma_0 \cap G_-$ with vertex space $V = (V_0 \cap G_-) \cup M_0$. Moreover, we always assume the following *middle condition* (which can always be performed) for the vertex space:

Let x and y be two vertices of Γ. Assume that x and y are in M_0 and correspond to the same canonical torus T of G. Since two Seifert pieces of G_+ cannot be adjacent, by Lemma 2.19 there exists at most one Seifert piece S of G_+ such that $T \subset \partial S$. Then in this case x and y are in the same fiber of S.

We consider the following equivalence relation on Γ. Let x and y be two vertices of Γ. Then $x \sim y$ iff either x and y are in M_0 and live the same canonical torus of G or x and y are in $V_0 \cap G_-$ and live in the same Seifert piece of G.

Denote by $\hat{\Gamma}$ the quotient space $\Gamma/\sim$ and by $q\colon \Gamma \to \hat{\Gamma}$ the projection. Note that the vertex space $V(\hat{\Gamma})$ of $\hat{\Gamma}$ is equal to $q(V)$.

We define a "quotient space" of $\Gamma \cup \mathcal{F}_+$ in the following way. Denote by $\Pi\colon G_+ \to \bar{\mathcal{O}}_+$ the Seifert projection. First note that, since $\Pi|G^o_+$ is compatible with $\sim$ by the middle condition, $\Pi(\Gamma \cap \mathcal{F}_+)$ gives a subset of $V(\hat{\Gamma})$. Then define the space $\bar{\mathcal{O}}^o_+ \cup \hat{\Gamma}$ as the attachment of $\hat{\Gamma}$ to $\bar{\mathcal{O}}^o_+$ along $\Pi(\Gamma \cap \mathcal{F}_+) = \hat{\Gamma} \cap \bar{\mathcal{O}}^o_+$. Hence we get a map $\Pi \cup q\colon \mathcal{F}_+ \cup \Gamma \to \bar{\mathcal{O}}^o_+ \cup \hat{\Gamma}$. Next, for each component O of $\bar{\mathcal{O}}^o_+$ we pick a base point x_O and we connect each point of $V(\hat{\Gamma}) \cap \partial O$ to x_O by an embedded arc and we denote by $\Gamma_{\mathcal{F}}$ the graph obtained from the union of $\hat{\Gamma}$ with these embedded arcs whose vertex space $V(\Gamma_{\mathcal{F}})$ is $V(\hat{\Gamma}) \cup \big(\bigcup_{O \in \bar{\mathcal{O}}^o_+} \{x_O\}\big)$. Now $\hat{\Gamma} \cup \bar{\mathcal{O}}^o_+ = \Gamma_{\mathcal{F}} \cup \bar{\mathcal{O}}^o_+$ but the second presentation is easier for some computations because $\Gamma_{\mathcal{F}}$ is connected.

A subset Γ_1 of $\Gamma_{\mathcal{F}}$ will be termed a *pseudo subgraph* if Γ_1 is a graph whose vertex space $V(\Gamma_1)$ is a subset of $V(\Gamma_{\mathcal{F}})$. We say that a connected pseudo subgraph Γ_1 of $\Gamma_{\mathcal{F}}$ is a *minimal connection pseudo subgraph* of $\Gamma_{\mathcal{F}}$ if

(i) $\Gamma_1 \cup \bar{\mathcal{O}}^o_+$ is connected,

(ii) for each component O of $\bar{\mathcal{O}}^o_+$, the set $\Gamma_1 \cap O$ is simply connected,

(iii) the edge and the vertex space of Γ_1 satisfy

$$\mathrm{Card}(V(\Gamma_1)) = \mathrm{Card}(\pi_0((G^o)^*)),$$
$$\mathrm{Card}(E(\Gamma_1)) \leq \sum_{S\in(G^o_-)^*} \mathrm{Card}(\pi_0(\partial S)),$$

where $(G^o)^*$, resp. $(G^o_-)^*$, denotes the disjoint union of the Seifert pieces of G^o, resp. G^o_-. Then the main result of this section is

Proposition 3.1. *Let $f\colon (G,\partial G) \to (\Sigma,\partial\Sigma)$ be a $\mathcal{T}$-injective, proper, non-zero degree map from a Haken graph manifold G with toral boundary to an orientable $\mathbb{S}^1$-bundle (Σ,ξ,F) over an orientable hyperbolic surface F with bundle projection $\xi\colon \Sigma \to F$. Assume moreover that for each component L of G_- there exists a maximal vertical torus T_L in Σ such that $f(L) \subset T_L$. Then there exists an efficient surface $\mathcal{F}$ in G and a minimal connection pseudo subgraph Γ_1 of $\Gamma_{\mathcal{F}}$ such that $\beta_1(\Gamma_1 \cup \bar{\mathcal{O}}^o_+) \geq \beta_1(F)$.*

In the following we will denote by $\mathcal{T}_-$ the union of vertical tori T_L, where L runs over the components of G_-, and by c_- the union of the corresponding curves $c_L = \xi(T_L)$ in F.

Remark 3.2. Let $\widehat{L}$ denote a characteristic submanifold of G such that $f_*(\pi_1 S) \simeq \mathbb{Z}\times\mathbb{Z}$ for each Seifert piece S of $\widehat{L}$ and which is maximal with respect to the natural inclusion. This means that $\widehat{L}$ contains a component L of G_- but $\widehat{L}$ can be larger than L. However, by Lemma 2.17 we may assume, after a homotopy, that $f(\widehat{L}) \subset T_L$.

3.2. Domination of the target via essential surfaces. In this section we prove the following result.

Lemma 3.3. *There exists an efficient surface $\mathcal{F}$ in G such that $\beta_1(\Gamma_{\mathcal{F}} \cup \bar{\mathcal{O}}^o_+) \geq \beta_1(F)$.*

First we construct an efficient surface in G by pull back. Since $f|G_+\colon G_+ \to \Sigma$ is a bundle homomorphism one can choose a fiber t in $\Sigma \setminus W(\mathcal{T}_-)$ such that $f^{-1}(t)$ is a finite union of regular fibers $h_1,\dots,h_l$ in $\mathrm{int}(G_+)$. In the following we set $\Sigma' = \Sigma$ when the Euler number $\mathbf{e}(\Sigma) = 0$ and $\Sigma' = \overline{\Sigma \setminus W(t)}$ when $\mathbf{e}(\Sigma) \neq 0$. Next, denote by G' the space $f^{-1}(\Sigma') = G \setminus \bigcup_{1\leq i\leq l} W(h_i) = G'$ and by $f'\colon G' \to \Sigma'$ the induced proper non-zero degree map. In any case, Σ' is a circle bundle over a surface F' with zero Euler number. Fix a section of the bundle $\xi'\colon \Sigma' \to F'$ so that F' can be seen as an incompressible well-embedded surface in Σ'. After changing f' by a homotopy so that each component of $f'^{-1}(F')$ is an incompressible, well-embedded surface in G', fix a component $\mathcal{F}$ of $f'^{-1}(F')$ such that $\deg(f'|\mathcal{F} : \mathcal{F} \to F') \neq 0$.

Denote b y π_G, resp. π_Σ, the natural quotient map $\pi_G\colon G' \to G$, resp. $\pi_\Sigma\colon \Sigma' \to \Sigma$, and denote still by $\mathcal{F}$ the surface $\pi_G(\mathcal{F})$. It is immediate from the construction that $\mathcal{F}$ is an efficient surface of G. In the following we denote by I the map $\xi \circ f$.

Lemma 3.4. *The efficient surface $\mathcal{F}$ satisfies the following properties:*

(i) *$I_*(\pi_1(\mathcal{F}))$ is a finite index subgroup of $\pi_1 F$, $I_*(H_1(\mathcal{F};\mathbb{Q})) = H_1(F;\mathbb{Q})$ and $\mathcal{F}_+$ is not empty.*

(ii) *The map $I|\mathcal{F}_+\colon \mathcal{F}_+ \to F$ factors through $\bar{\mathcal{O}}^o_+$ in such a way that there exists a continuous map $J\colon \bar{\mathcal{O}}^o_+ \to F$ such that $I|\mathcal{F}_+ = J \circ \Pi$.*

(iii) *Let d be a simple closed curve in $\partial G_+ \setminus \partial G$ which is not homotopic to a fiber of G_+. Then $I_*(H_1(d;\mathbb{Q})) = H_1(c_L;\mathbb{Q})$, where L denotes the component of G_- which contains d.*

Proof. First observe that it is a direct consequence of our construction that the map $\xi \circ \pi_\Sigma \circ i'\colon F' \hookrightarrow \Sigma' \to \Sigma \to F$ is π_1-surjective and that $f'|\mathcal{F}\colon \mathcal{F} \to F'$ is a proper non-zero degree map. Thus $(f')_*(\pi_1(\mathcal{F}))$ is a finite index subgroup of $\pi_1 F'$ and $(f')_*(H_1(\mathcal{F};\mathbb{Q})) = H_1(F';\mathbb{Q})$. Hence $I_*(\pi_1(\mathcal{F}))$ is a finite index subgroup of $\pi_1 F$ and $I_*(H_1(\mathcal{F};\mathbb{Q})) = H_1(F;\mathbb{Q})$.

If $\mathcal{F}_+ = \emptyset$ then by connexity $\mathcal{F} = \mathcal{F}_-$ and there exists a component L of G_- containing $\mathcal{F}$. Hence $I_*(\pi_1\mathcal{F}) < \langle [c_L] \rangle$ which implies that $\pi_1 F$ contains a finite index cyclic subgroup. This is a contradiction since F is a hyperbolic surface. Hence point (i) follows.

Next we check point (ii). Since $f|G_+$ is a bundle homomorphism there exists a continuous map $J\colon \bar{\mathcal{O}}_+ \to F$ such that $I|G_+ = J \circ \Pi$. This proves point (ii).

It remains to check point (iii). Denote by T the component of $\mathcal{T}_G$ which contains d and by S the Seifert piece of G_+ containing T in its boundary. Since T is in $\partial G_+ \setminus \partial G$, there exists a component L of G_- adjacent to S along T. Since $f|S$ is fiber preserving and since $f|T$ is π_1-injective it follows that the map $I|d\colon d \to c_L$ has non-zero degree. This shows (iii). □

Lemma 3.5. *The map $I|\mathcal{F}_+ \cup \Gamma\colon \mathcal{F}_+ \cup \Gamma \to F$ factors through $\bar{\mathcal{O}}^o_+ \cup \Gamma_{\mathcal{F}}$ in such a way that there exists a continuous map $J\colon \bar{\mathcal{O}}^o_+ \cup \Gamma_{\mathcal{F}} \to F$ such that $I|\mathcal{F}_+ \cup \Gamma = J \circ (\Pi \cup q)$.*

Proof. Deform slightly f so that for any vertices x and y of Γ living in the same Seifert piece of G_- we have $I(x) = I(y)$. Then the lemma follows directly from Lemma 3.4. □

Now the proof of Lemma 3.3 follows from Lemma 3.5 combined with the following assertion.

Claim 3.6. *The induced homomorphism*

$$I_*\colon H_1(\Gamma \cup \mathcal{F}_+;\mathbb{Q}) \to H_1(F;\mathbb{Q})$$

is surjective.

Proof. Recall that $I_*\colon H_1(\mathcal{F};\mathbb{Q}) \to H_1(F;\mathbb{Q})$ is an epimorphism by Lemma 3.4. Using the Mayer–Vietoris exact sequence one sees that $H_1(\mathcal{F};\mathbb{Q})$ is generated by $H_1(\Gamma \cup \mathcal{F}_+;\mathbb{Q})$ and $H_1(\mathcal{F}_-;\mathbb{Q})$. Furthermore, for each component V of $\mathcal{F}_-$ there exists a component L of G_- containing V and a component d of ∂V adjacent to a component of $\mathcal{F}_+$ in G_+. Hence it follows from Lemma 3.4 that $\langle I_*([d])\rangle = I_*(H_1(V;\mathbb{Q}))$. This completes the proof of the claim. □

3.3. Connection by a minimal graph. In this section we prove Proposition 3.1. To this purpose we need to check the following Elimination Lemma.

Lemma 3.7. *There exists a connected pseudo subgraph Γ' of $\Gamma_{\mathcal{F}}$ satisfying the following conditions:*

(i) $V(\Gamma') = V(\Gamma_{\mathcal{F}})$,

(ii) *for each component O of $\bar{\mathcal{O}}^o_+$, the set $\Gamma' \cap O$ is simply connected,*

(iii) *the valence $v(x)$ of x is 2 in Γ' for any $x \in q(M_0)$,*

(iv) *the space $\Gamma' \cup \bar{\mathcal{O}}^o_+$ is still connected and the map $J|\Gamma' \cup \bar{\mathcal{O}}^o_+\colon \Gamma' \cup \bar{\mathcal{O}}^o_+ \to F$ induces an epimorphism at the H_1-level (with coefficient $\mathbb{Q}$).*

Proof. First notice that $\Gamma_{\mathcal{F}}$ satisfies points (i), (ii) by construction and (iv) by Lemma 3.5 and Claim 3.6. Moreover, it follows from our construction that for any $x \in q(M_0)$ we have $v(x) \geq 2$.

Now assume that there exists a point $x \in q(M_0)$ such that $v(x) \geq 3$. Then there exists at least three edges e_1, e_2 and e_3 of $\Gamma_{\mathcal{F}}$ such that x is an end of e_i for $i = 1, 2, 3$. For each i denote by y_i the end of e_i such that $\partial e_i = \{x, y_i\}$. Note that each y_i is a point of $q(V_0 \cap G_-) \bigcup_{O \in \bar{\mathcal{O}}^o_+} \{x_O\}$ and thus each y_i corresponds to a unique Seifert piece of G^o. Denote by S_i, $i = 1, 2, 3$, the Seifert piece of G^o corresponding to y_i and by T the canonical torus corresponding to x. Since T is shared by S_1, S_2 and S_3, there exists $i, j \in \{1, 2, 3\}$ such that $S_i = S_j$. For simplicity assume that $S_2 = S_3$. Thus necessarily $y_2 = y_3$. Since $\Gamma_{\mathcal{F}} \cap O$ is simply connected for each component O of $\mathcal{O}^o_+$, obviously $S_2 = S_3$ is a Seifert piece of G^o_-. Denote by L the component of G_- containing $S_2 = S_3$ and denote by d the curve defined by $e_2 \cup e_3$. Then $J_*([d]) \in H_1(c_L;\mathbb{Q})$, where c_L is the simple closed curve in F corresponding to the torus T_L of $\mathcal{T}_-$ such that $f(L) \subset T_L$. Let U be a component of $\mathcal{T}_G$ which is shared by L and a component S of G^o_+, and let c be a component of $\mathcal{F} \cap U$. Then by Lemma 3.4, $I_*(\langle [c]\rangle) = H_1([c_L];\mathbb{Q})$. Then there exists an integer a such that

$I_*([c]^a) = J_*([d])$. Denote by s_U a cross section of U with respect to the Seifert fibration of S induced on U. Then there exists $(p,q) \in \mathbb{Z}\times\mathbb{Z}$ such that $[c]^a = s_U^p h_S^q$ with $p \neq 0$ and where h_S denotes the generic fiber of S.

Denote still by s_U the component of $\partial\bar{\mathcal{O}}_+^o$ corresponding to U. Then $J_*([s_U]^p) = J_*([d]))$ at the H_1-level with coefficient $\mathbb{Q}$.

Consider the graph Γ' obtained from $\Gamma_{\mathcal{F}}$ after removing $\mathrm{int}(e_2)$. Then Γ' satisfies points (i), (ii) and (iv) and the valence of x in Γ' is strictly less than the valence of x in $\Gamma_{\mathcal{F}}$. The proof of the lemma follows by repeating this operation finitely many of times. □

Proof of Proposition 3.1. Let x be an element of $q(M_0) \cap V(\Gamma')$. We know by Lemma 3.7 that $v(x) = 2$. Then there exist exactly two edges e_1, e_2 whose x is an end point. Hence one can replace the edges e_1, e_2 by a single edge $e_1 \cup_x e_2$. By performing this operation for all points of $q(M_0)$ we get a new graph Γ_1 satisfying the conclusion of Proposition 3.1. □

4. The volume decreases under non-zero degree maps

The main purpose of this section is to prove the following result.

Proposition 4.1. *Let $f: (G,\partial G) \to (\Sigma,\partial\Sigma)$ be a $\mathcal{T}$-injective, π_1-surjective, proper, non-zero degree map from a Haken graph manifold G with toral boundary to an orientable $\mathbb{S}^1$-bundle (Σ,ξ,F) over an orientable hyperbolic surface F with bundle projection $\xi: \Sigma \to F$. Assume moreover that for each component L of G_-, there exists a vertical torus T_L in Σ such that $f(L) \subset T_L$. Then* $\mathrm{Vol}(G) \geq \mathrm{Vol}(\Sigma)$ *and if $G_- \neq \emptyset$ then* $\mathrm{Vol}(G) > \mathrm{Vol}(\Sigma)$.

Remark 4.2. Roughly speaking, in the proof of Proposition 4.1 we establish the following inequality: $\mathrm{Vol}(\Sigma) \leq \mathrm{Vol}(G_+) + \varepsilon$, where $\varepsilon \ll \mathrm{Vol}(G_-)$ when $G_- \neq \emptyset$. This inequality is sufficient for our purpose. However the following question is natural: Is it true that $\mathrm{Vol}(\Sigma) \leq \mathrm{Vol}(G_+)$?

Throughout this section, we keep the same notations as in Section 3.1.

4.1. Domination of the target by the thick part of the domain

Lemma 4.3. *Let $f: (G,\partial G) \to (\Sigma,\partial\Sigma)$ be a map satisfying the same hypotheses as in Proposition 4.1. Assume moreover that either*

(i) *$\mathcal{T}_- = \emptyset$ (which is equivalent to the condition $G_- = \emptyset$), or*

(ii) *$\mathcal{T}_- \neq \emptyset$ and the homomorphism $H_1(c_-;\mathbb{Q}) \to H_1(F;\mathbb{Q})$ induced by the natural inclusion is surjective.*

Then

$$\mathrm{Vol}(G_+) \geq \mathrm{Vol}(\Sigma).$$

Proof. Assume first that condition (i) is satisfied. If $\mathcal{T}_- = \emptyset$ then necessarily $G_- = \emptyset$ and so $G = G_+$ is a Seifert fibered space by Lemma 2.19. Hence in this case $f|G : G \to \Sigma$ is a bundle homomorphism of non-zero degree and the inequality follows.

Assume now that condition (ii) is satisfied. Let $\widehat{L}$ denote a connected characteristic submanifold of G such that $f_*(\pi_1 S) \simeq \mathbb{Z} \times \mathbb{Z}$ for each Seifert piece S of $\widehat{L}$ and choose $\widehat{L}$ so that it is maximal with respect to the natural inclusion. Denote by $L_1, \dots, L_n$ the set of all such maximal characteristic submanifolds of G. It follows from Remark 3.2 that the submanifolds L_i satisfy the following properties:

(a) $\bigcup_{1 \leq i \leq n} L_i \supset G_-$.

(b) There exists a component T_i of $\mathcal{T}_-$ such that $f(L_i) \subset T_i$ and $\{T_1, \dots, T_n\} = \mathcal{T}_-$.

(c) There exists a canonical torus D_i of G shared by L_i and a Seifert piece S_i of G_+ such that $f_*(\pi_1 S_i)$ is non-abelian.

On the other hand, it follows from the $\mathcal{T}$-injectivity that $f|D_i : D_i \to T_i$ is a non-zero degree map. For each i, denote by d_i the simple closed curve obtained from D_i after killing the primitive curve of D_i corresponding to the generic fiber of the Seifert piece S_i adjacent to D_i. In this way, d_i can be seen as a boundary component of the base surface of S_i denoted by $\bar{\mathcal{O}}_{S_i}$. Then $f|D_i : D_i \to T_i$ descends to a non-zero degree map $\tilde{f} : d_i \to c_i$, where $c_i = \xi(T_i)$ is a component of c_-. Since $f|S_i$ is a fiber preserving map, it follows that $f|\bigcup_i S_i$ descends to the map $J : \bigcup_i \bar{\mathcal{O}}_{S_i} \to F$. The natural inclusion $c_- = \bigcup_L c_L \hookrightarrow F$ induces an epimorphism at the $H(\cdot, \mathbb{Q})$-level which implies that $J : \bigcup_i \partial\bar{\mathcal{O}}_{S_i} \to F$ induces an epimorphism at the $H(\cdot, \mathbb{Q})$-level by properties (a) and (b).

For convenience we index the components S_i by $S_1, \dots, S_q$ in such a way that $S_i \neq S_j$ when $i \neq j$. Denote by g_i the genus of the base surface of S_i and by p_i the number of components of ∂S_i. Then

$$\mathrm{Vol}(G_+) \geq \sum_{i=1}^{q} (2g_i + p_i - 2).$$

Since $J : \bigcup_i \partial\bar{\mathcal{O}}_{S_i} \to F$ induces an epimorphism at the $H(\cdot, \mathbb{Q})$-level, it follows that

$$\sum_{i=1}^{q} p_i \geq \beta_1 F.$$

On the other hand, we know that $\mathrm{Vol}(\Sigma) = \beta_1(F) - \varepsilon$, with $\varepsilon = 2$ or 1 depending on whether Σ is closed or not, and by Lemmas 2.5 and 2.8, we may assume that

$g_i \geq 2$ when $i = 1, \ldots, q$. Thus we get

$$\mathrm{Vol}(G_+) \geq \beta_1(F_\Sigma) + 2\sum_{i=1}^{q}(g_i - 1) > \mathrm{Vol}(\Sigma).$$

This proves the lemma. □

4.2. Increasing the genus of the base 2-orbifolds of the efficient thin part. In order to prove Proposition 4.1, we first state a technical lemma which allows us to construct suitable coverings which increase the genus of the base of the thin part of G.

Lemma 4.4. *Let $f: G \to \Sigma$ be a map satisfying the hypotheses of Proposition 4.1. Assume moreover that $\mathcal{T}_- \neq \emptyset$ and that*

(C_1) $$H_1(c_-;\mathbb{Q}) \to H_1(F;\mathbb{Q}) \text{ is not surjective.}$$

For any $n \in \mathbb{N}^*$ there exists a finite regular covering $f_n: G_n \to \Sigma_n$ of $f: G \to \Sigma$ satisfying the following properties:

(i) *Any Seifert piece of G_n over a Seifert piece of G_- admits a fibration over a 2-orbifold of genus at least n.*

(ii) *The covering $\Sigma_n \to \Sigma$ has fiber degree $\leq$ the fiber degree of $S_n \to S$ for any Seifert piece S of G and for any Seifert piece S_n of G_n over S.*

In order to prove this result it will be convenient to define a set of invariants which parametrize the map $f: G \to \Sigma$ satisfying the same hypotheses as in Proposition 4.1. Recall that we know, from Lemma 2.3, that each Seifert piece of G admits always a fibration over an orientable 2-orbifold. Given a Seifert piece S of G, we denote by h_S its generic fiber, by $c_1, \ldots, c_{r_S}$ its exceptional fibers and by $T_1(S), \ldots, T_{p_S}(S)$ its boundary components, and for each $i = 1, \ldots, p_S$ denote by $d_i(S)$ a section of $T_i(S)$ so that $d_1(S) + \cdots + d_{p_S}(S) + q_1 + \cdots + q_{r_S} = 0$ in $H_1(S;\mathbb{Z})$, where each q_i is a chosen section corresponding to the exceptional fiber c_i.

Recall that for each component S of G_- there exists a component T_S of $\mathcal{T}_-$ such that $f(S) \subset T_S$. Denote by $\bar{u}_S$ a simple closed curve in T_S such that $\pi_1 T_S = \langle [\bar{u}_S], t\rangle$. After a homotopy on $f|S$ we may assume that each component of $(f|S)^{-1}(\bar{u}_S)$ is a well-embedded incompressible surface in S.

In the following it will be convenient to decompose G_- into the union $G_{-,\mathrm{h}} \cup G_{-,\mathrm{v}}$, where $G_{-,\mathrm{h}}$, resp. $G_{-,\mathrm{v}}$, consists of the Seifert pieces S of G_- such that $(f|S)^{-1}(\bar{u}_S)$ is made of horizontal, resp. vertical surfaces.

If S denotes a Seifert piece of G_+, let q_S be the non-zero integer satisfying $f_*(h_S) = t^{q_S}$.

Suppose that S denotes a Seifert piece of $G_{-,\mathrm{h}}$. Let (β_S, α_S) be the integers such that $f_*(h_S) = \bar{u}_S^{\beta_S} t^{\alpha_S}$, where t denotes the fiber of Σ. Note that by definition of

$G_{-,\mathrm{h}}$ for each $i = 1, \dots, p_S$ there exists $\gamma_S^i \neq 0$ and coprime integers (a_S^i, n_S^i) with $a_S^i \neq 0$ such that $f_*(d_i^{a_S^i}(S) h_S^{-n_S^i}) = \bar{u}_S^{\gamma_S^i}$.

Note that $\beta_S \neq 0$ since $f|S$ is not fiber preserving. On the other hand, $a_S^i \neq 0$ implies that $\mathrm{Rank}(\langle d_i^{a_S^i}(S) h_S^{-n_S^i}, h_S \rangle) = 2$ and since f is non-degenerate we have $\alpha_S \neq 0$.

Suppose that S is a Seifert piece of $G_{-,\mathrm{v}}$. We denote by ν_S the non-zero integer such that $f_*(h_S) = \bar{u}_S^{\nu_S}$ and by (λ_S^i, μ_S^i) the integers such that $f_*(d_i(S)) = \bar{u}_S^{\lambda_S^i} t^{\mu_S^i}$, with $\mu_S^i \neq 0$.

Then we define the *parameters space* of the maps f by setting

$$\mathcal{M}(f) := \left\{ \begin{array}{ll} q_S & \text{when } S \in G_+, \\ (\alpha_S, \beta_S), \gamma_S^i, a_S^i & i = 1, \dots, p_S \text{ when } S \in G_{-,\mathrm{h}}, \\ \nu_S, (\lambda_S^i, \mu_S^i) & \text{when } S \in G_{-,\mathrm{v}} \end{array} \right\}.$$

To prove Lemma 4.4 we first check that we have the following reduction. The hypotheses are the same as in Lemma 4.4. More precisely:

Claim 4.5. *We may assume that the map $f: G \to \Sigma$ satisfies the following condition.*

There exists a prime number q such that $\mathbf{e}(\Sigma) \in q\mathbb{Z}$ and

(C_2)
$$q > \text{l.c.m} \left\{ \begin{array}{l} q_S, \ S \in G_+, \\ \gamma_S^i, \alpha_S, \ S \in G_{-,\mathrm{h}}, \ i = 1, \dots, p_S, \\ \nu_S, \mu_S^i \ S \in G_{-,\mathrm{v}} \end{array} \right\}.$$

Proof. First note that if $\mathbf{e}(\Sigma) = 0$ then the claim is obvious. Hence, let us assume that $\mathbf{e}(\Sigma) \neq 0$ which implies in particular that Σ is closed. By Lemma 2.6, passing to a finite covering with fiber degree one of the target we may assume that for each component c of c_-,

$(*)$
$$\mathrm{Im}\,(H_1(c; \mathbb{Z}) \to H_1(F; \mathbb{Z})) \neq \{0\}.$$

On the other hand, recall that the group $\pi_1 \Sigma$ has a presentation

$(\mathcal{P}_{\mathbf{e}})$
$$\langle t, a_1, b_1, \dots, a_g, b_g : a_i^{-1} t a_i = t, b_j^{-1} t b_j = t, [a_1, b_1] \dots [a_g, b_g] = t^n \rangle,$$

where $n = \mathbf{e}(\Sigma)$. The integer n also has the following interpretation: the group $\pi_1 \Sigma$ is obtained as a central extension of $\langle t \rangle = \mathbb{Z}$ by $\pi_1 F$ using the exact sequence of the fibration

$$\{1\} \to \langle t \rangle \simeq \mathbb{Z} \xrightarrow{i_*} \pi_1 \Sigma \xrightarrow{\xi_*} \pi_1 F \to \{1\}.$$

Recall that central extensions of $\mathbb{Z}$ by $\pi_1 F$ correspond to elements of $H^2(\pi_1 F, \mathbb{Z})$ and the integer n is the element of $\mathbb{Z} \simeq H^2(\pi_1 F, \mathbb{Z})$ corresponding to $\pi_1 \Sigma$.

Let q be a prime number. By condition (C_1), there exists an epimorphism $\varepsilon\colon H_1(F;\mathbb{Z}) \to \mathbb{Z}_q$ such that $\ker\varepsilon \supset H_1(c_-;\mathbb{Z})$. Consider the finite covering $\pi\colon \widetilde{\Sigma} \to \Sigma$ corresponding to $\xi_*^{-1}(\ker(\bar{\varepsilon}))$, where $\bar{\varepsilon}$ denotes the composition $\pi_1 F \to H_1(F;\mathbb{Z}) \to \mathbb{Z}_q$. It follows from the construction that π is trivial over $\mathcal{T}_-$. On the other hand, $\widetilde{\Sigma}$ is an $\mathbb{S}^1$-bundle over a surface $\widetilde{F}$ that is the covering of F corresponding to $\bar{\varepsilon}$. Note that the inclusion $\pi_1\widetilde{F} \to \pi_1 F$ gives a map

$$H^2(\pi_1 F, \mathbb{Z}) \simeq \mathbb{Z} \ni 1 \mapsto q \times 1 \in \mathbb{Z} \simeq H^2(\pi_1 \widetilde{F}, \mathbb{Z})$$

and thus the integer $\tilde{n}$ corresponding to the fibration of $\widetilde{\Sigma}$ satisfies the equation $\tilde{n} = qn$. Note that since the covering is trivial over $\mathcal{T}_-$, it follows that the covering $\widetilde{G} \to G$ corresponding to $f_*^{-1}(\xi_*^{-1}(\ker(\bar{\varepsilon})))$ is trivial over G_- and over h_S, where h_S denotes the generic fiber of S where S runs over the components of G_+. In particular for any Seifert piece S in $G_{-,\mathrm{h}}$, resp. G_+, resp. $G_{-,\mathrm{v}}$ and any component $\widetilde{S}$ over S in $\widetilde{G}^*$ we have $\alpha_{\widetilde{S}} = \alpha_S$, $\beta_{\widetilde{S}} = \beta_S$, $\gamma^i_{\widetilde{S}} = \gamma^i_S$, $a^i_{\widetilde{S}} = a^i_S$, resp. $q_{\widetilde{S}} = q_S$, resp. $\nu_{\widetilde{S}} = \nu_S$, $\mu^i_{\widetilde{S}} = \mu^i_S$ and $\lambda^i_{\widetilde{S}} = \lambda^i_S$. In other words, this covering does not affect the parameter space. This completes the proof of the claim. □

Lemma 4.6. *Let $f\colon G \to \Sigma$ be a map satisfying the hypotheses of Proposition 4.1 and conditions (C_1), (C_2) and $(*)$. Let S be a geometric piece of G_-. Let g be an element of $\pi_1 S$ which denotes either the homotopy class of an exceptional fiber or the homotopy class of a section of a boundary component of S. Then there exists a finite group H and an epimorphism $\varphi\colon \pi_1\Sigma \to H$ such that the following holds:*

(i) *Separation: $\varphi f_*(g) \notin \langle \varphi f_*(h_S)\rangle$.*

(ii) *Action on the fibers: Let $p\colon \widetilde{\Sigma} \to \Sigma$ denote the covering of Σ corresponding to φ and for any Seifert piece S of G denote by $\pi_S\colon \widetilde{S} \to S$ the finite covering of S corresponding to* $\ker(\varphi \circ (f|S)_*)$. *Then $G_h(\pi_S) \geq G_h(p)$.*

Proof. Let S be a geometric piece of G_- and let g be an element of $\pi_1 S$ satisfying the hypothesis of the lemma.

First assume that g is the homotopy class of an exceptional fiber c of S and denote by $\mu > 1$ the index of this fiber. Let $(\beta, \alpha) \in \mathbb{Z}^2$ such that $f_*(g) = \bar{u}_S^{\beta} t^{\alpha}$. In particular we have $\beta\mu = x_S \neq 0$, where $x_S = \beta_S$ if S is a Seifert piece of $G_{-,\mathrm{h}}$ or $x_S = \nu_S$ if S is in $G_{-,\mathrm{v}}$.

Let p be a prime number such that $p|\mu$. According to Lemma 2.7 there exists a finite group H_p and an epimorphism $\tau\colon \pi_1 F \to H_p$ such that $\tau(u_S^{\beta}) \neq 1$ and p divides the order of $\tau(u_S^{\beta})$, where $u_S = \xi(\bar{u}_S)$. Consider the homomorphism φ given by

$$\pi_1\Sigma \xrightarrow{\xi_*} \pi_1 F \xrightarrow{\tau} H_p.$$

This completes the proof when $g = c$. Indeed suppose that there exists $n \in \mathbb{Z}$ such that $\varphi f_*(g) = \varphi f_*(h_S^n)$. Then $\tau(u_S^\beta) = \tau(u_S^{n\beta\mu})$. Then p divides $1 - n\mu$. A contradiction since $p|\mu$. Moreover, the second point of the lemma is satisfied since the covering on the target corresponding to φ has fiber degree one.

Assume now that g denotes the homotopy class of a section d of a component of ∂S.

Case 1: *Assume that S is a Seifert piece of $G_{-,\mathrm{h}}$.* According to the notation of Paragraph 4.2 we know that there exists $i \in \{1, \dots, p_S\}$ such that $d = d_i(S)$. In particular we have $f_*(d^{a_S^i}) = \bar{u}_S^{\gamma_S^i + n_S^i \beta_S} t^{n_S^i \alpha_S}$ where $a_S^i \neq 0$.

From the presentation $(\mathcal{P}_\mathbf{e})$ of $\pi_1\Sigma$ and by condition (C_2) one sees that $H_1(\Sigma;\mathbb{Z}) \simeq \mathbb{Z}_n \oplus H_1(F;\mathbb{Z})$ where $n \in q\mathbb{Z}$. Since $n \in q\mathbb{Z}$ there exists an epimorphism $\lambda_q \colon \mathbb{Z}_n \to \mathbb{Z}_q$. On the other hand, it follows from condition $(*)$ that the $\xi(\bar{u}_S)$'s are non-trivial elements of $H_1(F;\mathbb{Q})$ (when S runs over the Seifert pieces of G_-). Then there exists a q-group $(F_q, +)$ and an epimorphism $\tau_q \colon H_1(F;\mathbb{Z}) \to F_q$ such that $\tau_q(u_S) \neq 0$ for any S in G_-. Consider now the homomorphism φ defined by

$$\pi_1\Sigma \to H_1(\Sigma;\mathbb{Z}) \simeq \mathbb{Z}_n \oplus H_1(F;\mathbb{Z}) \xrightarrow{\lambda_q \times \tau_q} \mathbb{Z}_q \times F_q.$$

Using condition (C_2) we claim that φ satisfies the conclusion of the lemma. First we check point (i). To see this it is sufficient to check that $\varphi f_*(d^{a_S^i}) \notin \langle \varphi f_*(h_S) \rangle$. Assume that there exists $m \in \mathbb{Z}$ such that $\varphi f_*(d^{a_S^i}) = \varphi f_*(h_S^m)$. Then using our notations this means that

$$(\gamma_S^i + n_S^i \beta_S)\tau_q(u_S) = m\beta_S\tau_q(u_S) \quad \text{and} \quad n_S^i \alpha_S \lambda_q(t) = m\alpha_S\lambda_q(t).$$

Then q divides $\gamma_S^i + \beta_S(n_S^i - m)$ and $(n_S^i - m)\alpha_S$. Since $(\alpha_S, q) = 1$, q divides $n_S^i - m$ and thus q divides γ_S^i. A contradiction. It remains to check the second point of the lemma. First it follows from the construction of φ that $G_h(p) = q$, where p is the finite covering corresponding to φ. On the other hand, for any Seifert piece S of G it follows from our construction and from condition (C_2) that $\varphi f_*(h_S)$ has order q^{r_S} with $r_S \geq 1$, since $f_*(h_S) = t^{q_S}$ and $(q, q_S) = 1$ or $f_*(h_S) = \bar{u}_S^{\beta_S} t^{\alpha_S}$ and $(\alpha_S, q) = 1$ or $\xi_* f_*(h_S) = \xi_*(\bar{u}_S^{\nu_S})$ with $(\nu_S, q) = 1$ depending on whether S is a Seifert piece of G_+, $G_{-,\mathrm{h}}$ or $G_{-,\mathrm{v}}$.

Case 2: *Assume that S is a Seifert piece of $G_{-,\mathrm{v}}$.* We use the same arguments as in the first case. Let q be a prime number satisfying condition (C_2). Then consider the epimorphism

$$H_1(\Sigma;\mathbb{Z}) \simeq \mathbb{Z}_n \oplus H_1(F;\mathbb{Z}) \xrightarrow{\lambda_q \times \tau_q} \mathbb{Z}_q \times F_q$$

constructed in the first case and denote by φ the composition $\pi_1\Sigma \to H_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}_q \times F_q$. Then

$$\varphi(f_*(h_S)) = (0, \nu_S\tau_q([\xi(\bar{u}_S)]))$$

and

$$\varphi(f_*(d)) = (\mu_S^i \lambda_q(t), \lambda_S^i \tau_q([\xi(\bar{u}_S)])).$$

Since $q|n$ and since $(q, \mu_S^i) = 1$, it follows that $\varphi(f_*(d)) \notin \langle \varphi(f_*(h_S)) \rangle$. On the other hand, it follows from our construction and from condition (C_2) that $G_h(p) = q$ and that for any Seifert piece S of G then $\varphi f_*(h_S)$ has order q^{r_S} with $r_S \geq 1$. This completes the proof of the lemma. □

Proof of Lemma 4.4. Let S be a Seifert piece of G_- and assume that the genus g_S of the base 2-orbifold $\mathcal{O}_S$ of S satisfies $g_S \geq 1$. Denote by $d_1, \dots, d_{p_S}$ the chosen section of ∂S (with respect to the fixed Seifert fibration of S) and let $c_1, \dots, c_{r_S}$ denote the homotopy class of the exceptional fibers of S with index $\mu_1, \dots, \mu_{r_S}$. Using Lemma 4.6 and Claim 4.5 we may assume that there exists a homomorphism $\varphi \colon \pi_1 \Sigma \to K$ onto a finite group such that

(i) $\varphi f_*(d_i) \notin \langle \varphi f_*(h_S) \rangle$, for $i = 1, \dots, p_S$ and $\varphi f_*(c_j) \notin \langle \varphi f_*(h_S) \rangle$ for $j = 1, \dots, r_S$.

Denote by $p \colon \widetilde{S} \to S$ the covering corresponding to $\varphi \circ (f|S)_*$. This covering induces a branched covering, whose degree is denoted by σ, between the underlying space of the base 2-orbifolds of S and $\widetilde{S}$. Let β_j be the order of $\varphi f_*(c_j)$ in K and for each $i = 1, \dots, p_S$ denote by r_i the number of component of $\partial \widetilde{S}$ over T_i and set $n_i = \sigma / r_i$. Then the Riemann–Hurwitz formula allows us the compute the genus of the base 2-orbifold of $\widetilde{S}$ in the following way:

$$2g_{\widetilde{S}} = 2 + \sigma \left(p_S + 2g_S + r_S - 2 - \sum_{i=1}^{i=p_S} \frac{1}{n_i} - \sum_{i=1}^{i=r_S} \frac{1}{(\mu_i, \beta_i)} \right).$$

By condition (i) one can check that $\sigma \geq 2$, $n_i \geq 2$ for $i = 1, \dots, p_S$ and $(\mu_i, \beta_i) \geq 2$ for $i = 1, \dots, r_S$. Then, since moreover $p_S \geq 1$ (because G_+ and G_- are non-empty), it is easy to check that $g_{\widetilde{S}} > g_S$ when $g_S \geq 1$. Note that condition (ii) of Lemma 4.4 is guaranteed by condition (ii) of Lemma 4.6.

Assume now that $g_S = 0$. We follow here the same construction as in the case of $g_S \geq 1$ using Lemma 4.6. The Riemann–Hurwitz formula gives

$$2g_{\widetilde{S}} \geq 2 + \sigma \left(\frac{p_S}{2} - 2 \right).$$

Hence if $p_S \geq 4$ then $g_{\widetilde{S}} \geq 1$ and we have a reduction to the first case. Assume that $p_S \leq 3$ and perform the same construction as above. Denote by $p_{\widetilde{S}}$ the number of boundary components of $\widetilde{S}$. Then the Riemann–Hurwitz formula gives

$$2g_{\widetilde{S}} = 2 - p_{\widetilde{S}} + \sigma \left(p_S + r_S - 2 - \sum_{i=1}^{i=r_S} \frac{1}{(\mu_i, \beta_i)} \right).$$

Assume $p_S = 3$. If $p_{\widetilde{S}} \geq 4$ then we have a reduction to the case above. If $p_{\widetilde{S}} = 3$ the Riemann–Hurwitz formula gives, since $\sigma \geq 2$, that $2g_{\widetilde{S}} \geq -1 + \sigma \geq 1$ and thus $g_{\widetilde{S}} \geq 1$.

Assume $p_S = 2$. Applying the same argument (since $g_S = 0$ and $p_S = 2$ imply $r_S \geq 1$) we get a reduction to the case $p_S = 3$ or $g_{\widetilde{S}} \geq 1$.

Note that the case $p_S = 1$ is impossible. Indeed it follows from the construction of G_- that $f_*(\pi_1 S) \simeq \mathbb{Z} \times \mathbb{Z}$ and from the non-degeneration condition that $f_*(\pi_1 T) \simeq \mathbb{Z} \times \mathbb{Z}$ for any component T of ∂S. Thus we get the following commutative diagram:

$$\begin{array}{ccccc} \pi_1 T & \longrightarrow & \pi_1 S & \longrightarrow & \mathbb{Z} \times \mathbb{Z} \subset \pi_1 \Sigma \\ \downarrow & & \downarrow & \nearrow & \\ H_1(T;\mathbb{Z}) & \longrightarrow & H_1(S;\mathbb{Z}). & & \end{array}$$

This implies that $\mathrm{Rank}(H_1(T;\mathbb{Z}) \to H_1(S;\mathbb{Z})) = 2$. If ∂S is connected then it follows from the exact sequence corresponding to the pair $(S, \partial S)$ that $\mathrm{Rk}(H_1(\partial S;\mathbb{Z}) \to H_1(S;\mathbb{Z})) = 1$. Hence ∂S cannot be connected. Next we perform this construction for each Seifert piece of G_-.

To complete the proof of the lemma it remains to check that one can find a regular covering. More precisely assume that there exists a finite covering $f_n \colon G_n \to \Sigma_n$ satisfying the conclusion of the lemma. Denote by $\pi_n \colon \Sigma_n \to \Sigma$ the associated covering of Σ, by H_n the finite index subgroup of $\pi_1 \Sigma$ corresponding to this covering and denote by $p_n \colon G_n \to G$ the finite covering corresponding to $f_*^{-1}(H_n)$. Denote by $\varepsilon_n \colon \widehat{\Sigma}_n \to \Sigma_n$ the finite covering so that $\pi_n \circ \varepsilon_n$ is the regular covering of Σ corresponding to the normal subgroup

$$K_n = \bigcap_{g \in \pi_1 \Sigma} g H_n g^{-1} \lhd \pi_1 \Sigma.$$

Then consider the corresponding regular covering of $f \colon G \to \Sigma$ denoted by $\hat{f}_n \colon \widehat{G}_n \to \widehat{\Sigma}_n$. Since G_n satisfies point (i) of Lemma 4.4 and since $\widehat{G}_n$ is a finite covering of G_n, point (i) also holds for $\widehat{G}_n$. On the other hand, since the fiber of Σ is central in $\pi_1 \Sigma$, it follows from the construction the fiber degree of $\pi_n \circ \varepsilon_n$ is equal to the fiber degree of π_n. This completes the proof of the lemma. □

4.3. Proof of Proposition 4.1. By Lemma 4.3, we may assume that $G_- \neq \emptyset$ and that the inclusion $c_- \hookrightarrow F$ induces a non-surjective homomorphism $H_1(c_-;\mathbb{Q}) \to H_1(F;\mathbb{Q})$.

Case 1: First suppose that $\mathrm{genus}(\mathcal{O}_S) \geq 1$ for any Seifert piece S of G_-, where $\mathcal{O}_S$ denotes the base 2-orbifold of S. Denote by Γ_1 the minimal connection graph given by Proposition 3.1. Consider the Mayer–Vietoris exact sequence corresponding to the decomposition of $\Gamma_1 \cup \bar{\mathcal{O}}_+^o$ given by $\left(\Gamma_1, \bar{\mathcal{O}}_+^o, \Gamma_1 \cap \bar{\mathcal{O}}_+^o\right)$. Denote by $O_1, \dots, O_k$ the components of $\bar{\mathcal{O}}_+^o$ and by $\Sigma_1, \dots, \Sigma_l$ the Seifert pieces of G_-^o. Then we get

$$\{0\} \to H_1\left(\bar{\mathcal{O}}_+^o\right) \oplus H_1(\Gamma_1) \to H_1\left(\Gamma_1 \cup \bar{\mathcal{O}}_+^o\right) \to H_0\left(\Gamma_1 \cap \bar{\mathcal{O}}_+^o\right) \to \cdots$$

$$\cdots \to H_0\left(\bar{\mathcal{O}}_+^o\right) \oplus H_0(\Gamma_1) \to H_0\left(\Gamma_1 \cup \bar{\mathcal{O}}_+^o\right) \to \{0\}.$$

Thus, since Γ_1 is connected and since $\bar{\mathcal{O}}_+^o$ and $\bar{\mathcal{O}}_+^o \cap \Gamma_1$ have the same number of components we get the following relation:

$$\beta_1\left(\bar{\mathcal{O}}_+^o\right) = \beta_1\left(\Gamma_1 \cup \bar{\mathcal{O}}_+^o\right) - \beta_1(\Gamma_1).$$

We know that

$$\operatorname{Vol}(G^o) \geq \beta_1\left(\bar{\mathcal{O}}_+^o\right) + \beta_1\left(\bar{\mathcal{O}}_-^o\right) - k - l,$$

where $\bar{\mathcal{O}}_-^o$ denotes the union of the base surfaces of the Seifert pieces of G_-^o. Thus we get

$$\operatorname{Vol}(G^o) \geq \beta_1\left(\Gamma_1 \cup \bar{\mathcal{O}}_+^o\right) - \beta_1(\Gamma_1) + \beta_1\left(\bar{\mathcal{O}}_-^o\right) - k - l$$

By Proposition 3.1, we know that $\beta_1\left(\Gamma_1 \cup \bar{\mathcal{O}}_+^o\right) \geq \beta_1(F)$ and we know that $\operatorname{Vol}(\Sigma) = \beta_1(F) - \varepsilon$, where $\varepsilon = 2$ or 1 depending on whether F is closed or not.

Moreover we know that $\beta_1(\Gamma_1) = \operatorname{Card}(E(\Gamma_1)) - \operatorname{Card}(V(\Gamma_1)) + 1$. This implies that

$$\operatorname{Vol}(G^o) \geq \operatorname{Vol}(\Sigma) + \varepsilon - 1 + \operatorname{Card}(V(\Gamma_1)) - k - l + \beta_1\left(\bar{\mathcal{O}}_-^o\right) - \operatorname{Card}(E(\Gamma_1)).$$

Again by Proposition 3.1, we have $\operatorname{Card}(V(\Gamma_1)) = k + l$. Note also that $\beta_1\left(\bar{\mathcal{O}}_-^o\right) = \sum_{i=1}^{l}(2g_i + r_i - 1)$, where g_i, resp. r_i, denotes the genus, resp. the number boundary components, of Σ_i, $i = 1, \dots, l$. Then

$$\operatorname{Vol}(G^o) \geq \operatorname{Vol}(\Sigma) + 2\sum_{i=1}^{l} g_i - l + \sum_{i=1}^{l} r_i - \operatorname{Card}(E(\Gamma_1)).$$

Finally, by Proposition 3.1, we know that $\sum_{i=1}^{l} r_i - \operatorname{Card}(E(\Gamma_1)) \geq 0$ and since $g_i \geq 1$ for $i = 1, \dots, l$ we then get

$$\operatorname{Vol}(G^o) \geq \operatorname{Vol}(\Sigma) + 2\sum_{i=1}^{l} g_i - l > \operatorname{Vol}(\Sigma).$$

This proves that $\operatorname{Vol}(G^o) > \operatorname{Vol}(\Sigma)$ since $l \geq 1$ by hypothesis. Hence this completes the proof in this case.

Case 2: If the condition on the genus of the base surfaces of the Seifert pieces in G_- is not satisfied then, since condition (C_1) is satisfied, we know from Lemma 4.4

that there exists a finite regular covering $f_1: G_1 \to \Sigma_1$ of $f: G \to \Sigma$ satisfying the following properties. Let $\pi: \Sigma_1 \to \Sigma$ and $p: G_1 \to G$ denote the finite regular coverings corresponding to f_1. Then

(i) any Seifert piece of $p^{-1}(G_-)$ admits a Seifert fibration over a 2-orbifold of genus ≥ 1,

(ii) for any Seifert piece S of G and for any component S_1 of $p^{-1}(S)$ it holds that $G_h(p|S_1) \geq G_h(\pi)$.

Since $(G_1)_- = p^{-1}(G_-)$ by Lemma 2.13, one can apply the above arguments to the map $f_1: G_1 \to \Sigma_1$. It follows from the paragraph above that we have $\mathrm{Vol}(G_1) > \mathrm{Vol}(\Sigma_1)$. Thus we get

$$\mathrm{Vol}\,(\Sigma_1) = \mathrm{Vol}\,(\Sigma)\,\frac{\deg(\pi)}{G_h(\pi)} < \mathrm{Vol}\,(G_1) = \mathrm{Vol}\left(p^{-1}\,(G)\right).$$

Denote by $Q_1, \dots, Q_l$ the geometric components of G and by p_i the induced covering $p|p^{-1}(Q_i): p^{-1}(Q_i) \to Q_i$. Then since p is a regular covering we have

$$\mathrm{Vol}\,(G) = \frac{1}{\deg(p)} \sum_{i=1}^{i=l} \mathrm{Vol}\left(p^{-1}\,(Q_i)\right) G_h(p_i).$$

Since $f: G \to \Sigma$ is π_1-surjective we have $\deg(\pi) = \deg(p)$ and by (ii) we get

$$\mathrm{Vol}\,(G) \geq \frac{G_h(\pi)}{\deg(\pi)} \mathrm{Vol}\left(p^{-1}\,(G)\right).$$

By combining this latter inequality with the first one we get $\mathrm{Vol}(\Sigma) < \mathrm{Vol}(G)$. This ends the proof of Proposition 4.1.

5. Proof of the theorems

5.1. Nonzero degree maps decreases the volume. In this section we prove Theorem 1.2.

Case 1: Assume that $\tau(N) = 0$. If $\tau(M) = 0$ then M is a virtual torus bundle and then f is homotopic to a finite covering by [W], in particular $f_*: \pi_1 M \to \pi_1 N$ is injective. In the other cases $\tau(M) \neq 0$ and thus $\mathrm{Vol}(M) > 0$.

Case 2: Assume now that $\tau(N) \neq 0$. Suppose that $f|\mathcal{T}_M: \mathcal{T}_M \to N$ is π_1-injective. By Lemmas 2.9 and 2.10, $G_\Sigma = f^{-1}(\Sigma)$ is a characteristic graph submanifold of M for any Seifert piece Σ of N. Choose a component G of $f^{-1}(\Sigma)$ so that $f|G: G \to \Sigma$ has non-zero degree. Let Σ_1 denote the finite covering of Σ such that $f|G$ has a π_1-surjective lift $f_1: G \to \Sigma_1$. By Lemmas 2.2, 2.14 and 2.5, we may assume that f_1 satisfies the hypothesis of Proposition 4.1. This proves that $\mathrm{Vol}(G_\Sigma) \geq \mathrm{Vol}(G) \geq \mathrm{Vol}(\Sigma_1) \geq \mathrm{Vol}(\Sigma)$. Hence $\mathrm{Vol}(M) \geq \mathrm{Vol}(N)$.

Suppose that $f|\mathcal{T}_M : \mathcal{T}_M \to N$ is not π_1-injective. Then using Lemma 2.11 we know that there exists a connected characteristic submanifold $M_1 \subset M$ which contains $\mathcal{H}(M)$ in its interior, a closed Haken manifold $\widehat{M}_1$ obtained from M_1 after Seifert Dehn fillings along ∂M_1 and a $\mathcal{T}$-injective non-zero degree extension $\hat{f}_1 : \widehat{M}_1 \to N$ of $f_1 = f|M_1 : M_1 \to N$ such that $\|\widehat{M}_1\| = |\deg(\hat{f}_1)| \|N\|$.

Since $f|\mathcal{T}_M : \mathcal{T}_M \to N$ is degenerate, there exists at least one Seifert piece $\widehat{S}$ in $\widehat{M}_1$ obtained from S after non-trivial (i.e. with slope $\neq \infty$) Seifert Dehn fillings. The base 2-orbifold $\mathcal{O}_{\widehat{S}}$ of $\widehat{S}$ is obtained from the base 2-orbifold $\mathcal{O}_S$ of S after gluing some cone points along some components of $\partial\mathcal{O}_S$. Note that S necessarily supports an $\mathbb{H}^2 \times \mathbb{R}$-geometry.

Indeed, if not then S is the twisted I-bundle over the Klein bottle and thus $\widehat{M}_1 = \widehat{S}$ is a closed Seifert fibered space whose base is a 2-sphere with cone points $(2, 2, n)$. Then $\widehat{M}_1$ is a Seifert fibered space whose base 2-orbifold admits a spherical geometry. This contradicts the fact that $\widehat{M}_1$ is a Haken manifold.

Then we get $\chi(\mathcal{O}_S) < \chi(\mathcal{O}_{\widehat{S}}) \leq 0$. This proves that $\mathrm{Vol}(\widehat{M}_1) < \mathrm{Vol}(M)$. On the other hand, since $\hat{f}_1 : \widehat{M}_1 \to N$ has non-zero degree and since $\|\widehat{M}_1\| = |\deg(\hat{f}_1)| \|N\|$, it follows that $\mathrm{Vol}(\widehat{M}_1) \geq \mathrm{Vol}(N)$ by the first case. This completes the proof of Theorem 1.2.

5.2. Proof of the rigidity theorem. In this paragraph we prove Theorem 1.3. Let $f : M \to N$ be a non-zero degree map between closed Haken manifolds satisfying the Volume Condition $\|M\| = |\deg(f)| \|N\|$ and $\mathrm{Vol}(M) = \mathrm{Vol}(N)$. Then it follows from Theorem 1.2 that $f|\mathcal{T}_M$ is π_1-injective.

Case 1: Assume that N admits a geometry E^3, Nil or Sol. This means that $\tau(M) = \tau(N) = 0$. Then M is a virtual torus bundle (in particular M is geometric) and since N is irreducible, the map f is homotopic to a $\deg(f)$-fold covering by a result of [W].

Case 2: Assume that N admits a geometry $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Then we check the following

Lemma 5.1. *Let $f : (M, \partial M) \to (N, \partial N)$ be a proper non-zero degree map from a Haken graph manifold M with toral boundary to an orientable Seifert manifold with geometry $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. If $\mathrm{Vol}(M) = \mathrm{Vol}(N)$ then f is homotopic to a covering map with $G_{\mathrm{ob}}(f) = 1$ and $G_h(f) = |\deg(f)|$.*

Proof. Denote by $f_1 : M \to N_1$ the π_1-surjective lift of f into the finite covering N_1 of N corresponding to $f_*(\pi_1 M)$.

We first check that M is a Seifert manifold. If not then we claim that $\mathrm{Vol}(M) > \mathrm{Vol}(N)$. Indeed, to see this, first note that by Lemmas 2.2, 2.14 and 2.5 we may assume that f_1 satisfies the hypothesis of Proposition 4.1. Hence, if M is not Seifert then $M_+ \neq \emptyset$ and $M_- \neq \emptyset$ by Lemma 2.19. This implies that $\mathrm{Vol}(M) > \mathrm{Vol}(N_1) \geq$

$\mathrm{Vol}(N)$, by Proposition 4.1. A contradiction. Thus $M = M_+$ which implies that M is Seifert and that f and f_1 are homotopic to fiber preserving maps.

Since f_1 is fiber preserving, by Lemma 2.2 there exists a finite covering $\tilde{f}_1 : \widetilde{M} \to \widetilde{N}$ of f_1 such that $\widetilde{M} \to M$ and $\widetilde{N} \to N_1$ have fiber degree ± 1 and such that $\widetilde{N}$ is an $\mathbb{S}^1$-bundle over an orientable hyperbolic surface $\widetilde{F}$. Note that it follows from our construction that $\mathrm{Vol}(\widetilde{M}) = \mathrm{Vol}(\widetilde{N})$. Then the map $\tilde{f}_1$ descends to a non-zero degree map $\pi : \overline{\mathcal{O}}_{\widetilde{M}} \to \widetilde{F}$, where $\overline{\mathcal{O}}_{\widetilde{M}}$ denotes the base surface of $\widetilde{M}$. Note that $-\chi(\mathcal{O}_{\widetilde{M}}) \geq -\chi(\overline{\mathcal{O}}_{\widetilde{M}}) \geq \deg(\pi)(-\chi(\widetilde{F})) > 0$ (where $\mathcal{O}_{\widetilde{M}}$ denotes the base 2-orbifold of $\widetilde{M}$) and from $\mathrm{Vol}(\widetilde{M}) = \mathrm{Vol}(\widetilde{N})$ we conclude that $\chi(\mathcal{O}_{\widetilde{M}}) = \chi(\overline{\mathcal{O}}_{\widetilde{M}}) = \chi(\widetilde{F}) < 0$. Thus $\widetilde{M}$ is an $\mathbb{S}^1$-bundle over an orientable hyperbolic surface $\widetilde{K} = \overline{\mathcal{O}}_{\widetilde{M}} = \mathcal{O}_{\widetilde{M}}$ and $\deg(\pi) = 1$ which implies that $\pi : \widetilde{K} \to \widetilde{F}$ is homotopic to a homeomorphism. Denote by h (resp. t) the homotopy class of the fiber in $\widetilde{M}$ (in $\widetilde{N}$ resp.) and let n denote the non-zero integer such that $(\tilde{f}_1)_*(h) = t^n$. Using the exact sequences

$$\begin{array}{ccccccccc}
\{1\} & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\widetilde{M}) & \longrightarrow & \pi_1(\widetilde{K}) & \longrightarrow & \{1\} \\
\downarrow & & \downarrow{\scriptstyle \times n} & & \downarrow{\scriptstyle \tilde{f}_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow \\
\{1\} & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_1(\widetilde{N}) & \longrightarrow & \pi_1(\widetilde{F}) & \longrightarrow & \{1\},
\end{array}$$

we check that $(\tilde{f}_1)_*$ is an isomorphism. Thus so is $(f_1)_*$ and finally, by [Wa], f is a covering map. Moreover we claim that $G_h(f) = \deg(f)$ and $G_{\mathrm{ob}}(f) = 1$. Indeed, by Lemma 2.1 we have $|\chi(\mathcal{O}_M)| = G_{\mathrm{ob}}(f)|\chi(\mathcal{O}_N)| > 0$ and from $\mathrm{Vol}(M) = |\chi(\mathcal{O}_M)| = \mathrm{Vol}(N) = |\chi(\mathcal{O}_N)| \neq 0$ we get $G_{\mathrm{ob}}(f) = 1$. Since $|\deg(f)| = G_h(f) \times G_{\mathrm{ob}}(f)$ our lemma is shown. □

Case 3: Assume that N is hyperbolic. In this case the condition on the volume implies that M is still a hyperbolic manifold and f is homotopic to a covering map by a rigidity result of Soma ([S1, Theorem 1]).

Case 4: Assume that N is a non-geometric Haken manifold. This means in particular that $\tau(N) \neq 0$. Let $q : \widehat{N} \to N$ be the finite covering of N corresponding to $f_*(\pi_1 M)$ and let $\hat{f} : M \to \widehat{N}$ denote the lifting of f. By Theorem 1.2 we know that $\mathrm{Vol}(M) = \mathrm{Vol}(\widehat{N})$ and $\|M\| = |\deg(\hat{f})|\|\widehat{N}\|$.

By Lemmas 2.5 and 2.2, we may assume that $\widehat{N}$ contains no embedded Klein bottle.

After adjusting $\hat{f} : M \to \widehat{N}$ by a homotopy, we may assume, using Lemmas 2.9 and 2.10 that $\hat{f}$ is characteristic and M is necessarily a non-geometric Haken manifold.

Assume that $\mathcal{S}(\widehat{N}) \neq \emptyset$. Fix a Seifert piece Σ in $\widehat{N}$. Then necessarily Σ admits a $\mathbb{H}^2 \times \mathbb{R}$-geometry. Consider a component G of $\hat{f}^{-1}(\Sigma)$ so that $\deg(\hat{f}|G : G \to \Sigma) \neq 0$. Applying, Lemmas 2.2, 2.14, 2.5 and Proposition 4.1 to $\hat{f}|G$ we see that

$\mathrm{Vol}(G) \geq \mathrm{Vol}(\Sigma)$. This implies, since $\mathrm{Vol}(M) = \mathrm{Vol}(\widehat{N})$, that $\mathrm{Vol}(G) = \mathrm{Vol}(\Sigma)$ and that any component G_i of $\hat{f}^{-1}(\Sigma) \setminus G$ is a twisted I-bundle over the Klein bottle. Then using Lemma 5.1 we know that $\hat{f}|G$ is a covering map such that $G_h(\hat{f}|G) = \deg(\hat{f}|G)$ and $G_{\mathrm{ob}}(\hat{f}|G) = 1$.

On the other hand, since $\hat{f}$ is characteristic, it follows from the construction that $\hat{f}|G_i \colon G_i \to \Sigma$ is a proper map. Denote by T the component of $\partial\Sigma$ such that $\hat{f}(\partial G_i) \subset T$. Since $\hat{f}$ is $\mathcal{T}$-injective, there then exists a non-zero integer n such that $\hat{f}_*([\partial G_i]) = n[T]$ at the $H_2(\cdot;\mathbb{Z})$-level. Since G_i has connected boundary, $[\partial G_i] = 0$ in $H_2(G_i;\mathbb{Z})$ and thus, since $H_2(\Sigma;\mathbb{Z})$ is torsion free, $[T] = 0$ in $H_2(\Sigma;\mathbb{Z})$. This proves that $\partial\Sigma$ is connected. Hence $\deg(\hat{f}|G_i) \neq 0$ which is impossible since Σ admits a $\mathbb{H}^2 \times \mathbb{R}$-geometry. Thus $\hat{f}^{-1}(\Sigma) = G$. This proves that $\hat{f}|\mathcal{S}(M) \colon \mathcal{S}(M) \to \mathcal{S}(\widehat{N})$ is a covering map.

Assume that $\mathcal{S}(\widehat{N}) = \emptyset$. In this case $\mathrm{Vol}(\widehat{N}) = 0$ and thus $\mathrm{Vol}(M) = 0$. This means that if $\mathcal{S}(M) \neq \emptyset$ then each Seifert piece of M is homeomorphic to a twisted I-bundle over the Klein bottle and that for each component K of $\mathcal{S}(M)$ there exists a canonical torus T of $\widehat{N}$ such that $\hat{f}(K) \subset W(T)$. Hence $\hat{f}|K$ is non-degenerate and $\hat{f}_*(\pi_1 K)$ is abelian. We get a contradiction since K admits a Seifert fibration over a non-orientable surface (see the proof of Lemma 2.3). This shows that $\mathcal{S}(M) = \emptyset$.

On the other hand $\hat{f}|\mathcal{H}(M) \colon \mathcal{H}(M) \to \mathcal{H}(\widehat{N})$ is a covering map by a result of Soma in [S1]. But since $\hat{f}$ is π_1-surjective, $\hat{f}$ actually is a homeomorphism, using [Wa], and hence f is a covering map. Note that the induced proper map $f|\mathcal{S}(M) \colon \mathcal{S}(M) \to \mathcal{S}(N)$ is a covering map such that $G_h(f|\mathcal{S}_{\mathrm{h}}(M)) = \deg(f)$ and $G_{\mathrm{ob}}(f|\mathcal{S}_{\mathrm{h}}(M)) = 1$. This completes the proof of Theorem 1.3.

5.3. Proof of Theorem 1.6. We consider here degree one maps between closed Haken manifolds. In view of Theorem 1.3, to prove Theorem 1.6 we have to check the following

Claim 5.2. *For any closed Haken manifold M there exists a constant $\eta_M \in (0,1)$, which depends only on M, such that for any degree one map $f \colon M \to N$ into a closed Haken manifold N satisfying $\tau(N) \geq \tau(M)(1-\eta_M)$ it holds that $\tau(M) = \tau(N)$.*

Proof. Suppose the contrary. Then there is a closed Haken manifold M_0 and a sequence of closed Haken manifolds N_n such that there are degree one maps $f_n \colon M_0 \to N_n$ satisfying $\tau(N_n) \geq \tau(M_0)(1-1/n)$ and $\tau(N_n) \neq \tau(M_0)$ for any $n \in \mathbb{N}$. This implies in particular that $\|M_0\| \geq \|N_n\| \geq \|M_0\|(1-1/n)$. Then $\lim_{n\to\infty} \|N_n\| = \|M_0\|$. Hence by [D] this implies that the sequence $\{N_n\}_{n\in\mathbb{N}}$ is finite up to homeomorphism. This contradicts the inequalities

$$\|M_0\|\left(1-\frac{1}{n}\right) \leq \tau\left(N_n\right) < \tau\left(M_0\right).$$

This completes the proof of the claim. □

Thus one can apply Theorem 1.3 with the hypothesis $\deg(f) = 1$. This completes the proof of the theorem.

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