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Bounding the symbol length in the Galois cohomology of function fields of p-adic curves

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Dedicated to my teacher Professor R. Parimala on her 60th birthday

Abstract. Let K be a function field of a p-adic curve and l a prime not equal to p. Assume that K contains a primitive l^{th} root of unity. We show that every element in the l-torsion subgroup of the Brauer group of K is a tensor product of two cyclic algebras over K.

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Introduction

Let k be a field and l a prime number not equal to the characteristic of k. Let μ_l be the group of l^{th} roots of unity and $\mu_l(m)$ the tensor product of m copies of μ_l. For $n \geq 0$, let $H^n(k, \mu_l)$ denote the n^{th} Galois cohomology group with coefficients in μ_l. Let $k^* = k \setminus \{0\}$. We have an isomorphism $k^*/k^{*^l} \to H^1(k, \mu_l)$. For $a \in k^*$, let (a) denote its image in $H^1(k, \mu_l)$. For $a_1, \dots, a_m \in k^*$, the cup product gives an element $(a_1) \cdot (a_2) \dots (a_m) \in H^n(k, \mu_l(m))$, which we call a *symbol*.

Assume that k contains a primitive l^{th} root of unity. Fix a primitive l^{th} root of unity $\zeta \in k$. Then we have isomorphisms $\mu_l \to \mu_l(m)$ of Galois groups. Hence we have isomorphisms $H^n(k, \mu_l(m)) \to H^n(k, \mu_l)$. A *symbol* in $H^n(k, \mu_l)$ is simply the image of a symbol under this map.

A classical theorem of Merkurjev ([M]) asserts that every element in $H^2(k, \mu_2)$ is a sum of symbols. A deep result of Merkurjev and Suslin ([MS]) says that every element in $H^2(k, \mu_l)$ is a sum of symbols. By a theorem of Voevodsky ([V]), every element in $H^n(k, \mu_2)$ is a sum of symbols. Suppose that k is a p-adic field. Local class field theory tells us that every element in $H^2(k, \mu_l)$ is a symbol and $H^n(k, \mu_l) = 0$ for $n \geq 3$. If k is a global field, then the global class field theory asserts that every element in $H^n(k, \mu_l)$ is a symbol.

Question 1. Do there exist integers $N_l(n)(k)$ such that every element in $H^n(k, \mu_l)$ is a sum of at most $N_l(n)(k)$ symbols?

Of course, the answer to the above question is negative in general. It can be shown that for $K = k(X_1, \dots, X_n, \dots)$, there is no such $N_l(n)(K)$ for $n \geq 2$. However we can restrict to some special fields. It is well-known that if $N_l(n)(k)$ exist for k, then $N_l(n)(k((t)))$ exist. We ask the following

Question 2. Suppose that $N_l(n)(k)$ exist for some field k. Do they exist for $k(t)$?

This is an open question. However we can restrict to fields of arithmetic interest. For example we consider the p-adic fields. The most important result in this direction is the following

Theorem (Saltman, [S1], (cf. [S2])). *Let k be a p-adic field and $K/k(t)$ be a finite extension. Suppose that $l \neq p$. If A is a central simple algebra over K representing an element in $H^2(K, \mu_l)$, then* $\mathrm{ind}(A)$ *divides l^2.*

Let K be as in the above theorem. Suppose $p \neq 2$. Let $\alpha \in H^2(K, \mu_2)$ and A a central simple algebra over K representing α. Then by the above theorem, we have $\mathrm{ind}(A) = 1, 2, 4$. If $\mathrm{ind}(A) = 1$, then α is a trivial element. If $\mathrm{ind}(A) = 2$, then it is well known that α is a symbol. Assume that $\mathrm{ind}(A) = 4$. By a classical theorem of Albert ([A]), α is a sum of two symbols. For $H^3(K, \mu_l)$, we have the following

Theorem ([PS2], 3.5, (cf. [PS1], 3.9)). *Let k be a p-adic field and $K/k(t)$ be a finite extension. Suppose that $l \neq p$. Every element in $H^3(K, \mu_l)$ is a symbol.*

Let k and K be as above. The field K is of cohomological dimension 3 and $H^n(K, \mu_l) = 0$ for $n \geq 4$. By the above theorem, $N_l(3)(K) = 1$ and the only case where $N_l(n)(K)$ is to be determined is for $n = 2$. It is known that $N_l(2)(K) \geq 2$ (cf. [S1], Appendix). In this article we prove the following

Theorem. *Let k be a p-adic field and $K/k(t)$ be a finite extension. Suppose that $l \neq p$. Every element in $H^2(K, \mu_l)$ is a sum of at most two symbols; in other words, $N_l(2)(K) = 2$.*

1. Some preliminaries

In this section we recall a few basic facts about Galois cohomology groups and divisors on arithmetic surfaces. We refer the reader to ([C]), ([Li1]), ([Li2]) and ([Se]).

Let k be a field and l a prime number not equal to the characteristic of k. Assume that k contains a primitive l^{th} root of unity. Let $\zeta \in k$ be a primitive l^{th} root of unity. Let μ_l be the group of l^{th} roots of unity. Since k contains a primitive l^{th} root of unity, the absolute Galois group of k acts trivially on μ_l. For $m \geq 1$, let $\mu_l(m)$ denote the tensor product of m copies of μ_l. By fixing a primitive l^{th} root of unity ζ in k, we have isomorphisms of Galois modules $\mu_l(m) \to \mu_l$. Throughout this paper we fix a primitive l^{th} root of unity and identify $\mu_l(m)$ with μ_l.

Let $H^n(k, A)$ be the n^{th} Galois cohomology group of the absolute Galois group Γ of k with values in a discrete Γ-module A. The identification of $\mu_l(m)$ with μ_l gives an identification of $H^n(k, \mu_l(m))$ with $H^n(k, \mu_l)$. In the rest of this paper we use this identification.

Let $k^* = k \setminus \{0\}$. For $a, b, c \in k^*$ we have the following relations in $H^2(k, \mu_l)$.

(1) $(a) \cdot (bc) = (a) \cdot (b) + (a) \cdot (c)$;

(2) $(a) \cdot (b) = -((b) \cdot (a))$;

(3) $(a) \cdot (b^l) = 0$;

(4) $(a) \cdot (-a) = 0$.

If $l \geq 3$, we have $(a) \cdot (a) = (a) \cdot ((-1)^l a) = (a) \cdot (-a) = 0$.

Let K be a field and l a prime number not equal to the characteristic of K. Let v be a discrete valuation of K. The residue field of v is denoted by $\kappa(v)$. Suppose $\text{char}(\kappa(v)) \neq l$. Then there is a *residue* homomorphism $\partial_v \colon H^n(K, \mu_l(m)) \to H^{n-1}(\kappa(v), \mu_l(m-1))$. Let $\alpha \in H^n(K, \mu_l(m))$. We say that α is *unramified* at v if $\partial_v(\alpha) = 0$; otherwise it is said to be *ramified* at v.

Let $\mathcal{X}$ be a regular integral scheme of dimension d, with function field K. Let $\mathcal{X}^1$ be the set of points of $\mathcal{X}$ of codimension 1. A point $x \in \mathcal{X}^1$ gives rise to a discrete valuation ν_x on K. The residue field of this discrete valuation ring is denoted by $\kappa(x)$. The corresponding residue homomorphism is denoted by ∂_x. We say that an element $\zeta \in H^n(K, \mu_l(m))$ is *unramified* at x if $\partial_x(\zeta) = 0$; otherwise it is said to be *ramified* at x. We define the ramification divisor $\text{ram}_{\mathcal{X}}(\zeta) = \sum x$ as x runs over points in $\mathcal{X}^1$ where ζ is ramified. Suppose C is an irreducible subscheme of $\mathcal{X}$ of codimension 1. Then the generic point x of C belongs to $\mathcal{X}^1$ and we set $\partial_C = \partial_x$. If $\alpha \in H^n(K, \mu_l(m))$ is unramified at x, then we say that α is *unramified* at C. We say that α is *unramified* on $\mathcal{X}$ if it is unramified at every point of $\mathcal{X}^1$.

Let k be a p-adic field and K the function field of a smooth projective geometrically integral curve X over k. By the resolution of singularities for surfaces (cf. [Li1] and [Li2]), there exists a regular projective model $\mathcal{X}$ of X over the ring of integers $\mathcal{O}_k$ of k. We call such an $\mathcal{X}$ a *regular projective model of* K. Since the generic fibre X of $\mathcal{X}$ is geometrically integral, it follows that the special fibre $\overline{\mathcal{X}}$ is connected. Further if D is a divisor on $\mathcal{X}$, there exists a proper birational morphism $\mathcal{X}' \to \mathcal{X}$ such that the total transform of D on $\mathcal{X}'$ is a divisor with normal crossings (cf. [Sh], Theorem, p. 38 and Remark 2, p. 43). We use this result throughout this paper without further reference. If $P \in \mathcal{X}$ is a closed point and $f \in K$ is a unit at P, then we denote the image of f in the residue field at P by $f(P)$.

Let k be a p-adic field and K the function field of a smooth projective geometrically integral curve over k. Let l be a prime number not equal to p. Assume that k contains a primitive l^{th} root of unity. Let $\alpha \in H^2(K, \mu_l)$. Let $\mathcal{X}$ be a regular projective model of K such that $\text{ram}_{\mathcal{X}}(\alpha) = C + E$, where C and E are regular curves with normal crossings. We have the following

Theorem 1.1 (Saltman [S1]). *Let K, α, $\mathcal{X}$, C and E be as above and $P \in C \cup E$. Let R be the local ring at P. Let $\pi, \delta \in R$ be local equations of C and E respectively at P.*

(1) *If $P \in C \setminus E$ (or $E \setminus C$), then $\alpha = \alpha' + (\pi)\cdot(u)$ (or $\alpha = \alpha' + (\delta)\cdot(u)$) for some unit $u \in R$, $\alpha' \in H^2(K, \mu_l)$ unramified on R.*
(2) *If $P \in C \cap E$, then either $\alpha = \alpha' + (\pi)\cdot(u) + (\delta)\cdot(v)$ or $\alpha = \alpha' + (\pi)\cdot(u\delta^i)$ for some units $u, v \in R$, $\alpha' \in H^2(K, \mu_l)$ unramified on R.*

Let $P \in C \cap E$. Suppose that $\alpha = \alpha' + (\pi)\cdot(u) + (\delta)\cdot(v)$ for some units $u, v \in R$, $\alpha' \in H^2(K, \mu_l)$ unramified on R and π, δ are local equations of C and E respectively. Then $u(P) = \partial_C(\alpha)(P)$ and $v(P) = \partial_E(\alpha)(P)$. Note that $u(P)$ and $v(P)$ are uniquely defined modulo l^{th} powers. Following Saltman ([S3], §2), we say that P is a *hot point* of α if $u(P)$ and $v(P)$ do not generate the same subgroup of $\kappa(P)^*/\kappa(P)^{*^l}$.

We have the following

Theorem 1.2 (Saltman ([S3], 5.2). *Let k, K, α, $\mathcal{X}$ be as above. Then α is a symbol if and only if there are no hot points of α.*

2. The main theorem

Let k be a p-adic field and $K/k(t)$ be a finite extension. Let $l \geq 3$ be a prime number not equal to p. Assume that k contains a primitive l^{th} root of unity. Let $\beta \in H^2(K, \mu_l)$ and $\mathcal{X}$ a regular proper model of K. Let $\phi\colon \mathcal{X}' \to \mathcal{X}$ be a blow-up such that $\mathcal{X}'$ is a regular proper model of K and $\text{ram}_{\mathcal{X}'}(\beta) = C' + E'$, where C' and E' are two regular curves with normal crossings (cf. §1 or [S1], Proof of 2.1). Let $Q \in C' \cap E'$. Let $C'_1 \subset C'$ and $E'_1 \subset E'$ be the irreducible curves containing Q. Let $R' = \mathcal{O}_{\mathcal{X}',Q}$ be the regular local ring at Q and m_Q its maximal ideal. We have $m_Q = (\pi', \delta')$, where π' and δ' are local equations of C'_1 and E'_1 at Q respectively. Let $\nu_{C'_1}$ and $\nu_{E'_1}$ be the discrete valuations on K at C'_1 and E'_1 respectively.

Let $P = \phi(Q)$. Let R be the regular local ring at P and m_P its maximal ideal. We have the induced homomorphism $\phi^*\colon R \to R'$ of local rings, which is injective. Let $\pi, \delta \in R$ be such that $m_P = (\pi, \delta)$.

Lemma 2.1. *Suppose that $\beta = \beta' + (f')\cdot(g')$ for some $f', g' \in K$ and β' unramified on R'. Then Q is not a hot point of β.*

Proof. Since β' is unramified on R', the ramification data of β on R' is same as that of $(f')\cdot(g')$. Since $(f')\cdot(g')$, being a symbol, has no hot points ([S3], cf. 1.2), Q is not a hot point of β. □

Lemma 2.2. *Suppose that $\beta = \beta' + (\delta)\cdot(gv) + (f)\cdot(g)$, where β' is unramified on R', $f \in R$ is not divisible by δ and $v, g \in R$ are units with $g(P) = v(P)$. Then Q is not a hot point of β.*

Proof. We have $m_Q = (\pi', \delta')$ and β has ramification on R' only at π' and δ'. Since $R/m_P \hookrightarrow R'/m_Q$, we have $g(Q) = g(P) = v(P) = v(Q)$. If C_1' (or E_1') is the strict transform of a curve on $\mathcal{X}$, then either δ is a local equation of C_1' or $\nu_{C_1'}(\delta) = 0$. In fact, if C_1' is the strict transform of C_1 on $\mathcal{X}$, then $\nu_{C_1}(\delta) = \nu_{C_1'}(\delta)$ and δ itself being a prime in R, the assertion follows.

Suppose that C_1' and E_1' are strict transforms of two irreducible curves on $\mathcal{X}$. If δ is not a local equation for either C_1' or E_1', we claim that $(\delta)\cdot(gv)$ is unramified on R'. In fact, since g and v are units in R, $(\delta)\cdot(gv)$ is unramified on R except possibly at δ. Since f is not divisible by δ, $(f)\cdot(g)$ is unramified at δ. Since β is ramified on R' only at π' and δ' and δ is not one of them, $(\delta)\cdot(gv)$ is unramified on R'. By (2.1), Q is not a hot point of β. Assume that δ is a local equation for one of them, say C_1'. Since δ does not divide f, we have $\partial_{C_1'}(\beta) = \overline{vg}$ and $\partial_{E_1'}(\beta) = \tilde{g}^{\nu_{E_1'}(f)}$, where *bar* denotes the image in the residue field of C_1' and *tilde* denotes the image in the residue field of E_1'. Since β is ramified at E_1', $\nu_{E_1'}(f)$ is not a multiple of l. We have $\partial_{C_1'}(\beta)(Q) = v(Q)g(Q) = g(Q)^2$ and $\partial_{E_1'}(\beta)(Q) = g(Q)^{\nu_{E_1'}(f)}$. Since $l \neq 2$ and $\nu_{E_1'}(f)$ is not a multiple of l, $g(Q)^2$ and $g(Q)^{\nu_{E_1'}(f)}$ generate the same subgroup modulo l^{th} powers. Hence Q is not a hot point of β.

Suppose that C_1' is a strict transform of an irreducible curve on $\mathcal{X}$ and E_1' is an exceptional curve on $\mathcal{X}'$. We have $\partial_{E_1'}(\beta) = (\tilde{gv})^{\nu_{E_1'}(\delta)}\tilde{g}^{\nu_{E_1'}(f)}$. Since E_1' is an exceptional fibre in $\mathcal{X}'$, the residue field of R is contained in the residue field at E_1'. Hence $\partial_{E_1'}(\beta) = (g(P)v(P))^{\nu_{E_1'}(\delta)}g(P)^{\nu_{E_1'}(f)} = g(P)^{2\nu_{E_1'}(\delta)+\nu_{E_1'}(f)}$. Since β is ramified at E_1', $2\nu_{E_1'}(\delta) + \nu_{E_1'}(f)$ is not a multiple of l. Suppose δ is a local equation of C_1' at Q. Since δ does not divide f and $\nu_{C_1'}(\delta) = 1$, we have $\partial_{C_1'}(\beta) = \overline{gv}$. Thus $\partial_{C_1'}(\beta)(Q) = g(P)v(P) = g(P)^2$. Since $l \neq 2$ and $2\nu_{E_1'}(\delta) + \nu_{E_1'}(f)$ is not a multiple of l, the subgroups generated by $g(Q)^2$ and $g(P)^{2\nu_{E_1'}(\delta)+\nu_{E_1'}(f)}$ are equal modulo l^{th} powers. Hence Q is not a hot point of β. Suppose δ is not a local equation of C_1' at Q. We have $\partial_{C_1'}(\beta) = \bar{g}^{\nu_{C_1'}(f)}$. Since β is ramified at C_1', $\nu_{C_1'}(f)$ is not a multiple of l. Thus as above Q is not a hot point of β.

The case E_1' is a strict transform of an irreducible curve on $\mathcal{X}$ and C_1' is an exceptional curve in $\mathcal{X}'$ follows on similar lines.

Suppose that both C_1' and E_1' are exceptional curves in $\mathcal{X}'$. Then as above we have $\partial_{C_1'}(\beta) = g(P)^{2\nu_{C_1'}(\delta)+\nu_{C_1'}(f)}$ and $\partial_{E_1'}(\beta) = g(P)^{2\nu_{E_1'}(\delta)+\nu_{E_1'}(f)}$. Since β is ramified at C_1' and E_1', $2\nu_{C_1'}(\delta)+\nu_{C_1'}(f)$ and $2\nu_{E_1'}(\delta)+\nu_{E_1'}(f)$ are not multiples of l. In particular, the subgroups generated by $g(P)^{2\nu_{C_1'}(\delta)+\nu_{C_1'}(f)}$ and $g(P)^{2\nu_{E_1'}(\delta)+\nu_{E_1'}(f)}$ are equal modulo the l^{th} powers. Thus Q is not a hot point of β. □

Lemma 2.3. *Suppose that $\beta = \beta' + (\pi)\cdot(u) + (\delta)\cdot(v)$, where β' unramified on R' and $u, v \in R$ units with $u(P) = v(P)$. Then Q is not a hot point of β.*

Proof. Since β is ramified at C_1', either $\nu_{C_1'}(\pi)$ or $\nu_{C_1'}(\delta)$ is not divisible by l. In particular their sum $\nu_{C_1'}(\pi\delta)$ is non-zero. We have

$$\partial_{C_1'}(\beta)(Q) = u(P)^{\nu_{C_1'}(\pi)} v(P)^{\nu_{C_1'}(\delta)} = u(P)^{\nu_{C_1'}(\pi\delta)}.$$

Suppose that $\nu_{C_1'}(\pi\delta)$ is a multiple of l. Since $\nu'_{C_1}(\pi\delta)$ is non-zero, C_1' is an exceptional curve. As in the proof of (2.2), we see that $\partial_{C_1'}(\beta) = u(P)^{\nu_{C_1'}(\pi\delta)} = 1$. Which is a contradiction, as β is ramified at C_1'. Hence $\nu_{C_1'}(\pi\delta)$ is not a multiple of l. Similarly, we have $\partial_{E_1'}(\beta)(Q) = u(P)^{\nu_{E_1'}(\pi\delta)}$ and $\nu_{E_1'}(\pi\delta)$ is not a multiple of l. Hence $u(P)^{\nu_{C_1'}(\pi\delta)}$ and $u(P)^{\nu_{E_1'}(\pi\delta)}$ generate the same subgroup of $\kappa(P)^*$ modulo $\kappa(P)^{*^l}$ and Q is not a hot point of β. □

Theorem 2.4. *Let k be a p-adic field and $K/k(t)$ be a finite extension. Let l be a prime number not equal to p. Suppose that k contains a primitive l^{th} root of unity. Then every element in $H^2(K, \mu_l)$ is a sum of at most two symbols.*

Proof. If $l = 2$, then, as we mentioned before, by ([A]), α is a sum of at most two symbols. Assume that $l \geq 3$. Let $\alpha \in H^2(K, \mu_l)$. Let $\mathcal{X}$ be a regular proper model of K such that $\mathrm{ram}_{\mathcal{X}}(\alpha) = C + E$, where C and E are regular curves with normal crossings.

Let $P \in C \cup E$ be a closed point of $\mathcal{X}$. Let R_P be the regular local ring at P on $\mathcal{X}$ and m_P be its maximal ideal.

Let T be a finite set of closed points of $\mathcal{X}$ containing $C \cap E$ and at least one closed point from each irreducible curve in C and E. Let A be the semi-local ring at T on $\mathcal{X}$. Let $\pi_1, \dots \pi_r, \delta_1, \dots, \delta_s \in A$ be prime elements corresponding to irreducible curves in C and E respectively. Let $f_1 = \pi_1 \dots \pi_r \delta_1 \dots \delta_s \in A$. Let $P \in C \cap E$. Then $P \in C_i \cap E_j$ for unique irreducible curves C_i in C and E_j in E. Then $\pi = \pi_i$ and $\delta = \delta_j$ are local equations of C and E at P. We have $\alpha = \alpha' + (\pi)\cdot(u_P) + (\delta)\cdot(v_P)$

or $\alpha = \alpha' + (\pi)\cdot(u_P\delta^i)$ for some units, $u_P, v_P \in R$ and α' unramified on R ([S1], cf. 1.1). By the choice of f_1, we have $f_1 = \pi\delta w_P$ for some $w_P \in A$ which is a unit at P. Let $u \in A$ be such that $u(P) = w_P(P)^{-1}u_P(P)$ for all $P \in C \cap E$. Let $f = f_1 u \in A$. Then, we have $(f) = C + E + F$, where F is a divisor on $\mathcal{X}$ which avoids C, E and all the points of $C \cap E$. Further, for each $P \in C_i \cap E_j$, we have $f = \pi_i\delta_j w_{ij}$ for some $w_{ij} \in A$ such that $w_{ij}(P) = u_P(P)$.

By a similar argument, choose $g \in K$ satisfying

(1) $(g) = C + G$, where G is a divisor on $\mathcal{X}$ which avoids C, E, F and also avoids the points of $C \cap E$, $C \cap F$, $E \cap F$;

(2) if $P \in E \cap F$ and $\alpha = \alpha' + (\delta)\cdot(v)$ for some unit $v \in R_P$ and α' is unramified at P, then $g(P) = v(P)$.

Since $C \cap E \cap F = \emptyset$, such a g exists.

We claim that $\beta = \alpha + (f)\cdot(g)$ is a symbol.

Let $\phi\colon \mathcal{X}' \to \mathcal{X}$ be a blow up of $\mathcal{X}$ such that $\mathcal{X}'$ is a regular proper model of K and $\mathrm{ram}_{\mathcal{X}'}(\beta) = C' + E'$, where C' and E' are regular curves with normal crossings.

To show that β is a symbol, it is enough to show that β has no hot points ([S3], cf. 1.2). Let $Q \in C' \cap E'$. Let $P = \phi(Q)$. Then P is a closed point of $\mathcal{X}$, $R = \mathcal{O}_{\mathcal{X},P} \subset \mathcal{O}_{\mathcal{X}',Q} = R'$ and the maximal ideal m_P of R is contained in the maximal ideal m_Q of R'. Let $m_Q = (\pi', \delta')$, with π' and δ' be local equations of C' and E' at Q respectively.

Suppose that $P \notin C \cup E$. Then α is unramified at P and hence unramified at Q. By (2.1), Q is not a hot point of β.

Assume that $P \in C \cup E$.

Suppose that $P \in C \cap E$. Let π and δ be local equations of C and E at P respectively. Then $m_P = (\pi, \delta)$. By the choice of f and g, we have $f = \pi\delta w_1$ and $g = \pi w_2$ for some units $w_1, w_2 \in R$. In particular, β is ramified on R only at π and δ. Suppose that $\alpha = \alpha' + (\pi)\cdot(u) + (\delta)\cdot(v)$ for some units $u, v \in R$ and α' unramified on R. We have

$$\begin{aligned}\beta &= \alpha + (f)\cdot(g)\\ &= \alpha' + (\pi)\cdot(u) + (\delta)\cdot(v) + (\pi\delta w_1)\cdot(\pi w_2)\\ &= \alpha' + (\pi)\cdot(u) + (\delta w_1)\cdot(v) + (w_1^{-1})\cdot(v) + (\pi)\cdot(\pi w_2) + (\delta w_1)\cdot(\pi w_2)\\ &= \alpha' + (w_1^{-1})\cdot(v) + (\pi)\cdot(u\pi w_2) + (\delta w_1)\cdot(\pi w_2 v)\\ &= \alpha' + (w_1^{-1})\cdot(v) + (\pi)\cdot(u w_2) + (\delta w_1)\cdot(\pi w_2 v)\\ &= \alpha' + (w_1^{-1})\cdot(v) + (\pi w_2 v)\cdot(u w_2) + (w_2^{-1}v^{-1})\cdot(u w_2) + (\delta w_1)\cdot(\pi w_2 v)\\ &= \alpha' + (w_1^{-1})\cdot(v) + (w_2^{-1}v^{-1})\cdot(u w_2) + (\pi w_2 v)\cdot(u w_2\delta^{-1}w_1^{-1}).\end{aligned}$$

Since $\alpha' + (w_1^{-1})\cdot(v) + (w_2^{-1}v^{-1})\cdot(u w_2)$ is unramified on R, by (2.1), Q is not a hot point of β.

Suppose that $\alpha = \alpha' + (\pi)\cdot(u\delta^i)$ for some units, $u, v \in R$ and α' unramified on R. Then we have

$$\begin{aligned}\beta &= \alpha + (f)\cdot(g)\\ &= \alpha' + (\pi)\cdot(u\delta^i) + (\pi\delta w_1)\cdot(\pi w_2)\\ &= \alpha' + (\pi)\cdot(u\delta^i) + (\delta w_1 w_2^{-1})\cdot(\pi w_2)\\ &= \alpha' + (\pi)\cdot(u\delta^i(\delta w_1 w_2^{-1})^{-1}) + (\delta w_1 w_2^{-1})\cdot(w_2)\\ &= \alpha' + (\pi)\cdot(\delta^{i-1} u w_1^{-1} w_2) + (\delta w_1 w_2^{-1})\cdot(w_2).\end{aligned}$$

If $i = 1$, then $\beta = \alpha' + (\pi)\cdot(u w_1^{-1} w_2) + (\delta w_1 w_2^{-1})\cdot(w_2)$. Since, by the choice of f, $u(P) = w_1(P)$, by (2.3), Q is not a hot point of β. Assume that $i > 1$. Then $1 \le i-1 < l-1$. Let i' be the inverse of $1-i$ modulo l. We have

$$\begin{aligned}\beta &= \alpha' + (\pi)\cdot(\delta^{i-1} u w_1^{-1} w_2) + (\delta w_1 w_2^{-1})\cdot(w_2)\\ &= \alpha' + (\delta^{1-i} u^{-1} w_1 w_2^{-1})\cdot(\pi) + (\delta w_1 w_2^{-1})\cdot(w_2)\\ &= \alpha' + ((\delta(u^{-1}w_1 w_2^{-1})^{i'})^{1-i})\cdot(\pi) + (\delta(u^{-1}w_1 w_2^{-1})^{i'})\cdot(w_2)\\ &\quad + ((u^{-1}w_1 w_2^{-1})^{-i'})\cdot(w_2) + (w_1 w_2^{-1})\cdot(w_2)\\ &= \alpha' + ((u^{-1}w_1 w_2^{-1})^{-i'})\cdot(w_2) + (w_1 w_2^{-1})\cdot(w_2)\\ &\quad + ((\delta(u^{-1}w_1 w_2^{-1})^{i'})^{1-i})\cdot(\pi^{1-i} w_2).\end{aligned}$$

Since $\alpha' + ((u^{-1}w_1 w_2^{-1})^{-i'})\cdot(w_2) + (w_1 w_2^{-1})\cdot(w_2)$ is unramified on R, by (2.1), Q is not a hot point of β.

Suppose that $P \in C \setminus E$. We have $\alpha = \alpha' + (\pi)\cdot(u)$ for some unit u in R and α' unramified on R. We also have $f = \pi f_1$ for some $f_1 \in R$ which is not divisible by π. We have

$$\begin{aligned}\beta &= \alpha + (f)\cdot(g)\\ &= \alpha' + (\pi)\cdot(u) + (\pi f_1)\cdot(g)\\ &= \alpha' + (f_1^{-1})\cdot(u) + (\pi f_1)\cdot(u) + (\pi f_1)\cdot(g)\\ &= \alpha' + (f_1^{-1})\cdot(u) + (\pi f_1)\cdot(gu).\end{aligned}$$

If f_1 is a unit in R, then $\alpha' + (f_1^{-1})\cdot(u)$ is unramified on R, by (2.1), Q is not a hot point of β. Assume that f_1 is not a unit in R. Then $P \in C \cap F$ and $g = \pi g_1$ for some unit $g_1 \in R$. We have

$$\begin{aligned}\beta &= \alpha + (f)\cdot(g)\\ &= \alpha' + (\pi)\cdot(u) + (\pi f_1)\cdot(\pi g_1)\\ &= \alpha' + (\pi g_1)\cdot(u) + (g_1^{-1})\cdot(u) + (\pi f_1)\cdot(\pi g_1)\\ &= \alpha' + (g_1^{-1})\cdot(u) + (\pi g_1)\cdot(u(\pi f_1)^{-1}).\end{aligned}$$

Since $\alpha' + (g_1^{-1}) \cdot (u)$ is unramified on R, by (2.1), Q is not a hot point of β.

Suppose that $P \in E \setminus C$. Then $\alpha = \alpha' + (\delta) \cdot (v)$ for some unit $v \in R$ and $f = \delta f_1$ for some $f_1 \in R$ which is not divisible by δ. Suppose that f_1 is a unit in R. Then, as above, Q is not a hot point of β. Assume that f_1 is not a unit in R. Then $P \in E \cap F$ and g is a unit in R. We have

$$\begin{aligned}\beta &= \alpha + (f) \cdot (g)\\ &= \alpha' + (\delta) \cdot (v) + (\delta f_1) \cdot (g)\\ &= \alpha' + (\delta) \cdot (vg) + (f_1) \cdot (g).\end{aligned}$$

Since α' is unramified on R and by the choice of g, $g(P) = v(P)$, by (2.2), Q is not a hot point of β.

By ([S3], cf. 1.2), β is symbol. Thus $\alpha = (f) \cdot (g) - \beta$ is a sum of at most two symbols. □

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