

Maslov class rigidity for Lagrangian submanifolds via Hofer’s geometry

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Abstract. In this work, we establish new rigidity results for the Maslov class of Lagrangian submanifolds in large classes of closed and convex symplectic manifolds. Our main result establishes upper bounds for the minimal Maslov number of displaceable Lagrangian submanifolds which are product manifolds whose factors each admit a metric of negative sectional curvature. Such Lagrangian submanifolds exist in every symplectic manifold of dimension greater than six or equal to four.

The proof utilizes the relations between closed geodesics on the Lagrangian, the periodic orbits of geometric Hamiltonian flows supported near the Lagrangian, and the length minimizing properties of these flows with respect to the negative Hofer length functional.

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1. Introduction

Among the fundamental rigidity phenomena in symplectic topology are restrictions on Lagrangian submanifolds that are undetectable by topological methods. In this work we establish such restrictions which are expressible in terms of the Maslov class. All the Lagrangian submanifolds we consider are assumed to be connected, compact and without boundary.

The first *symplectic* restrictions on Lagrangian submanifolds were discovered in [Gr], where Gromov proves that there are no exact Lagrangian submanifolds of \mathbb{R}^{2n} equipped with its standard symplectic form ω_{2n} . This result can be rephrased as the fact that the symplectic area class

$$\omega_{2n}^L : \pi_2(\mathbb{R}^{2n}, L) \rightarrow \mathbb{R},$$

which is defined by integrating ω_{2n} over smooth representatives, is nontrivial for any Lagrangian submanifold L of $(\mathbb{R}^{2n}, \omega_{2n})$.

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Another type of restriction on Lagrangian submanifolds involves the Maslov class. Recall that for a Lagrangian submanifold L of a symplectic manifold (M, ω) , the Maslov class is a homomorphism

$$\mu_{\text{Maslov}}^L : \pi_2(M, L) \rightarrow \mathbb{Z},$$

that measures the *winding* of TL in TM along loops in L .¹ The minimal Maslov number of L is the smallest nonnegative integer N_L such that $\mu_{\text{Maslov}}^L(\pi_2(M, L)) = N_L \mathbb{Z}$. As an application of his proof of the Weinstein conjecture for $(\mathbb{R}^{2n}, \omega_{2n})$, Viterbo established the first restrictions on the Maslov class in the following result.

Theorem 1.1 ([V11]). *Let L be a closed Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_{2n})$. If L is a torus, then N_L is in $[2, n + 1]$. If L admits a metric with negative sectional curvature then $N_L = 2$.*

A Lagrangian submanifold $L \subset (M, \omega)$ is called *monotone* if the symplectic area class and the Maslov class are proportional with a nonnegative constant of proportionality, i.e., $\omega^L = \lambda \mu_{\text{Maslov}}^L$ for some $\lambda \geq 0$. In this case, restrictions on the two classes are related and one can relax the assumptions on the intrinsic geometry of L needed in Theorem 1.1. The first rigidity results for the Maslov class of monotone Lagrangians were obtained for $(\mathbb{R}^{2n}, \omega_{2n})$ by Polterovich in [Po1]. Later, in [Oh], Oh introduced a spectral sequence for Lagrangian Floer theory and used it to establish similar rigidity results for displaceable monotone Lagrangian submanifolds in more general ambient symplectic manifolds.² Indeed, monotone Lagrangian submanifolds are the natural setting for the construction of Lagrangian Floer homology. This homology vanishes if the Lagrangian submanifold is displaceable, and its grading is determined by the Maslov class. Oh observed that if the Lagrangian Floer homology of $L \subset (M, \omega)$ exists, e.g., if $N_L \geq 2$, then it can not vanish unless N_L is less than $\frac{1}{2} \dim M + 2$. This strategy to obtain rigidity results for the Maslov class has been generalized, refined and exploited in many recent works, see for example [AI], [AI], [Bi], [BCi], [BCo], [Bu], [FOOO].

The primary goal of the present work is to obtain rigidity results for the Maslov class away from settings where Lagrangian Floer homology can be used. In particular, our assumptions on the symplectic area class and the Maslov class only involve their values on $\pi_2(M)$ rather than $\pi_2(M, L)$. We also avoid the use of holomorphic discs with boundary on the Lagrangian submanifold. Hence, we do not need to manage or avoid codimension one bubbling phenomena. Instead of Lagrangian Floer theory, our argument utilizes two results from the study of Hofer's length functional for Hamiltonian paths. The first of these results, due to Sikorav, is the fact that

¹The precise definition is given in §4.2.1.

²Note that a symplectic manifold which admits a monotone Lagrangian submanifold is itself monotone in the sense of [FI].

the Hamiltonian flow generated by an autonomous Hamiltonian with displaceable support, does not minimize the Hofer length for all time. In particular, Sikorav's result implies that the flow of an autonomous Hamiltonian which is supported near a displaceable Lagrangian submanifold is, eventually, not length minimizing. The second result from Hofer's geometry involves the following theme: if a Hamiltonian path is not length minimizing then it has a contractible periodic orbit with additional *special* properties, see for example [Ho2], [Ke1], [KL], [LMcD], [McD], [McDSI]. Here, we use the version of this phenomena established in [Ke1] which states that if a Hamiltonian flow is not length minimizing then it has contractible periodic orbits with spanning discs such that the corresponding action values and Conley–Zehnder indices lie in certain intervals. By applying these results to reparameterizations of perturbed cogeodesic flows that are supported near a displaceable Lagrangian submanifold, we detect *nonconstant* periodic orbits with spanning discs for which the corresponding Conley–Zehnder index is $\frac{1}{2} \dim M$. Utilizing an identity which relates this Conley–Zehnder index to the Morse index of the corresponding perturbed geodesic and the Maslov index of the spanning disc, we obtain the rigidity results for the Maslov class stated below.

1.1. Statement and discussion of results. Throughout this work, we will denote our ambient symplectic manifold by (M, ω) and its dimension will be $2n$. Lagrangian submanifolds will be denoted by L and will always be assumed to be connected, compact and without boundary.

We will consider symplectic manifolds which are either closed or convex and have the following two additional properties. We assume that the *index of rationality* of (M, ω) ,

$$r(M, \omega) = \inf_{A \in \pi_2(M)} \{\omega(A) \mid \omega(A) > 0\},$$

is positive.³ Such a manifold is said to be *rational*. We will also assume that there is a (possibly negative) constant ν such that $c_1(A) = \nu\omega(A)$ for every A in $\pi_2(M)$. A symplectic manifold with this property will be called *proportional*.

Recall that L is *displaceable* if there is a Hamiltonian diffeomorphism ϕ of (M, ω) such that $\phi(L) \cap L = \emptyset$.⁴ For example, for any symplectic manifold (M, ω) , every Lagrangian submanifold of the product $(M \times \mathbb{R}^2, \omega \oplus \omega_2)$, is displaceable.⁵ A Lagrangian submanifold of a rational symplectic manifold (M, ω) is said to be *easily displaceable* if it can be displaced by a Hamiltonian diffeomorphism ϕ_H^1 that is the

³Our convention is that the infimum over the empty set is equal to infinity.

⁴Here, Hamiltonian diffeomorphisms on convex symplectic manifolds are generated by compactly supported Hamiltonian flows, see §2.2.

⁵Note that if (M, ω) is compact, then $(M \times \mathbb{R}^2, \omega \oplus \omega_2)$ is convex. This basic example is a primary motivation for our consideration of the class of convex symplectic manifolds.

time one flow of a Hamiltonian H whose Hofer norm,

$$\|H\| = \int_0^1 \max_{p \in M} H(t, p) dt - \int_0^1 \min_{p \in M} H(t, p) dt,$$

is less than $r(M, \omega)/2$. In other words, L is easily displaceable if its displacement energy,

$$e(L, M, \omega) = \inf_H \{ \|H\| \mid \phi_H^1(L) \cap L = \emptyset \},$$

is less than $r(M, \omega)/2$.

If (M, ω) is weakly exact, i.e., $\omega|_{\pi_2(M)} = 0$, then the index of rationality is infinite and the notions of *displaceable* and *easily displaceable* Lagrangian submanifolds are equivalent. The simplest example of a symplectic manifold which is not weakly exact is the two-sphere with an area form. In this case, every Lagrangian submanifold is either monotone and nondisplaceable or is easily displaceable. The following example shows that being easily displaceable is actually a stronger condition than being displaceable.

Example 1.2. Let ϑ be an area form on S^2 with total area equal to one. Consider the symplectic manifold

$$(W, \omega_j) = (S^2 \times S^2, \vartheta \oplus (1 + \alpha_j)\vartheta),$$

where α_j is an increasing sequence of positive rational numbers which converges to an irrational number α . Let $\hat{L} \subset S^2$ be an embedded circle which divides S^2 into two regions of unequal ϑ -area. For every j , the Lagrangian submanifold $L = \hat{L} \times \hat{L}$ of (W, ω_j) is displaceable. We will show that the indices of rationality $r(W, \omega_j)$ converge to zero, whereas the displacement energy of L in (W, ω_j) is uniformly bounded away from zero. Hence, for sufficiently large values of j the Lagrangian submanifold $L \subset (W, \omega_j)$ is displaceable but not easily displaceable.

For all j we have $0 < r(W, \omega_j) \leq \alpha_j < \alpha$. Passing to a subsequence, if necessary, we may therefore assume that $\lim_{j \rightarrow \infty} r(W, \omega_j)$ exists. Since $\omega_\infty := \vartheta \oplus (1 + \alpha)\vartheta$, and α is irrational, the limit $r(W, \omega_\infty)$ is zero.

To obtain the uniform lower bound on the displacement energies $e(L, W, \omega_j)$, we use a deep theorem from [Ch]. Assume that the symplectic manifold (M, ω) admits an ω -compatible almost complex structure J such that the metric $\omega(\cdot, J\cdot)$ is complete. Denote by \mathcal{S} , the space of nonconstant J -holomorphic spheres in M , and let \mathcal{D} be the set of nonconstant J -holomorphic discs in M with boundary on L . The quantities

$$\hbar(M, \omega, J) = \min_{u \in \mathcal{S}} \int_{S^2} u^* \omega$$

and

$$\hbar_D(M, L, \omega, J) = \min_{v \in \mathcal{D}} \int_{D^2} v^* \omega$$

are well defined and positive. In [Ch], Chekanov proves that

$$e(L, M, \omega) \geq \min\{\hbar(M, \omega, J), \hbar_D(M, L, \omega, J)\}. \tag{1}$$

Now choose a ϑ -compatible almost complex structure \hat{J} on S^2 and set $J = \hat{J} \oplus \hat{J}$. Note that J is ω_j -compatible for each j and the spaces \mathcal{S} and \mathcal{D} do not depend on j . Hence,

$$\lim_{j \rightarrow \infty} \hbar(W, \omega_j, J) = \hbar(W, \omega_\infty, J) > 0$$

and

$$\lim_{j \rightarrow \infty} \hbar_D(W, L, \omega_j, J) = \hbar_D(W, L, \omega_\infty, J) > 0.$$

It then follows from inequality (1) that the displacement energies $e(L, M, \omega_j)$ are bounded uniformly away from zero.

As in Theorem 1.1, we need to make some assumptions on the metrics which our Lagrangian submanifolds admit. A manifold L is said to be *split hyperbolic* if it is diffeomorphic to a product manifold

$$L = P_1 \times \cdots \times P_k,$$

such that each of the factors P_j admits a metric with negative sectional curvature. Our convention will be to label the factors of L so that $\dim P_j \leq \dim P_{j+1}$.

Note that the set of split hyperbolic manifolds is strictly larger than the set of manifolds which admit metrics of negative sectional curvature. This follows from a theorem of Preissmann [Pr], which states that no (nontrivial) product manifold admits a metric of negative sectional curvature. More importantly, Lagrangian submanifolds which are both displaceable and split hyperbolic, are somewhat ubiquitous.

Lemma 1.3. *There is an (easily) displaceable, split hyperbolic Lagrangian submanifold in every (rational) symplectic manifold with dimension four or with dimension greater than six.*

Proof. The product of a Lagrangian embedding into \mathbb{R}^{2m} and a Lagrangian immersion into \mathbb{R}^{2n} is homotopic to a Lagrangian embedding into $\mathbb{R}^{2(m+n)}$ (see, [AL], Proposition 1.2.3, p. 275). In [Gi], Givental constructs Lagrangian embeddings, into \mathbb{R}^4 , of compact nonorientable surface with Euler characteristic equal to zero modulo four. The Gromov–Lees Theorem implies that one can find a Lagrangian immersion of any hyperbolic three-manifold into (\mathbb{R}^6, ω_6) . By taking products of such examples, it follows that there is a split hyperbolic Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_{2n})$ for $n = 2$ and all $n > 3$. The lemma then follows easily from Darboux’s theorem. \square

Remark 1.4. In contrast to the split hyperbolic case, there are more subtle obstructions to the existence of Lagrangian submanifolds which admit metrics of negative sectional curvature. A result of Viterbo states that there can be no such Lagrangian submanifold in any uniruled symplectic manifold of dimension greater than or equal to six, [Vi3], [EGH]. Hence, there are no split hyperbolic Lagrangian submanifolds of (\mathbb{R}^6, ω_6) . It is not known to the authors whether there exists a displaceable hyperbolic Lagrangian submanifold in any six-dimensional symplectic manifold.

We are now in a position to state the main result of the paper.

Theorem 1.5. *Let (M, ω) be a rational and proportional symplectic manifold of dimension $2n$ which is either closed or convex. If L is an easily displaceable Lagrangian submanifold of (M, ω) which is split hyperbolic, then $N_L \leq n + 2$. If, in addition, L is orientable, then $N_L \leq n + 1$.*

More precisely, we prove that if $L = P_1 \times \dots \times P_k$ is (un)orientable, then there is an element $[w] \in \pi_2(M, L)$ such that

$$\dim P_1 - 1 \leq \mu_{\text{Maslov}}^L([w]) \leq n + 1(+1).$$

One can not expect similar bounds to hold for the minimal Maslov number of general Lagrangian submanifolds. Consider, for example, the quadric

$$M = \{[z_0, \dots, z_{n+1}] \subset \mathbb{C}P^{n+1} \mid z_0^2 + \dots + z_n^2 = z_{n+1}^2\},$$

which is both rational and proportional when equipped with the symplectic form inherited from $\mathbb{C}P^{n+1}$. The real quadric $L \subset M$ is a Lagrangian sphere with minimal Maslov number $N_L = 2n$.⁶

On the other hand, for displaceable Lagrangian submanifolds, the bounds in Theorem 1.5 are nearly sharp. In particular, in [Po1], Polterovich constructs examples of orientable monotone Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_{2n})$ with minimal Maslov number k for every integer k in $[2, n]$.

For the special class of displaceable Lagrangians considered in Theorem 1.5, one might expect stronger restrictions on the Maslov class similar to those obtained for $(\mathbb{R}^{2n}, \omega_{2n})$ in Theorem 1.1 and by Fukaya in [Fu]. Indeed, this is the case if one makes further assumptions on (M, ω) . In the sequel to this paper, we will study Maslov class rigidity for displaceable Lagrangians in closed symplectic manifolds which are symplectically aspherical. In that setting, we will obtain finer rigidity statements as well as rigidity statements for a larger class of Lagrangian submanifolds which includes Lagrangian tori.

⁶The authors are grateful to Paul Biran for pointing this example out to them.

1.2. Organization. In the next section, we recall the definitions and basic results concerning convex symplectic manifolds, Hamiltonian flows, the Hofer length functional, and Sikorav's curve shortening method. In Section 3, we construct and study a special Hamiltonian H_L which is supported in a small neighborhood of our Lagrangian submanifold L . We then establish a relation between the Conley–Zehnder indices of the nonconstant contractible periodic orbits of H_L and the Maslov indices of their spanning discs in Section 4. In Section 5, we state a result, Theorem 5.1, which relates the length minimizing properties of H_L to its periodic orbits. Together with the previous results, this is shown to imply Theorem 1.5. The proof of Theorem 5.1 is contained in Section 6, and some generalizations of our main result are described in Section 7.

2. Preliminaries

2.1. Convex symplectic manifolds. A compact symplectic manifold (M, ω) is said to be *closed* if the boundary of M , ∂M , is empty. It is called *convex* if ∂M is nonempty and there is a 1-form α on ∂M such that $d\alpha = \omega|_{\partial M}$ and the form $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on ∂M which induces the outward orientation. Equivalently, a compact symplectic manifold (M, ω) is convex if there is a vector field X defined in a neighborhood of ∂M which is transverse to ∂M , outward pointing, and satisfies $\mathcal{L}_X \omega = \omega$. For some $\epsilon_0 > 0$, the vector field X can be used to symplectically identify a neighborhood of ∂M with the submanifold

$$M_{\epsilon_0} = \partial M \times (-\epsilon_0, 0],$$

equipped with coordinates (x, τ) and the symplectic form $d(e^\tau \alpha)$. We will use the notation $M_\epsilon = \partial M \times (-\epsilon, 0]$ for $0 < \epsilon \leq \epsilon_0$.

A noncompact symplectic manifold (M, ω) is called convex if there is an exhausting sequence of compact convex submanifolds M_j of M , i.e., $M_1 \subset M_2 \subset \dots \subset M$ and

$$\bigcup_j M_j = M.$$

2.2. Hamiltonian flows. A function $H \in C^\infty(S^1 \times M)$ will be referred to as a *Hamiltonian* on M . Here, we identify the circle S^1 with \mathbb{R}/\mathbb{Z} and parameterize it with the coordinate $t \in [0, 1]$. Set $H_t(\cdot) = H(t, \cdot)$ and let $C_0^\infty(S^1 \times M)$ be the space of Hamiltonians H such that the support of dH_t is compact and does not intersect ∂M for all $t \in [0, 1]$. Each $H \in C_0^\infty(S^1 \times M)$ determines a 1-periodic time-dependent Hamiltonian vector field X_H via Hamilton's equation

$$i_{X_H} \omega = -dH_t.$$

The time- t flow of X_H, ϕ_H^t , is defined for all $t \in [0, 1]$ (in fact, for all $t \in \mathbb{R}$). The group of Hamiltonian diffeomorphisms of (M, ω) is the set of time-1 flows obtained in this manner,

$$\text{Ham}(M, \omega) = \{\phi = \phi_H^1 \mid H \in C_0^\infty(S^1 \times M)\}.$$

Remark 2.1. In this work, we are only concerned with compact Lagrangian submanifolds which are displaceable by Hamiltonian diffeomorphisms. By definition, Hamiltonian diffeomorphisms are trivial away from a compact set. Hence, it suffices for us to prove Theorem 1.5 for closed symplectic manifolds and *compact* convex symplectic manifolds. In particular, the noncompact convex case can be reduced to the compact convex case by restricting attention to some element M_j of an exhausting sequence for M , for sufficiently large j .

In the case of a compact convex symplectic manifold we will also consider Hamiltonian flows which are nontrivial near ∂M but which are still defined for all $t \in \mathbb{R}$. A function $f \in C^\infty(M)$ is said to be *admissible* if for some ϵ in $(0, \epsilon_0]$ we have

$$f|_{M_\epsilon}(x, \tau) = ae^{-\tau} + b, \tag{2}$$

where a and b are arbitrary constants and $a < 0$.

A Hamiltonian $H \in C^\infty(S^1 \times M)$ is called *pre-admissible* if for some ϵ in $(0, \epsilon_0]$ we have

$$H|_{S^1 \times M_\epsilon}(t, x, \tau) = ae^{-\tau} + b(t), \tag{3}$$

where a is again a negative constant and $b(t)$ is a smooth 1-periodic function. We will refer to a as the *slope* of H . The prescribed behavior of H on M_ϵ implies that its Hamiltonian flow is defined for all $t \in \mathbb{R}$. In particular, consider the Reeb vector field R on ∂M which is defined uniquely by the following conditions:

$$\omega(R(x), v(x)) = 0 \quad \text{and} \quad \omega(X(x), R(x)) = 1$$

for all $x \in \partial M$ and $v \in T_x \partial M$. If H has the form (3) on M_ϵ , then

$$X_H(t, x, \tau) = -aR(x) \quad \text{for all } (t, x, \tau) \in S^1 \times M_\epsilon,$$

and so the level sets $\{\tau = \text{constant}\}$ in M_ϵ are preserved by ϕ_H^t .

Let T_R be the minimum period of the closed orbits of R . We say that a pre-admissible Hamiltonian H is *admissible* if its slope a satisfies $-a < T_R$. If (M, ω) is closed, then the space of admissible functions on M is simply $C^\infty(M)$ and the space of admissible Hamiltonians is $C^\infty(S^1 \times M)$. For simplicity, in either the closed or convex case, we will denote the space of admissible Hamiltonians by $\widehat{C}^\infty(S^1 \times M)$.

For any Hamiltonian flow ϕ_H^t defined for $t \in [0, 1]$, we will denote the set of contractible 1-periodic orbits of ϕ_H^t by $\mathcal{P}(H)$. Note that if (M, ω) is compact and

convex and H is in $C_0^\infty(S^1 \times M)$ or in $\widehat{C}^\infty(S^1 \times M)$, then the elements of $\mathcal{P}(H)$ are contained in the complement of M_ϵ for some $\epsilon \in (0, \epsilon_0]$.

An element $x(t) \in \mathcal{P}(H)$ is said to be nondegenerate if the linearized time-1 flow $d\phi_H^1: T_{x(0)}M \rightarrow T_{x(1)}M$ does not have one as an eigenvalue. If every element of $\mathcal{P}(H)$ is nondegenerate we will call H a *Floer Hamiltonian*.

2.3. The Hofer length functional and Sikorav curve-shortening. Following [Ho1], the Hofer length of a Hamiltonian path ϕ_H^t is defined to be

$$\begin{aligned} \text{length}(\phi_H^t) &= \|H\| \\ &= \int_0^1 \max_M H_t \, dt - \int_0^1 \min_M H_t \, dt \\ &= \|H\|^+ + \|H\|^-. \end{aligned}$$

When M is compact, then a Hamiltonian H in $C_0^\infty(S^1 \times M)$ or $\widehat{C}^\infty(S^1 \times M)$ is *normalized* if

$$\int_M H_t \omega^n = 0$$

for every t in $[0, 1]$. If the generating Hamiltonian H is normalized, then the quantities $\|H\|^+$ and $\|H\|^-$ provide different measures of the length of ϕ_H^t called the positive and negative Hofer lengths, respectively.

For a path of Hamiltonian diffeomorphisms ψ_t with $\psi_0 = \text{id}$, let $[\psi_t]$ be the class of Hamiltonian paths which are homotopic to ψ_t relative to its endpoints. Denote the set of normalized Hamiltonians which lie in $C_0^\infty(S^1 \times M)$ and generate the paths in $[\psi_t]$ by

$$C_0^\infty([\psi_t]) = \{H \in C_0^\infty(S^1 \times M) \mid \int_M H_t \omega^n = 0, [\phi_H^t] = [\psi_t]\}.$$

The Hofer semi-norm of $[\psi_t]$ is then defined by

$$\rho([\psi_t]) = \inf_{H \in C_0^\infty([\psi_t])} \{\|H\|\}.$$

The positive and negative Hofer semi-norms of $[\psi_t]$ are defined similarly as

$$\rho^\pm([\psi_t]) = \inf_{H \in C_0^\infty([\psi_t])} \{\|H\|^\pm\}.$$

Note that if

$$\|H\|^{(\pm)} > \rho^{(\pm)}([\phi_H^t]),$$

then ϕ_H^t fails to minimize the (positive/negative) Hofer length in its homotopy class.

The displacement energy of a subset $U \subset (M, \omega)$ is the quantity

$$e(U, M, \omega) = \inf_{\psi_t} \{\rho([\psi_t]) \mid \psi_0 = \text{id} \text{ and } \psi_1(U) \cap \bar{U} = \emptyset\},$$

where \bar{U} denotes the closure of U . The following result relates the negative Hofer semi-norm and the displacement energy. It is a direct application of Sikorav’s curve shortening technique and the reader is referred to Lemma 4.2 of [Ke2] for the entirely similar proof of the analogous result for the positive Hofer semi-norm.

Proposition 2.2. *Let $H \in C_0^\infty(S^1 \times M)$ be a time-independent normalized Hamiltonian that is constant and equal to its maximum value on the complement of an open set $U \subset M$. If U has finite displacement energy and $\|H\|^- > 2e(U)$, then*

$$\|H\|^- > \rho^-([\phi_H^t]) + \frac{1}{2}\|H\|^+.$$

In other words, the Hamiltonian path ϕ_H^t does not minimize the negative Hofer length in its homotopy class.

3. A special Hamiltonian flow near L

In this section we construct a special Hamiltonian H_L whose Hamiltonian flow is supported in a tubular neighborhood of L . The nonconstant contractible periodic orbits of this flow project to perturbed geodesics on L . As well, the Hamiltonian path $\phi_{H_L}^t$ fails to minimize the negative Hofer length in its homotopy class.

3.1. Perturbed geodesic flows. We begin by recalling some relevant facts about geodesic flows on L . For now, we assume only that L is a closed manifold without boundary. Let g be a Riemannian metric on L , and consider the energy functional of g which is defined on the space of smooth loops $C^\infty(S^1, L)$ by

$$\mathcal{E}_g(q(t)) = \int_0^1 \frac{1}{2} \|\dot{q}(t)\|^2 dt.$$

The critical points of \mathcal{E}_g , $\text{Crit}(\mathcal{E}_g)$, are the closed geodesics of g with period equal to one. The closed geodesics of g with any positive period $T > 0$ correspond to the 1-periodic orbits of the metric $\frac{1}{T}g$, and are thus the critical points of the functional $\mathcal{E}_{\frac{1}{T}g}$.

The Hessian of \mathcal{E}_g at a critical point $q(t)$ will be denoted by $\text{Hess}(\mathcal{E}_g)_q$. As is well known, the space on which $\text{Hess}(\mathcal{E}_g)_q$ is negative definite is finite-dimensional. It’s dimension is, by definition, the Morse index of q and will be denoted here by $I_{\text{Morse}}(q)$. The kernel of $\text{Hess}(\mathcal{E}_g)_q$ is also finite-dimensional, and is always nontrivial unless L is a point.

A submanifold $D \subset C^\infty(S^1, L)$ which consists of critical points of \mathcal{E}_g is said to be Morse–Bott nondegenerate if the dimension of the kernel of $\text{Hess}(\mathcal{E}_g)_q$ is equal to the dimension of D for every $q \in D$. An example of such a manifold is the

set of constant geodesics of any metric on L . This is a Morse–Bott nondegenerate submanifold which is diffeomorphic to L . The energy functional \mathcal{E}_g is said to be Morse–Bott if all the 1-periodic geodesics are contained in Morse–Bott nondegenerate critical submanifolds of \mathcal{E}_g . Note that if \mathcal{E}_g is Morse–Bott, then so is $\mathcal{E}_{\frac{1}{T}g}$ for any $T > 0$.

Example 3.1. If g is a metric with negative sectional curvature, then the nonconstant 1-periodic geodesics occur in Morse–Bott nondegenerate S^1 -families. In particular, the unparameterized geodesics of g are isolated. The Morse index of every 1-periodic geodesic is zero.

Example 3.2. Let $L = P_1 \times \cdots \times P_k$ be a split hyperbolic manifold as defined in Section 1.1. Let g_j be a metric on P_j with negative sectional curvature and set

$$g = g_1 + \cdots + g_k.$$

The nonconstant 1-periodic geodesics of g occur in Morse–Bott nondegenerate critical submanifolds whose dimension is no greater than $1 + \dim P_2 + \cdots + \dim P_k$. The Morse index of these closed geodesics is again zero.

3.1.1. Potential perturbations. It will be useful for us to perturb a Morse–Bott energy functional \mathcal{E}_g so that the critical points of the resulting functional are nondegenerate. We restrict ourselves to perturbations of the following classical form

$$\mathcal{E}_{g,V}(q) = \int_0^1 \left(\frac{1}{2} \|\dot{q}(t)\|^2 - V(t, q(t)) \right) dt,$$

where the function $V : S^1 \times L \rightarrow \mathbb{R}$ is assumed to be smooth. The critical points of $\mathcal{E}_{g,V}$ are solutions of the equation

$$\nabla_t \dot{q} + \nabla_g V(t, q) = 0, \tag{4}$$

where ∇_t denotes covariant differentiation in the \dot{q} -direction with respect to the Levi-Civita connection of g , and $\nabla_g V$ is the gradient vector field of V with respect to g . We refer to solutions of (4) as perturbed geodesics.

Theorem 3.3 ([We]). *There is a dense set $\mathcal{V}_{\text{reg}}(g) \subset C^\infty(S^1 \times L)$ such that for $V \in \mathcal{V}_{\text{reg}}(g)$ the critical points of $\mathcal{E}_{g,V}$ are nondegenerate.*

When a Morse–Bott functional \mathcal{E}_g is perturbed, the critical submanifolds break apart into critical points. For a small perturbation V it is possible to relate each critical point of $\mathcal{E}_{g,V}$ to a specific critical submanifold of \mathcal{E}_g , and to relate their indices. Here is the precise statement.

Lemma 3.4. *Let \mathcal{E}_g be Morse–Bott and let $\text{Crit}^a(\mathcal{E}_g) = \coprod_{j=1}^\ell D_j$ be the (finite, disjoint) union of all critical submanifolds of \mathcal{E}_g with energy less than a . Let $\epsilon > 0$ be small enough so that the ϵ -neighborhoods of the D_j in $C^\infty(S^1, L)$ are disjoint. If $V \in \mathcal{V}_{\text{reg}}(g)$ is sufficiently small, then each nondegenerate critical point $q(t)$ of $\mathcal{E}_{g,V}$, with action less than a , lies in the ϵ -neighborhood of exactly one component D_j of $\text{Crit}^a(\mathcal{E}_g)$. Moreover,*

$$I_{\text{Morse}}(q) \in [I_{\text{Morse}}(D_j), I_{\text{Morse}}(D_j) + \dim(D_j)].$$

Proof. For a proper Morse–Bott function on a finite-dimensional manifold the proof of the lemma is elementary and requires only the generalized Morse Lemma which yields a normal form for a function near a Morse–Bott nondegenerate critical submanifold. As we now describe, the present case is essentially identical to the finite-dimensional one. Let $\widehat{\mathcal{E}}_g$ be the extension of \mathcal{E}_g to the space $\Lambda^1(S^1, M)$ of loops in L of Sobolev class $W^{1,2}$. This space of loops is a Hilbert manifold and the critical point sets $\text{Crit}(\mathcal{E}_g)$ and $\text{Crit}(\widehat{\mathcal{E}}_g)$ coincide, as do the corresponding Morse indices. Let q be a critical point of $\widehat{\mathcal{E}}_g$ which belongs to a critical submanifold D . By the Morse–Bott assumption, the nullity of $\text{Hess}(\widehat{\mathcal{E}}_g)_q$, is equal to the dimension of D . Moreover, the only possible accumulation point for the eigenvalues of $\text{Hess}(\widehat{\mathcal{E}}_g)_q$ is one, [KI], Theorem 2.4.2. Hence, there are only finitely many negative eigenvalues and the positive eigenvalues are uniformly bounded away from zero. Using the generalized Morse Lemma for Hilbert manifolds from [Me], see also [KI], Lemma 2.4.7, the result then follows exactly as in the finite-dimensional case. \square

3.2. Hamiltonian geodesic flows. Consider the cotangent bundle of L , T^*L , equipped with the symplectic structure $d\theta$ where θ is the canonical Liouville 1-form. We will denote points in T^*L by (q, p) where q is in L and p belongs to T_q^*L . In these local coordinates, $\theta = pdq$ and so $d\theta = dp \wedge dq$.

The metric g on L induces a bundle isomorphism between TL and the cotangent bundle T^*L and hence a cometric on T^*L . Let $K_g : T^*L \rightarrow \mathbb{R}$ be the function $K_g(q, p) = \frac{1}{2}\|p\|^2$. The Legendre transform yields a bijection between the critical points of the perturbed energy functional $\mathcal{E}_{g,V}$ and the critical points of the action functional

$$\mathcal{A}_{K_g+V} : C^\infty(S^1, T^*L) \rightarrow \mathbb{R}$$

defined by

$$\mathcal{A}_{K_g+V}(x) = \int_0^1 (K_g + V)(t, x(t)) dt - \int_{S^1} x^* \theta.$$

The critical points of \mathcal{A}_{K_g+V} are the 1-periodic orbits of the Hamiltonian $K_g + V$ on T^*L . If $x(t) = (q(t), p(t))$ belongs to $\mathcal{P}(K_g + V)$, then its projection to L is a closed 1-periodic solution of (4) with initial velocity $\dot{q}(0)$ determined by $g(\dot{q}(0), \cdot) = p(0)$. Moreover, $x(t)$ is nondegenerate if and only if $q(t)$ is nondegenerate.

3.3. A perturbed geodesic flow supported near L . We now assume that L is a Lagrangian submanifold of (M, ω) as in the statement of Theorem 1.5. In particular, L is easily displaceable and split hyperbolic. We equip L with the metric g from Example 3.2 and remind the reader that the energy functional for this metric is Morse–Bott.

Consider a neighborhood of the zero section in T^*L of the following type

$$U_r = \{(q, p) \in T^*L \mid \|p\| < r\}.$$

By Weinstein's neighborhood theorem, for sufficiently small $r > 0$, there is a neighborhood of L in (M, ω) which is symplectomorphic to U_r . We only consider values of r for which this holds, and will henceforth identify U_r with a neighborhood of L in (M, ω) . For a subinterval $I \subset [0, r)$, we will use the notation

$$U_I = \{(q, p) \in U_r \mid \|p\| \in I\}.$$

Proposition 3.5. *For sufficiently small $r > 0$, there is a normalized and admissible Hamiltonian $H_L : S^1 \times M \rightarrow \mathbb{R}$ with the following properties:*

- (H1) *The constant 1-periodic orbits of H_L correspond to the critical points of an admissible Morse function F on M . Near these points the Hamiltonian flows of H_L and $c_0 F$ are identical for some arbitrarily small constant $c_0 > 0$.*
- (H2) *The nonconstant 1-periodic orbits of H_L are nondegenerate and contained in $U_{(r/5+\delta, 2r/5-\delta)} \cup U_{(3r/5+\delta, 4r/5-\delta)}$ for some $0 < \delta < r/5$. Each such orbit projects to a nondegenerate closed perturbed geodesic $q(t)$. Moreover, if T is the period of q , then q can be associated to exactly one critical submanifold D of $\mathcal{E}_{\frac{1}{T}g}$ and*

$$I_{\text{Morse}}(q) \in [I_{\text{Morse}}(D), I_{\text{Morse}}(D) + \dim(D)].$$

- (H3) *There is a point $Q \in L \subset M$ which is the unique local minimum of $H_L(t, \cdot)$ for all $t \in [0, 1]$. Moreover,*

$$\|H_L\|^- = - \int_0^1 H_L(t, Q) dt > 2e(U_r). \tag{5}$$

- (H4) *The Hofer norm of H_L satisfies*

$$2e(U_r) < \|H_L\| < r(M, \omega). \tag{6}$$

Proof. The construction of H_L is elementary but somewhat involved. We divide the process into four steps. Some of the constituents of H_L which are defined in these steps will be referred to in later sections.

Step 1: A geodesic flow supported near L .

Let $v = v(A, B, C, r): [0, +\infty) \rightarrow \mathbb{R}$ be a smooth function with the following properties:

- $v = -A$ on $[0, r/5]$;
- $v', v'' > 0$ on $(r/5, 2r/5)$;
- $v' = C$ on $[2r/5, 3r/5]$;
- $v' > 0$ and $v'' < 0$ on $(3r/5, 4r/5)$;
- $v = B$ on $[4r/5, +\infty)$.

Define the function K_v on T^*L by

$$K_v(q, p) = \begin{cases} v(\|p\|) & \text{if } (q, p) \text{ is in } U_r, \\ B & \text{otherwise.} \end{cases}$$

Since L is easily displaceable, for sufficiently small $r > 0$ we have

$$e(U_r) < r(M, \omega)/2.$$

Choose the constant A so that

$$2e(U_r) < A < r(M, \omega). \tag{7}$$

Restricting r again, if necessary, we can choose the constant B so that

$$0 < B < A \frac{\text{Vol}(U_r)}{\text{Vol}(M \setminus U_r)}.$$

For these choices of r , A and B , we can construct v , as above, so that K_v is normalized and satisfies

$$2e(U_r) < \|K_v\| < r(M, \omega). \tag{8}$$

We may also choose the positive constant C so that it is not the length of any closed geodesic of g .

The Hamiltonian flow of K_v is trivial in both $U_{r/5}$ and the complement of $U_{4r/5}$. Hence, each nonconstant 1-periodic orbit $x(t) = (q(t), p(t))$ of K_v is contained in $U_{(r/5, 4r/5)}$ where

$$X_{K_v}(q, p) = \left(\frac{v'(\|p\|)}{\|p\|} \right) X_{K_g}(q, p).$$

Our choice of C implies that all nonconstant orbits of K_v occur on the level sets contained in $U_{(r/5, 2r/5)}$ or $U_{(3r/5, 4r/5)}$, where v is convex or concave, respectively. In fact, these nonconstant orbits lie in

$$U_{(r/5+\delta, 2r/5-\delta)} \cup U_{(3r/5+\delta, 4r/5-\delta)}$$

for some $r/5 > \delta > 0$. This follows from the fact that dK_ν equals zero along the boundary of $U_{(r/5, 2r/5)} \cup U_{(3r/5, 4r/5)}$.

Step 2: A Morse function isolating L .

Let $F_0: M \rightarrow \mathbb{R}$ be an admissible Morse–Bott function with the following properties:

- The submanifold L is a critical submanifold with index equal to 0.
- On U_r , we have $F_0 = f_0(\|p\|)$ for some increasing function $f_0: [0, r] \rightarrow \mathbb{R}$ which is strictly convex on $[0, 2r/5)$, linear on $[2r/5, 3r/5]$, and strictly concave on $(3r/5, r]$.
- All critical submanifolds other than L are isolated nondegenerate critical points with strictly positive Morse indices.

Such a function is easily constructed by starting with the square of the distance function from L with respect to a metric which coincides, in the normal directions, with the cometric of g inside U_r . This distance function can then be deformed within U_r to obtain the first and second properties above. Perturbing the resulting function away from U_r , one can ensure that it is Morse there. Then performing some elementary handle slides and cancelations away from U_r , as in [Mi], one can get rid of all other local minima.

Let $F_L: L \rightarrow \mathbb{R}$ be a Morse function with a unique local minimum at a point Q in L . Choose a bump function $\hat{\sigma}: [0, +\infty) \rightarrow \mathbb{R}$ such that $\hat{\sigma}(s) = 1$ for s near zero and $\hat{\sigma}(s) = 0$ for $s \geq r/5$. Let $\sigma = \hat{\sigma}(\|p\|)$ be the corresponding function on M with support in $U_{r/5}$ and set

$$F = F_0 + \epsilon_L \cdot \sigma \cdot F_L.$$

For a sufficiently small choice of $\epsilon_L > 0$, F is a Morse function whose critical points away from $U_{r/5}$ agree with those of F_0 and whose critical points in $U_{r/5}$ are precisely the critical points of F_L on $L \subset M$ (see, for example, [BH], p. 87).

Step 3: Pre-perturbation.

For $c_0 > 0$, consider the function

$$H_0 = K_\nu + c_0 F.$$

By construction, we have

$$H_0 = \begin{cases} -A + c_0 F & \text{on } U_{r/5}, \\ (v + c_0 f_0)(\|p\|) & \text{on } U_{[r/5, 4r/5]}, \\ B + c_0 F & \text{elsewhere.} \end{cases}$$

From this expression it is clear that each H_0 is a Morse function with $\text{Crit}(H_0) = \text{Crit}(F)$. As well, Q is the unique local minimum of H_0 . Moreover, when c_0 is

sufficiently small, the nonconstant 1-periodic orbits of H_0 , like those of K_ν , are contained in

$$U_{(r/5+\delta, 2r/5-\delta)} \cup U_{(3r/5+\delta, 4r/5-\delta)}$$

for some $r/5 > \delta > 0$.

Step 4: Perturbation of H_0 .

The function H_0 has properties (H1), (H3) and (H4). To obtain a function with property (H2) we must perturb H_0 so that the 1-periodic orbits of the resulting Hamiltonian are nondegenerate.

Let N^ρ be a critical submanifold of \mathcal{A}_{H_0} which is contained in the level set $\|p\| = \rho$. Denote the projection of N^ρ to L by D^ρ . Then D^ρ is a Morse–Bott nondegenerate set of periodic geodesics with period $((v' + c_0 f'_0)(\rho))^{-1}$. Alternatively, D^ρ can be viewed as a collection of 1-periodic geodesics of the metric $(v' + c_0 f'_0)(\rho)g$. We will adopt this latter point of view.

There are finitely many critical submanifolds of \mathcal{A}_{H_0} . We label the submanifolds of 1-periodic closed geodesics which appear as their projections by

$$\{D_j^{\rho_j} \mid j = 1, \dots, \ell\}.$$

Theorem 3.3 implies that the set of potentials

$$\bigcap_{j=1, \dots, \ell} \mathcal{V}_{\text{reg}}((v' + c_0 f'_0)(\rho_j)g)$$

is dense in $C^\infty(S^1 \times L)$. We can choose a V in this set which is arbitrarily small with respect to the C^∞ -metric. By Lemma 3.4, the projection, $q(t)$, of each 1-periodic orbit $x(t)$ of $\nu(K_g + V)$ is then nondegenerate and lies arbitrarily close to (within a fixed distance of) exactly one of the $D_j^{\rho_j}$ and satisfies

$$I_{\text{Morse}}(q) \in [I_{\text{Morse}}(D_j^{\rho_j}), I_{\text{Morse}}(D_j^{\rho_j}) + \dim(D_j^{\rho_j})].$$

Given such a V we define $V_0: S^1 \times M \rightarrow \mathbb{R}$ so that

$$V_0 = \begin{cases} V(t, q) & \text{in } U_{(2r/5+\delta, 3r/5-\delta)}, \\ 0 & \text{on the complement of } U_{(2r/5, 3r/5)}. \end{cases}$$

Clearly, the function V_0 , like V itself, can be chosen to be arbitrarily small. We then define H_L by

$$H_L(t, q, p) = \begin{cases} (v + c_0 f_0) \left(\sqrt{\frac{1}{2} \|p\|^2 + V_0(t, q, p)} \right) & \text{for } (q, p) \text{ in } U_{(2r/5, 3r/5)}, \\ H_0(q, p) & \text{otherwise.} \end{cases}$$

This function has properties (H1)–(H4) as desired. It only remains to normalize H_L . Adding the function

$$- \int_0^1 H_L(t, \cdot) \omega^n$$

to H_L we get a normalized Hamiltonian. Since H_L is a small perturbation of K_ν , this new Hamiltonian still has properties (H1)–(H4). For simplicity, we still denote it by H_L . □

3.4. Additional properties of H_L . Let H be a Floer Hamiltonian. A spanning disc for a 1-periodic orbit x in $\mathcal{P}(H)$ is a smooth map $w: D^2 \subset \mathbb{C} \rightarrow M$ such that $w(e^{2\pi i t}) = x(t)$. The action of x with respect to w is defined as

$$\mathcal{A}_H(x, w) = \int_0^1 H(t, x(t)) dt - \int_{D^2} w^* \omega.$$

A spanning disc also determines a homotopy class of trivializations of x^*TM and hence a Conley–Zehnder index $\mu_{CZ}(x, w)$. This index is described in §4.2.2. We only mention here that it is normalized so that for a critical point p of a C^2 -small Morse function, and the constant spanning disc $w_p(z) = p$, we have

$$\mu_{CZ}(p, w_p) = \frac{1}{2} \dim M - I_{\text{Morse}}(p).$$

The following result provides a simple criterion for recognizing nonconstant periodic orbits of the Hamiltonian H_L on (M^{2n}, ω) constructed in Proposition 3.5.

Lemma 3.6. *If x is a contractible 1-periodic orbit of H_L which admits a spanning disc w such that $-\|H_L\|^- < \mathcal{A}_{H_L}(x, w) \leq \|H_L\|^+$ and $\mu_{CZ}(x, w) = n$, then x is nonconstant.*

Proof. Arguing by contradiction, we assume that $x(t) = P$ for some point P in M . The spanning disc w then represents an element $[w]$ in $\pi_2(M)$, and we have

$$\mathcal{A}_{H_L}(x, w) = \int_0^1 H_L(t, P) dt - \omega([w]).$$

By property (H1), the point P corresponds to a critical point of F and the Hamiltonian flow of H_L and $c_0 F$ are identical near P . The normalization of the Conley–Zehnder index then yields

$$\mu_{CZ}(x, w) = n - I_{\text{Morse}}(P) + 2c_1([w]). \tag{9}$$

If $\omega([w]) = 0$, then the proportionality of (M, ω) implies that $c_1([w]) = 0$.⁷ It then follows from (9) that the Morse index of P must be zero. This implies that

⁷This is the only point where we use the assumption that (M, ω) is proportional.

$P = Q$, since Q is the unique fixed local minimum of H_L . However, the action of Q with respect to a spanning disc w with $\omega([w]) = 0$ is equal to $-\|H_L\|^-$. This is outside the assumed action range and hence a contradiction.

We must therefore have $\omega([w]) \neq 0$ and thus, by the definition of $r(M, \omega)$ and property (H4),

$$|\omega([w])| \geq r(M, \omega) > \|H_L\|.$$

If $\omega([w]) > 0$, then

$$\begin{aligned} \mathcal{A}_{H_L}(x, w) &\leq \int_0^1 H_L(t, P) dt - \|H_L\| \\ &= \int_0^1 H_L(t, P) dt - \|H_L\|^+ - \|H_L\|^- \\ &\leq -\|H_L\|^- \end{aligned}$$

which is again outside the assumed action range.

For the remaining case, $\omega([w]) < 0$, we have

$$\mathcal{A}_{H_L}(x, w) \geq \int_0^1 H_L(t, P) dt + \|H_L\| = \int_0^1 H_L(t, P) dt + \|H_L\|^+ + \|H_L\|^-.$$

Hence, either $\mathcal{A}_{H_L}(P, w) > \|H_L\|^+$ or $P = Q$. Both of these conclusions again contradict our hypotheses, and so x must be nonconstant. □

Periodic orbits meeting the previous criteria will be detected using the techniques developed in [Ke1] to study length minimizing Hamiltonian paths. The following intermediate result will be crucial in applying these methods.

Lemma 3.7. *There is a normalized admissible Hamiltonian G_L on M such that*

- (1) *the admissible Hamiltonian path $\phi_{G_L}^t$ is homotopic to $\phi_{H_L}^t$ relative to its end points;*
- (2) $\|G_L\|^- < \|H_L\|^-$.

Proof. First we note that inequality (8) and Proposition 2.2, together imply that the path $\phi_{K_v}^t$ does not minimize the negative Hofer length in its homotopy class. Hence, there is a normalized Hamiltonian G_v in $C_0^\infty([\phi_{K_v}^t])$ such that

$$\|K_v\|^- \geq \|G_v\|^- + 2\zeta \tag{10}$$

for some $\zeta > 0$.

Now consider the Hamiltonian flow $\phi_{H_L}^t \circ (\phi_{K_v}^t)^{-1}$ which is generated by the Hamiltonian

$$F_v = H_L - K_v \circ \phi_{K_v}^t \circ (\phi_{H_L}^t)^{-1}.$$

By construction, H_L is arbitrarily close to K_v in the C^∞ -topology. Hence, F_v is also arbitrarily close to zero in the C^∞ -topology.

The Hamiltonian flow

$$\phi_{H_L}^t \circ (\phi_{K_v}^t)^{-1} \circ \phi_{G_v}^t \tag{11}$$

is generated by the normalized Hamiltonian

$$G_L = F_v + G_v \circ (\phi_{F_v}^t)^{-1}.$$

Since $F_v = H_L - B$ near the boundary of M and G_v belongs to $C_0^\infty(S^1 \times M)$, we see that G_L is also admissible. The flow of G_L is homotopic to $\phi_{H_L}^t$ via the homotopy of admissible Hamiltonian flows

$$s \mapsto \phi_{H_L}^t \circ (\phi_{sK_v}^t)^{-1} \circ \phi_{sG_v}^t.$$

Finally, by choosing $\|H_L - K_v\|_{C^\infty}$ to be sufficiently small, we have

$$\begin{aligned} \|G_L\|^- &\leq \|F_v\|^- + \|G_v\|^- \\ &\leq \|F_v\|^- + \|K_v\|^- - 2\zeta \\ &\leq \|H_L\|^- + \|H_L - K_v\|^- + \|F_v\|^- - 2\zeta \\ &\leq \|H_L\|^- - \zeta. \end{aligned} \tag{□}$$

4. Index relations

Let x be a 1-periodic orbit of H_L with spanning disc w . In this section we establish an identity, (21), which relates the Conley–Zehnder index of x with respect to w , the Maslov index of w , and the Morse index of the perturbed geodesic determined by x . For a split hyperbolic Lagrangian submanifold L , this identity yields crucial bounds for the Maslov index of w which depend on the dimensions of the factors of L and the Conley–Zehnder index of x with respect to w . The results presented in this section are not new, but we were unable to find a reference for all of them which was suitable for our purposes.

4.1. Basic indices. There are two classical versions of the Maslov index in symplectic linear algebra (see, for example, [McDSa1]). The first of these indices is defined for continuous loops in Λ_{2n} , the set of Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_{2n})$. We denote this index by μ_{Maslov} . As noted by Arnold in [Ar], it can be defined as an intersection number. We now recall the generalization of this interpretation from [RS].

Let $\eta: S^1 \rightarrow \Lambda_{2n}$ be a loop of Lagrangian subspaces and let $V \in \Lambda_{2n}$ be a fixed reference space. One calls $t_0 \in S^1$ a *crossing* of η (with respect to V) if $\eta(t_0)$ and

V intersect nontrivially. At a crossing t_0 , one can define a crossing form $Q(t_0)$ on $\eta(t_0) \cap V$ as follows. Let $W \in \Lambda_{2n}$ be transverse to $\eta(t_0)$. For each v in $\eta(t_0) \cap V$ we define, for t near t_0 , the path $w(t)$ in W by

$$v + w(t) \in \eta(t).$$

We then set

$$Q(t_0)(v) = \left. \frac{d}{dt} \right|_{t=t_0} \omega(v, w(t)).$$

The crossing t_0 is said to be *regular* if $Q(t_0)$ is nondegenerate. If all the crossings of η are regular then they are isolated and the Maslov index is defined by

$$\mu_{\text{Maslov}}(\eta; V) = \sum_{t \in S^1} \text{sign}(Q(t)), \tag{12}$$

where ‘sign’ denotes the signature and the sum is over all crossings. This integer is independent of the choice of V (as well as the choices of W at each crossing).

The second classical Maslov index is defined for continuous loops $\gamma: S^1 \rightarrow \text{Sp}(2n)$ where $\text{Sp}(2n)$ is the group of $2n \times 2n$ real matrices which preserve ω_{2n} . We denote this integer by $m(\gamma)$ and refer the reader to [McDSa1] for its definition. We note that it is related to the Maslov index for loops of Lagrangian subspaces in the following way. Recall that the graph of a matrix $A \in \text{Sp}(2n)$, Γ_A , is a Lagrangian subspace of the product $\mathbb{R}^{4n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ equipped with the symplectic form $\omega_{2n} \times (-\omega_{2n})$. To a loop $\gamma(t)$ in $\text{Sp}(2n)$ one can then associate a loop of Lagrangian subspaces $\Gamma_{\gamma(t)}$ in \mathbb{R}^{4n} . In this case,

$$2m(\gamma) = \mu_{\text{Maslov}}(\Gamma_\gamma). \tag{13}$$

One can also define a Maslov-type index for certain paths in $\text{Sp}(2n)$. This was first defined by Conley and Zehnder in [CZ]. Let $\text{Sp}^*(2n)$ be the subset of $\text{Sp}(2n)$ which consists of matrices which do not have one as an eigenvalue. Set

$$\mathbf{Sp}(2n) = \{\Phi \in C^0([0, 1], \text{Sp}(2n)) \mid \Phi(0) = \text{id}, \Phi(1) \in \text{Sp}^*(2n)\}.$$

The Conley–Zehnder index associates an integer, $\mu_{\text{CZ}}(\Phi)$, to any path Φ in $\mathbf{Sp}(2n)$. It can be defined axiomatically, and the relevant axioms for the present work are as follows.

Homotopy invariance. The index μ_{CZ} is constant on the components of $\mathbf{Sp}(2n)$;

Loop property. If $\gamma \in \mathbf{Sp}(2n)$ satisfies $\gamma(0) = \gamma(1)$, then

$$\mu_{\text{CZ}}(\gamma \circ \Phi) = 2m(\gamma) + \mu_{\text{CZ}}(\Phi)$$

for every $\Phi \in \mathbf{Sp}(2n)$;

Normalization. The Conley–Zehnder index of the path $e^{t\pi i}$ in $\mathbf{Sp}(2)$ is one.⁸

⁸This normalization differs by a minus sign from the one used in [Ke1].

4.2. Nonlinear versions of the basic indices. The basic indices described above can be used to define useful indices in a variety of nonlinear settings. We recall three such examples which will be used in the proof of Theorem 1.5.

4.2.1. The Maslov class of a Lagrangian submanifold. We begin with the definition of the Maslov class, $\mu_{\text{Maslov}}^L : \pi_2(M, L) \rightarrow \mathbb{Z}$, of a Lagrangian submanifold L of (M, ω) . Any continuous representative $w : (D^2, \partial D^2) \rightarrow (M, L)$ of a class in $\pi_2(M, L)$ determines a symplectic trivialization of $q^*(TM) = q^*(TT^*L)$, where $q(t) = w(e^{2\pi it})$. Let

$$\Phi_w : S^1 \times \mathbb{R}^{2n} \rightarrow q^*(TT^*L).$$

be such a trivialization. Recall that the vertical subbundle Vert of $T(T^*L)$ is a Lagrangian subbundle. The trivialization Φ_w then yields a loop

$$\eta_w(t) = \Phi_w(t)^{-1}(\text{Vert}(q(t)))$$

of Lagrangian subspaces of \mathbb{R}^{2n} . One then defines

$$\mu_{\text{Maslov}}^L([w]) = \mu_{\text{Maslov}}(\eta_w).$$

4.2.2. Contractible periodic orbits of Hamiltonian flows. One can also define a Conley–Zehnder index for the contractible nondegenerate periodic orbits of a general Hamiltonian flow. Let H be a Hamiltonian on (M, ω) and let $x : S^1 \rightarrow M$ be a contractible and nondegenerate 1-periodic orbit of X_H . A spanning disc for x , $w : D^2 \rightarrow M$, determines a symplectic trivialization

$$\Phi_w : S^1 \times \mathbb{R}^{2n} \rightarrow x^*(TM).$$

The Conley–Zehnder index of x with respect to w is then defined by

$$\mu_{\text{CZ}}(x, w) = \mu_{\text{CZ}}(\Phi_w(t)^{-1} \circ (d\phi_H^t)_{x(0)} \circ \Phi_w(0)).$$

By the homotopy invariance property of the Conley–Zehnder index, $\mu_{\text{CZ}}(x, w)$ depends only on the homotopy class of the spanning disc w , relative its boundary. Changing this homotopy class by gluing a representative of the class $A \in \pi_2(M)$ to the map w , in the obvious way, has the following effect

$$\mu_{\text{CZ}}(x, A\#w) = \mu_{\text{CZ}}(x, w) + 2c_1(A).$$

The normalization of μ_{CZ} implies that if p is a critical point of a C^2 -small Morse function H , and $w_p(D^2) = p$ is the constant spanning disc, then

$$\mu_{\text{CZ}}(p, w_p) = \frac{1}{2} \dim M - I_{\text{Morse}}(p).$$

This fact was used in the proof of Lemma 3.6.

4.2.3. Closed perturbed geodesics.. Finally, we recall the definition of a Conley–Zehnder index associated to closed orbits of perturbed geodesic flows. Consider a Hamiltonian $H : S^1 \times T^*L \rightarrow \mathbb{R}$ of the form

$$H(t, q, p) = \frac{1}{2} \|p\|^2 + V(t, q),$$

and let $x(t) = (q(t), p(t))$ be a nondegenerate 1-periodic orbit of H . Recall from Section 3.1 that the projection $q(t)$ is a perturbed geodesic, i.e., a critical point of the energy functional

$$\mathcal{E}_g(q) = \int_0^1 \left(\frac{1}{2} \|\dot{q}(t)\|^2 + V(t, q(t)) \right) dt.$$

As such, q has a finite Morse index, $I_{\text{Morse}}(q)$ which is the number of negative eigenvalues of the Hessian of \mathcal{E}_g at q , counted with multiplicity.

To define an index of Conley–Zehnder type for x , one does not use a symplectic trivialization of $x^*(T^*L)$ determined by a spanning disc. Instead, as we describe below, one uses an intrinsic class of trivializations which are determined by a global Lagrangian splitting of TT^*L . This yields a Conley–Zehnder index for both contractible and noncontractible orbits in T^*L which we will refer to as the *internal Conley–Zehnder* and will denote by $\mu_{\text{CZ}}^{\text{int}}$.

For any point $x_0 = (q_0, p_0) \in T^*L$, the Levi-Civita connection for the metric g determines a splitting

$$T_{x_0}(T^*L) = \text{Hor}(x_0) \oplus \text{Vert}(x_0)$$

into horizontal and vertical subbundles. The vertical bundle $\text{Vert}(x_0)$ is a Lagrangian subbundle and is canonically isomorphic to $T_{q_0}^*L$. The horizontal bundle $\text{Hor}(x_0)$ is canonically isomorphic to $T_{q_0}L$ and is also Lagrangian, [K1]. Hence we can identify $x^*(TT^*L)$ with $q^*(TL) \oplus q^*(T^*L)$. Note that while the symplectic vector bundle $x^*(TT^*L)$ is always trivial, the factors $q^*(TL)$ and $q^*(T^*L)$ need not be.

If $q^*(TL)$ is trivial, e.g., if L is orientable, then a trivialization $\psi : S^1 \times \mathbb{R}^n \rightarrow q^*(TL)$ determines a trivialization Ψ of $T_{x(t)}T^*L = T_{q(t)}L \oplus T_{q(t)}^*L$ as follows,

$$\begin{aligned} \Psi : S^1 \times \mathbb{R}^n \times \mathbb{R}^n &\rightarrow T_{q(t)}L \oplus T_{q(t)}^*L \\ (t, v, w) &\mapsto \begin{pmatrix} \psi(t) & 0 \\ 0 & (\psi(t)^*)^{-1} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \end{aligned}$$

The *internal Conley–Zehnder index* of x is then defined by

$$\mu_{\text{CZ}}^{\text{int}}(x) = \mu_{\text{CZ}}(\Psi(t)^{-1} \circ (d\phi_H^t)_{x(0)} \circ \Psi(0)).$$

This index does not depend on the choice of the trivialization ψ (see Lemma 1.3 of [AS]).

When $q^*(TL)$ is nontrivial, we proceed as in [We]. Consider a map $\psi: [0, 1] \times \mathbb{R}^n \rightarrow q^*(TL)$ such that

$$\psi(1) = \psi(0) \circ E_1, \tag{14}$$

where $E_1: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the diagonal $n \times n$ matrix with diagonal $(-1, 1, \dots, 1)$. Equip $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$, and extend ψ to a symplectic trivialization

$$\begin{aligned} \Psi: S^1 \times \mathbb{R}^n \times \mathbb{R}^n &\rightarrow T_{q(t)}L \oplus T_{q(t)}^*L \\ (t, v, w) &\mapsto \begin{pmatrix} \psi(t) & 0 \\ 0 & (\psi(t)^*)^{-1} \end{pmatrix} U(t) \begin{pmatrix} v \\ w \end{pmatrix}, \end{aligned}$$

where $U(t)$ is the rotation of the $x_1 y_1$ -plane in $\mathbb{R}^n \times \mathbb{R}^n$ by $-\pi$ radians. Again, we set

$$\mu_{CZ}^{\text{int}}(x) = \mu_{CZ}(\Psi(t)^{-1} \circ (d\phi_H^t)_{x(0)} \circ \Psi(0)),$$

and note that this index is also independent of the choice of the initial trivialization ψ satisfying (14).

4.3. Relations between indices. The first relation we discuss is between the internal Conley–Zehnder index of a nondegenerate 1-periodic orbit $x(t) = (q(t), p(t))$ of the Hamiltonian $H(t, q, p) = \frac{1}{2}\|p\|^2 + V(t, q)$ on $(T^*L, d\theta)$, and the Morse index of the closed perturbed geodesic $q(t)$. The following result was proven by Duistermaat in [Du]. An alternative proof, as well as the extension to the nonorientable case, is contained in [We].

Theorem 4.1 ([Du], [We]). *Let $x(t) = (q(t), p(t))$ be a nondegenerate 1-periodic orbit of $H(t, q, p) = \frac{1}{2}\|p\|^2 + V(t, q)$. If $q^*(TL)$ is trivial then*

$$\mu_{CZ}^{\text{int}}(x) = I_{\text{Morse}}(q). \tag{15}$$

Otherwise,

$$\mu_{CZ}^{\text{int}}(x) = I_{\text{Morse}}(q) - 1. \tag{16}$$

By construction, each nonconstant 1-periodic orbit x of H_L is also a reparameterization of a closed orbit of a Hamiltonian on T^*L of the form $H(t, q, p) = \frac{1}{2}\|p\|^2 + V(t, q)$. Hence, one can associate to x an internal Conley–Zehnder index as well as a Morse index for its projection to L . Recall that x lies either in $U_{(r/5+\delta, 2r/5-\delta)}$ where v is convex, or $U_{(3r/5+\delta, 4r/5-\delta)}$ where v is concave. In the latter case, the identities above must be shifted in the following way.

Corollary 4.2. *If $x(t) = (q(t), p(t))$ is a 1-periodic orbit of H_L in $U_{(r/5+\delta, 2r/5-\delta)}$, then equations (15) and (16) hold. If x is contained in $U_{(3r/5+\delta, 4r/5-\delta)}$, then we have*

$$\mu_{CZ}^{\text{int}}(x) = I_{\text{Morse}}(q) - 1, \tag{17}$$

if $q^*(TL)$ is trivial, and

$$\mu_{CZ}^{\text{int}}(x) = I_{\text{Morse}}(q) - 2 \tag{18}$$

otherwise.

Proof. A proof of the shift in the concave case is contained in Proposition 2.1 of [Th]. □

Suppose that $x \in \mathcal{P}(H_L)$ is contractible in M and w is a spanning disc for x . One can then define $\mu_{CZ}(x, w)$ as well as $\mu_{CZ}^{\text{int}}(x)$. Moreover, w determines a unique class in $\pi_2(M, L)$ and hence a Maslov index $\mu_{\text{Maslov}}^L([w])$. These indices are related by the following identity which was first established by Viterbo in [Vi1] for $(M, \omega) = (\mathbb{R}^{2n}, \omega_{2n})$. We include a (different) proof, in the general case, for the sake of completeness.

Proposition 4.3. *Let $x(t) = (q(t), p(t))$ be a nonconstant 1-periodic orbit of H_L . Then for any spanning disc w of x we have*

$$\mu_{CZ}(x, w) = \mu_{CZ}^{\text{int}}(x) + \mu_{\text{Maslov}}^L([w]).$$

Proof. Let

$$\Phi_w : S^1 \times \mathbb{R}^{2n} \rightarrow x^*(TT^*L)$$

be a trivialization determined by the spanning disc w . Let

$$\Psi : S^1 \times \mathbb{R}^{2n} \rightarrow q^*(TT^*L).$$

be a trivialization of the type necessary to compute $\mu_{CZ}^{\text{int}}(x)$. That is,

$$\mu_{CZ}^{\text{int}}(x) = \mu_{CZ}(\Psi(t)^{-1} \circ (d\phi_H^t)_{x(0)} \circ \Psi(0)).$$

We may assume that

$$\Phi(0) = \Psi(0).$$

Using the loop property of the Conley–Zehnder index and identity (13), we get

$$\begin{aligned} \mu_{CZ}(x, w) &= \mu_{CZ}(\Phi_w(t)^{-1} \circ (d\phi_H^t)_{x(0)} \circ \Phi_w(0)) \\ &= \mu_{CZ}(\Phi_w(t)^{-1} \circ \Psi(t) \circ \Psi(t)^{-1} \circ (d\phi_H^t)_{x(0)} \circ \Psi(0) \circ \Psi(0)^{-1} \circ \Phi_w(0)) \\ &= \mu_{CZ}(\Phi_w(t)^{-1} \circ \Psi(t) \circ \Psi(t)^{-1} \circ (d\phi_H^t)_{x(0)} \circ \Psi(0)) \\ &= \mu_{CZ}^{\text{int}}(x) + 2m(\Phi_w(t)^{-1} \circ \Psi(t)) \\ &= \mu_{CZ}^{\text{int}}(x) + \mu_{\text{Maslov}}(\Gamma_{\Phi_w(t)^{-1} \circ \Psi(t)}). \end{aligned}$$

Since $\mu_{\text{Maslov}}(\Phi_w(t)^{-1}(\text{Vert}(q(t)))) = \mu_{\text{Maslov}}^L([w])$, it remains to prove that

$$\mu_{\text{Maslov}}(\Gamma_{\Phi_w(t)^{-1} \circ \Psi(t)}) = \mu_{\text{Maslov}}(\Phi_w(t)^{-1}(\text{Vert}(q(t)))).$$

Choose $V_0 = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ and note that for the trivialization Ψ we have $\Psi(t)(V_0) = \text{Vert}(q(t))$. Hence, it suffices to show that

$$\mu_{\text{Maslov}}(\Gamma_{\Phi_w(t)^{-1} \circ \Psi(t)}) = \mu_{\text{Maslov}}(\Phi_w(t)^{-1} \circ \Psi(t)(V_0)). \tag{19}$$

Using the recipe for the Maslov index described in §4.2.1, we will verify (19) by proving that

$$\mu_{\text{Maslov}}(\Gamma_{\Phi_w(t)^{-1} \circ \Psi(t)}; V_0 \times V_0) = \mu_{\text{Maslov}}(\Phi_w(t)^{-1} \circ \Psi(t)(V_0); V_0).$$

By homotoping the trivializations Φ_w and Ψ , if necessary, we may assume that all of the relevant crossings are regular. Note that t is a crossing of $\Gamma_{\Phi_w(t)^{-1} \circ \Psi(t)}$ with respect to $V_0 \times V_0$ if

$$(v, \Phi_w(t)^{-1} \circ \Psi(t)(v)) \in V_0 \times V_0$$

for some nonzero $v \in \mathbb{R}^{2n}$. Similarly, t is a crossing of $\Phi_w(t)^{-1} \circ \Psi(t)(V_0)$ with respect to V_0 if there is some nonzero $v \in V_0$ such that

$$\Phi_w(t)^{-1} \circ \Psi(t)(v) \in V_0.$$

Hence, the crossings are identical.

It remains to show that at each crossing the signatures of the relevant crossing forms are equal. For simplicity we fix a crossing t_0 and set $\Pi(t) = \Phi_w(t)^{-1} \circ \Psi(t)$. We also choose a $v \in V_0$ such that $\Pi(t_0)(v)$ is in V_0 .

Fix a Lagrangian complement W of V_0 . The first crossing form evaluated at $(v, \Pi(t_0)v)$ is

$$Q(t_0)(v, \Pi(t_0)v) = -\omega_{2n} \oplus \omega_{2n}((v, \Pi(t_0)v), \dot{\hat{w}}(t_0)),$$

where $\hat{w}(t)$ is a path in $W \times W$ defined by the condition

$$(v, \Pi(t_0)v) + \hat{w}(t) = (v(t), \Pi(t)v(t)) \tag{20}$$

for some path $v(t)$ in \mathbb{R}^{2n} with $v(0) = v$. From (20) we get

$$\begin{aligned} Q(t_0)(v, \Pi(t_0)v) &= -\omega_{2n}(v, \dot{v}(0)) + \omega_{2n}(\Pi(t_0)v, \dot{\Pi}(t_0)v) \\ &\quad + \omega_{2n}(\Pi(t_0)v, \Pi(t_0)\dot{v}(0)) \\ &= \omega_{2n}(\Pi(t_0)v, \dot{\Pi}(t_0)v). \end{aligned}$$

Similarly, the second crossing form at t_0 , when evaluated at $\Pi(t_0)v$, yields

$$Q(t_0)(\Pi(t_0)v) = \omega_{2n}(\Pi(t_0)v, \dot{v}(0)).$$

Here, $w(t)$ is a path in W defined, for t near zero, by the condition

$$\Pi(t_0)v + w(t) = \Pi(t)v(t),$$

and $v(t)$ is now a path in V_0 . In this case

$$\begin{aligned} Q(t_0)(\Pi(t_0)v) &= \omega_{2n}(\Pi(t_0)v, \dot{\Pi}(t_0)v) + \omega_{2n}(\Pi(t_0)v, \Pi(t_0)\dot{v}(0)) \\ &= \omega_{2n}(\Pi(t_0)v, \dot{\Pi}(t_0)v) \end{aligned}$$

since both $\Pi(t_0)v$ and $\Pi(t_0)\dot{v}(0)$ belong to the Lagrangian subspace $\Pi(t_0)(V_0)$. Clearly, the isomorphism of domains $(v, \Pi(t_0)v) \mapsto \Pi(t_0)v$ takes the first crossing form to the second and so they have the same signature, as desired. \square

4.4. Cumulative bounds on the Maslov class. Let $x(t) = (q(t), p(t))$ be a non-constant 1-periodic orbit of H_L , and let w be a spanning disc for x . Using the relations above, we now obtain bounds on $\mu_{\text{Maslov}}^L([w])$. By Corollary 4.2 and Proposition 4.3 we have

$$\mu_{\text{Maslov}}^L([w]) = \mu_{\text{CZ}}(z, w) - I_{\text{Morse}}(q)(+1)(+1),$$

where the first $(+1)$ contributes only if q^*TM is not orientable and the second $(+1)$ contributes if x is contained in $U_{(3r/5+\delta, 4r/5-\delta)}$.

Applying Lemma 3.4 to Example 3.2 we see that q is a closed perturbed geodesic whose Morse index satisfies

$$I_{\text{Morse}}(q) \in [0, 1 + \dim P_2 + \dots + \dim P_k]. \tag{21}$$

Overall, we then have

Proposition 4.4. *Let $x(t) = (q(t), p(t))$ be a contractible 1-periodic orbits of H_L and let w be a spanning disc for x . Then*

$$\mu_{\text{CZ}}(x, w) - 1 - \sum_{j=2}^k \dim P_j \leq \mu_{\text{Maslov}}^L([w]) \leq \mu_{\text{CZ}}(x, w)(+1)(+1), \tag{22}$$

where the first $(+1)$ contributes only if q^*TM is not orientable and the second $(+1)$ contributes if x is contained in $U_{(3r/5+\delta, 4r/5-\delta)}$.

5. Hamiltonian paths which are not length minimizing and the proof of

Theorem 1.5

Theorem 1.5 follows from the index inequalities of Proposition 4.4 and the fact, established in Lemma 3.7, that H_L does not minimize the negative Hofer length

functional. The missing ingredient, which we describe in this section and prove in the next, is a theorem which relates the failure of an admissible Hamiltonian path to minimize the negative Hofer length to its 1-periodic orbits.

Let \mathcal{J} be the space of smooth compatible almost complex structures on (M, ω) . Recall that for a compact, convex symplectic manifold (M, ω) , there is a vector field X defined in a neighborhood of ∂M which is transverse to ∂M , outward pointing, and satisfies $\mathcal{L}_X \omega = \omega$, see §2.1. In this case, we say that $J \in \mathcal{J}$ is *admissible* if

(J1) $\omega(X(x), J(x)v) = 0$ for all $x \in \partial M$ and $v \in T_x \partial M$;

(J2) $\omega(X(x), J(x)X(x)) = 1$ for all $x \in \partial M$;

(J3) $\mathcal{L}_X J = 0$ on M_ϵ for some $\epsilon > 0$.

If (M, ω) is closed then every $J \in \mathcal{J}$ is admissible. In either case, the space of admissible almost complex structures will be denoted by $\hat{\mathcal{J}}$.

For J in \mathcal{J} , let $\hat{h}(J)$ be the infimum over the symplectic areas of nonconstant J -holomorphic spheres in M . Set

$$\hat{h} = \sup_{J \in \hat{\mathcal{J}}} \hat{h}(J).$$

The constant \hat{h} is strictly positive and is greater than or equal to $r(M, \omega)$.

Theorem 5.1. *Let (M, ω) be a symplectic manifold of dimension $2n$ which is either closed or compact and convex. Let H be an admissible Floer Hamiltonian on M , such that $\|H\| < \hat{h}$. If ϕ_H^t does not minimize the negative Hofer seminorm in its homotopy class, then there is a 1-periodic orbit x of H which admits a spanning disc w such that*

$$\mu_{CZ}(x, w) = n$$

and

$$-\|H\|^- < \mathcal{A}_H(x, w) \leq \|H\|^+.$$

When (M, ω) is closed, this result follows immediately from the main theorem of [Ke1]. In the next section, we present the proof for the case when (M, ω) is convex. This proof also works in the closed case but yields weaker results than those in [Ke1].

5.1. Proof of Theorem 1.5 assuming Theorem 5.1. Before proving Theorem 5.1, we first show that it implies Theorem 1.5. In Lemma 3.7 we proved that $\phi_{H_L}^t$ does not minimize the negative Hofer length in its homotopy class. By construction, we also have

$$\|H_L\| < r(M, \omega) \leq \hat{h}.$$

Hence, Theorem 5.1 implies that there is a 1-periodic orbit x of H_L which admits a spanning disc w such that

$$\mu_{CZ}(x, w) = n$$

and

$$-\|H_L\|^- < \mathcal{A}_H(x, w) \leq \|H_L\|^+.$$

It follows from Lemma 3.6 that x must be nonconstant. By Proposition 4.4, the Maslov index of the class $[w]$ then satisfies

$$\dim P_1 - 1 \leq \mu_{\text{Maslov}}^L([w]) \leq n(+1)(+1). \tag{23}$$

The lower bound $\dim P_1 - 1$ is greater than zero since the dimension of P_1 must be at least two in order for it to admit a metric of negative sectional curvature. The upper bound is at most $n + 1$ if L is orientable and at most $n + 2$ otherwise. Hence, inequality (23) implies the desired bounds on N_L .

6. Proof of Theorem 5.1

In this section we prove Theorem 5.1 under the assumption that (M, ω) is compact and convex.

6.1. Overview. Let $f : M \rightarrow \mathbb{R}$ be a Morse function on M which is admissible in the sense of Section 2.2. Fix a metric h on M so that the Morse complex $(\text{CM}_*(f), \partial_h)$ is well defined. Here, $\text{CM}_*(f)$ is the vector space over \mathbb{Z}_2 which is generated by the critical points of f and is graded by the Morse index. The boundary map ∂_h is defined by counting solutions of the negative gradient equation

$$\dot{\gamma} = -\nabla_h f(\gamma). \tag{24}$$

More precisely, ∂_h counts, modulo two, the elements of the spaces

$$m(p, q)/\mathbb{R} := \{\gamma : \mathbb{R} \rightarrow M \mid \dot{\gamma} = -\nabla_h f(\gamma), \gamma(-\infty) = p, \gamma(+\infty) = q\}/\mathbb{R},$$

where p and q are critical points of f with $I_{\text{Morse}}(p) = I_{\text{Morse}}(q) + 1$, and \mathbb{R} acts by translation on the argument of γ . The homology of the Morse complex is independent of both the admissible Morse function f and the metric h , and is isomorphic to $H_*(M, \partial M : \mathbb{Z}_2)$.

For a Floer Hamiltonian H we will define a chain map

$$\Phi : \text{CM}(f) \rightarrow \text{CM}(f)$$

which counts rigid configurations that consist of solutions of (24) and perturbed holomorphic cylinders which are asymptotic at one end to elements of $\mathcal{P}(H)$. If $\|H\| < \hat{h}$, then one can prove that Φ is chain homotopic to the identity. The fact that the Morse homology group $H_0(M, \partial M : \mathbb{Z}_2)$ is \mathbb{Z}_2 will then yield an element $x \in \mathcal{P}(H)$ which contributes to one of the configurations counted by Φ . This will be

the desired periodic orbit of H . In particular, if the path ϕ_H^t does not minimize the negative Hofer length in its homotopy class, then this orbit will admit a spanning disc with respect to which its Conley–Zehnder index is equal to $\frac{1}{2} \dim M$, and its action satisfies the required bounds.

In the next five sections we recall the relevant Floer theoretic tools following the presentation of [Ke1]. The proof of Theorem 5.1 is then contained in Section 6.7.

6.2. Homotopy triples and curvature. Let $\widehat{\mathcal{J}}_{S^1}$ be the space of smooth S^1 -families of admissible almost complex structures in $\widehat{\mathcal{J}}$. A smooth \mathbb{R} -family of Hamiltonians F_s in $C^\infty(S^1 \times M)$ or elements in $\widehat{\mathcal{J}}_{S^1}$ will be called a *compact homotopy* from F^- to F^+ , if there is an $\eta > 0$ such that

$$F_s = \begin{cases} F^-, & \text{for } s \leq -\eta, \\ F^+, & \text{for } s \geq \eta. \end{cases}$$

A compact homotopy H_s of Hamiltonians with $H^- = 0$ is called *admissible* if H^+ is admissible in the sense of §2.2, and for some $\epsilon > 0$ and all $s > -\eta$ we have

$$H_s|_{S^1 \times M_\epsilon}(t, x, \tau) = a(s)e^{-\tau} + b(s, t) \tag{25}$$

with

$$a(s) < 0$$

and

$$\frac{da}{ds} \leq 0.$$

Fix an admissible Floer Hamiltonian H and an admissible family of almost complex structures J in $\widehat{\mathcal{J}}_{S^1}$. A *homotopy triple* for the pair (H, J) is a collection of compact homotopies

$$\mathcal{H} = (H_s, K_s, J_s),$$

such that

- H_s is an admissible compact homotopy from the zero function to H ;
- K_s is a compact homotopy of Hamiltonians in $C_0^\infty(S^1 \times M)$ from the zero function to itself;
- J_s is a compact homotopy in $\widehat{\mathcal{J}}_{S^1}$ from some J^- to J .

The *curvature* of the homotopy triple $\mathcal{H} = (H_s, K_s, J_s)$ is the function $\kappa(\mathcal{H}): \mathbb{R} \times S^1 \times M \rightarrow \mathbb{R}$ defined by

$$\kappa(\mathcal{H}) = \partial_s H_s - \partial_t K_s + \{H_s, K_s\}.$$

The positive and negative norms of the curvature are, respectively,

$$|||\kappa(\mathcal{H})|||^{+} = \int_{\mathbb{R} \times S^1} \max_{p \in M} \kappa(\mathcal{H})(s, t, p) \, ds \, dt,$$

and

$$|||\kappa(\mathcal{H})|||^{-} = - \int_{\mathbb{R} \times S^1} \min_{p \in M} \kappa(\mathcal{H})(s, t, p) \, ds \, dt.$$

6.3. Floer caps. Given a homotopy triple $\mathcal{H} = (H_s, K_s, J_s)$ for (H, J) , we consider smooth maps $u: \mathbb{R} \times S^1 \rightarrow M$, which satisfy the equation

$$\partial_s u - X_{K_s}(u) + J_s(u)(\partial_t u - X_{H_s}(u)) = 0. \tag{26}$$

The energy of a solution u of (26) is defined as

$$E(u) = \int_{\mathbb{R} \times S^1} \omega(u)(\partial_s u - X_{K_s}(u), J_s(\partial_s u - X_{K_s}(u))) \, ds \, dt.$$

If this energy is finite, then

$$u(+\infty) := \lim_{s \rightarrow \infty} u(s, t) = x(t) \in \mathcal{P}(H)$$

and

$$u(-\infty) := \lim_{s \rightarrow -\infty} u(s, t) = p \in M,$$

where the convergence is in $C^\infty(S^1, M)$ and the point p in M is identified with the constant map $t \mapsto p$. This asymptotic behavior implies that if a solution u of (26) has finite energy, then it determines an asymptotic spanning disc for the 1-periodic orbit $u(+\infty) = x$. More precisely, for sufficiently large $s > 0$, one can complete and reparameterize $u|_{[-s, s]}$ to be a spanning disc for x in a homotopy class which is independent of s .

The set of *left Floer caps* of $x \in \mathcal{P}(H)$ with respect to \mathcal{H} is

$$\mathcal{L}(x; \mathcal{H}) = \{u \in C^\infty(\mathbb{R} \times S^1, M) \mid u \text{ satisfies (26), } E(u) < \infty, u(+\infty) = x\}.$$

For each $u \in \mathcal{L}(x; \mathcal{H})$ we define the action of x with respect to u by

$$\mathcal{A}_H(x, u) = \int_0^1 H(t, x(t)) \, dt - \int_{\mathbb{R} \times S^1} u^* \omega.$$

A straight forward computation yields

$$0 \leq E(u) = -\mathcal{A}_H(x, u) + \int_{\mathbb{R} \times S^1} \kappa(\mathcal{H})(s, t, u(s, t)) \, ds \, dt. \tag{27}$$

Each left Floer cap $u \in \mathcal{L}(x; \mathcal{H})$ also determines a unique homotopy class of trivializations of $x^*(T^*M)$ and hence a Conley–Zehnder index $\mu_{CZ}(x, u)$.

For any function of the form $F(s, \cdot)$, we set

$$\overleftarrow{F}(s, \cdot) = F(-s, \cdot).$$

Given a homotopy triple $\mathcal{H} = (H_s, K_s, J_s)$, we will also consider maps $v: \mathbb{R} \times S^1 \rightarrow M$ which satisfy the equation

$$\partial_s v + X_{\overleftarrow{K}_s}(v) + \overleftarrow{J}_s(v)(\partial_t v - X_{\overleftarrow{H}_s}(v)) = 0. \tag{28}$$

In this way, we obtain for each $x \in \mathcal{P}(H)$ the space of *right Floer caps*,

$$\mathcal{R}(x; \mathcal{H}) = \{ v \in C^\infty(\mathbb{R} \times S^1, M) \mid v \text{ satisfies (28), } E(v) < \infty, v(-\infty) = x \}.$$

Every right Floer cap $v \in \mathcal{R}(x; \mathcal{H})$ also determines an asymptotic spanning disc for x ,

$$\overleftarrow{v}(s, t) = v(-s, t).$$

The action of x with respect to \overleftarrow{v} is defined as

$$\mathcal{A}_H(x, \overleftarrow{v}) = \int_0^1 H(t, x(t)) dt - \int_{\mathbb{R} \times S^1} \overleftarrow{v}^* \omega,$$

and it satisfies the inequality

$$0 \leq E(v) = \mathcal{A}_H(x, \overleftarrow{v}) + \int_{\mathbb{R} \times S^1} \kappa(\mathcal{H})(s, t, \overleftarrow{v}(s, t)) ds dt. \tag{29}$$

The Conley–Zehnder index of x with respect to \overleftarrow{v} is denoted by $\mu_{CZ}(x, \overleftarrow{v})$.

6.4. Cap data and compactness. For the starting data (H, J) , we will choose a pair of homotopy triples

$$\mathbf{H} = (\mathcal{H}_L, \mathcal{H}_R).$$

This will be referred to as our *cap data*. The norm of the curvature of \mathbf{H} is defined to be

$$\|\kappa(\mathbf{H})\| = \|\kappa(\mathcal{H}_R)\|^- + \|\kappa(\mathcal{H}_L)\|^{+}.$$

We will use \mathbf{H} to define three classes of perturbed holomorphic cylinders. Two of these classes are the left Floer caps with respect to \mathcal{H}_L , $\mathcal{L}(x, \mathcal{H}_L)$, and the right Floer caps with respect to \mathcal{H}_R , $\mathcal{R}(x, \mathcal{H}_R)$. These are used to construct the map Φ . The third class of perturbed holomorphic cylinders that we consider are called *Floer spheres*. These are defined in Section 6.6 where they are used to construct the desired chain homotopy between Φ and the identity map. For each of the three classes of

perturbed holomorphic cylinders there are three possible sources of noncompactness. We need to avoid two of these sources, and in this section we describe how this is accomplished.

The first source of noncompactness to be avoided is the possibility that a sequence of curves can approach the boundary of M . This possibility has already been precluded by the admissibility conditions on \mathcal{H}_L and \mathcal{H}_R . Consider the case of left Floer caps. For $\mathcal{H}_L = (H_s, K_s, J_s)$, we note that the admissibility conditions on H_s and J_s , and the fact that K_s belongs to $C_0^\infty(S^1 \times M)$ for all $s \in \mathbb{R}$, imply that for some $\epsilon > 0$ equation (26) restricts to $S^1 \times M_\epsilon$ as

$$\partial_s u + J_s(u)(\partial_t u + a(s)R(u)) = 0. \tag{30}$$

Here, each J_s satisfies conditions **(J1)**-**(J3)**. The function $a(s)$ is determined by H_s , as in (25), and so we have $a(s) \leq 0$ and $\frac{da}{ds} \leq 0$. If $T: M_\epsilon \rightarrow \mathbb{R}$ is the function $T(\tau, x) = e^\tau$ and u is a solution of (30), then a straight forward computation yields

$$\Delta(T \circ u) = \omega(\partial_s u, J(u)\partial_s u) - \frac{da}{ds}(T \circ u),$$

(see [Vi2] or Theorem 2.1 of [FS]). Since the right hand side is nonnegative, the Strong Maximum Principle implies that if $T \circ u$ attains its maximum then it is constant, [GT]. Hence, no left Floer cap $u \in \mathcal{L}(x; \mathcal{H}_L)$ enters M_ϵ and no sequence of left Floer caps can approach ∂M . The arguments for right Floer caps and the Floer spheres defined in Section 6.6, are entirely similar and are left to the reader.

The other source of noncompactness that we wish to avoid is bubbling. To achieve this we will exploit the following fact: the energy of the Floer caps and Floer spheres that we consider is bounded above by $\|\kappa(\mathbf{H})\|$. Using the assumption from Theorem 5.1 that $\|H\| < \hat{h}$ we will construct cap data \mathbf{H} for which we have the curvature bound

$$\|\kappa(\mathbf{H})\| \leq \|H\| < \hat{h}. \tag{31}$$

As we now describe, this condition allows us to avoid bubbling by simply restricting our choices of the almost complex structures which appear as part of the cap data \mathbf{H} .

The following result is a simple consequence of Gromov’s compactness theorem for holomorphic curves.

Lemma 6.1 ([Kel]). *For every $\delta > 0$ there is a nonempty open subset $\hat{\mathcal{J}}^\delta \subset \hat{\mathcal{J}}(M, \omega)$ such that for every $J \in \hat{\mathcal{J}}^\delta$ we have $\hbar(J) \geq \hat{h} - \delta$.*

Since $\|H\| < \hat{h}$, we can set

$$\delta_H = \frac{\hat{h} - \|H\|}{2}$$

and let $\widehat{\mathcal{J}}^{\delta H}$ be an open set in $\widehat{\mathcal{J}}$ as described in Lemma 6.1. We will assume from now on that the almost complex structures which appear in the families in \mathbf{H} all belong to $\widehat{\mathcal{J}}^{\delta H}$. This implies that any bubble which forms from a sequence of Floer spheres, must be a J -holomorphic sphere $w: S^2 \rightarrow M$ for some $J \in \widehat{\mathcal{J}}^{\delta H}$. Inequality (31) implies that for this J we have

$$\hbar(J) \geq \hat{\hbar} - \frac{1}{2}(\hat{\hbar} - \|H\|) > \|\kappa(\mathbf{H})\|.$$

As mentioned above, the energy of our Floer spheres is bounded above by $\|\kappa(\mathbf{H})\|$. This implies that the energy of the bubble w is less than $\|\kappa(\mathbf{H})\|$ and hence $\hbar(J)$. By the definition of $\hbar(J)$, the bubble w must therefore be trivial. Thus, if \mathbf{H} satisfies (31) and if the families of almost complex structures appearing in \mathbf{H} take values in $\widehat{\mathcal{J}}^{\delta H}$, then no bubbling occurs for sequences of Floer caps or Floer spheres that are defined using \mathbf{H} .

We also note, that since $\widehat{\mathcal{J}}^{\delta H}$ is open we can achieve transversality for our spaces of Floer caps and Floer spheres, which can therefore be assumed to be manifolds of the expected dimensions, [FHS].

6.5. Specific cap data. We now specify cap data $\mathbf{H} = (\mathcal{H}_L, \mathcal{H}_R)$ for (H, J) , where J takes values in $\widehat{\mathcal{J}}^{\delta H}$ and H is a Hamiltonian as in Theorem 5.1, i.e., $\|H\| < \hat{\hbar}$ and ϕ_H^t does not minimize the negative Hofer length in its homotopy class. This cap data will satisfy the curvature bound (31). It will be assumed throughout, that the families of almost complex structures appearing in \mathbf{H} take values in $\widehat{\mathcal{J}}^{\delta H}$.

Let $b: \mathbb{R} \rightarrow [0, 1]$ be a smooth nondecreasing function such that $b(s) = 0$ for $s \leq -1$ and $b(s) = 1$ for $s \geq 1$. Let \mathcal{H}_L be a linear homotopy of the form

$$\mathcal{H}_L = (b(s)H, 0, J_{L,s}).$$

For this choice, $\kappa(\mathcal{H}_L) = b'(s)H$ and we have

$$\|\kappa(\mathcal{H}_L)\|^+ = \|H\|^+. \tag{32}$$

Fix an admissible Hamiltonian G such that ϕ_G^t is homotopic to ϕ_H^t , relative its endpoints, and $\|G\|^- < \|H\|^-$. Note that H and G have the same slope, a_0 , because they generate the same time one map and are admissible.

We now use the Hamiltonian G to construct \mathcal{H}_R . We start with a linear homotopy triple for G of the form

$$\mathcal{E} = (b(s)G, 0, J_s).$$

Let F_s be the normalized Hamiltonian which generates the Hamiltonian flow $\phi_G^t \circ (\phi_{b(s)G}^t)^{-1} \circ \phi_{b(s)H}^t$. A straightforward computation shows that the slope of F_s does

not depend on s . Set

$$\tilde{H}_s = \begin{cases} 0 & \text{for } s \leq -1; \\ b(s)G & \text{for } -1 \leq s \leq 1; \\ F_{s-2}, & \text{for } 1 \leq s \leq 3; \\ H, & \text{for } s \geq 3. \end{cases}$$

This is an admissible compact homotopy from the zero function to H . In particular, we have

$$\frac{da}{ds}(s) = \dot{b}(s)a_0 \leq 0,$$

where again a_0 is the common slope of G and H .

Let $\varrho_t = \phi_H^t \circ (\phi_G^t)^{-1}$. The map $\Upsilon: C^\infty(S^1, M) \rightarrow C^\infty(S^1, M)$, defined by

$$\Upsilon(x(t)) = \varrho_t(x(t)), \tag{33}$$

takes contractible loops to contractible loops and hence $\Upsilon(\mathcal{P}(G)) = \mathcal{P}(H)$. Now consider the family of contractible Hamiltonian loops

$$\varrho_{s,t} = \phi_{\tilde{H}_s}^t \circ (\phi_{b(s)G}^t)^{-1}.$$

For each value of s , $\varrho_{s,t}$ is a loop based at the identity, and

$$\varrho_{s,t} = \begin{cases} \text{id}, & \text{for } s \leq 1; \\ \phi_{F_{s-2}}^t \circ (\phi_G^t)^{-1}, & \text{for } 1 \leq s \leq 3; \\ \phi_H^t \circ (\phi_G^t)^{-1}, & \text{for } s \geq 3. \end{cases}$$

From $\varrho_{s,t}$ we obtain the family of normalized Hamiltonians, $A_s \in C_0^\infty(S^1 \times M)$, defined by

$$\partial_s(\varrho_{s,t}(p)) = X_{A_s}(\varrho_{s,t}(p)).$$

Set

$$\mathcal{H}_R = (\tilde{H}_s, \tilde{K}_s, \tilde{J}_{R,s}) = (\tilde{H}_s, A_s, d\varrho_{s,t} \circ J_s \circ d(\varrho_t^{-1})).$$

It is easy to verify that \mathcal{H}_R is a homotopy triple for (H, J) where $J = \varrho_t \circ J^+ \circ d(\varrho_t^{-1})$ and $J^+ = \lim_{s \rightarrow \infty} J_s$. It also follows from Proposition 2.7 of [Kel] that

$$\|\kappa(\mathcal{H}_R)\|^\pm = \|\kappa(\mathcal{G})\|^\pm. \tag{34}$$

The following result is proved as Propositions 2.6 in [Kel].

Proposition 6.2. *The map $\tilde{\Upsilon}$ defined on $\mathcal{R}(x; \mathcal{G})$ by $\tilde{\Upsilon}(v(s, t)) = \varrho_{-s,t}(v(s, t))$ is a bijection onto $\mathcal{R}(\Upsilon(x); \mathcal{H}_R)$ for every $x \in \mathcal{P}(G)$, and*

$$\mu_{\text{CZ}}(x, \overleftarrow{v}) = \mu_{\text{CZ}}(\Upsilon(x), \overleftarrow{\tilde{\Upsilon}(v)})$$

and

$$\mathcal{A}_G(x, \overleftarrow{v}) = \mathcal{A}_H(\Upsilon(x), \overleftarrow{\Upsilon(v)}).$$

The curvature identity (34) yields

$$\begin{aligned} \|\kappa(\mathcal{H}_R)\|^- &= \|\kappa(\mathcal{G})\|^- \\ &= - \int_{\mathbb{R} \times S^1} \dot{b}(s) (\min_{p \in M} G(t, p)) ds dt \\ &= \|G\|^-. \end{aligned}$$

Hence, by (32) we have

$$\begin{aligned} \|\kappa(\mathbf{H})\| &= \|\kappa(\mathcal{H}_L)\| + \|\kappa(\mathcal{H}_R)\|^- \\ &= \|H\|^+ + \|G\|^- \\ &< \|H\| \\ &< \hat{h}. \end{aligned}$$

6.6. The map Φ . Let $\mathbf{H} = (\mathcal{H}_L, \mathcal{H}_R)$ be the cap data for (H, J) constructed in the previous section. For each critical point p of f we consider the following two spaces of half-trajectories of the negative gradient equation (24):

$$\ell(p) = \{\alpha: (-\infty, 0] \rightarrow M \mid \dot{\alpha} = -\nabla_h f(\alpha), \alpha(-\infty) = p\}$$

and

$$r(p) = \{\beta: [0, +\infty) \rightarrow M \mid \dot{\beta} = -\nabla_h f(\beta), \beta(+\infty) = p\}.$$

Let $\mathcal{N}(p, x, q; f, \mathbf{H})$ be the set of tuples

$$(\alpha, u, v, \beta) \in \ell(p) \times \mathcal{L}(x; \mathcal{H}_L) \times \mathcal{R}(x; \mathcal{H}_R) \times r(q)$$

such that

$$\begin{aligned} \alpha(0) &= u(-\infty), \\ v(+\infty) &= \beta(0) \end{aligned}$$

and

$$[u\#v] = 0.$$

It follows from [PSS] that for generic choices of the families of almost complex structures appearing in \mathbf{H} , the dimension of $\mathcal{N}(p, x, q; f, \mathbf{H})$ is $I_{\text{Morse}}(p) - I_{\text{Morse}}(q)$.

The map $\Phi: \text{CM}_*(f) \rightarrow \text{CM}_*(f)$, is then defined by setting the coefficient of q in the image of p to be the number of elements, modulo two, of the set

$$\bigcup_{x \in \mathcal{P}(H)} \mathcal{N}(p, x, q; f, \mathbf{H}).$$

To show that Φ is well defined, we must verify that the zero-dimensional spaces $\mathcal{N}(p, x, q; f, \mathbf{H})$ are compact. The only possible source of noncompactness are bubbles which appear on the Floer caps of the configurations in $\mathcal{N}(p, x, q; f, \mathbf{H})$. To avoid this, we only need to show that for each tuple (α, u, v, β) in $\mathcal{N}(p, x, r; f, \mathbf{H})$, both u and v have energy less than or equal to $\|\kappa(\mathbf{H})\|$. Then, as described in Section 6.4, our choice of almost complex structures in \mathbf{H} precludes such bubbling.

Equations (27) and (29) imply that

$$0 \leq E(u) \leq -\mathcal{A}_H(x, u) + \|\kappa(\mathcal{H}_L)\|^+$$

and

$$0 \leq E(v) \leq \mathcal{A}_H(x, \overleftarrow{v}) - \|\kappa(\mathcal{H}_R)\|^-$$

Since $[u\#v] = 0$, we have $\mathcal{A}_H(x, u) = \mathcal{A}_H(x, \overleftarrow{v})$ and

$$\begin{aligned} E(u) &\leq -\mathcal{A}_H(x, \overleftarrow{v}) + \|\kappa(\mathcal{H}_L)\|^+ \\ &\leq \|\kappa(\mathcal{H}_R)\|^- + \|\kappa(\mathcal{H}_L)\|^+ \\ &= \|\kappa(\mathbf{H})\|. \end{aligned}$$

A similar computation implies that $E(v) \leq \|\kappa(\mathbf{H})\|$, and so Φ is well defined.

Proposition 6.3. *The map $\Phi: \text{CM}(f) \rightarrow \text{CM}(f)$ is chain homotopic to the identity.*

Proof. The specific cap data, $\mathbf{H} = (\mathcal{H}_L, \mathcal{H}_R)$, is used to construct a homotopy of homotopy triples $\mathcal{H}^\lambda = (H_s^\lambda, K_s^\lambda, J_s^\lambda)$ for $\lambda \in [0, +\infty)$. The desired chain homotopy is then defined using maps w in $\mathbb{C}^\infty(\mathbb{R} \times S^1, M)$ which satisfy the equation

$$\partial_s w - X_{K_s^\lambda}(w) + J_s^\lambda(w)(\partial_t w - X_{H_s^\lambda}(w)) = 0. \tag{35}$$

To define \mathcal{H}^λ we introduce the notation $\mathcal{H} = \mathcal{H}(s) = (H_s, K_s, J_s)$ to emphasize the s -dependence of \mathcal{H} . Set

$$\mathcal{H}^1(s) = \begin{cases} \mathcal{H}_L(s + 3) & \text{when } s \leq 0, \\ \mathcal{H}_R(s - 3) & \text{when } s \geq 0. \end{cases}$$

For $\lambda \in [1, +\infty)$, we then define

$$\mathcal{H}^\lambda(s) = \begin{cases} \mathcal{H}_L(s + C(\lambda)) & \text{when } s \leq 0, \\ \mathcal{H}_R(s - C(\lambda)) & \text{when } s \geq 0, \end{cases}$$

where $C(\lambda)$ is a smooth nondecreasing function which equals λ for $\lambda \gg 1$ and is equal to 3 for λ near 1. Finally, for $\lambda \in [0, 1]$ we set

$$\mathcal{H}^\lambda(s) = (D(\lambda)H_s^1, D(\lambda)K_s^1, J_s^\lambda)$$

for a smooth nondecreasing function $D : [0, 1] \rightarrow [0, 1]$ which equals zero near $\lambda = 0$ and equals one near $\lambda = 1$. The compact homotopies J_s^λ are chosen so that they equal J_s^0 for λ near zero and equal J_s^1 for λ near one.

The fact that J_s takes values in $\widehat{\mathcal{J}}^{\delta H}$ implies the same for the families J_s^λ with $\lambda \geq 1$. For $\lambda \in [0, 1)$, we choose the families J_s^λ so that they also take values in $\widehat{\mathcal{J}}^{\delta H}$.

The following properties of \mathcal{H}^λ are easily verified:

- (1) For each $\lambda \in [0, +\infty)$, H_s^λ is an admissible compact homotopy from the zero function to itself.
- (2) For each $\lambda \in [0, +\infty)$, K_s^λ is a compact homotopy in $C_0^\infty(S^1 \times M)$ from the zero function to itself.
- (3) There is an $\epsilon > 0$ such that equation (35) restricts to $S^1 \times M_\epsilon$ as

$$\partial_s w + J_s^\lambda(w)(\partial_t w + a(\lambda, s)R(w)) = 0, \tag{36}$$

where $a(\lambda, s)$ is determined by $H_s^\lambda|_{M_\epsilon}$ and satisfies $a(\lambda, s) \leq 0$ and $\frac{\partial a}{\partial s} \leq 0$.

- (4) $|||\kappa(\mathcal{H}^\lambda)||| + = D(\lambda)|||\kappa(\mathbf{H})|||$ for all $\lambda \in [0, +\infty)$.

Solutions of (35) with finite energy are perturbed holomorphic cylinders which are asymptotic, at both ends, to points in M . In particular, since the perturbations are compact, each such w can be uniquely completed to a perturbed holomorphic sphere. We define the space of *Floer spheres* for \mathbf{H} as

$$\mathcal{LR}(\mathbf{H})$$

$$= \{(\lambda, w) \in [0, +\infty) \times C^\infty(\mathbb{R} \times S^1, M) \mid w \text{ satisfies (35), } [w] = 0 \in \pi_2(M)\}.$$

Invoking again the Strong Maximum Principle, the third property of \mathcal{H}^λ , listed above, implies that the distance between the images of the maps w in $\mathcal{LR}(\mathbf{H})$ and the boundary of M is bounded away from zero by a positive constant which is independent of λ .

For every $(\lambda, w) \in \mathcal{LR}(\mathbf{H})$, we also have the uniform energy bound

$$\begin{aligned} E(\lambda, w) &= \int_{\mathbb{R} \times S^1} \omega(\partial_s w - X_{K_s^\lambda}(w), J_s^\lambda(\partial_s w - X_{K_s^\lambda}(w))) ds dt \\ &= \int_{\mathbb{R} \times S^1} (\omega(X_{H_s^\lambda}(w), \partial_s w) - \omega(X_{K_s^\lambda}(w), \partial_t w) \\ &\quad + \omega(X_{K_s^\lambda}(w), X_{H_s^\lambda}(w))) ds dt \\ &= \int_{\mathbb{R} \times S^1} \kappa(\mathcal{H}^\lambda)(s, t, w) ds dt \\ &\leq |||\kappa(\mathcal{H}^\lambda)||| + \\ &= D(\lambda)|||\kappa(\mathbf{H})||| \\ &< \hat{h}. \end{aligned}$$

As described in Section 6.4, this allows us to rule out the possibility of bubbling for sequences in $\mathcal{LR}(\mathbf{H})$.

For a pair of critical points p and q of f , we define $\mathcal{N}_\lambda(p, q)$ by

$$\{(\alpha, (\lambda, w), \beta) \in \ell(p) \times \mathcal{LR}(\mathbf{H}) \times r(q) \mid \alpha(0) = w(-\infty), w(+\infty) = \beta(0)\}$$

For generic data, $\mathcal{N}_\lambda(p, q)$ is a manifold of dimension $I_{\text{Morse}}(p) - I_{\text{Morse}}(q) + 1$, [PSS]. We define the map $\chi: \text{CM}_*(f) \rightarrow \text{CM}_{*+1}(f)$ by setting

$$\chi(p) = \sum_{\mu_{\text{CZ}}(q)=\mu_{\text{CZ}}(p)+1} \#_2(\mathcal{N}_\lambda(p, q))q,$$

where $\#_2(\mathcal{N}_\lambda(p, q))$ is the number of components in the compact zero-dimensional manifold $\mathcal{N}_\lambda(p, q)$, modulo two. To prove that χ is the desired chain homotopy, it suffices to show that for every pair of critical points p and r of f such that $I_{\text{Morse}}(p) = I_{\text{Morse}}(r)$, the coefficient of r in

$$(\text{id} - \Phi + \chi \circ \partial_h + \partial_h \circ \chi)(p) \tag{37}$$

is zero.

Consider the compactification $\bar{\mathcal{N}}_\lambda(p, r)$ of the one dimensional moduli space $\mathcal{N}_\lambda(p, r)$. Since the elements of $\mathcal{N}_\lambda(p, r)$ can not approach the boundary of M and we have precluded bubbling, it follows from Floer’s gluing and compactness theorems that the boundary of $\bar{\mathcal{N}}_\lambda(p, r)$ can be identified with the union of the following zero-dimensional manifolds:

- (i) $m(p, r)$,
- (ii) $\bigcup_{x \in \mathcal{P}(H)} \mathcal{N}(p, x, r; f, \mathbf{H})$,
- (iii) $\bigcup_{\mu_{\text{CZ}}(q)=\mu_{\text{CZ}}(p)-1} m(p, q)/\mathbb{R} \times \mathcal{N}_\lambda(q, r)$,
- (iv) $\bigcup_{\mu_{\text{CZ}}(q)=\mu_{\text{CZ}}(p)+1} \mathcal{N}_\lambda(p, q) \times m(q, r)/\mathbb{R}$.

By definition, the number of elements in the sets (ii), (iii) and (iv), modulo two, are the coefficients of r in $\Phi(p)$, $\chi \circ \partial_h(p)$ and $\partial_h \circ \chi(p)$, respectively. The first set $m(p, r)$ is empty if $p \neq r$, and consists only of the constant map when $p = r$. So, the number of elements in $m(p, r)$, modulo two, is equal to the coefficient of r in $\text{id}(p)$. As the boundary of the one-dimensional manifold $\mathcal{N}_\lambda(p, r)$, the total number of these boundary terms is zero, modulo two. Hence, the coefficient of r in (37) is zero, and χ is the desired chain homotopy. \square

6.7. Completion of the proof of Theorem 5.1. For simplicity, we choose the admissible Morse function f on M so that it has a unique local, and hence global,

minimum at a point P in M . (For a proof that this is always possible see, for example, Theorem 8.1 of [Mi].) Since P is the only critical point of f with Morse index zero, P is a cycle in the Morse complex of f and is the only representative of $H_0(\text{CM}(f), \partial_h) = \mathbb{Z}_2$. By Proposition 6.3 we then have $\Phi(P) = P$ at the level of chains. Hence, for some $x \in \mathcal{P}(H)$ there is a tuple (α, u, v, β) in the zero-dimensional space $\mathcal{N}(P, x, P; f, H)$. The pairs (α, u) and (v, β) each belong to moduli spaces with the same asymptotic behavior. In fact, these moduli spaces are used in the definition of the Piunikhin–Salamon–Schwarz maps from [PSS], where their dimensions are computed to be $n - \mu_{\text{CZ}}(x, u)$ and $\mu_{\text{CZ}}(x, \overleftarrow{v}) - n$, respectively. Since $\mathcal{N}(P, x, P; f, H)$ has dimension zero, the moduli spaces containing (α, u) and (v, β) must also be zero-dimensional and so

$$\mu_{\text{CZ}}(x, u) = \mu_{\text{CZ}}(x, \overleftarrow{v}) = n. \tag{38}$$

The condition that $[u\#v] = 0$ implies that

$$\mathcal{A}_H(x, u) = \mathcal{A}_H(x, \overleftarrow{v}).$$

By Proposition 6.2 we have

$$\mathcal{A}_H(x, \overleftarrow{v}) = \mathcal{A}_G(\Upsilon^{-1}(x), \overleftarrow{\tilde{\Upsilon}^{-1}(v)}),$$

where $\tilde{\Upsilon}^{-1}(v)$ is a right Floer cap in $\mathcal{R}(\Upsilon^{-1}(x); \mathcal{G})$ and \mathcal{G} is a linear homotopy triple of the form $(b(s)G, 0, J_s)$. Inequality (29) then implies that

$$\mathcal{A}_H(x, \overleftarrow{v}) \geq -\|\kappa(\mathcal{G})\|^- = -\|G\|^- > -\|H\|^-.$$

On the other hand, inequality (27) yields

$$\mathcal{A}_H(x, u) \leq \|\kappa(\mathcal{H}_L)\|^+ = \|H\|^+,$$

since \mathcal{H}_L is the linear homotopy triple $(b(s)H, 0, J_{L,s})$ and

$$\|\kappa(\mathcal{H}_L)\|^+ = \int_{\mathbb{R} \times S^1} \dot{b}(s) \left(\max_{p \in M} H(t, p) \right) ds dt = \|H\|^+.$$

Altogether, we have

$$\|H\|^- < \mathcal{A}_H(x, \overleftarrow{v}) = \mathcal{A}_H(x, u) \leq \|H\|^+. \tag{39}$$

Let w be a genuine spanning disc for x which is obtained by reparameterizing the asymptotic spanning disc u . By (38) and (39), we have $\mu_{\text{CZ}}(x, w) = n$ and $\|H\|^- < \mathcal{A}_H(x, w) \leq \|H\|^+$, as required.

7. Generalizations

We end this note by describing some simple generalizations of Theorem 1.5. To begin with, one expects that these rigidity results should hold for more general classes of ambient symplectic manifolds. Indeed, one can immediately extend Theorem 1.5 to products of convex symplectic manifolds or the more general class of *split convex* manifolds defined in [FS].

Theorem 1.5 also holds for more general classes of Lagrangian submanifolds. In particular, the methods used here only detect those closed geodesics on a Lagrangian submanifold which are contractible in the ambient symplectic manifold. Hence, the theorem also holds for easily displaceable Lagrangian submanifolds of the form $L = L_1 \times L_2$, where L_1 is split hyperbolic and L_2 admits a metric which has no nonconstant contractible geodesics and is incompressible in the sense that the map $\pi_1(L_2) \rightarrow \pi_1(M)$, induced by inclusion, is an injection.

Combining these observations, we get

Corollary 7.1. *Suppose that the symplectic manifold (M, ω) is rational, proportional and either closed or split convex. Let $L = L_1 \times L_2$ be an easily displaceable Lagrangian submanifold of (M, ω) such that L_1 is split hyperbolic and L_2 is incompressible and admits a metric which has no nonconstant contractible geodesics. Then $N_L \leq \frac{1}{2} \dim M + 2$, and if L is orientable we have $N_L \leq \frac{1}{2} \dim M + 1$.*

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