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Partial Differential Equations — *On a Semilinear Parabolic Equation with Inverse-Square Potential*, by Fabio Punzo and Alberto Tesei, communicated on 14 May 2010.

Abstract. — We study existence and uniqueness, nonexistence and nonuniqueness of nonnegative solutions to a semilinear parabolic equation with inverse-square potential. Analogous existence and nonexistence results for the companion elliptic equation were proved in [4]. Concerning nonuniqueness, we extend the results proved in [16] for the case without potential.

Key words: Inverse-square potential, semigroup estimates, nonnegative solutions, well-posedness, instantaneous blow-up.

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1. Introduction

We investigate existence and uniqueness of nonnegative solutions to the problem

$$\begin{cases} u_t - \Delta u - \frac{c}{|x|^2} u = u^\nu & \text{in } \Omega \times (0,T) =: Q_T \\ u = 0 & \text{in } \partial\Omega \times (0,T] \\ u = u_0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{1.1}$$

Here $\Omega \ni 0$ is an open bounded subset of $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary $\partial\Omega$, $\nu > 1$ and $c \in (0, c_0)$, $c_0 := \frac{(n-2)^2}{4}$ being the best constant in the Hardy inequality. We always suppose $u_0 \geq 0$. Special attention will be paid to nonexistence and nonuniqueness phenomena.

Let us motivate our interest in the above problem and outline our results.

(i) The Schrödinger operator with inverse-square potential

$$H = \Delta + \frac{c}{|x|^2} \quad (c > 0) \tag{1.2}$$

has a number of peculiar features, which have attracted much attention [1], [9], [10], [19]. Accordingly, semilinear elliptic and parabolic problems where it appears, or related to it, have been widely investigated in recent years (in particular, see [2]–[4], [6], [8], [14], [18]–[21]).

Consider a realization $\overset{o}{H}$ of the operator (1.2) in $L^2(\mathbb{R}^n)$ with domain $C_0^\infty(\Omega\backslash\{0\})$. It is known that

(a) $\overset{o}{H}$ is essentially self-adjoint if and only if $c \leq c_0$;
(b) $\overset{o}{H} \geq 0$ if $c \leq c_0$, but $\overset{o}{H}$ not semibounded if $c > c_0$

(see [1] and references therein). By these spectral properties, an extension of $\overset{o}{H}$ will generate a contraction semigroup in $L^2(\mathbb{R}^n)$ if and only if $c \leq c_0$. Hence the Cauchy problem for the parabolic equation

$$v_t = \Delta v + \frac{c}{|x|^2} v. \tag{1.3}$$

is expectedly well-posed for $c \leq c_0$, but ill-posed for $c > c_0$.

This point was addressed in the pioneering paper [2], where nonnegative solutions of the equation

$$v_t - \Delta v - \frac{c}{|x|^2} v = f \tag{1.4}$$

were studied; here $f = f(x,t)$, $f \geq 0$ is a given integrable function. It turned out that nontrivial nonnegative solutions of (1.4) (defined in a suitable weak sense) do exist when $c \leq c_0$. However, no such a solution, even in the weakest possible sense, can exist if $c > c_0$.

Interestingly, the above nonexistence result is related with the lack of regularity at the origin of solutions of (1.4)—an effect of the strongly singular potential $V(x) = \frac{c}{|x|^2}$. Observe preliminarily that equation (1.3) admits the explicit solution

$$V(x,t;c) = \frac{|x|^{\lambda_+/2}}{t^{1+\sqrt{c_0-c}}} e^{-|x|^2/4t}, \tag{1.5}$$

where

$$\lambda_+ \equiv \lambda_+(c) := 2 - n + 2\sqrt{c_0 - c}$$

is the largest root of the equation

$$\lambda^2 + 2(n-2)\lambda + 4c = 0 \quad (c \in (0, c_0]). \tag{1.6}$$

Therefore the solution (1.5) exhibits a standing singularity at $x = 0$, in contrast with the case $c = 0$ (yet in agreement with the sharp estimates of the heat kernel associated with (1.3); see [13], [15]). More generally, let $c \leq c_0$ and $v \geq 0$ solve the Cauchy problem for equation (1.4), with Cauchy data $v_0 \geq 0$, $v_0 \not\equiv 0$ or $f \geq 0$, $f \not\equiv 0$. Then, as proven in [2], for any $\varepsilon \in (0,T)$ and $R > 0$ there exists $C > 0$ such that

$$v(x,t) \geq C|x|^{\lambda_+/2} \quad \text{if } |x| < R,\ t \in (\varepsilon, T). \tag{1.7}$$

The above inequality is easily understood considering radial stationary supersolutions of (1.4). In fact, if $v = k|x|^{\alpha}$ $(k > 0)$ satisfies

$$-\Delta v - \frac{c}{|x|^2} v \geq 0,$$

then there holds

$$\{\alpha^2 + (n-2)\alpha + 4c\} k|x|^{\alpha-2} \leq 0 \quad \Rightarrow \quad \alpha \leq \frac{\lambda_+}{2}. \tag{1.8}$$

Inequality (1.7) can be regarded as a necessary condition for existence. A further necessary condition is

$$\int_0^{T-\varepsilon} \int_{|x|<R} f(x,t)|x|^{\lambda_+/2}\, dx\, dt < \infty \tag{1.9}$$

for any ε, R as above (see [2]). Using (1.7) and (1.9), nonexistence for $c > c_0$ can be heuristically explained as follows. Let v be a nontrivial nonnegative solution when $c > c_0$, then it also solves the Cauchy problem for the equation

$$v_t - \Delta v - \frac{c_0}{|x|^2} v = \frac{c - c_0}{|x|^2} + f.$$

Hence there holds inequality (1.9), which now reads

$$\int_0^{T-\varepsilon} \int_{|x|<R} \left\{ \frac{c - c_0}{|x|^2} v + f(x,t) \right\} |x|^{(2-n)/2}\, dx\, dt < \infty$$

(observe that $\lambda_+/2 = 2 - n$ for $c = c_0$). However, condition (1.7) then implies

$$\int_0^{T-\varepsilon} \int_{|x|<R} \frac{v}{|x|^2} |x|^{(2-n)/2}\, dx\, dt \geq (T - 2\varepsilon) \int_0^R r^{-1}\, dr = \infty,$$

a contradiction.

(ii) In the light of the above remarks, nonexistence results can be expected also for the semilinear parabolic equation

$$u_t - \Delta u - \frac{c}{|x|^2} u = u^{\nu} \tag{1.10}$$

which appears in (1.1). It is obviously so for $c > c_0$. However, nonexistence can be expected in this case even if $c \leq c_0$, provided that the exponent ν is "too large". In fact, this happens for the semilinear elliptic equation associated with (1.1), namely

$$-\Delta v - \frac{c}{|x|^2} v = v^\nu. \tag{1.11}$$

As proven in [4], the value

$$\nu_+ \equiv \nu_+(c) := 1 + \frac{4}{|\lambda_+|}$$

is a dividing line with respect to existence or nonexistence of nonnegative solutions to equation (1.11) (see [4, Theorems 1.1 and 1.2]; observe that $\lambda_+ \to 0$, thus $\nu_+ \to \infty$ as $c \to 0^+$). Concerning nonexistence, we prove below that a similar situation holds for problem (1.1) (see Subsection 3.1 and Section 4, in particular Theorem 3.9, Theorems 4.2 and 4.3).

This nonexistence result is again made plausible by a heuristic argument, if we consider radial stationary supersolutions. In fact, let $v = k|x|^\alpha$ $(k > 0)$ be a supersolution of (1.10). Then we have

$$-\{\alpha^2 + (n-2)\alpha + 4c\}k|x|^{\alpha-2} \geq k^\nu |x|^{\alpha\nu},$$

which requires

$$\alpha - 2 \leq \alpha\nu \quad \Leftrightarrow \quad \alpha \geq -\frac{2}{\nu - 1}.$$

Hence the compatibility condition

$$\frac{\lambda_+}{2} \geq -\frac{2}{\nu - 1} \quad \Leftrightarrow \quad \nu \leq \nu_+$$

(see (1.8)) arises as a necessary condition for existence.

Let us observe that inequalities (1.7) and (1.9) with $f = u^\nu$ give the necessary condition for existence

$$\nu < -1 + \frac{2n}{|\lambda_+|},$$

as is easily checked. However, our nonexistence results improve on this condition, since $\nu_+ < -1 + \frac{2n}{|\lambda_+|}$.

(iii) Beside nonexistence, we also address nonuniqueness of nonnegative solutions of problem (1.1) (see Subsection 3.2). Let us recall that the initial-boundary value problem

$$\begin{cases} v_t = \Delta v + v^{n/(n-2)} & \text{in } B \times (0, T] \\ v = 0 & \text{on } \partial B \times (0, T] \\ v = v_0 & \text{in } B \times \{0\}, \end{cases} \tag{1.12}$$

(B denoting the unit ball in $\mathbb{R}^n$) is known to have more than one solution for infinitely many $v_0 \in L^{n/(n-2)}(B)$ (see [16]). A related nonuniqueness result holds for the linear problem

$$\begin{cases} v_t = \Delta v + \frac{c}{|x|^2} v & \text{in } B \times (0,T] \\ v = 0 & \text{on } \partial B \times (0,T] \\ v = v_0 & \text{in } B \times \{0\} \end{cases} \tag{1.13}$$

(see [12, Theorem 4.2], [21]).

Theorem 3.13 below generalizes this nonuniqueness result to the case $c \neq 0$, pointing out the role of the exponent $\frac{2n}{|\lambda_-|}$ (see Remark 3.15). Here

$$\lambda_- \equiv \lambda_-(c) := 2 - n - 2\sqrt{c_0 - c} \tag{1.14}$$

denotes the smallest root of equation (1.6). Observe that for any $c \in (0, c_0)$ there holds $\lambda_- < \lambda_+ < 0$ and

$$\lambda_+ \in (2-n, 0), \quad \lambda_- \in (2(2-n), 2-n); \tag{1.15}$$

moreover,

$$\lim_{c \to 0^+} \frac{2n}{|\lambda_-(c)|} = \frac{n}{n-2}.$$

To investigate existence and uniqueness we make use of the results proved in [22], regarding problem (1.1) as an abstract Cauchy problem (see (3.1)). To this purpose we need estimates of the semigroup generated by a realization of the operator $\Delta + \frac{c}{|x|^2}$ in some suitable Lebesgue space. Such estimates, which are of independent interest, are proved in Section 2. Let us mention that they cannot be derived from the heat kernel estimates proved in [13].

Nonexistence is studied by two different methods. When $c \in (0, c_0)$ and $v > v_+$, we adapt a method used in [20] for the elliptic case. On the other hand, in the limiting cases $c \in (0, c_0]$, $v > v_+$, or $c \in (0, c_0)$, $v = v_+$ we generalize some results in [18]. It should be noted that the notion of solution used in [18] was stronger, whereas our concept of solution is the weakest possible (see Definition 4.1).

2. Semigroup estimates

We make use of the Lebesgue spaces $L^p(\Omega) \equiv L^p(\Omega, dx)$, $L^p_\lambda(\Omega) \equiv L^p(\Omega, |x|^\lambda\, dx)$ ($p \in [1, \infty]$, $\lambda \in (2-n, 0)$); their norms will be denoted by $\|\cdot\|_{p,\lambda}$ and $\|\cdot\|_p$, respectively.

It is known that the change of unknown $u \to v := \frac{u}{|x|^{\lambda_+/2}}$ formally recasts the equation

$$u_t - \Delta u - \frac{c}{|x|^2} u = 0 \quad \text{in } \Omega \times (0,T) \tag{2.1}$$

into the equation

$$v_t = \frac{1}{|x|^{\lambda}}\, div(|x|^{\lambda}\nabla v) \quad \text{in } \Omega \times (0,T) \tag{2.2}$$

with $\lambda = \lambda_+$. Equation (2.2) is the heat equation for the weighted Laplacian

$$\Delta_\lambda \equiv \frac{1}{|x|^{\lambda}}\, div(|x|^{\lambda}\nabla\, \cdot)$$

on the weighted manifold $(\Omega, |x|^{\lambda}\, dx)$. This operator is properly defined in the weighted space $L^2_\lambda(\Omega)$, $\lambda \in (2-n, 0)$ as follows. Denote by $H^1_{0,\lambda}(\Omega)$ the closure of $C_0^\infty(\Omega\backslash\{0\})$ in the norm

$$v \to \|v\|_{H^1_\lambda} := \left\{ \int_\Omega (|\nabla v|^2 + v^2)|x|^{\lambda}\, dx \right\}^{1/2} \tag{2.3}$$

(since $\lambda > 2-n$, this is also the closure of $C_0^\infty(\Omega)$ in the same norm). Then the weighted Laplacian Δ_λ (complemented with Dirichlet homogeneous boundary conditions) is defined in $L^2_\lambda(\Omega)$ as the opposite of the generator of the symmetric form in $L^2_\lambda(\Omega)$:

$$\mathscr{H}_\lambda[v_1, v_2] := \int_\Omega \nabla v_1 \nabla v_2 |x|^{\lambda}\, dx \tag{2.4}$$

with domain $D(\mathscr{H}_\lambda) := H^1_{0,\lambda}(\Omega)$. Therefore,

$$\begin{cases} D(\Delta_\lambda) := \left\{ v \in H^1_{0,\lambda}(\Omega) \mid \frac{1}{|x|^{\lambda}}\, div(|x|^{\lambda}\nabla v) \in L^2_\lambda(\Omega) \right\} \\ \Delta_\lambda v := \frac{1}{|x|^{\lambda}}\, div(|x|^{\lambda}\nabla v) \quad \text{for any } v \in D(\Delta_\lambda). \end{cases} \tag{2.5}$$

Here use is made of the equality

$$\int_\Omega \nabla v_1 \nabla v_2 |x|^{\lambda}\, dx = -\int_\Omega \frac{1}{|x|^{\lambda}}\, div(|x|^{\lambda}\nabla v_2) v_1 |x|^{\lambda}\, dx,$$

which holds for any $v_1 \in H^1_{0,\lambda}(\Omega)$, $v_2 \in D(\Delta_\lambda)$, and of general characterization results (*e.g.*, see [11, Theorem VI.2.1]). We shall denote by $\{e^{\Delta_\lambda t}\}_{t\geq 0}$ the contraction holomorphic semigroup generated by Δ_λ in $L^2_\lambda(\Omega)$.

The Schrödinger operator $H = -\Delta - \frac{c}{|x|^2}$, $c \in (0, c_0)$ (with Dirichlet homogeneous boundary conditions) is similarly defined in $L^2(\Omega)$ as the generator of the symmetric form

$$\mathscr{H}[u_1, u_2] := \int_\Omega \left(\nabla u_1 \nabla u_2 - \frac{c}{|x|^2} u_1 u_2 \right) dx$$

with domain $D(\mathscr{H}) := H_0^1(\Omega)$, namely

$$\text{(2.6)}\qquad \begin{cases} D(H) := \left\{ u \in H_0^1(\Omega) \mid \Delta u + \frac{c}{|x|^2} u \in L^2(\Omega) \right\} \\ Hu := -\Delta u - \frac{c}{|x|^2} u \quad \text{for any } u \in D(H). \end{cases}$$

In view of the Hardy inequality, the form $\mathscr{H}$ is nonnegative and $C_0^\infty(\Omega \setminus \{0\})$ is a core for it. The operator H is nonnegative and self-adjoint, so that $-H$ is the generator of a contraction holomorphic semigroup $\{e^{-Ht}\}_{t \geq 0}$ on $L^2(\Omega)$.

The relationship between equations (2.1), (2.2) can now be made rigorous. Consider the unitary map

$$\text{(2.7)}\qquad \Phi : L^2_{\lambda_+}(\Omega) \to L^2(\Omega), \qquad (\Phi v)(x) := |x|^{\lambda_+/2} v(x) \qquad (v \in L^2_{\lambda_+}(\Omega)).$$

Define the nonnegative, self-adjoint operator

$$H_{\lambda_+} := \Phi^* H \Phi$$

in $L^2_{\lambda_+}(\Omega)$. Clearly,

$$e^{-H_{\lambda_+} t} = \Phi^* e^{-Ht} \Phi \quad \text{for any } t \geq 0,$$

$\{e^{-H_{\lambda_+} t}\}_{t \geq 0}$ denoting the semigroup on $L^2_\varphi(\Omega)$ generated by $-H_{\lambda_+}$. It is easily checked that $H_{\lambda_+} = -\Delta_{\lambda_+}$ (*e.g.*, see [15]). Then the above equalities read

$$\text{(2.8)}\qquad \Delta_{\lambda_+} = \Phi^* \left(\Delta + \frac{c}{|x|^2} \right) \Phi,$$

$$\text{(2.9)}\qquad e^{\Delta_{\lambda_+} t} = \Phi^* e^{-Ht} \Phi \quad \text{for any } t \geq 0.$$

The link (2.9) between the semigroups $\{e^{\Delta_\lambda t}\}_{t \geq 0}$ and $\{e^{-Ht}\}_{t \geq 0}$ will be used in the following. In the next two propositions we recall some properties of these semigroups (see [7], [15], [21]), concerning in particular their extensions to different Lebesgue spaces.

Proposition 2.1. *For any $\lambda \in (2-n, 0)$ the semigroup $\{e^{\Delta_\lambda t}\}_{t \geq 0}$ on $L^2_\lambda(\Omega)$:*

(i) *is positivity preserving;*
(ii) *can be extended to L^p_λ for any $p \in [1, \infty]$. Moreover, there exists $\omega \in \mathbb{R}$ such that*

$$\text{(2.10)}\qquad \|e^{\Delta_\lambda t} \phi\|_{p,\lambda} \leq e^{\omega t} \|\phi\|_{p,\lambda} \quad (p \in [1, \infty], t \geq 0);$$

(iii) *is ultracontractive. In fact, there exists $C_1 > 0$ such that*

$$\text{(2.11)}\qquad \|e^{\Delta_\lambda t} \phi\|_{\infty,\lambda} \leq C_1 t^{-n/2} e^{\kappa t} \|\phi\|_{1,\lambda}$$

for any $\phi \in L^1_\lambda$, $t > 0$, $\kappa > \max\{\omega, 0\}$.

PROPOSITION 2.2. *Let $c \in (0, c_0)$. Then the semigroup $\{e^{-Ht}\}_{t\geq 0}$ in $L^2(\Omega)$ can be extended to a contraction semigroup in $L^p(\Omega)$, for any $p \in \left[\frac{2(n-2)}{|\lambda_-|}, \infty\right)$.*

PROOF. Observe that

$$p \geq \frac{2(n-2)}{|\lambda_-|} \quad \Leftrightarrow \quad \frac{c}{(n-2)^2} \leq \frac{1}{p} - \frac{1}{p^2} \qquad (c \in [0, c_0)), \tag{2.12}$$

Then the claim follows from [21, Proposition 11.1] and subsequent Remarks. □

For the above extension we will use the same notation $\{e^{-Ht}\}_{t\geq 0}$ already used in $L^2(\Omega)$.

Let us prove the following result.

PROPOSITION 2.3. *Let $\lambda \in (2-n, 0)$. Then there exists a positive function $K_\lambda(x, y, t) = K_\lambda(y, x, t) (x, y \in \Omega; t > 0)$, $K_\lambda(\cdot, \cdot, t) \in L^\infty(\Omega \times \Omega)$ for any $t > 0$, such that*

$$(e^{\Delta_\lambda t}\phi)(x) = \int_\Omega K_\lambda(x, y, t)\phi(y)|y|^\lambda \, dy \tag{2.13}$$

for any $t > 0$, for almost every $x \in \Omega$ and any $\phi \in L^p_\lambda(\Omega)$, $p \in [1, \infty)$.

PROOF. If $p \geq 2$, equality (2.13) follows from Proposition 2.1-(iii) by the Dunford–Pettis theorem. Let $p \in [1, 2)$; take a sequence $\{\phi_m\} \subseteq L^2_\lambda(\Omega)$ such that $\phi_m \to \phi$ in $L^p_\lambda(\Omega)$. We have

$$\begin{aligned}
&\left\| e^{\Delta_\lambda t}\phi - \int_\Omega K_\lambda(\cdot, y, t)\phi(y)|y|^\lambda \, dy \right\|_{p,\lambda} \\
&\quad \leq \|e^{\Delta_\lambda t}\phi - e^{\Delta_\lambda t}\phi_m\|_{p,\lambda} + \left\| e^{\Delta_\lambda t}\phi_m - \int_\Omega K_\lambda(\cdot, y, t)\phi(y)|y|^\lambda \, dy \right\|_{p,\lambda} \\
&\quad \leq e^{\omega T}\|\phi - \phi_m\|_{p,\lambda} + \left\| \int_\Omega K_\lambda(\cdot, y, t)[\phi_m(y) - \phi(y)]|y|^\lambda \, dy \right\|_{p,\lambda} \\
&\quad \leq e^{\omega T}\|\phi - \phi_m\|_{p,\lambda} + \Big(\sup_{(x,y)\in\Omega\times\Omega} K_\lambda(x, y, t) \Big) \Big(\int_\Omega |x|^\lambda \, dx \Big)^{1/p} \|\phi - \phi_m\|_{p,\lambda}.
\end{aligned}$$

Letting $m \to \infty$ in the above inequality, the conclusion follows. □

Now we address the following

PROPOSITION 2.4. *Let $\lambda \in (2-n, 0)$, $1 < p \leq q \leq \infty$. Then for any $T > 0$ there exists $C_2 > 0$ such that*

$$\|e^{\Delta_\lambda t}\phi\|_{q,\lambda} \leq C_2 t^{-(n/2)(1/p-1/q)} \|\phi\|_{p,\lambda} \quad (t \in (0, T)) \tag{2.14}$$

for any $\phi \in L^p_\lambda(\Omega)$.

PROOF. Using the interpolation inequality, inequality (2.10) with $p = 1$ and (2.11), we obtain for any $q \in [1, \infty]$

$$\|e^{\Delta_\lambda t}\phi\|_{q,\lambda} \le \|e^{\Delta_\lambda t}\phi\|_{1,\lambda}^{1/q}\|e^{\Delta_\lambda t}\phi\|_{\infty,\lambda}^{1-1/q} \le C_1^{1-1/q}e^{\kappa T}t^{-(n/2)(1-1/q)}\|\phi\|_{1,\lambda}.$$

By the above inequality, inequality (2.10) with $p = q$ and the Riesz-Thorin theorem, we obtain (2.14) with $C_2 := C_1^{1/p-1/q}e^{\omega T}$. □

Define

$$\beta \equiv \beta(c,p) := \begin{cases} \frac{2(n-2)+p\lambda_+}{2(n-2+\lambda_+)} & \text{if } p \in \left[\frac{2(n-2)}{|\lambda_-|}, 2\right] \\ 1 & \text{if } p \in (2,\infty). \end{cases} \tag{2.15}$$

LEMMA 2.5. *Let $p \in \left[\frac{2(n-2)}{|\lambda_-|}, \infty\right)$. If $\phi \in L^p(\Omega)$, then $\psi := |x|^{-\lambda_+/2}\phi \in L^{p/\beta}_{\lambda_+}(\Omega)$.*

PROOF. Observe that $\beta \ge 1$; moreover, $p \ge \frac{2(n-2)}{|\lambda_-|} \Rightarrow \frac{p}{\beta} \ge 1$. We have

$$\|\psi\|_{p/\beta,\lambda_+} = \left\{\int_\Omega |\psi(x)|^{p/\beta}|x|^{\lambda_+}\,dx\right\}^{\beta/p} = \left\{\int_\Omega |\phi(x)|^{p/\beta}|x|^{\lambda_+(1-p/2\beta)}\,dx\right\}^{\beta/p}.$$

If $p \ge 2$, the conclusion follows from the above inequality. Otherwise, by the Hölder inequality

$$\left\{\int_\Omega |\phi(x)|^{p/\beta}|x|^{\lambda_+(1-p/2\beta)}\,dx\right\}^{\beta/p} \le \left\{\int_\Omega |\phi(x)|^p\,dx\right\}^{1/p}\left\{\int_\Omega |x|^{(\lambda_+/2)((\beta-p)/(\beta-1))}\,dx\right\}^{(\beta-1)/p} =: C_3\|\phi\|_p.$$

Hence the result follows. □

A representation of the semigroup $\{e^{-Ht}\}_{t\ge0}$, similar to (2.13) for the semigroup $\{e^{\Delta_\lambda t}\}_{t\ge0}$, is the content of the following

PROPOSITION 2.6. *Let $c \in (0, c_0)$. Define*

$$K(x,y,t) := (|x|\,|y|)^{\lambda_+/2}K_{\lambda_+}(x,y,t) \quad (x,y \in \Omega; t > 0), \tag{2.16}$$

where K_{λ_+} denotes the heat kernel in (2.9) *with $\lambda = \lambda_+$. Then*

$$(e^{-Ht}\phi)(x) = \int_\Omega K(x,y,t)\phi(y)\,dy \tag{2.17}$$

for any $t > 0$, for almost every $x \in \Omega$ and any $\phi \in L^p(\Omega)$, $p \in \left[\frac{2(n-2)}{|\lambda_-|}, \infty\right)$.

To prove Proposition 2.6 we need a preliminary result. Set

$$\gamma \equiv \gamma(c,q) := \begin{cases} 1 & \text{if } q \in [1,2] \\ \frac{2(n-2)+q\lambda_+}{2(n-2+\lambda_+)} & \text{if } q \in \left(2, \frac{2(n-2)}{|\lambda_+|}\right), \end{cases} \tag{2.18}$$

$$a(c,p,q) := \frac{n}{2}\left(\frac{\beta}{p} - \frac{\gamma}{q}\right) \tag{2.19}$$

(when unimportant, we shall disregard the dependence of a on c and write $a(p,q) \equiv a(c,p,q)$).

Remark 2.7. It is easily checked that

(a) the map $a(\cdot,p,q)$ is nondecreasing;
(b) the map $a(c,\cdot,q)$ is decreasing;
(c) the map $a(c,p,\cdot)$ is increasing.

Lemma 2.8. *Let*

$$\frac{2(n-2)}{|\lambda_-|} < p \le q < \frac{2(n-2)}{|\lambda_+|}. \tag{2.20}$$

Then for any $T > 0$ there exists a constant $C_4 > 0$ such that

$$\left\| \int_\Omega K(\cdot,y,t)\phi(y)\,dy \right\|_q \le C_4 t^{-a(p,q)} \|\phi\|_p \quad (t \in (0,T)) \tag{2.21}$$

for any $\phi \in L^p(\Omega)$.

Proof. Observe that:

(i) $\gamma \le 1$, and $\frac{2(n-2)}{|\lambda_-|} < q < \frac{2(n-2)}{|\lambda_+|} \Rightarrow \frac{q}{\gamma} \ge 1$;
(ii) by Lemma 2.5 $\phi \in L^p(\Omega) \Rightarrow \psi := |x|^{-\lambda_+/2}\phi \in L^{p/\beta}_{\lambda_+}(\Omega)$.

Then by equality (2.16) and Lemma 2.3 we have

$$\begin{aligned} \int_\Omega K(x,y,t)\phi(y)\,dy &= |x|^{\lambda_+/2} \int_\Omega |y|^{\lambda_+/2} K_{\lambda_+}(x,y,t)\phi(y)\,dy \\ &= |x|^{\lambda_+/2}(e^{\Delta_{\lambda_+}t}\psi)(x), \end{aligned} \tag{2.22}$$

whence

$$\begin{aligned} \left\| \int_\Omega K(\cdot,y,t)\phi(y)\,dy \right\|_q &= \left\{ \int_\Omega |(e^{\Delta_{\lambda_+}t}\psi)(x)|^q |x|^{q(\lambda_+/2)}\,dx \right\}^{1/q} \\ &= \left\{ \int_\Omega |(e^{\Delta_{\lambda_+}t}\psi)(x)|^q |x|^{(\lambda_+/2)(q-2)} |x|^{\lambda_+}\,dx \right\}^{1/q} \le M_1 \|e^{\Delta_{\lambda_+}t}\psi\|_{q/\gamma,\lambda_+}, \end{aligned} \tag{2.23}$$

where

$$M_1 := \left\{ \max_{x \in \bar{\Omega}} |x|^{(\lambda_+/2)(q-2)} \right\}^{1/q} \quad \text{if } q \in [1, 2],$$

or

$$M_1 := \left\{ \int_\Omega \left(\frac{\lambda_+}{2} q - \lambda_+ \right) s |x|^{\lambda_+} |x|^{2-n} \, dx \right\}^{(1-\gamma)/q} \quad \text{if } q \in \left(2, \frac{2(n-2)}{|\lambda_+|}\right).$$

Since by assumption $q < \frac{2(n-2)}{|\lambda_+|}$, it is easily seen that $\frac{q}{\gamma} \geq \frac{p}{\beta}$. Hence by inequality (2.14) and Lemma 2.5 we obtain

$$\|e^{\Delta_{\lambda_+} t} \psi\|_{q/\gamma, \lambda_+} \leq C_2 t^{-(n/2)(\beta/p - \gamma/q)} \|\psi\|_{p/\beta, \lambda_+} \leq C_2 C_3 t^{-(n/2)(\beta/p - \gamma/q)} \|\phi\|_p. \tag{2.24}$$

From (2.23) and (2.24) the conclusion follows with $C_4 := C_2 C_3 M_1$. □

Now we can prove Proposition 2.6.

PROOF OF PROPOSITION 2.6. If $p \in [2, \infty)$, equality (2.17) follows from (2.7), (2.9) and (2.13). If $p \in \left[\frac{2(n-2)}{|\lambda_-|}, 2\right)$, let $\phi \in L^p(\Omega)$, $\{\phi_m\} \subseteq L^2(\Omega)$ such that $\phi_m \to \phi$ as $m \to \infty$ in $L^p(\Omega)$. Then for any $t > 0$

$$\begin{aligned} &\left\| e^{-Ht}\phi - \int_\Omega K(x, y, t)\phi(y) \, dy \right\|_p \\ &\quad \leq \|e^{-Ht}\phi - e^{-Ht}\phi_m\|_p + \left\| e^{-Ht}\phi_m - \int_\Omega K(x, y, t)\phi(y) \, dy \right\|_p \\ &\quad \leq \|\phi - \phi_m\|_p + \left\| \int_\Omega K(x, y, t)(\phi_m(y) - \phi(y)) \, dy \right\|_p \\ &\quad \leq \{1 + C_4 t^{-a(p,q)}\} \|\phi - \phi_m\|_p; \end{aligned}$$

here use of Proposition 2.2 and of (2.21) has been made (observe that $\frac{2(n-2)}{|\lambda_+|} > 2$ for any $c \in (0, c_0)$). Letting $m \to \infty$ in the above inequality the conclusion follows. □

The next proposition is an immediate consequence of Proposition 2.6 and Lemma 2.8.

PROPOSITION 2.9. *Let inequality* (2.20) *be satisfied. Then for any* $T > 0$ *there exists a constant* $C_4 > 0$ *such that*

$$\|e^{-Ht}\phi\|_q \leq C_4 t^{-a(p,q)} \|\phi\|_p \quad (t \in (0, T)) \tag{2.25}$$

for any $\phi \in L^p(\Omega)$, *with* $a(p, q)$ *defined by* (2.19).

3. Existence and nonuniqueness results

3.1. Existence

To address problem (1.1) we shall think of it as an abstract Cauchy problem, namely

$$(3.1)\qquad \begin{cases} u' + Hu = u^{\nu} & \text{in } (0,T) \\ u(0) = u_0, \end{cases}$$

where H is the operator introduced in Section 2 (in particular, see definition (2.6), Proposition 2.2 and following remarks).

Let us make the following

Definition 3.1. *Let* $u_0 \in L^p(\Omega)$ *with* $p \in \left[\frac{2(n-2)}{|\lambda_-|}, \infty\right)$, $u_0 \geq 0$, $\nu > 1$. *By a* mild solution *to problem* (1.1) *in* $L^p(\Omega)$ *we mean any nonnegative function* $u \in C([0,T); L^p(\Omega)) \cap C((0,T); L^{p\nu}(\Omega)) \cap L^1((0,T); L^{p\nu}(\Omega))$ *such that*

$$u(t) = e^{-Ht}u_0 + \int_0^t e^{-H(t-s)}u^{\nu}(s)\,ds \quad \textit{for any } t \in [0,T).$$

Define

$$(3.2)\qquad b(c,p,q) := \frac{p}{q-p}[1 - a(c,p,q)],$$

where $a(p,q)$ is defined by (2.19). As for $a(p,q)$, we usually write $b(p,q)$ instead of $b(c,p,q)$. Concerning existence of solutions to problem (1.1), let us prove the following preliminary result.

Proposition 3.2. *Let* $c \in (0, c_0)$, $\nu > 1$. *Let the following assumptions be satisfied:*

$$(3.3)\qquad \frac{2(n-2)}{|\lambda_-|} \leq p < p\nu < \frac{2(n-2)}{|\lambda_+|},$$

$$(3.4)\qquad 0 < a(p, p\nu) < 1,$$

$$(3.5)\qquad 0 < b(p, p\nu) < a(p, p\nu).$$

Moreover, let there exist $\bar{p} \in (p, p\nu)$ *such that*

$$(3.6)\qquad a(\bar{p}, p\nu) = b(p, p\nu).$$

Then for any $u_0 \in L^{\bar{p}}(\Omega)$, $u_0 \geq 0$ *there exists a unique mild solution of problem* (1.1) *in* $L^p(\Omega)$.

Proof. Let us show that under the present assumptions the existence results in [22] (in particular, [22, Theorem 2]) apply to the abstract Cauchy problem (3.1). Consider the map

$$J := L^{pv}(\Omega) \to L^{p}(\Omega), \quad u \to J(u) := u^{v}.$$

For any $\phi, \psi \in L^{pv}(\Omega)$ such that $\|\phi\|_{pv} \le r$, $\|\psi\|_{pv} \le r$ $(r > 0)$, there holds

$$\|J(\phi) - J(\psi)\|_{p} \le l(r)\|\phi - \psi\|_{pv}$$

with $l : \mathbb{R}_{+} \to \mathbb{R}_{+}$ such that

$$l(r) = O(|x|^{v-1}) = O(|x|^{(1-a(p,pv))/b(p,pv)}) \quad \text{as } r \to \infty \tag{3.7}$$

(see (3.2)). On the other hand,

- since $\bar{p} \in (p, pv)$ and (3.3) holds, by Proposition 2.9 we obtain

$$\|e^{-Ht}u_0\|_{pv} \le C_4 t^{-a(p,pv)}\|u_0\|_{p} \tag{3.8}$$

  for any $u_0 \in L^{\bar{p}}(\Omega)$;
- there holds

$$\limsup_{t \to 0^+} t^{b(p,pv)}\|e^{-Ht}u_0\|_{pv} = 0. \tag{3.9}$$

  In fact, let $\{u_{0,m}\}$ be any sequence in $L^{pv}(\Omega)$ such that $u_{0,m} \to u_0$ in $L^{\bar{p}}(\Omega)$ as $m \to \infty$. Since $pv > \frac{2(n-2)}{|\lambda_-|}$ (see (3.3)), $\{e^{-Ht}\}_{t \ge 0}$ is a continuous semigroup of contractions in $L^{pv}(\Omega)$. Using this fact and inequality (2.25) with $p = \bar{p}$, $q = pv$, we obtain

$$\begin{aligned}
\limsup_{t \to 0^+} t^{b(p,pv)}\|e^{-Ht}u_0\|_{pv} &\le \limsup_{t \to 0^+} t^{b(p,pv)}\{\|e^{-Ht}(u_0 - u_{0,m})\|_{pv} + \|e^{-Ht}u_{0,m}\|_{pv}\} \\
&\le C_4 \limsup_{t \to 0^+} t^{b(p,pv)}\{t^{-a(\bar{p},pv)}\|u_0 - u_{0,m}\|_{\bar{p}} + \|u_{0,m}\|_{pv}\} \\
&= C_4\|u_0 - u_{0,m}\|_{pv};
\end{aligned}$$

  here use of (3.6) and of the left inequality in (3.5) has been made. Sending $m \to \infty$ we have (3.9).

In view of (3.7)–(3.9), we can apply Theorem 2 in [22]. Hence the conclusion follows. □

Remark 3.3. As $c \to 0^+$, the limiting form of conditions (3.3)–(3.5) is $1 \le p < pv < \infty$ and

$$\frac{n(v-1)}{2v} < p < \frac{n(v-1)}{v}. \tag{3.10}$$

As for equality (3.6), it is satisfied by $\bar{p} = \frac{n(\nu-1)}{\nu}$. Then in this limiting case Proposition 3.2 reduces to [22, Theorem 1], showing that for any $u_0 \in L^{n(\nu-1)/\nu}(\Omega)$, $u_0 \geq 0$ the problem

$$\begin{cases} u_t - \Delta u = u^\nu & \textit{in } Q_T \\ u = 0 & \textit{in } \partial\Omega \times (0, T] \\ u = u_0 & \textit{in } \Omega \times \{0\}. \end{cases}$$

has a unique mild solution in $L^p(\Omega)$.

Let us consider for completeness the simpler case when condition (3.5) is not satisfied. In this case we have the following existence result.

Proposition 3.4. *Let $c \in (0, c_0)$, $\nu > 1$. Let inequality* (3.3) *be satisfied, and*

$$0 < a(p, p\nu) < \frac{1}{\nu} < b(p, p\nu). \tag{3.11}$$

Then for any $u_0 \in L^p(\Omega)$, $u_0 \geq 0$ there exists a unique mild solution of problem (1.1) *in $L^p(\Omega)$.*

Proof. Inequality (3.11) implies that

$$\int_1^\infty r^{-1/a(p,p\nu)+\nu-1}\, dr < \infty.$$

Then the conclusion follows from [22, Theorem 2-(a)]. □

Propositions 3.2 and 3.4 can be directly applied for any given $p \in \left[\frac{2(n-2)}{|\lambda_-|}, \infty\right)$. This is made below for $p = 2$ (see Theorem 3.5). Since the semigroup $\{e^{-Ht}\}_{t\geq 0}$ is holomorphic in $L^2(\Omega)$, in this case the mild solution is *classical, i.e*, $u \in C^1((0, T); L^2(\Omega)) \cap C((0, T); D(H))$ ($D(H)$ being defined in (2.6)) and system (3.1) is satisfied.

Theorem 3.5. (i) *Let*

$$1 < \nu < 1 + \frac{4(n - 2 + \lambda_+)}{n(n-2)}. \tag{3.12}$$

Then for any $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ there exists a unique classical solution of problem (1.1) *in $L^2(\Omega)$.*

(ii) *Let*

$$\frac{n(n-2)}{4(n-2+\lambda_+)}\left(1 - \frac{1}{\nu}\right) < 1, \tag{3.13}$$

$$\nu > 1 + \frac{4(n-2+\lambda_+)}{n(n-2)}. \tag{3.14}$$

If $n = 3$, assume also

$$v < 1 + \frac{4(n-2+\lambda_+)}{n|\lambda_+|}. \tag{3.15}$$

Let

$$\bar{p} := \frac{n}{2}\left[\frac{1}{v-1} - \frac{n|\lambda_+|}{4(n-2+\lambda_+)}\right]^{-1}. \tag{3.16}$$

Then for any $u_0 \in L^{\bar{p}}(\Omega)$, $u_0 \geq 0$ there exists a unique classical solution of problem (1.1) *in $L^2(\Omega)$.*

Proof. It is easily seen that

$$a(2,2v) = \frac{n(n-2)}{4(n-2+\lambda_+)}\left(1-\frac{1}{v}\right), \quad b(2,2v) = \frac{1}{v-1} - \frac{n(n-2)}{4(n-2+\lambda_+)}\frac{1}{v}.$$

It is also easily checked that inequality (3.12) ensures condition (3.11) with $p = 2$. Therefore by Proposition 3.4 for any $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ there exists a unique mild solution of problem (1.1) in $L^2(\Omega)$.

Similarly, inequalities (3.13)–(3.14) (and (3.15) if $n = 3$) ensure conditions (3.4), (3.5) with $p = 2$. The value $\bar{p}$ defined by (3.16) is the unique solution of (3.6) with $p = 2$. Hence by Proposition 3.2 for any $u_0 \in L^{\bar{p}}(\Omega)$, $u_0 \geq 0$ there exists a unique mild solution of (1.1) in $L^2(\Omega)$.

It remains to prove that in any event the above solution is classical (recall that $\bar{p} > 2$, thus $L^{\bar{p}}(\Omega) \subseteq L^2(\Omega)$). To this purpose, observe that by [22, Theorem 2]:

(a) in case (i) there exists $k_1 > 0$ such that

$$\|u^v(t)\|_2 \leq k_1 t^{-av} \quad \text{for any } t \in (0,\tau) \quad (\tau > 0);$$

(b) in case (ii) there exists $k_2 > 0$ such that

$$\|u^v(t)\|_2 \leq k t^{-bv} \quad \text{for any } t \in (0,T).$$

Therefore in case (i) the function $t \to \|u^v(t)\|_2$ is integrable with exponent $\frac{\rho}{av} > 1$ for any $\rho \in (av, 1)$ (observe that $av < 1$ by assumption (3.12)). Similarly, in case (ii) the same function is integrable with exponent $\frac{\rho}{bv} > 1$ for any $\rho \in (bv, 1)$ (in fact, it is easily checked that the assumption (3.14) ensures $bv < 1$). As a consequence, since the semigroup $\{e^{-Ht}\}_{t\geq 0}$ is holomorphic in $L^2(\Omega)$, by [17, Theorem 4.3.1] the function $u^v : (0,T) \to L^2(\Omega)$ is Hölder continuous in $[\varepsilon, T)$ for any $\varepsilon > 0$ (with exponent $\frac{av}{\rho}$ in case (i), $\frac{bv}{\rho}$ in case (ii)). Then the conclusion follows by [17, Corollary 4.3.3]. □

In the general case it is interesting to investigate which restriction on p, v derive from the compatibility of conditions (3.3)–(3.6) of Proposition 3.2 (a similar but simpler study can be made for conditions (3.3)–(3.11) of Proposition 3.4; we omit the details). This is the content of Theorem 3.9, whose proof requires the preliminary Lemmata 3.6, 3.7 and 3.8.

The conditions (3.3)–(3.6) depend on the quantities n, c, p, ν—or, alternatively, on n, c, p and $q := p\nu$. In the following we suppose $n \geq 3$ arbitrarily fixed, and investigate the compatibility of (3.3)–(3.6) as depending on p, q, regarding $c \in (0, c_0)$ as a parameter.

Set

$$I := \left[\frac{2(n-2)}{|\lambda_-|}, \frac{2(n-2)}{|\lambda_+|}\right], \quad D := \{(p,q) \in I \times I \,|\, p \leq q\}.$$

Observe that the interval I and the set D depend on n and c. In particular, the measure of I is a decreasing function of c and $I = (1, \infty)$ for $c = 0$, $I = \emptyset$ for $c = c_0$.

Concerning condition (3.4) we have the following

LEMMA 3.6. *Let $c \in (0, c_0)$. For any $(p,q) \in D$ there holds $a(c,p,q) > 0$. Moreover, there exists a nonempty subset $D_1 \subseteq D$, D_1 depending on n and c, such that $a(c,p,q) < 1$ for any $(p,q) \in D_1$. More precisely, there holds*

$$D_1 := \{(p,q) \in D \,|\, \hat{p}_-(c) < p < \hat{p}_+(c), p < q < \tau(c,p)\},$$

where $\hat{p}_\pm$ and τ are as follows.

(i) *If $n = 3, 4$, then $\hat{p}_\pm = \hat{p}_\pm(c) := \frac{2(n-2)}{|\lambda_\pm|}$ for any $c \in (0, c_0)$. In these cases there exists a unique root $\tilde{p} = \tilde{p}(c) \in I$ of the equation*

$$a\left(c, p, \frac{2(n-2)}{|\lambda_+|}\right) \equiv \frac{n}{2}\frac{\beta(p)}{p} = 1. \tag{3.17}$$

There holds $\tilde{p} = \frac{6}{4+\lambda_+}$ if $n = 3$, $\tilde{p} = 2$ if $n = 4$.

The function $\tau = \tau(c,p)$ is implicitly defined by the equation

$$a(c,p,q) = 1 \tag{3.18}$$

for any $p \in \left(\frac{2(n-2)}{|\lambda_-|}, \tilde{p}\right)$, and $\tau \equiv \frac{2(n-2)}{|\lambda_+|}$ for any $p \in \left(\tilde{p}, \frac{2(n-2)}{|\lambda_+|}\right)$.

(ii) *The same situation prevails if $n \geq 5$ and $c \in (0, \hat{c}]$ with*

$$\hat{c} := \frac{2(n-2)^3}{n^2} = \frac{8(n-2)}{n^2}c_0;$$

however, $\tilde{p} = \frac{n}{2}$ in this case.

If $n \geq 5$ and $c \in (\hat{c}, c_0)$, there holds

$$\hat{p}_\pm = \hat{p}_\pm(c) := \frac{2n|\lambda_+|}{n(n-2) - 2(n \pm 4)\sqrt{c_0 - c}},$$

and the function τ is defined by (3.18) in $(\hat{p}_-(c), \hat{p}_+(c))$.

In all cases the function τ is continuous, nondecreasing with respect to p (increasing when defined by (3.18)) *and nonincreasing with respect to c.*

Concerning condition (3.5) the following holds.

LEMMA 3.7. *Let $c \in (0, c_0)$. For any $(p,q) \in D_1$ there holds $b(c,p,q) > 0$. Moreover, there exists a nonempty subset $D_2 \subseteq D_1$, D_2 depending on c, such that $b(c,p,q) < a(c,p,q)$ for any $(p,q) \in D_2$. More precisely, there holds*

$$D_2 := \{(p,q) \in D_1 \mid \hat{p}_- < p < p_+^*, \rho(c,p) < q < \tau(c,p)\},$$

where $\hat{p}_-$ and the function τ are as in Lemma 3.6, *and p_+^* and the function ρ are as follows.*

(i) *If either $n = 3, 4$ and $c \in (0, c_0)$, or $n \geq 5$ and $c \in (0, \hat{c}]$ (with $\hat{c}$ defined in Lemma* 3.6), *p_+^* is the unique solution of the equation*

$$a\Big(c, p, \frac{2(n-2)}{|\lambda_+|}\Big) \equiv \frac{n}{2}\frac{\beta(p)}{p} = \frac{|\lambda_+|}{2(n-2)}p. \tag{3.19}$$

If $n = 3$, then $p_+^* \leq \frac{3}{4}$. If $n \geq 4$, then $p_+^* = \sqrt{\frac{n(n-2)}{|\lambda_+|}}$. In all cases there holds $\tilde{p} < p_+^*$, $\tilde{p}$ being defined in Lemma 3.6.

(ii) *The function $\rho = \rho(c,p)$ is implicitly defined by the equation*

$$a(c,p,q) = b(c,p,q) \quad (p \in (\hat{p}_-, p_+^*)). \tag{3.20}$$

It is continuous, increasing with respect to p and nonincreasing with respect to c.

As for condition (3.6), we have:

LEMMA 3.8. *Let $c \in (0, c_0)$. Then there exists a nonempty subset $D_3 \subseteq D_2$, D_3 depending on c, such that for any $(p,q) \in D_3$ there exists a unique $\bar{p} = \bar{p}(c) \in (p,q)$ satisfying*

$$a(c, \bar{p}, q) = b(c,p,q). \tag{3.21}$$

More precisely, there holds

$$D_3 := \{(p,q) \in D_2 \mid \rho(c,p) < q < \sigma(c,p)\},$$

where:

(i) *the function ρ is as in Lemma* 3.7;
(ii) *the function $\sigma = \sigma(c,p)$ is implicitly defined by the equation*

$$a(c,q,q) = b(c,p,q) \quad (p \in (\hat{p}_-, p_+^*)).$$

It is continuous, increasing with respect to p and nonincreasing with respect to c.

In view of the above lemmata, conditions (3.3)–(3.6) are satisfied for any couple $\nu > 1$, $p > 1$ such that $(p, p\nu) \in D_3$. Therefore we have the following existence result.

THEOREM 3.9. *Let $c \in (0, c_0)$. Let $\hat{p}_- = \hat{p}_-(c)$, $p_+^* = p_+^*(c)$ and the functions ρ, σ be defined as in Lemmata* 3.6–3.8. *Assume*

$$\hat{p}_- < p < p_+^*, \quad \frac{\rho(p)}{p} < \nu < \frac{\sigma(p)}{p}.$$

Then for any $u_0 \in L^{\bar{p}}(\Omega)$, $u_0 \geq 0$ (with $\bar{p} = \bar{p}(c, \nu)$ defined by equation (3.6)) *there exists a mild solution of problem* (1.1) *in $L^p(\Omega)$.*

REMARK 3.10. Observe that $\hat{p}_\pm(c) \to 2$ as $c \to c_0^-$, so that the assumptions of Theorem 3.9 cannot be satisfied for $c > c_0$. This is in agreement with the nonexistence result in [2, Theorem 2.2].

We complete this subsection by proving Lemmata 3.6–3.8.

PROOF OF LEMMA 3.6. (a) A simple calculation shows that

$$a(c, p, p) = \frac{n|\lambda_+|}{4(n - 2 + \lambda_+)} \left|1 - \frac{2}{p}\right| \quad (c \in (0, c_0), p \in I). \tag{3.22}$$

Since $p \leq q$ in D and $a(c, p, \cdot)$ is increasing (see Remark 2.7), there holds $a(c, p, q) > 0$ for any $(p, q) \in D$, $c \in (0, c_0)$.

(b) It is immediately seen that:

- there holds

$$a\Big(c, \frac{2(n-2)}{|\lambda_-|}, \frac{2(n-2)}{|\lambda_+|}\Big) = \frac{n}{2} > 1 \quad (c \in (0, c_0));$$

- for any fixed $c \in (0, c_0)$ the function $p \to a(c, p, p)$ takes its maximum for $p = \frac{2(n-2)}{|\lambda_\pm|}$ and there holds

$$\begin{aligned} a\Big(c, \frac{2(n-2)}{|\lambda_-|}, \frac{2(n-2)}{|\lambda_-|}\Big) &= a\Big(c, \frac{2(n-2)}{|\lambda_+|}, \frac{2(n-2)}{|\lambda_+|}\Big) \\ &= \frac{n|\lambda_+|}{4(n-2)} \quad (c \in (0, c_0)). \end{aligned} \tag{3.23}$$

It is also easy to check that the inequality

$$\frac{n|\lambda_+|}{4(n-2)} < 1 \tag{3.24}$$

is satisfied for any $c \in (0, c_0)$ if $n = 3, 4$, or for any $c \in (0, \hat{c})$ if $n \geq 5$ (observe that $\hat{c} < c_0$ in the latter case). It is easily seen that in these cases there exists a unique root $\tilde{p} \in I$ of equation (3.17), and

$$\tilde{p} = \begin{cases} \frac{6}{4+\lambda_+} & \text{if } n = 3, \\ 2 & \text{if } n = 4, \\ \frac{n}{2} & \text{if } n \geq 5. \end{cases}$$

Observe that $\tilde{p} < 2$ if $n = 3$, whereas for $n \geq 5$ there holds $\tilde{p} < \frac{2(n-2)}{|\lambda_+|}$ if and only if $c \in (0, \hat{c})$ (see (3.24)).

On the other hand, if $n \geq 5$ and $c \in [\hat{c}, c_0)$ there holds

$$a(c, p, p) < 1 \quad \Leftrightarrow \quad \hat{p}_-(c) < p < \hat{p}_+(c),$$

and

$$\hat{p}_\pm(\hat{c}) = \frac{2(n-2)}{|\lambda_\pm(\hat{c})|} = \frac{n}{n-2}.$$

Then the conclusion follows easily from Remark 2.7 by the Implicit Function Theorem. □

Proof of Lemma 3.7. Observe preliminarily that $b(c, p, q) > 0$ for any $(p, q) \in D_1$, since in this set $a(c, p, q) < 1$ by Lemma 3.6 ($c \in (0, c_0)$). Further observe that

$$b(c, p, q) - a(c, p, q) = -\frac{q}{q - p} A(c, p, q),$$

where

$$A(c, p, q) := a(c, p, q) - \frac{p}{q}.$$

Clearly,

$$b(c, p, q) < a(c, p, q) \quad \Leftrightarrow \quad A(c, p, q) > 0,$$

and the function A has the same monotonicity properties as a.

In view of Lemma 3.6-(iii), we have

$$A(c, p, \tau(c, p)) = a(c, p, \tau(c, p)) - \frac{p}{\tau(c, p)} = \frac{\tau(c, p) - p}{\tau(c, p)} > 0 \quad (c \in (0, c_0)).$$

Moreover, the function $p \to A(c, p, p) = a(c, p, p) - 1$ takes its maximum for $p = \frac{2(n-2)}{|\lambda_\pm|}$ and there holds

$$A\Big(c, \frac{2(n-2)}{|\lambda_-|}, \frac{2(n-2)}{|\lambda_-|}\Big) = A\Big(c, \frac{2(n-2)}{|\lambda_+|}, \frac{2(n-2)}{|\lambda_+|}\Big) = \frac{n|\lambda_+|}{4(n-2)} - 1 \quad (c \in (0, c_0)).$$

Hence the proof of Lemma 3.6 shows that:

- if $n = 3, 4$ and $c \in (0, c_0]$, or $n \geq 5$ and $c \in (0, \hat{c}]$,

$$A(c, p, p) < 0 \quad \text{for any } p \in I \quad (c \in (0, c_0));$$

- if $n \geq 5$ and $c \in (\hat{c}, c_0)$,

$$A(c, p, p) < 0 \quad \Leftrightarrow \quad \hat{p}_-(c) < p < \hat{p}_+(c).$$

In addition, equation (3.19) is equivalent to $A\big(c, p, \frac{2(n-2)}{|\lambda_+|}\big) = 0$. It is easily seen that it has a unique solution p_+^* with the asserted properties when $c \in (0, c_0)$ if $n = 3$, or when $c \in (0, \hat{c})$ if $n \geq 4$. In this connection, observe that if $n \geq 4$

$$p_+^* = \sqrt{\frac{n(n-2)}{|\lambda_+|}} < \frac{2(n-2)}{|\lambda_+|} \quad \Leftrightarrow \quad c < \hat{c}.$$

In all cases there holds $A\big(c, \tilde{p}, \frac{2(n-2)}{|\lambda_+|}\big) > 0$, thus $\tilde{p} < p_+^*$ by the monotonicity of $A(c, \cdot, q)$.

In the light of the above remarks, the conclusion follows from the monotonicity properties as A by the Implicit Function Theorem. □

Proof of Lemma 3.8. In view of Lemma 3.7, there holds $b(c, p, q) < a(c, p, q)$ for any $(p, q) \in D_2$. Therefore, since $a(c, \cdot, q)$ is decreasing, a solution $\bar{p} \in (p, q)$ of equation (3.21) exists if and only if

$$b(c, p, q) > a(c, q, q). \tag{3.25}$$

Let us show that the above inequality is satisfied for a suitable subset $D_3 \subseteq D_2$. Set

$$B(c, p, q) := b(c, p, q) - a(c, q, q) \quad ((p, q) \in D_2).$$

Observe that

$$D_2 = \{(p, q) \in D_1 \,|\, \tau^{-1}(c, q) < p < \rho^{-1}(c, q)\}.$$

When $p = \tau^{-1}(c, q)$ we have

$$B(c, \tau^{-1}(c, q), q) = b(c, \tau^{-1}(c, q), q) - a(c, q, q) = -a(c, q, q) < 0,$$

for $b(c, \tau^{-1}(c,q), q) = 0$ (see (3.18)). On the other hand,

$$\begin{aligned} B(c, \rho^{-1}(c,q), q) &= b(c, \rho^{-1}(c,q), q) - a(c,q,q) \\ &= a(c, \rho^{-1}(c,q), q) - a(c,q,q) > 0, \end{aligned}$$

since

$$b(c, \rho^{-1}(c,q), q) = a(c, \rho^{-1}(c,q), q)$$

(see (3.20)), $a(c, \cdot, q)$ is decreasing and $\rho^{-1}(c,q) < q$.

In view of the above remarks and of the increasing character of $B(c, \cdot, q)$, for any $q \in (\rho(\hat{p}_-), \rho(\hat{p}_+))$ there exists a unique $p = h(c,q) \in (\tau^{-1}(c,q), \rho^{-1}(c,q))$ such that

$$B(c, h(c,q), q) = 0, \qquad B(c,p,q) > 0 \quad \Leftrightarrow \quad p \in (h(c,q), \rho^{-1}(c,q)).$$

Then in the set

$$D_3 := \{(p,q) \in D_2 \,|\, q \in (\rho(\hat{p}_-), \rho(\hat{p}_+)), p \in (h(c,q), \rho^{-1}(c,q))\}$$

inequality (3.25) is satisfied. Since $D_3 \subseteq D_2$, there also holds $b(c,p,q) < a(c,p,q)$ for any $(p,q) \in D_3$. Therefore for any $(p,q) \in D_3$ there exists a unique $\bar{p} = \bar{p}(c) \in (p,q)$ satisfying (3.21), and the first statement holds true.

Let us prove that $h(c, \cdot)$ is increasing, and $h(\cdot, q)$ nondecreasing; then the remaining statements will follow defining $\sigma(c, \cdot) := h^{-1}(c, \cdot)$. Since by definition

$$B(c, h(c,q), q) = b(c, h(c,q), q) - a(c,q,q) = 0$$

and $B(c, \cdot, q)$ is increasing, the first claim follows from the Implicit Function Theorem, if we prove that $B(c,p,\cdot)$ is decreasing. This is immediate if $q > 2$, for $b(c,p,\cdot)$ is decreasing and $\frac{\partial}{\partial q} a(c,q,q) > 0$ in this case (see (3.22)). If $q < 2$, a direct calculation using (3.22) and the definition of $b(c,p,q)$ shows that

$$\frac{\partial}{\partial q} B(c,p,q) < -\frac{n}{q^2}.$$

Hence the first claim follows in this case, too. The last claim concerning the monotonicity of $h(\cdot, q)$ also follows from the Implicit Function Theorem and the fact that $B(c, \cdot, q)$ is increasing whereas $a(\cdot, p, q)$ is nondecreasing (see Remark 2.7), thus $B(\cdot, p, q)$ is nonincreasing. This completes the proof. □

3.2. Nonuniqueness

In the following we denote by $C^{\infty;1}_{x,t}(U)$ $(U \subseteq \mathbb{R}^{n+1})$ the space of functions of class C^∞ with respect to x and C^1 with respect to t.

Let us make the following definitions.

Definition 3.11. (i) *By a* weak solution *of the equation*

$$u_t - \Delta u - \frac{c}{|x|^2} u = u^\nu \quad in\ Q_T \tag{3.26}$$

we mean any nonnegative function $u \in L^\nu_{loc}(\Omega \times [0,T))$ *such that* $\frac{u}{|x|^2} \in L^1_{loc}(\Omega \times [0,T))$ *and there holds*

$$-\iint_{Q_\tau} u\left\{\zeta_t + \Delta\zeta + \frac{c}{|x|^2}\zeta\right\} dx\,dt = \iint_{Q_\tau} u^\nu \zeta\, dx\, dt \tag{3.27}$$

for any $\tau \in (0,T)$ *and any test function* $\zeta \in C^{\infty;1}_{x,t}(Q_\tau)$, $\zeta(\cdot,t) \in C^\infty_0(\Omega)$ *for any* $t \in [0,\tau]$, $\zeta(\cdot,\tau) = 0$.

(ii) *Let* $u_0 \in L^1_{loc}(\Omega)$. *By a* weak solution *to problem* (1.1) *we mean any nonnegative function* $u \in C([0,T]; L^1_{loc}(\Omega)) \cap L^\nu_{loc}(\Omega \times [0,T))$ *such that* $\frac{u}{|x|^2} \in L^1_{loc}(\Omega \times [0,T))$, *which is smooth out of the origin for any* $t \in (0,T]$, *vanishes on* $\partial\Omega \times (0,T]$ *and satisfies*

$$-\iint_{Q_\tau} u\left\{\zeta_t + \Delta\zeta + \frac{c}{|x|^2}\zeta\right\} dx\,dt = \int_\Omega u_0 \zeta(x,0)\,dx + \iint_{Q_\tau} u^\nu \zeta\, dx\, dt \tag{3.28}$$

for any $\tau \in (0,T)$ *and any test function* $\zeta \in C^{\infty;1}_{x,t}(\bar{Q}_\tau)$, $\zeta(\cdot,t) \in C^\infty_0(\Omega)$ *for any* $t \in [0,\tau]$, $\zeta(\cdot,\tau) = 0$.

Definition 3.12. (i) *By a* weak stationary solution *of equation* (3.26) *we mean any nonnegative function* $u \in L^\nu_{loc}(\Omega)$ *such that* $\frac{u}{|x|^2} \in L^1_{loc}(\Omega)$ *and there holds*

$$-\int_\Omega u\left\{\Delta\eta + \frac{c}{|x|^2}\eta\right\} dx = \int_\Omega u^\nu \eta\, dx \tag{3.29}$$

for any test function $\eta \in C^\infty_0(\Omega)$.

(ii) *By a* weak stationary solution *of problem* (1.1) *we mean any weak stationary solution of equation* (3.26) *which is smooth out of the origin and vanishes on* $\partial\Omega \times (0,T]$.

In the following of this subsection we take $\Omega = B$, B denoting the unit ball in $\mathbb{R}^n$. By strengthening the assumptions of Theorem 3.9, we can prove the following nonuniqueness result.

Theorem 3.13. *Let* $c \in (0, c_0)$, $\nu \in (1, \nu_-]$, *and the assumptions of Theorem* 3.9 *be satisfied. Moreover, let*

$$p < \bar{p} < \frac{2n}{|\lambda_-|} \le p\nu \tag{3.30}$$

(*where* $\bar{p} = \bar{p}(c,\nu)$ *is defined by equation* (3.6) *and* $|\lambda_-| = |\lambda_-|(c)$). *Then for some* $u_0 \in L^{\bar{p}}(B)$, $u_0 \ge 0$ *there exist two weak solutions of problem* (1.1) *from the space* $C([0,T]; L^p(B))$.

PROOF. As observed in [20], for any $c \in (0, c_0)$ and $v \in (1, v_-]$ there exists a weak stationary solution $\tilde{u}$ of problem (1.1) such that $\tilde{u}(x) \sim |x|^{\lambda_-/2}$ as $|x| \to 0$. It is easily checked that $\tilde{u} \in L^q(B)$ if and only if $q \in \left(1, \frac{2n}{|\lambda_-|}\right)$, thus $\tilde{u} \in L^{\bar{p}}(B) \backslash L^{pv}(B)$ by assumption (3.30).

On the other hand, let u denote the mild solution of problem (1.1) in $L^p(B)$ with Cauchy data $u_0 = \tilde{u}$, which exists by Theorem 3.9. It is shown below that u is also a weak solution. Since $u \in C([0, T]; L^{pv}(B))$, it cannot coincide with $\tilde{u}$. Hence the conclusion follows.

To complete the proof, let u be the mild solution referred to above. Then, in view of [22, Theorem 2-(vii)], there exists $C_1 > 0$ such that

$$\|t^{b(p,pv)}u(t)\|_{pv} \le C_1 \quad \text{for any } t \in (0, T). \tag{3.31}$$

Since $0 < b(p, pv) < 1$, we obtain

$$\int_0^T \|u(t)\|_{pv}\, dt \le C_2$$

with $C_2 := C_1 \frac{T^{1-b(p,pv)}}{1-b(p,pv)}$. Since by assumption (3.30)

$$pv > \frac{2n}{|\lambda_-|} > \frac{n}{n-2},$$

we obtain easily

$$\iint_{Q_T} \frac{u(x,t)}{|x|^2}\, dx\, dt \le C_3, \tag{3.32}$$

where $C_3 := C_2 \left(\int_\Omega |x|^{2pv/(1-pv)}\, dx \right)^{(pv-1)/pv}$. Moreover, inequality (3.31) also implies

$$\iint_{Q_T} |u(x,t)|^v\, dx\, dt \le C_4 \tag{3.33}$$

with $C_4 := C_1^{v-1} C_2 |\Omega|^{(p-1)v/p}$; here use of the inequality $b(p, pv)v < 1$ (or equivalently $b(p, pv) < a(p, pv)$) has been made. By (3.32)–(3.33) the conclusion follows. □

The compatibility of conditions of Theorem 3.13 can be investigated as already done for those of Proposition 3.2. However, this leads to a complicated set of constraints for p, v and c. Therefore we limit ourselves to consider sufficiently small values of c, proving the following result.

THEOREM 3.14. *Let $p > 1$, $v > 1$ satisfy*

$$v < \frac{n}{n-2} < pv < n(v-1)\min\left\{1, \frac{v}{2}\right\}. \tag{3.34}$$

Then there exists $\tilde{c} \in (0, c_0)$ *such that for any* $c \in (0, \tilde{c})$ *the assumptions of Theorem* 3.13 *are satisfied. Therefore for any* $c \in (0, \tilde{c})$ *and for some* $u_0 \in L^{\bar{p}}(B)$ ($\bar{p}$ *depending on* c), $u_0 \geq 0$ *there exist two weak solutions of problem* (1.1) *from the space* $C([0, T]; L^p(B))$.

Proof. It has been already observed that inequality (3.10), namely

$$\frac{n(v-1)}{2v} < p < \frac{n(v-1)}{v}$$

corresponds to (3.4)–(3.5) when $c = 0$; moreover $\bar{p}(0, v) = \frac{n(v-1)}{v}$ (see Remark 3.3). Similarly, the limiting form of (3.30) as $c \to 0^+$ is

$$p < \frac{n(v-1)}{2} < \frac{n}{n-2} \leq pv. \tag{3.35}$$

It is easily seen that the inequalities in (3.34) imply (3.10) and (3.35) (with strict inequality). In fact,

- $v < \frac{n}{n-2} \Rightarrow \frac{n(v-1)}{2} < \frac{n}{n-2}$;
- $v < \frac{n}{n-2} < pv \Rightarrow \frac{n(v-1)}{2v} < p$;
- $pv < n(v-1)\min\{1, \frac{v}{2}\} \Rightarrow p < \frac{n(v-1)}{v}$ and $p < \frac{n(v-1)}{2}$

(in this connection, observe that $2 < v < \frac{n}{n-2}$ is compatible if $n = 3$).

Therefore all conditions of Theorem 3.13 are satisfied at $c = 0$ with strict inequality. Since they depend on c continuously, the conclusion follows. □

Remark 3.15. Set

$$v_- \equiv v_-(c) := 1 + \frac{4}{|\lambda_-|}. \tag{3.36}$$

Observe that

$$1 < \frac{n}{n-2} < v_- < \frac{n+2}{n-2} < v_+ \quad (c \in (0, c_0)),$$

$$\lim_{c \to 0^+} v_- = \frac{n}{n-2}.$$

It was conjectured in [20] that the problem

$$\begin{cases} u_t - \Delta u - \frac{c}{|x|^2} u = u^{v_-} & \text{in } B \times (0, T] \\ u = 0 & \text{in } \partial B \times (0, T] \\ u = u_0 & \text{in } B \times \{0\} \end{cases} \tag{3.37}$$

has two weak solutions from $u \in C([0,T]; L^{\nu_-}(B))$. In this connection, observe that:

(i) if (3.34) holds, the assumptions of Theorem 3.13, yet with (3.30) replaced by

$$p < \bar{p} < \nu_- \leq p\nu, \tag{3.38}$$

are still satisfied. This depends on the fact that

$$\lim_{c\to 0^+} \nu_-(c) = \lim_{c\to 0^+} \frac{2n}{|\lambda_-(c)|} = \frac{n}{n-2}.$$

However, the conclusion of Theorem 3.14 is more precise, for $\nu_-(c) < \frac{2n}{|\lambda_-(c)|}$ for any $c \in (0, c_0)$;

(ii) in Theorem 3.14 we cannot recover the limiting case $p = \frac{2n}{|\lambda_-|}$, nor could we recover the case $p = \nu_-$ if (3.30) were replaced by (3.38). In fact, the argument used in [16] in the limiting case $c = 0$—namely, to prove nonuniqueness in $C([0,T]; L^{n/(n-2)}(B))$ of solutions to the problem

$$\begin{cases} u_t - \Delta u = u^{n/(n-2)} & \text{in } B \times (0,T] \\ u = 0 & \text{in } \partial B \times (0,T] \\ u = u_0 & \text{in } B \times \{0\}, \end{cases}$$

makes use of the boundedness of the solution for any $t \in (0,T]$. On the other hand, it is known that every weak solution of problem (3.37) diverges at the origin at least as $|x|^{\lambda_+/2}$ [2].

4. Nonexistence results

Set $\hat{Q}_\tau := (\Omega \backslash \{0\}) \times [0,\tau]$ $(\tau \in [0,T])$. Let us make the following definition.

Definition 4.1. *Let $u_0 \in L^1_{loc}(\Omega\backslash\{0\})$. By a* very weak solution *to problem* (1.1) *we mean any smooth nonnegative function $u \in C([0,T]; L^1_{loc}(\Omega\backslash\{0\})) \cap L^\nu_{loc}(\hat{Q}_T)$, which vanishes on $\partial\Omega \times (0,T]$ and satisfies equality* (3.28) *for any $\tau \in (0,T)$ and any test function $\zeta \in C^{\infty;1}_{x,t}(\hat{Q}_\tau)$, $\zeta(\cdot,t) \in C^\infty_0(\Omega\backslash\{0\})$ for any $t \in [0,\tau]$, $\zeta(\cdot,\tau) = 0$.*

Concerning nonexistence of very weak solutions to problem (1.1), we shall prove the following result.

Theorem 4.2. *Let $c \in (0, c_0]$, $\nu > \nu_+$, or $c \in (0,c_0)$, $\nu = \nu_+$. Suppose that*

$$\liminf_{x\to 0}[|x|^{-(\gamma+|\lambda_+|/2)} u_0(x)] > 0 \quad \text{for some } \gamma < 0. \tag{4.1}$$

Then no very weak solution to problem (1.1) *exists.*

Theorem 4.2 follows from the results proven in [18], if a stronger concept of solution is used (see [18] for details). The concept of solution used above is the weakest possible. In this respect, Theorem 4.2 is the parabolic counterpart of the elliptic nonexistence result in [4].

Observe that by assumption (4.1) there exist $k > 0$, $r > 0$ such that for any $|x| < r$ there holds $u_0(x) \geq k|x|^{\gamma+|\lambda_+|/2}$. If $|\gamma| \geq n + \frac{|\lambda_+|}{2}$, this implies $u_0 \notin L^1_{loc}(\Omega)$, whereas assumption (4.1) is compatible with $u_0 \in L^1_{loc}(\Omega)$ if $|\gamma| < n + \frac{|\lambda_+|}{2}$. A direct proof of nonexistence of very weak solutions, if $u_0 \in L^1_{loc}(\Omega)$, is the content of the following theorem.

THEOREM 4.3. *Let $c \in (0, c_0)$, $\nu > \nu_+$. Let $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$, $u_0 \not\equiv 0$. Then no very weak solution to problem* (1.1) *exists.*

4.1. Proof of Theorem 4.2

If we introduce the new unknown $v(x,t) := |x|^{\lambda_+/2}u(x,t)$, problem (1.1) reads:

$$\text{(4.2)} \qquad \begin{cases} v_t - \Delta v - \frac{\lambda_+}{|x|^2}\langle x, \nabla v\rangle = |x|^{((\nu-1)/2)\lambda_+}v^\nu & \text{in } Q_T \\ v = 0 & \text{in } \partial\Omega \times (0, T] \\ v = v_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $v_0(x) := |x|^{\lambda_+/2}u_0(x)$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. Very weak solutions of problem (4.2) are defined exactly as those of problem (1.1) (see Definition 4.1), yet with equality (3.28) replaced by

$$\text{(4.3)} \qquad -\iint_{Q_\tau} v\zeta_t \,dx\,dt - \iint_{Q_\tau} v\left\{\Delta\zeta - \lambda_+ \, div\left(\frac{x}{|x|^2}\zeta\right)\right\} dx\,dt$$
$$= \int_\Omega v_0(x)\zeta(x,0)\,dx + \iint_{Q_\tau} |x|^{((\nu-1)/2)\lambda_+}v^\nu\zeta \,dx\,dt.$$

It is easily seen that u is a very weak solution of problem (1.1) if and only if $v = |x|^{\lambda_+/2}u$ is a very weak solution of problem (4.2).

To prove Theorem 4.2 we modify the proof given in [18] in a suitable way, by making a proper choice of the test function ζ in inequality (4.3). For this purpose some preliminary remarks are needed.

Let $\Omega_1 \subseteq \Omega$ be any neighbourhood containing the origin, $0 < \varepsilon < \eta$, $\eta > 2\varepsilon$ so small that $A_{\varepsilon,\eta} := \{x \in \mathbb{R}^n \,|\, \varepsilon < |x| < \eta\} \subseteq \Omega_1 \backslash \{0\}$. For $r \in [\varepsilon, \eta]$ define

$$\phi_0(r) := (r^\sigma - \eta^\sigma)^\alpha,$$

where $\alpha \in \left(1 + \frac{1}{\nu}, 2\right)$ and $\sigma < 0$ will be fixed later. Define also

$$\phi_1(r) := \bar{\phi}\left(\frac{r}{\varepsilon}\right) \quad (r \in [\varepsilon, \eta]),$$

where $\bar{\phi} \in C^\infty\left(\left[0, \frac{\eta}{\varepsilon}\right]\right)$ is nondecreasing, such that

$$\bar{\phi}(s) := \begin{cases} 0 & \text{if } s \in (0,1) \\ 1 & \text{if } s \in \left(2, \frac{\eta}{\varepsilon}\right). \end{cases}$$

Finally, set

$$\bar{\zeta}(r) := r^\rho \phi_0(r)\phi_1(r) \quad (r \in [\varepsilon, \eta]),$$

where ρ is a real parameter to be chosen later, and

$$\tilde{\zeta}(x) := \bar{\zeta}(|x|) \quad (x \in \bar{A}_{\varepsilon,\eta}). \tag{4.4}$$

It is immediately seen that the function $\bar{\zeta}$ has the following properties:

(i) there holds

$$\bar{\zeta}(\varepsilon) = \bar{\zeta}(\eta) = \frac{d\bar{\zeta}}{dr}(\varepsilon) = \frac{d\bar{\zeta}}{dr}(\eta) = 0;$$

(ii) there exists a sequence $\{\zeta_k\} \subseteq C_0^\infty(A_{\varepsilon,\eta})$, $\zeta_k \geq 0$ for any k, such that $\zeta_k \to \tilde{\zeta}$ in $W_0^{2,p}(A_{\varepsilon,\eta})$ for any $p \in \left(1, \frac{1}{2-\alpha}\right)$.

In fact, claim (i) follows from the very definition of $\bar{\zeta}$. Concerning (ii), observe that $\tilde{\zeta} \in C_0^1(\bar{A}_{\varepsilon,\eta}) \cap W^{2,p}(A_{\varepsilon,\eta})$, thus $\tilde{\zeta} \in W_0^{2,p}(A_{\varepsilon,\eta})$ for any $p \in \left(1, \frac{1}{2-\alpha}\right)$; then the claim follows.

The proof of Theorem 4.2 makes use of the following

Proposition 4.4. *Let v be a very weak solution to problem* (4.2). *Let either of the following assumptions be satisfied:*

(i) $\rho \leq 2 - n \leq \lambda_+$, $\sigma < 0$;
(ii) $\rho = \lambda_+ > 2 - n$, $\sigma = 2 - n - \rho$.

Then for any $0 < \varepsilon < \eta$, η sufficiently small and any $\tau \in (0, T)$ there holds:

$$\begin{aligned} &\int_0^\tau (\tau - t)^\beta \, dt \int_{A_{\varepsilon,\eta}} |x|^{((\nu-1)/2)\lambda_+} v^\nu(x,t)\tilde{\zeta}(x)\, dx \\ &\quad \leq -\int_0^\tau (\tau - t)^\beta \, dt \int_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \chi(|x|) v(x,t)\, dx \\ &\qquad + \beta \int_0^\tau (\tau - t)^{\beta-1} \, dt \int_{A_{\varepsilon,\eta}} v(x,t)\tilde{\zeta}(x)\, dx - \tau^\beta \int_{A_{\varepsilon,\eta}} v_0(x)\tilde{\zeta}(x)\, dx, \end{aligned} \tag{4.5}$$

where $\beta > \max\left\{1, \frac{1}{\nu-1}\right\}$, the function $\tilde{\zeta}$ is defined in (4.4) *and*

$$\chi(r) := r^{n-1} \frac{d}{dr}[r^\rho \phi_0] \frac{d\phi_1}{dr} - \lambda_+ r^{n-2+\rho}\phi_0 + \frac{d}{dr}\left[r^{n-1+\rho}\phi_0 \frac{d\phi_1}{dr}\right] \quad (r \in [\varepsilon, \eta]). \tag{4.6}$$

PROOF. Let $\tau \in (0,T]$; set

$$\hat{\phi}(t) = \begin{cases} (\tau - t)^\beta & \text{if } t \in (0,\tau) \\ 0 & \text{if } t \in (\tau, T), \end{cases}$$

with $\beta > \max\{1, \frac{1}{\nu-1}\}$. Let $\{\zeta_k\} \subseteq C_0^\infty(A_{\varepsilon,\eta})$, $\zeta_k \geq 0$ for any k, such that $\zeta_k \to \tilde{\zeta}$ in $W_0^{2,p}(A_{\varepsilon,\eta})$, $p \in \left(1, \frac{1}{2-\alpha}\right)$. Choosing $\zeta(x,t) = \zeta_k(x)\hat{\phi}(t)$ in (4.3), then sending $k \to \infty$ we obtain:

$$\begin{aligned}(4.7)\qquad &\int_0^\tau (\tau - t)^\beta \, dt \int_{A_{\varepsilon,\eta}} |x|^{((\nu-1)/2)\lambda_+} v^\nu(x,t)\tilde{\zeta}(x)\, dx \\ &= \beta \int_0^\tau (\tau - t)^{\beta - 1}\, dt \int_{A_{\varepsilon,\eta}} v(x,t)\tilde{\zeta}(x)\, dx \\ &- \int_0^\tau (\tau - t)^\beta \, dt \int_{A_{\varepsilon,\eta}} v(x,t)\left\{\Delta\tilde{\zeta} - \lambda_+ \, div\left(\frac{x}{|x|^2}\tilde{\zeta}\right)\right\} dx \\ &- \tau^\beta \int_{A_{\varepsilon,\eta}} v_0(x)\tilde{\zeta}(x)\, dx.\end{aligned}$$

An elementary calculation shows that

$$(4.8)\qquad \Delta\tilde{\zeta} - \lambda_+ \, div\left(\frac{x}{|x|^2}\tilde{\zeta}\right) = |x|^{-(n-1)}\frac{d\psi}{dr}(|x|),$$

where

$$(4.9)\qquad \psi(r) := r^{n-1}\frac{d\bar{\zeta}}{dr}(r) - \lambda r^{n-2}\bar{\zeta}(r).$$

Defining

$$\psi_1 = \psi_1(r) := r^{n-1}\frac{d}{dr}[r^\rho \phi_0(r)] - \lambda r^{n-2+\rho}\phi_0(r) \quad (r \in [\varepsilon, \eta]),$$

we obtain

$$(4.10)\qquad \psi(r) = \phi_1\psi_1 + r^{n-1+\rho}\phi_0\frac{d\phi_1}{dr}.$$

From (4.7)–(4.10) the conclusion immediately follows, if we prove the following

CLAIM. *Let either assumption* (i)–(ii) *be satisfied. Then there exists* $\eta_0 > 0$ (*depending on* ρ, σ) *such that for any* $\eta < \eta_0$ *there holds*

$$\frac{d\psi_1}{dr} \geq 0 \quad in\ (\varepsilon, \eta).$$

To prove the Claim, observe that

$$(r^\sigma - \eta^\sigma)^{2-\alpha}\frac{d\psi_1}{dr} = \alpha(\alpha-1)\sigma^2 r^{n+\rho+2\sigma-3} + (r^\sigma - \eta^\sigma)r^{n+\rho+\sigma-3} \times \Big\{\alpha\sigma[(\rho-\lambda_+)+n+\rho+\sigma-2] + (\rho-\lambda_+)(n+\rho-2)\Big[1-\Big(\frac{r}{\eta}\Big)^{|\sigma|}\Big]\Big\},$$

as an elementary calculation shows. It is easily checked that the right-hand side of the above equality is nonnegative, if either assumption (i) or (ii) is satisfied. Then the Claim and the conclusion follow. □

Proposition 4.4 allows us to to prove Theorem 4.2, repeating the argument used in [18, Theorem 2.7]. We outline the proof for convenience of the reader.

Proof of Theorem 4.2. (i) Let u be any very weak solution of problem (1.1), and $v = |x|^{\lambda_+/2}u$ the corresponding very weak solution to problem (4.2). Arguing as in [18, Proposition 4.2], from inequality (4.5) we obtain for any $\varepsilon \in (0,\eta)$ sufficiently small and any $\tau \in (0,T]$

$$\int_0^\tau (\tau-t)^\beta\, dt\Big\{\int_{A_{\varepsilon,\eta}} v(x,t)\tilde{\zeta}(x)\,dx\Big\}^\nu \le M\tau^\beta C_1(\varepsilon,\eta)\Big\{C_1(\varepsilon,\eta)^{1/(\nu-1)}\tau^{-1/(\nu-1)} + C_2(\varepsilon)\tau - \int_{A_{\varepsilon,\eta}} v_0(x)\tilde{\zeta}(x)\,dx\Big\} \tag{4.11}$$

with some constant $M = M(\beta,\nu) > 0$. Here

$$C_1(\varepsilon,\eta) := \Big\{\int_\varepsilon^\eta r^{|\lambda_+|/2+n-1}\bar{\zeta}(r)\,dr\Big\}^{\nu-1}, \tag{4.12}$$

$$C_2(\varepsilon) := \int_{A_{\varepsilon,2\varepsilon}} |x|^{-(n-1)\nu'}[|x|^{((\nu-1)/2)\lambda_+}\tilde{\zeta}(x)]^{-\nu'-1}[\chi(|x|)]^{\nu'}\,dx, \tag{4.13}$$

where $\nu' := \frac{\nu-1}{\nu}$ (observe that $\chi(|x|) \equiv 0$ if $|x| > 2\varepsilon$).

In view of the definition of the function $\bar{\zeta}$, we have

$$C_1(\varepsilon,\eta) \le \Big\{\int_\varepsilon^\eta r^{|\lambda_+|/2+\rho+\sigma+n-1}\,dr\Big\}^{\nu-1},$$

thus by monotonicity

$$\lim_{\varepsilon\to 0^+} C_1(\varepsilon,\eta) \le K_1\eta^{(|\lambda_+|/2+\rho+\sigma+n)(\nu-1)}$$

for some $K_1 > 0$. Moreover, arguing as in [18, Proposition 3.4], we obtain for some constant $K_2 > 0$

$$C_2(\varepsilon) \le K_2 C \varepsilon^{\theta},$$

where

$$\theta := n + \rho + \sigma - 2 + \frac{|\lambda_+|}{2(\nu - 1)}(\nu - \nu_+). \tag{4.14}$$

(ii) In the following we choose the parameters ρ, σ of the test function $\tilde{\zeta}$ as in Proposition 4.4. More precisely, we assume

(a) $\rho \le 2 - n$, $\sigma < 0$ if $\nu > \nu_+$, or
(b) $\rho = \lambda_+ > 2 - n$, $\sigma = 2 - n - \rho$ if $\nu = \nu_+$.

Observe that the choice $c = c_0$ is allowed in case (a), but forbidden in case (b).

By Proposition 4.4 and the first sentence of this proof, in both cases inequality (4.11) holds, thus we take its limit as $\varepsilon \to 0^+$. To this purpose, observe preliminarly that for any $t \in [0, \tau]$:

$$\lim_{\varepsilon \to 0^+} \int_{A_{\varepsilon,\eta}} v(x,t)\tilde{\zeta}(x)\,dx = \int_{B_\eta} |x|^{\rho+\sigma}\left[1 - \left(\frac{|x|}{\eta}\right)^{|\sigma|}\right] v(x,t)\,dx$$

by monotonicity, due to the choice of the function $\tilde{\zeta}$. In view of assumption (4.1), there exist $k > 0$ and $\eta_1 > 0$ such that for any $|x| < \eta < \eta_1$ there holds $v_0(x) \ge k|x|^{\gamma}$. Hence

$$\int_{B_\eta} |x|^{\rho+\sigma}\left[1 - \left(\frac{|x|}{\eta}\right)^{|\sigma|}\right] v_0(x)\,dx \ge k \int_0^{\eta} r^{\gamma+\rho+\sigma+n-1}\left[1 - \left(\frac{r}{\eta}\right)^{|\sigma|}\right] dr,$$

where $B_\eta := \{x \in \mathbb{R}^n \mid |x| < \eta\}$. If $\gamma \le -2$, the integral in the right-hand side of the above inequality diverges. On the other hand, if $\gamma > -2$ we obtain:

$$\int_{B_\eta} |x|^{\rho+\sigma}\left[1 - \left(\frac{|x|}{\eta}\right)^{|\sigma|}\right] v_0(x)\,dx \ge K_3 \eta^{\gamma+\rho+\sigma+n}$$

for some $K_3 > 0$.

(iii) Assume first $\nu > \nu_+$. We claim that in this case the above choice (a) of the parameters ρ, σ is consistent with the inequality $\theta > 0$. If so, letting $\varepsilon \to 0^+$ in inequality (4.11) gives

$$\int_0^{\tau} (\tau - t)^{\beta}\,dt \left\{ \int_{B_\eta} |x|^{\rho+\sigma}\left[1 - \left(\frac{|x|}{\eta}\right)^{|\sigma|}\right] v(x,t)\,dx \right\}^{\nu} \le M \tau^{\beta} \eta^{(|\lambda_+|/2+\rho+\sigma+n)\nu} \{\tau^{-1/(\nu-1)} - K_3 \eta^{\gamma+\lambda_+/2}\} \tag{4.15}$$

for any $\tau \in (0, T)$, if $\gamma > -2$. In this case the right-hand side of the above inequality is negative for any $\tau > \tau_* = \tau_*(\eta) := K_3^{-(\nu-1)} \eta^{(|\lambda_+|/2-\gamma)(\nu-1)}$; since $\tau_*(\eta) \to 0^+$ as $\eta \to 0^+$, the conclusion follows in this case. On the other hand, if $\gamma \leq -2$ the right-hand side of inequality (4.15) tends to $-\infty$ as $\varepsilon \to 0^+$, thus a contradiction follows in this case, too. This proves that $v \equiv 0$, thus $u \equiv 0$. By the arbitrariness of u the conclusion follows in this case.

To prove the above claim, observe that the requirement $\theta > 0$ reads

$$\rho + \sigma > 2 - n + \frac{\lambda_+}{2(\nu - 1)} (\nu - \nu_+)$$

(see (4.14)), whereas the choice (a) implies

$$\rho + \sigma < 2 - n.$$

The above inequalities are compatible since $\nu > \nu_+$. This completes the proof in the present case.

(iv) Finally, let $\nu = \nu_+$. Choosing the parameters ρ, σ as in (b) gives $\rho + \sigma + n = 2$, thus $\theta = 0$. Taking the limit of inequality (4.15) as $\varepsilon \to 0^+$ now we obtain

$$\int_0^\tau (\tau - t)^\beta \, dt \left\{ \int_{B_\eta} |x|^{\rho+\sigma} \left[1 - \left(\frac{|x|}{\eta} \right)^{|\sigma|} \right] v(x,t) \, dx \right\}^\nu \leq M \tau^\beta \eta^{2\nu} \{ f(\eta, \tau) - K_3 \eta^{\gamma+2} \}, \tag{4.16}$$

where

$$f(\eta, \tau) := \eta^{2\nu/(\nu-1)} \tau^{-1/(\nu-1)} + K_2 \tau.$$

It is easily seen that the function $f(\eta, \cdot)$ has a unique minimum $\tau_* = \tau_*(\eta) := [(\nu - 1)K_2]^{-(\nu-1)/\nu} \eta^2$ in $[0, T]$; moreover, $f(\eta, \tau_*) = \nu K_2 \tau_*$. Then by inequality (4.16) there holds:

$$\int_0^\tau (\tau - t)^\beta \, dt \left\{ \int_{B_\eta} |x|^{\rho+\sigma} \left[1 - \left(\frac{|x|}{\eta} \right)^{|\sigma|} \right] v(x,t) \right\}^\nu \leq M' \tau^{\beta+2} \eta^{2\nu} \{ K_2 - K_3 \eta^\gamma \} \tag{4.17}$$

for some $M' > 0$. Since $\gamma < 0$ and $\tau_*(\eta) \to 0^+$ as $\eta \to 0^+$, the conclusion follows also in this case. This completes the proof. □

4.2. Proof of Theorem 4.3

The proof of Theorem 4.3 makes use of the following

PROPOSITION 4.5. *Let $0 < c \leq c_0$, $\nu > \nu_+$. Suppose $u_0 \in L^1_{loc}(\Omega)$. Then every very weak solution to problem* (1.1) *is also a weak solution.*

To prove Proposition 4.5 we need two preliminary results.

LEMMA 4.6. *Let u be a very weak solution of the problem*

$$
\begin{cases} u_t - \Delta u = g & \text{in } Q_T \\ u = 0 & \text{in } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases} \tag{4.18}
$$

with $g \in L^1_{loc}(\Omega \times [0,T))$, $u_0 \in L^1_{loc}(\Omega)$. *Suppose that* $\frac{u}{|x|^2} \in L^1_{loc}(\Omega \times [0,T))$. *Then u is a weak solution to problem* (4.18).

PROOF. By assumption we have

$$
-\iint_{Q_\tau} u\{\hat{\zeta}_t + \Delta\hat{\zeta}\}\,dx\,dt = \int_\Omega u_0\hat{\zeta}(x,0)\,dx + \iint_{Q_\tau} g\hat{\zeta}\,dx\,dt \tag{4.19}
$$

for any $\tau \in (0,T)$ and any test function $\hat{\zeta} \in C^{\infty,1}_{x,t}(\hat{Q}_\tau)$, $\hat{\zeta}(\cdot,t) \in C^\infty_0(\Omega\backslash\{0\})$ for any $t \in [0,\tau]$, $\hat{\zeta}(\cdot,\tau) = 0$. Let $\zeta \in C^{\infty,1}_{x,t}(\bar{Q}_\tau)$, $\zeta(\cdot,t) \in C^\infty_0(\Omega)$ for any $t \in [0,\tau]$, $\zeta(\cdot,\tau) = 0$. Set $\chi_k(x) := \chi(k|x|)$ $(k \in \mathbb{N})$, where $\chi \in C^\infty((0,\infty))$, $0 \le \chi \le 1$ and

$$
\chi(s) = \begin{cases} 0, & \text{if } s \in [0,1] \\ 1, & \text{if } s \in [2,\infty). \end{cases}
$$

Choosing $\hat{\zeta} = \hat{\zeta}_k = \zeta\chi_k$ in (4.19) gives

$$
\begin{aligned}
-\iint_{Q_\tau} u\chi_k[\zeta_t + \Delta\zeta]\,dx\,dt - 2\iint_{Q_\tau} u\langle\nabla\zeta,\nabla\chi_k\rangle\,dx\,dt \\
-\iint_{Q_\tau} u\zeta\Delta\chi_k\,dx\,dt = \iint_{Q_\tau} g\zeta\chi_k\,dx\,dt + \int_\Omega u_0(x)\zeta\chi_k\,dx
\end{aligned}
$$

for any $\tau \in (0,T)$. This yields the conclusion as $k \to \infty$, if we prove that

$$
\lim_{k\to\infty}\iint_{Q_\tau} u\langle\nabla\zeta,\nabla\chi_k\rangle\,dx\,dt = \lim_{k\to\infty}\iint_{Q_\tau} u\zeta\Delta\chi_k\,dx\,dt = 0.
$$

This follows from the inequalities

$$
\begin{aligned}
\iint_{Q_\tau} |u|\,|\nabla\chi_k|\,|\nabla\zeta|\,dx\,dt &\le Ck\int_0^\tau\int_{\{1/k<|x|<2/k\}} |u|\,dx\,dt \\
&\le 4C\int_0^\tau\int_{\{1/k<|x|<2/k\}} \frac{u}{|x|^2}\,dx\,dt, \\
\iint_{Q_\tau} |u|\,|\zeta|\,|\Delta\chi_k|\,dx\,dt &\le Ck^2\int_0^\tau\int_{\{1/k<|x|<2/k\}} |u|\,dx\,dt \\
&\le 4C\int_0^\tau\int_{\{1/k<|x|<2/k\}} \frac{u}{|x|^2}\,dx\,dt
\end{aligned}
$$

(which hold for some $C > 0$), for by assumption $\frac{u}{|x|^2} \in L^1_{loc}(\Omega \times [0, T))$. Hence the conclusion. □

LEMMA 4.7. *Let $0 < c \le c_0$, $v > 1$. Let u be a very weak solution to problem (1.1). Then both $\frac{u}{|x|^2}$ and u^v belong to $L^1_{loc}(\Omega \times [0, T))$.*

PROOF. Consider the family of functions

$$\xi_\varepsilon(x) := \begin{cases} \xi\left(\left(\frac{\varepsilon}{r}\right)^{n-2}\right) & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases} \qquad (\varepsilon > 0),$$

where $\xi \in C_0^\infty([0, \infty))$ satisfies:

- $0 \le \xi \le 1$ in $(0, \infty)$, $\xi(0) = 1$, $\xi \equiv 0$ in $[1, \infty)$;
- $\xi' \le 0$, $\xi'' \ge 0$ in $(0, \infty)$.

Then for any $\varepsilon > 0$

- $0 \le \xi_\varepsilon \le 1$, $\xi_\varepsilon \equiv 0$ in B_ε, $\xi_\varepsilon \to 1$ as $\varepsilon \to 0$ for any $r > 0$;
- $|\nabla \xi_\varepsilon| = (n-2)\varepsilon^{n-2} r^{1-n} |\xi'| \to 0$ as $\varepsilon \to 0$, uniformly on the compact subsets of $\overline{\Omega} \setminus \{0\}$;
- $\Delta \xi_\varepsilon = (n-2)^2 \varepsilon^{2(n-2)} r^{2(1-n)} \xi'' \ge 0$ in Ω.

Moreover, take $\eta \in C^\infty(\overline{\Omega})$ such that $0 \le \eta \le 1$, $\eta \equiv 1$ in B_{ε_0} for some $\varepsilon_0 > 0$, $\eta = 0$ on $\partial\Omega$.

Observe that $\xi_\varepsilon \eta \in C_0^\infty(\Omega \setminus \{0\})$, thus we can choose

$$\zeta(x, t) = (\tau - t)^\beta \xi_\varepsilon(x) \eta(x) \quad ((x, t) \in \overline{Q}_\tau)$$

with $\beta > \frac{v}{v-1}$ in (3.28) (see Definition 4.1). We obtain

$$\begin{aligned}
(4.20) \qquad & \iint_{Q_\tau} u^v (\tau - t)^\beta \xi_\varepsilon \eta \, dx \, dt + c \iint_{Q_\tau} \frac{u}{|x|^2} (\tau - t)^\beta \xi_\varepsilon \eta \, dx \, dt \\
& \le \beta \iint_{Q_\tau} u(\tau - t)^{\beta - 1} \xi_\varepsilon \eta \, dx \, dt \\
& \quad - \iint_{Q_\tau} u(\tau - t)^\beta [\eta \Delta \xi_\varepsilon + \xi_\varepsilon \Delta \eta + 2\langle \nabla \xi_\varepsilon, \nabla \eta \rangle] \, dx \, dt \\
& \le \beta \iint_{Q_\tau} u(\tau - t)^{\beta - 1} \xi_\varepsilon \eta \, dx \, dt \\
& \quad - \int_0^\tau \int_{\Omega \setminus B_{\varepsilon_0}} u(\tau - t)^\beta [\xi_\varepsilon \Delta \eta + 2\langle \nabla \xi_\varepsilon, \nabla \eta \rangle] \, dx \, dt \\
& \le \beta \iint_{Q_\tau} u(\tau - t)^{\beta - 1} \xi_\varepsilon \eta \, dx \, dt + C_1
\end{aligned}$$

for some $C_1 > 0$. On the other hand,

$$\begin{aligned}\beta u(\tau - t)^{\beta-1}\xi_\varepsilon\eta &= u[(\tau - t)^\beta\xi_\varepsilon\eta]^{1/\nu}[\beta(\tau - t)^{\beta-1-\beta/\nu}(\xi_\varepsilon\eta)^{1-1/\nu}] \\ &\le \frac{1}{\nu}u^\nu(\tau - t)^\beta\xi_\varepsilon\eta + \frac{\nu - 1}{\nu}\beta^{\nu/(\nu-1)}(\tau - t)^{\beta-\nu/(\nu-1)}\eta;\end{aligned} \tag{4.21}$$

here the inequality $0 \le \xi_\varepsilon \le 1$ has been used. From (4.20)–(4.21) we obtain

$$\frac{\nu - 1}{\nu}\iint_{Q_\tau} u^\nu(\tau - t)^\beta\xi_\varepsilon\eta\,dx\,dt + c\iint_{Q_\tau}\frac{u}{|x|^2}(\tau - t)^\beta\xi_\varepsilon\eta\,dx\,dt \le C_2$$

for some $C_2 > 0$. In view of the Fatou lemma, letting $\varepsilon \to 0$ in the above inequality gives

$$\frac{\nu - 1}{\nu}\iint_{Q_\tau} u^\nu(\tau - t)^\beta\eta\,dx\,dt + c\iint_{Q_\tau}\frac{u}{|x|^2}(\tau - t)^\beta\eta\,dx\,dt \le C_2$$

whence the conclusion easily follows. □

We can now prove Proposition 4.5.

Proof of Proposition 4.5. Let u be a very weak solution to problem (1.1). In view of Lemma 4.7, $g := \frac{c}{|x|^2}u + u^\nu$ belongs to $L^1_{loc}(\Omega \times [0, T))$. Then by Lemma 4.6 u is a weak solution to problem (1.1), and the result follows. □

Finally, let us prove Theorem 4.3.

Proof of Theorem 4.3. Define

$$\gamma_0 := \frac{|\lambda_+|}{2}, \quad \gamma_k := \nu\gamma_{k-1} - 2 \quad (k \in \mathbb{N}). \tag{4.22}$$

We shall prove that the sequence $\{\gamma_k\}$ is increasing and diverging as $k \to \infty$. In fact, observe that

$$\gamma_1 - \gamma_0 = (\nu - 1)\gamma_0 - 2 > 0$$

since by assumption $\nu > \nu_+$. Moreover, assuming

$$\gamma_k - \gamma_{k-1} = (\nu - 1)\gamma_{k-1} - 2 > 0$$

for some $k \in \mathbb{N}$, we have

$$\gamma_{k+1} - \gamma_k = (\nu - 1)\gamma_k - 2 > (\nu - 1)\gamma_{k-1} - 2.$$

Then, by induction, we have that $\{\gamma_k\}$ is increasing. Suppose that $l := \lim_{k\to\infty} \gamma_k \in \mathbb{R}$. Then from (4.22) we get

$$l = \frac{2}{\nu - 1} < \frac{|\lambda_+|}{2},$$

since $\nu > \nu_+$. On the other hand, $l \geq \gamma_0 := \frac{|\lambda_+|}{2}$, since the sequence $\{\gamma_k\}$ is increasing. The contradiction proves that $l = \infty$.

As a consequence, there exists a unique $\bar{k} \in \mathbb{N}$ such that $\gamma_{\bar{k}} \geq n - 2$ and $\gamma_{\bar{k}-1} < n - 2$. We shall prove the following

Claim. Let there exists a very weak solution u to problem (1.1). Then for any $j = 1, \dots, \bar{k} - 1$ there exist $C_j > 0$, $R_j > 0$ such that

$$u(x,t) \geq C_j(t-\varepsilon)|x|^{-\gamma_j} \quad \text{for a.e. } (x,t) \in B_{R_j} \times (\varepsilon, T) \quad (\varepsilon \in (0,T)). \tag{4.23}$$

From the above claim the result follows plainly. In fact, inequality (4.23) with $j = \bar{k} - 1$ implies

$$u^\nu(x,t) \geq C_{\bar{k}-1}^\nu (t-\varepsilon)^\nu |x|^{-\gamma_{\bar{k}}-2} \quad \text{for a.e. } (x,t) \in B_{R_{\bar{k}-1}} \times (\varepsilon, T).$$

Since $\gamma_{\bar{k}} \geq n-2$ by definition, there holds $|x|^{-\gamma_{\bar{k}}-2} \notin L^1(B_{R_{\bar{k}-1}})$, hence $u \notin L^\nu(B_{R_{\bar{k}-1}} \times (\varepsilon, \tau))$ $(\tau \in (\varepsilon, T))$. On the other hand, by Proposition 4.5 u is a weak solution to problem (1.1), thus $u \in L^\nu_{loc}(Q_\tau)$ for any $\tau \in (0,T)$ (see Definition 3.11). This is a contradiction, since $B_{R_{\bar{k}-1}} \times (\varepsilon, \tau) \subseteq Q_\tau$ $(\tau \in (\varepsilon, T))$. Hence the conclusion follows.

It remains to prove the claim. To this purpose, recall that for any fixed $R \in (0,1)$ and $\varepsilon \in (0,T)$ there exists $C_0 > 0$ such that

$$u(x,t) \geq C_0 |x|^{\lambda_+/2} \quad \text{for a.e. } x \in B_R \times [\varepsilon, T] \tag{4.24}$$

(see [2, Theorem 2.2]). Set

$$U_j(x) := \frac{|x|^{-\gamma_j} - R^{-\gamma_j}}{\gamma_j(n-2-\gamma_j)}, \quad (j = 1, \dots, \bar{k} - 1)$$

(observe that $U_j > 0$ for any j, for $\gamma_0 > 0$, $\gamma_{\bar{k}-1} < n-2$ and $\{\gamma_k\}$ is increasing). It is easily checked that U_j satisfies

$$-\int_{B_R} U_j \Delta\eta \, dx\, dt = \int_{B_R} |x|^{-\gamma_j - 2} \eta \, dx \tag{4.25}$$

for any $\eta \in C^\infty(\bar{B}_R)$, $\eta \geq 0$, $\eta = 0$ on ∂B_R $(j = 1, \dots, \bar{k} - 1)$.

Fix any $\tau \in (\varepsilon, T)$. Under the present assumptions every very weak solution u to problem (1.1) is also a weak solution (see Proposition 4.5). Therefore, we can choose in (3.28) a test function $\zeta \in C^{\infty,1}_{x,t}(\bar{B}_R \times [\varepsilon, \tau])$, $\zeta \geq 0$, $\zeta = 0$ in $\partial B_R \times [\varepsilon, \tau]$, $\zeta(\cdot, \tau) = 0$. In view of (4.24), this yields

$$\begin{aligned} -\int_\varepsilon^\tau\int_{B_R} u\Big(\zeta_t + \Delta\zeta + \frac{c}{|x|^2}\zeta\Big)\,dx\,dt &= \int_{B_R} u_0\zeta(x,0)\,dx + \int_\varepsilon^\tau\int_{B_R} u^\nu\zeta\,dx\,dt \\ &\geq C_0^\nu \int_\varepsilon^\tau\int_{B_R} |x|^{(\lambda_+/2)\nu}\zeta\,dx\,dt \\ &= C_0^\nu \int_\varepsilon^\tau\int_{B_R} |x|^{-\gamma_1-2}\zeta\,dx\,dt, \end{aligned}$$

whence

$$-\int_\varepsilon^\tau\int_{B_R} u(\zeta_t + \Delta\zeta)\,dx\,dt \geq C_0^\nu \int_{B_R} |x|^{-\gamma_1-2}\zeta\,dx\,dt. \tag{4.26}$$

Define for any $j = 1, \ldots, \bar{k} - 1$

$$u_j := C_0^\nu(t-\varepsilon)\frac{|x|^{-\gamma_j} - R^{-\gamma_j}}{(T-\varepsilon)\gamma_j(n-2-\gamma_j)+1} \quad \text{in } B_R \times (\varepsilon, T).$$

Observe that

$$u_j = K_j(t-\varepsilon)U_j,$$

with

$$K_j := \frac{\gamma_j(n-2-\gamma_j)}{(T-\varepsilon)\gamma_j(n-2-\gamma_j)+1}C_0^\nu.$$

Let us first prove inequality (4.23) for $j = 1$. By equality (4.25) we have

$$\begin{aligned} -\int_\varepsilon^\tau\int_{B_R} u_1(\zeta_t + \Delta\zeta)\,dx\,dt &= -K_1\int_\varepsilon^\tau\int_{B_R}(t-\varepsilon)U_1(\zeta_t + \Delta\zeta)\,dx\,dt \\ &= K_1\int_\varepsilon^\tau\int_{B_R}[U_1 + (t-\varepsilon)|x|^{-\gamma_1-2}]\zeta\,dx\,dt \\ &\leq K_1\int_\varepsilon^\tau\int_{B_R}\left[\frac{1}{\gamma_1(n-2-\gamma_1)} + (t-\varepsilon)\right]|x|^{-\gamma_1-2}\zeta\,dx\,dt \\ &\leq C_0^\nu\int_\varepsilon^\tau\int_{B_R}|x|^{-\gamma_1-2}\zeta\,dx\,dt. \end{aligned} \tag{4.27}$$

Inequalities (4.26) and (4.27) imply

$$-\int_\varepsilon^\tau\int_{B_R}(u-u_1)(\zeta_t + \Delta\zeta)\,dx\,dt \geq 0$$

—namely, $u - u_1$ is a weak supersolution of the problem

$$\begin{cases} u_t - \Delta u = 0 & \text{in } B_R \times (\varepsilon, T) \\ u = 0 & \text{in } (\partial B_R \times (\varepsilon, T)) \cup (B_R \times \{\varepsilon\}) \end{cases}$$

Then by comparison principles we have

$$u \geq u_1 \quad \text{a.e. in } B_R \times (\varepsilon, T).$$

This proves the claim for $j = 1$, by a proper choice of the constant $C_1 > 0$ and $R_1 = R$. The argument can be iterated for the remaining values of j. Hence the claim follows, and the proof is complete. □

References

[1] P. Baras - J. Goldstein, *Remarks on the inverse square potential in quantum mechanics*, in *Differential Equations*, I. W. Knowles - R. T. Lewis Eds., pp. 31–35 (Elsevier Science Editor, 1984).

[2] P. Baras - J. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. 284 (1984), 121–139.

[3] H. Brezis - X. Cabré, *Some simple nonlinear PDE's without solutions*, Boll. Un. Mat. Ital. 1-B (1998), 223–262.

[4] H. Brezis, L. Dupaigne - A. Tesei, *On a semilinear elliptic equation with inverse-square potential*, Selecta Math. 11 (2005), 1–7.

[5] H. Brezis - P. L. Lions, *A note on isolated singularities for elliptic equations*, Math. Anal. Appl. Part A, Adv. Math. Suppl. Studies 7A (1981), 263–266.

[6] X. Cabré - Y. Martel, *Existence versus explosion instantanée pour des equations de la chaleur linéaires avec potentiel singulier*, C. R. Acad. Sci. Paris 329 (1999), 973–978.

[7] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics 92 (Cambridge University Press, 1989).

[8] L. Dupaigne, *A nonlinear elliptic PDE with the inverse square potential*, J. Anal. Math. 86 (2002), 359–398.

[9] C. L. Fefferman, *The uncertainty principle*, AMS Colloquium Lectures, Denver, January 1983 (Amer. Math. Soc., 1983).

[10] H. Kalf, U.-W. Schmincke, J. Walter - R. Wüst, *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials*, in *Spectral Theory and Differential Equations*, W. N. Everitt Ed., Lecture Notes in Mathematics 448, pp. 182–226 (Springer, 1975).

[11] T. Kato, *Perturbation Theory for Linear Operators* (Springer, 1980).

[12] R. Kersner - A. Tesei, *Well-posedness of initial value problems for singular parabolic equations*, J. Differential Equations 199 (2004), 47–76.

[13] L. Moschini, S. Filippas - A. Tertikas, *Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains*, Comm. Math. Phys. 273 (2007), 237–281.

[14] L. Moschini, G. Reyes - A. Tesei, *Nonuniqueness of solutions to semilinear parabolic equations with singular coefficients*, Comm. Pure Appl. Anal. 5 (2006), 155–179.

[15] L. Moschini - A. Tesei, *A parabolic Harnack inequality for the heat equation with inverse-square potential*, Forum Math. 19 (2007), 407–427.

[16] W.-M. Ni - P. Sacks, *Singular behavior in nonlinear parabolic equations*, Trans. Amer. Math. Soc. 287 (1985), 657–671.
[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences 44 (Springer, 1983).
[18] S. I. Pohozhaev - A. Tesei, *Nonexistence of local solutions to semilinear partial differential inequalities*, Ann. Inst. H. Poincaré Anal. Nonlin. 21 (2004), 487–502.
[19] M. Reed - B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis & Self-Adjointness* (Academic Press, 1975).
[20] A. Tesei, *Local properties of solutions of a semilinear elliptic equation with an inverse-square potential*, J. Math. Sciences 149 (2008), 1729–1740.
[21] J. L. Vazquez - E. Zuazua, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal. 173 (2000), 103–153.
[22] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in* L^p, Indiana Univ. Math. J. 29 (1980), 79–102.

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