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On the convergence of the modified Kähler–Ricci flow and solitons*

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Abstract. We investigate the Kähler–Ricci flow modified by a holomorphic vector field. We find equivalent analytic criteria for the convergence of the flow to a Kähler–Ricci soliton. In addition, we relate the asymptotic behavior of the scalar curvature along the flow to the lower boundedness of the modified Mabuchi energy.

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1. Introduction

Let M be a compact Kähler manifold of complex dimension n with $c_1(M) > 0$. A Kähler–Ricci soliton on M is a Kähler metric $\omega = \frac{i}{2} g_{\bar{k}j}\, dz^j \wedge d\overline{z^k}$ in the cohomology class $\pi\, c_1(M)$ together with a holomorphic vector field X such that

$$\mathrm{Ric}(\omega) - \omega = \mathcal{L}_X \omega, \tag{1.1}$$

or $R_{\bar{k}j} - g_{\bar{k}j} = \nabla_j X_{\bar{k}}$, in coordinate notation with $X_{\bar{k}} = g_{\bar{k}\ell} X^{\ell}$. Let Φ_t be the 1-parameter group of automorphisms of M generated by $\mathrm{Re}\, X$. The family of metrics $g_{\bar{k}j}(t) \equiv \Phi^*_{-t}(g_{\bar{k}j})$ provides then a solution of the Kähler–Ricci flow, $\dot{g}_{\bar{k}j}(t) = -R_{\bar{k}j} + g_{\bar{k}j}$, where the evolution in time is just by reparametrization.

If X is the zero vector field then (1.1) reduces to the Kähler–Einstein equation. Kähler–Ricci solitons are in many ways similar to extremal metrics, which generalize constant scalar curvature Kähler metrics and are characterized by the condition that the vector field $\nabla^i R$ is holomorphic. A classic conjecture of Yau [Y2] asserts that the existence of constant scalar curvature metrics in a given integral Kähler class should be equivalent to the stability of the polarization in the sense of geometric invariant theory. Notions of K-stability for constant scalar curvature metrics have

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been proposed by Tian [T] and Donaldson [D2], and extended to the case of extremal metrics by Szekelyhidi [Sz1] (see also [M]). Similarly, the existence of Kähler–Ricci solitons is expected to be equivalent to a suitable notion of stability.

Kähler–Ricci solitons are the stationary points of the *modified Kähler–Ricci flow*

$$\dot{g}_{\bar{k}j} = -R_{\bar{k}j} + g_{\bar{k}j} + \nabla_j X_{\bar{k}} \tag{1.2}$$

which is the Kähler–Ricci flow reparametrized by the automorphisms Φ_t generated by Re X. Similar reparametrizations of Hamilton's original flow [H] had been introduced by DeTurck [DeT] to simplify the proof of the short-time existence of the flow.

The modified Kähler–Ricci flow appears in the work of Tian–Zhu [TZ2] as part of their study of the Kähler–Ricci flow assuming *a priori* the presence of a Kähler–Ricci soliton. They make use of a Moser–Trudinger type inequality from [CTZ] to deduce the Cheeger–Gromov convergence of the flow. (When there are no nontrivial holomorphic vector fields, it is known by the work of Perelman, [TZ2], [PSSW1], that the existence of a Kähler–Einstein metric implies the exponential convergence of the Kähler–Ricci flow to that metric.)

In this paper, we study the long-time behavior of the modified Kähler–Ricci flow without assuming the existence of a Kähler–Ricci soliton. We give analytic conditions which are both necessary and sufficient for the convergence of the flow to a Kähler–Ricci soliton. These conditions are analogous to the ones given in [PSSW1] for the convergence of the Kähler–Ricci flow. As explained in [PS1] and [PSSW1] they can be interpreted as stability conditions in an infinite-dimensional geometric invariant theory, where the orbits are those of the diffeomorphism group acting on the space of almost-complex structures.[1] The arguments and viewpoint in this paper are parallel to the case $X = 0$ treated in [PSSW1]. In the proofs, we emphasize only the main changes due to non-vanishing X.

More precisely, let M be a compact Kähler manifold with $c_1(M) > 0$ and X a holomorphic vector field whose imaginary part Im X induces an S^1 action on M. Write $\mathcal{K}_X$ for the space of Kähler metrics in $\pi c_1(M)$ which are invariant under Im X. Given $\omega = \frac{i}{2} g_{\bar{k}j} dz^j \wedge dz^{\bar{k}} \in \mathcal{K}_X$, define the Hamiltonian $\theta_{X,\omega}$ as the real-valued function satisfying

$$X^j g_{\bar{k}j} = \partial_{\bar{k}} \theta_{X,\omega}, \quad \int_M e^{\theta_{X,\omega}} \omega^n = \int_M \omega^n =: V.$$

The Ricci potential $f = f(\omega)$ is given by $g_{\bar{k}j} - R_{\bar{k}j} = \partial_{\bar{k}} \partial_j f$, $\int_M e^{-f} \omega^n = V$ (we note that in the Kähler geometry literature, f often has the opposite sign). Define the modified Ricci potential $u_{X,\omega}$ by

$$u_{X,\omega} = f + \theta_{X,\omega}.$$

[1] In [D1], Donaldson also considers an infinite-dimensional geometric invariant theory, with the group of symplectomorphisms acting on the space of almost complex structures.

If M admits a Kähler–Ricci soliton $\omega \in \pi c_1(M)$ with respect to X, then ω is necessarily in $\mathcal{K}_X$ and $u_{X,\omega} = 0$. Let $g_{\bar{k}j}(t)$ evolve by the modified Kähler–Ricci flow and set

$$Y_X(t) = \int_M |\nabla u_{X,\omega}|^2 e^{\theta_{X,\omega}} \omega^n. \tag{1.3}$$

The modified Kähler–Ricci flow preserves the Kähler class, and can be expressed as a flow of Kähler potentials. Identify (modulo constants) $\mathcal{K}_X$ with

$$\mathcal{P}_X(M, \omega_0) = \{\varphi \in C^\infty(M) \mid \omega = \omega_0 + \tfrac{i}{2}\partial\bar{\partial}\varphi > 0,\ \mathrm{Im}\, X(\varphi) = 0\}.$$

Let $\varphi = \varphi(t) \in \mathcal{P}_X(M, \omega_0)$ be the solution of the equation

$$\dot{\varphi} = \log \frac{\omega^n}{\omega_0^n} + \varphi + \theta_{X,\omega} + f(\omega_0), \quad \varphi(0) = c_0. \tag{1.4}$$

Then the Kähler metrics $\omega = \omega_0 + \frac{i}{2}\partial\bar{\partial}\varphi$ evolve by the modified Kähler–Ricci flow (1.2). The initial constant c_0 can affect the growth of φ for large time, and has to be chosen with some care. We choose it to be given by the value (2.5) described in Section §2 below.

Our first theorem is a characterization of the convergence of the modified Kähler–Ricci flow, which shows in particular that if convergence occurs, it is always exponential:

Theorem 1. *Let $\omega_0 \in \mathcal{K}_X$, $\omega_0 := \frac{i}{2} g^0_{\bar{k}j} dz^j \wedge d\bar{z}^k$, and consider the modified Kähler–Ricci flow (1.2) with initial metric ω_0. Then the following conditions are equivalent:*

(i) *The modified Kähler–Ricci flow $g_{\bar{k}j}(t)$ converges in C^∞ to a Kähler–Ricci soliton $g_{\bar{k}j}(\infty)$ with respect to X.*

(ii) *The function $\|R - n - \nabla_j X^j\|_{C^0}$ is integrable, i.e.,*

$$\int_0^\infty \|R - n - \nabla_j X^j\|_{C^0}\, dt < \infty.$$

(iii) *Let $\varphi(t)$ evolve by (1.4), with initial value c_0 as specified in (2.5) below. Then*

$$\sup_{t \geq 0} \|\varphi(t)\|_{C^0} < \infty.$$

(iv) *Let $Y_X(t)$ be defined by (1.3). Then there exist constants $\kappa, C > 0$ so that*

$$Y_X(t) \leq C\, e^{-\kappa t}.$$

(v) *The modified Kähler–Ricci flow $g_{\bar{k}j}(t)$ converges exponentially fast in C^∞ to a Kähler–Ricci soliton $g_{\bar{k}j}(\infty)$ with respect to X.*

We remark that our method does not obviously extend to the case where condition (i) is weakened to sequential convergence of the flow.

A criterion for the convergence of the Kähler–Ricci flow in terms of a uniform bound for volume forms has been given by Pali [Pa]. Indeed, such a bound implies immediately that $\|\varphi\|_{C^0}$ is uniformly bounded, in view of the defining equation $\log(\omega(t)^n/\omega_0^n) = -f(\omega_0) - \varphi + \dot\varphi$ for the Kähler–Ricci flow and Perelman's uniform bound for $\|\dot\varphi\|_{C^0}$. A similar observation is used in the proof of (ii) implies (iii) below.

Theorem 1 relates the convergence of the flow rather to the growth of $Y_X(t)$ or $\|R - n - \nabla_j X^j\|_{C^0}(t)$. Our next result addresses the behavior of these quantities under a stability assumption. Following [TZ1], we define the modified Mabuchi K-energy $\mu_X : \mathcal{P}_X(M, \omega_0) \to \mathbb{R}$ by

$$\delta\mu_X(\varphi) = -\frac{1}{V}\int_M \delta\varphi\left(R - n - \nabla_j X^j - X u_{X,\omega}\right) e^{\theta_{X,\omega}}\omega^n, \quad \mu_X(0) = 0.$$

Since $R - n - \nabla_j X^j - Xu = -(\Delta + \mathrm{Re}\, X) u_{X,\omega}$, the integrand is real and μ_X does map into $\mathbb{R}$. For a proof that $\mu_X(\varphi)$ is independent of choice of path in $\mathcal{P}_X(M, \omega_0)$, see [TZ1].

We consider the following condition:

(A_X) $\quad\mu_X$ is bounded from below on $\mathcal{P}_X(M, \omega_0)$.

In [TZ1] it is shown that (A_X) is a necessary condition for the existence of a Kähler–Ricci soliton ω with respect to X. Here we shall establish the following theorem:

Theorem 2. *Assume that Condition (A_X) holds, and let $\omega_0 \in \mathcal{K}_X$. Then we have, along the modified Kähler–Ricci flow (1.2) starting at ω_0,*

$$Y_X(t) \to 0 \quad \textit{and} \quad \|R - n - \nabla_j X^j\|_{C^0} \to 0, \quad \textit{as } t \to \infty.$$

Furthermore, for any $p > 2$, we have

$$\int_0^\infty \|R - n - \nabla_j X^j\|_{C^0}^p dt < \infty.$$

Note that a metric $\omega \in \mathcal{K}_X$ satisfies $R - n - \nabla_j X^j = 0$ if and only if ω is a Kähler–Ricci soliton with respect to X. However, the convergence $\|R - n - \nabla_j X^j\|_{C^0} \to 0$ is of course weaker than the convergence of the metrics $g_{\bar{k}j}(t)$ themselves to a Kähler–Ricci soliton. This is to be expected, since the condition (A_X) is only a semi-stability condition.

It was shown in [CTZ] using the continuity method that the 'properness' of μ_X in a certain sense is equivalent to the existence of a Kähler–Ricci soliton. The properness condition can be thought of as a strong Moser–Trudinger inequality, while Condition (A_X) corresponds to a weaker form of the Moser–Trudinger inequality.

Associated to the modified K-energy is the modified Futaki invariant F_X (see [TZ1]),

$$F_X(Z) = -\int_M (Zu_{X,\omega})\, e^{\theta_{X,\omega}} \omega^n,$$

defined for holomorphic vector fields Z. The modified Futaki invariant F_X is independent of the choice of $\omega \in \mathcal{K}_X$. It follows immediately that $F_X \equiv 0$ is a necessary condition for the existence of a Kähler–Ricci soliton in $\mathcal{K}_X$.

In the unmodified case, corresponding to $X = 0$, the condition (A_X) reduces to the condition (A) from [PS1] of lower boundedness of the Mabuchi K-energy. It is then easy to show that (A) implies that the unmodified Futaki invariant $F_{X=0}(Z)$ vanishes for all holomorphic vector fields $Z \in H^0(M, T^{1,0})$, by differentiating the functional along the integral paths of Z. We show how to rework this argument to prove the analogous statement when $X \neq 0$ (to our knowledge, this result is not in the literature).

Proposition 1. *If (A_X) holds then $F_X(Z) = 0$ for all holomorphic vector fields Z.*

Our third theorem shows that (A_X) together with an eigenvalue condition give necessary and sufficient conditions for the convergence of the metrics $g_{\bar{k}j}(t)$ themselves. Set

$$\lambda(t) = \inf_{V \perp H^0(M,T^{1,0})} \frac{\|\bar{\partial} V\|^2}{\|V\|^2},$$

where $H^0(M, T^{1,0})$ is the space of holomorphic vector fields on M and we are using the natural L^2 inner product induced by $g_{\bar{k}j}(t)$. This quantity was first introduced in the context of the Kähler–Ricci flow in [PS1]. Recall the following condition from [PSSW1]:

(S) $\quad \inf_{t \geq 0} \lambda(t) > 0.$

Theorem 3. *The modified Kähler–Ricci flow* (1.2)*, starting at an arbitrary metric $\omega_0 \in \mathcal{K}_X$, converges exponentially fast in C^∞ to a Kähler–Ricci soliton with respect to the holomorphic vector field X if and only if the conditions (A_X) and (S) are satisfied.*

Since condition (S) is invariant under automorphisms, a consequence of Theorem 3 is that convergence modulo automorphisms implies full convergence, i.e., if $g_{\bar{k}j}(t)$ is a solution of the modified Kähler–Ricci flow starting at $\omega_0 \in \mathcal{K}_X$ and $\Psi_t^*(g_{\bar{k}j})$ converges to a Kähler–Ricci soliton with respect to X for some family of automorphisms $\{\Psi_t\}_{t\in[0,\infty)}$, then $g_{\bar{k}j}(t)$ converges exponentially fast to a Kähler–Ricci soliton with respect to X.

It would be interesting to determine whether condition (S) by itself is sufficient.

Finally we discuss in more detail the behavior of $Y_X(t)$ which, as can be seen from Theorem 1, is key to the convergence of the Kähler–Ricci flow. It is convenient to introduce a quantity λ_X which is uniformly equivalent to the eigenvalue λ described above (see Lemma 4 below). Equip the spaces $T^{1,0}$ and $T^{1,0} \otimes (T^*)^{0,1}$ with the Hilbert space norms

$$\|V\|_\theta^2 = \int_M g_{\bar{k}j} V^j \overline{V^k} \, e^{\theta_{X,\omega}} \omega^n, \quad \|W\|_\theta^2 = \int_M g_{\bar{k}j} W^j_{\bar{p}} \overline{W^k_{\bar{q}}} g^{q\bar{p}} \, e^{\theta_{X,\omega}} \omega^n.$$

Define the eigenvalue $\lambda_X(t)$ by $\lambda_X(t) = \inf_{V \perp H^0(M,T^{1,0})} \|\bar\partial V\|_\theta^2 / \|V\|_\theta^2$, where the notion of perpendicularity is taken with respect to the norm $\|\cdot\|_\theta$. Then we have:

Theorem 4. *Consider the modified Kähler–Ricci flow (1.2) with initial metric $\omega_0 \in \mathcal{K}_X$. Then there exist $C > 0$ depending only on ω_0, and N, $\delta_j \geq 0$, $0 \leq j \leq N$, depending only on n and satisfying $\sum_{j=0}^N \delta_j > 2$, so that for all $t \geq 2N$,*

$$\begin{aligned} \dot{Y}_X(t) \leq & -2\,\lambda_X(t)\, Y_X(t) - 2\,\lambda_X(t)\, F_X(\pi(\overline{\nabla}(u_{X,\omega}))) \\ & + C \prod_{j=0}^N Y_X(t-2j)^{\frac{\delta_j}{2}}. \end{aligned} \tag{1.5}$$

Here $\overline{\nabla} u_{X,\omega} = g^{j\bar{k}} \partial_{\bar{k}} u_{X,\omega}$, and π is the orthogonal projection, with respect to the norm $\|\cdot\|_\theta$, of the space of $T^{1,0}$ vector fields onto the subspace of holomorphic vector fields.

The main point of the estimate (1.5) is to relate the convergence of the modified Kähler–Ricci flow to three issues, namely the vanishing of the modified Futaki invariant F_X; the convergence of $Y_X(t)$ to 0 as $t \to \infty$; and the existence of a strictly positive uniform lower bound for $\lambda_X(t)$ (or equivalently, to $\lambda(t)$, cf. Lemma 4).

As mentioned above, our results extend those of the paper [PSSW1] which considered the case $X = 0$. Accordingly, some of our proofs are brief, and we focus on those changes due to non-vanishing X.

2. Preliminaries

In this section, we give a proof of Proposition 1, and determine an initial value c_0 for the modified Kähler–Ricci flow so that $\dot\varphi$ is bounded.

2.1. Proof of Proposition 1. We first show that $F_X(Z) = 0$ for all holomorphic vector fields Z satisfying $\mathcal{L}_{\mathrm{Im}\,X} Z = 0$. Fix a Kähler metric $\omega_0 \in \mathcal{K}_X$. Write Ψ_t for the 1-parameter family of automorphisms of M induced by Re Z. Define ω_t and ψ_t by

$$\Psi_t^*\omega_0 = \omega_t = \omega_0 + \frac{i}{2}\partial\bar{\partial}\psi_t.$$

Note that $\psi_t \in \mathcal{P}_X(M,\omega_0)$ is defined only up to the addition of a constant. Also, $\frac{i}{2}\partial\bar{\partial}\dot{\psi} = \mathcal{L}_{\mathrm{Re}\,Z}\omega$, where we are now dropping the t subscript. On the other hand, there exists a complex-valued function $\theta_{Z,\omega}$, invariant under Im X, such that $\iota_Z\omega = \frac{i}{2}\bar{\partial}\theta_{Z,\omega}$. Indeed, all manifolds M with $c_1(M) > 0$ are simply connected so the $\bar{\partial}$-closed $(0,1)$-form $Z^j g_{\bar{k}j} d\overline{z^k}$ must be $\bar{\partial}$-exact. Since $\partial\bar{\partial}\dot{\psi} = \partial\bar{\partial}\mathrm{Re}\,\theta_{Z,\omega}$ we can assume that $\dot{\psi} = \mathrm{Re}\,\theta_{Z,\omega}$.

Compute

$$\begin{aligned}
\frac{d}{dt}\mu_X(\psi) &= -\mathrm{Re}\Big(\frac{n}{V}\int_M \frac{i}{2}\partial u_{X,\omega}\wedge\bar{\partial}\dot{\psi}\wedge e^{\theta_{X,\omega}}\omega^{n-1}\Big)\\
&= \frac{n}{V}\int_M u_{X,\omega}e^{\theta_{X,\omega}}\Big(\frac{i}{2}\partial\bar{\partial}\dot{\psi} + \mathrm{Re}\Big(\frac{i}{2}\partial\theta_{X,\omega}\wedge\bar{\partial}\dot{\psi}\Big)\Big)\wedge\omega^{n-1}\\
&= \frac{1}{V}\int_M u_{X,\omega}e^{\theta_{X,\omega}}\Big(\Delta\dot{\psi}\,\omega^n + n\mathrm{Re}\Big(\frac{i}{2}\partial\theta_{X,\omega}\wedge\bar{\partial}\theta_{Z,\omega}\wedge\omega^{n-1}\Big)\Big) \qquad (2.1)\\
&= \frac{1}{V}\int_M u_{X,\omega}\mathcal{L}_{\mathrm{Re}\,Z}(e^{\theta_{X,\omega}}\omega^n) = -\frac{1}{V}\int_M(\mathrm{Re}\,Z)(u_{X,\omega})e^{\theta_{X,\omega}}\omega^n\\
&= \frac{1}{V}\mathrm{Re}(F_X(Z)).
\end{aligned}$$

To go from the 2nd to the 3rd line, we have used the fact that $\dot{\psi} = \overline{\theta_{Z,\omega}} - i\,\mathrm{Im}\,\theta_{Z,\omega}$ and $n\,\mathrm{Re}\left(\frac{i}{2}\partial\theta_{X,\omega}\wedge\bar{\partial}\mathrm{Im}\,\theta_{Z,\omega}\wedge\omega^{n-1}\right) = -n\,\mathrm{Re}\left(\frac{i}{2}\partial\mathrm{Im}\,\theta_{Z,\omega}\wedge\bar{\partial}\theta_{X,\omega}\wedge\omega^{n-1}\right) = -(\mathrm{Im}\,X)(\mathrm{Im}\,\theta_{Z,\omega})\,\omega^n = 0$, since $\mathrm{Im}\,\theta_{Z,\omega}$ is invariant under Im X.

Condition (A_X) implies from (2.1) that $\mathrm{Re}(F_X(Z)) = 0$. Replacing Z by iZ shows that $F_X(Z) = 0$ for all holomorphic vector fields Z invariant under Im X. If Z is now an arbitrary holomorphic vector field and $\hat{Z}$ its average over the S^1 orbit we obtain $F_X(Z) = F_X(\hat{Z}) = 0$ as required.

2.2. Choice of initial value c_0. We show how to choose c_0 so that $\sup_{t\geq 0}\|\dot{\varphi}\|_{C^0} < \infty$. This bound was proved in [TZ2] assuming the existence of a Kähler–Ricci soliton. Here we only require the invariance of the initial metric ω_0 under Im X.

Fix $\omega_0 = \frac{i}{2}g^0_{\bar{k}j}dz^j\wedge d\overline{z^k} \in \mathcal{K}_X$. The Kähler–Ricci and modified Kähler–Ricci flows are

$$\frac{\partial}{\partial t}\tilde{g}_{\bar{k}j}(t) = -\tilde{R}_{\bar{k}j} + \tilde{g}_{\bar{k}j}, \quad \tilde{g}_{\bar{k}j}(0) = g^0_{\bar{k}j} \qquad (2.2)$$

$$\frac{\partial}{\partial t}g_{\bar{k}j}(t) = -R_{\bar{k}j} + g_{\bar{k}j} + \nabla_j X_{\bar{k}}, \quad g_{\bar{k}j}(0) = g^0_{\bar{k}j}, \qquad (2.3)$$

respectively. Note that if $\{\Phi_t\}_{t\in[0,\infty)}$, $\Phi_0 = \mathrm{id}$, is the subgroup of automorphisms of M generated by $\mathrm{Re}\, X$, then the solutions to (2.2) and (2.3) are related by $g_{\bar{k}j}(t) = \Phi_t^*(\tilde{g}_{\bar{k}j})$. The Kähler–Ricci flow preserves the S^1 action induced by $\mathrm{Im}\, X$ and so the Kähler forms $\tilde{\omega}(t)$ and $\omega(t)$ lie in $\mathcal{K}_X$. In the sequel, we will often drop the t. Also, we will denote by $\tilde{f}$, $\tilde{\nabla}$ and $\tilde{\Delta}$ the Ricci potential, covariant derivative and Laplacian with respect to $\tilde{g}_{\bar{k}j}$.

Next, recall Perelman's estimates (see [ST]) for the Kähler–Ricci flow: with all norms taken with respect to $\tilde{g}_{\bar{k}j}(t)$, there exists a constant C depending only on ω_0 so that

$$\|\tilde{f}\|_{C^0} + \|\tilde{\nabla}\tilde{f}\|_{C^0} + \|\tilde{\Delta}\tilde{f}\|_{C^0} \leq C.$$

Furthermore, the diameters $\mathrm{diam}_{\tilde{g}(t)} M$ are uniformly bounded by a constant depending only on ω_0, and for any $\rho > 0$, there exists $c > 0$ depending only on ω_0 and ρ such that for all $x \in M$ and all r with $0 < r \leq \rho$ we have $\int_{B_r(x)} \tilde{\omega}^n(t) \geq c\, r^{2n}$, where $B_r(x)$ is the geodesic ball centered at x of radius r with respect to $\tilde{g}_{\bar{k}j}(t)$ ("non-collapsing"). Uniform bounds for the Sobolev constant have now been established by Zhang [Zha] and Ye [Ye].

These statements make no reference to the vector field X and indeed do not require the initial metric ω_0 to be invariant under $\mathrm{Im}\, X$. Moreover, they are all invariant under automorphisms and hence the analogous statements hold also for the metrics $g_{\bar{k}j}$.

We now describe (2.2), (2.3) in terms of potentials. Define $\tilde{\varphi} = \tilde{\varphi}(t)$ and $\varphi = \varphi(t)$ by

$$\frac{\partial\tilde{\varphi}}{\partial t} = \log\frac{\tilde{\omega}^n}{\omega_0^n} + \tilde{\varphi} + f(\omega_0), \quad \tilde{\varphi}(0) = \tilde{c}_0,$$
$$\frac{\partial\varphi}{\partial t} = \log\frac{\omega^n}{\omega_0^n} + \varphi + \theta_{X,\omega} + f(\omega_0), \quad \varphi(0) = c_0,$$

The constant $\tilde{c}_0$ is chosen to be the value (2.10) in [PSS] (see also [CT]), so that $\|\partial_t\tilde{\varphi}\|_{C^0} \leq C$, and the constant c_0 will be defined shortly. One can check that $\tilde{\omega} = \omega_0 + \frac{i}{2}\partial\bar{\partial}\tilde{\varphi}$ and $\omega = \omega_0 + \frac{i}{2}\partial\bar{\partial}\varphi$ satisfy (2.2) and (2.3) respectively. We need the following well-known properties of the Hamiltonian $\theta_{X,\omega}$:

Lemma 1. (a) (See e.g. [FM] or [Zhu1].) *For all $\omega' \in \mathcal{K}_X$, we have $\|\theta_{X,\omega'}\|_{C^0} = \|\theta_{X,\omega_0}\|_{C^0}$.*

(b) ([TZ1], p. 301) *For $\omega' \in \mathcal{K}_X$ with $\omega' = \omega_0 + \frac{i}{2}\partial\bar{\partial}\varphi'$, we have $\theta_{X,\omega'} = \theta_{X,\omega_0} + X(\varphi')$.*

For example, to see (a), we can apply Moser's theorem and obtain a diffeomorphism $\Psi\colon M \to M$ with $\Psi^*(\omega') = \omega_0$ and $\Psi^*(\mathrm{Im}\, X) = \mathrm{Im}\, X$. But then $d\Psi^*\theta_{X,\omega'} = d\theta_{X,\omega_0}$, and $\Psi^*\theta_{X,\omega'} = \theta_{X,\omega_0} + c$, for a constant c which must vanish by the normalization conditions. Thus $\theta_{X,\omega'}$, $\theta_{X,\omega}$ have the same image in $\mathbb{R}$, and (a) is proved.

We can show now that along the modified Kähler–Ricci flow,

$$\int_M |X|^2 e^{\theta_{X,\omega}} \omega^n \leq C. \tag{2.4}$$

Indeed, from the definition of the modified Futaki invariant and of $\theta_{X,\omega}$,

$$\int_M |X|^2 e^{\theta_{X,\omega}} \omega^n = -\int_M (Xf) e^{\theta_{X,\omega}} \omega^n - F_X(X).$$

Hence, since $F_X(X)$ is independent of choice of metric, we have

$$\begin{aligned}\int_M |X|^2 e^{\theta_{X,\omega}} \omega^n &\leq \int_M |X||\nabla f| e^{\theta_{X,\omega}} \omega^n + C \\ &\leq \frac{1}{2}\int_M |X|^2 e^{\theta_{X,\omega}} \omega^n + \frac{1}{2}\int_M |\nabla f|^2 e^{\theta_{X,\omega}} \omega^n + C,\end{aligned}$$

and the claim follows from Perelman's estimates and Lemma 1. Define now c_0 by

$$c_0 := \frac{1}{V}\int_0^\infty e^{-t}\int_M |\nabla u_{X,\omega}|^2 e^{\theta_{X,\omega}} \omega^n \, dt - \frac{1}{V}\int_M u_{X,\omega_0} e^{\theta_{X,\omega_0}} \omega_0^n. \tag{2.5}$$

To see that c_0 is finite, observe that $|\nabla u_{X,\omega}|^2 \leq 2(|\nabla f|^2 + |X|^2) \leq C + 2|X|^2$, and hence by Lemma 1 and (2.4), $\int_M |\nabla u_{X,\omega}|^2 e^{\theta_{X,\omega}} \omega^n \leq C$. We can now prove

Lemma 2. *There exists a uniform constant C such that along the flow,*

$$\|\dot{\varphi}\|_{C^0} \leq C.$$

Proof. Define $\alpha(t) = \frac{1}{V}\int_M \dot{\varphi} e^{\theta_{X,\omega}} \omega^n$. From Lemma 1 and the fact that $\dot{\varphi}$ and $u_{X,\omega}$ differ only by a time-dependent constant, we have $\frac{d}{dt}\alpha = \alpha - \frac{1}{V}\int_M |\nabla u_{X,\omega}|^2 e^{\theta_{X,\omega}} \omega^n$. Integrating this ODE (cf. the argument in [PSS]) shows that

$$0 \leq \alpha(t) = \frac{1}{V}\int_t^\infty e^{-(s-t)}\int_M |\nabla u_{X,\omega}|^2(s) e^{\theta_{X,\omega(s)}} \omega^n(s) \, ds \leq C. \tag{2.6}$$

From (2.2) and (2.3) we obtain $\partial_t \varphi = \Phi_t^* \partial_t \tilde{\varphi} + \theta_{X,\omega} + m(t)$, for some constant $m(t)$. The lemma follows from the boundedness of $\|\partial_t \tilde{\varphi}\|_{C^0}$, (2.6), and Lemma 1. □

3. Estimates for the modified Kähler–Ricci flow

In this section, we establish some key estimates for the modified Kähler–Ricci flow, namely the analogue of Perelman's estimates for the Ricci potential, the estimates for the Laplacian of the Hamiltonian function $\theta_{X,\omega}$, an L^2/C^0 Poincaré inequality, and a smoothing lemma.

Proposition 2. *Along the flow, the quantities*

$$\|\nabla u_{X,\omega}\|_{C^0}, \quad \|\Delta u_{X,\omega}\|_{C^0}, \ \|X\|_{C^0}, \quad \text{and} \quad \|\Delta\theta_{X,\omega}\|_{C^0}$$

are uniformly bounded by a constant depending only on the initial data. Here, all norms, covariant derivatives and Laplacians are taken with respect to the evolving metric $g_{\bar{k}j}(t)$.

Proof. It is convenient to work with $v := -\dot{\varphi}$, which differs from $-u_{X,\omega} = -(f + \theta_{X,\omega})$ only by a time-dependent constant, so that $|\nabla v| = |\nabla u_{X,\omega}|$, $|\Delta v| = |\Delta u_{X,\omega}|$. First, we need the evolution of v, which can be obtained by a straightforward calculation (cf. [CTZ]),

$$\begin{aligned} \frac{\partial v}{\partial t} &= (\Delta + X)v + v, \\ \frac{\partial}{\partial t}|\nabla v|^2 &= (\Delta + X)|\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\overline{\nabla} v|^2 + |\nabla v|^2. \\ \frac{\partial}{\partial t}(\Delta + X)v &= (\Delta + X)(\Delta + X)v + (\Delta + X)v + |\nabla\overline{\nabla} v|^2. \end{aligned} \tag{3.1}$$

Boundedness of $\|\nabla v\|_{C^0}$: This is a straightforward modification of Perelman's maximum principle argument for the bound of the gradient of the Ricci potential (see [ST], Proposition 6). Since v is uniformly bounded along the flow by Lemma 2, we may choose a constant B such that $v + B \geq 0$. Define $H = \frac{|\nabla v|^2}{v+2B}$, and compute, using (3.1),

$$(\Delta + X - \partial_t)H = \frac{H(H-2B)}{v+2B} - \frac{2\mathrm{Re}\big(g^{j\bar{k}}\partial_j H \partial_{\bar{k}} v\big)}{v+2B} + \frac{|\nabla\nabla v|^2 + |\nabla\overline{\nabla} v|^2}{v+2B}. \tag{3.2}$$

Fix $T > 0$. At a maximum point of $H(x,t)$ for $(x,t) \in M \times (0,T]$, the middle term on the right side of (3.2) vanishes and the left-hand side of (3.2) is nonpositive. It follows that H is uniformly bounded from above and hence so is $\|\nabla v\|_{C^0}$.

Boundedness of $\|X\|_{C^0}$: Since $u_{X,\omega} = f + \theta_{X,\omega}$, the uniform bound on $|X| = |\nabla\theta_{X,\omega}|$ follows from the bound on $|\nabla u_{X,\omega}| = |\nabla v|$ and Perelman's bound on $|\nabla f|$.

Boundedness of $\|\Delta v\|_{C^0}$ *and* $\|\Delta\theta_{X,\omega}\|_{C^0}$: First note that $|Xv| \leq |X||\nabla v| \leq C$ by the preceding bounds. From (3.1) we have

$$\begin{aligned}(\Delta + X - \partial_t)((\Delta + X)v) &= -\Delta v - Xv - |\nabla\overline{\nabla}v|^2 \\ &\leq -(\Delta v)\Big(1 + \frac{\Delta v}{n}\Big) + C,\end{aligned} \tag{3.3}$$

where we have used the elementary inequality $|\Delta v|^2 \leq n|\nabla\overline{\nabla}v|^2$. Fix an arbitrary $T > 0$. At a minimum point of $(\Delta + X)v$ on $M \times (0, T]$ the left-hand side of (3.3) is nonnegative and hence Δv is bounded uniformly from below at this point. This gives the lower bound of $(\Delta + X)v$ along the flow, depending only on the initial data.

To estimate $\|\Delta v\|_{C^0}$, it suffices to prove a uniform upper bound for $(\Delta + X)v$. As in Perelman's estimate of the scalar curvature (see [ST]), define $G = \frac{(\Delta+X)v+2|\nabla v|^2}{v+2B}$, where B is chosen as in the proof of the boundedness of $\|\nabla v\|_{C^0}$. Compute

$$(\Delta + X - \partial_t)\, G = -2\mathrm{Re}\Big(\frac{\nabla G \cdot \overline{\nabla} v}{v + 2B}\Big) + \frac{|\nabla\overline{\nabla}v|^2 + 2|\nabla\nabla v|^2}{v + 2B} - \frac{2BG}{(v + 2B)}.$$

Since $1/(v + 2B)$, $|Xv|$ and $|\nabla v|$ are uniformly bounded, we have

$$(\Delta + X - \partial_t)\, G \geq -2\mathrm{Re}\Big(\frac{\nabla G \cdot \overline{\nabla} v}{v + 2B}\Big) + C_1|\nabla\overline{\nabla}v|^2 - C_2|\Delta v| - C_3,$$

for uniform constants $C_1, C_2, C_3 > 0$ with C_1 uniformly bounded from below away from 0. By the maximum principle and a similar argument to the one above, we have $(\Delta + X)v \leq C$ for some uniform constant C. This gives the estimate for Δv. Notice that $\Delta(v + \theta_{X,\omega}) = -\Delta f$, which is uniformly bounded by Perelman's estimates. It follows that $\Delta\theta_{X,\omega}$ is uniformly bounded. □

Proposition 3. *Define* $b = b(t) = \frac{1}{V}\int_M u_{X,\omega} e^{-f}\omega^n$. *Then there exists a uniform constant* C *so that*

$$\|u_{X,\omega} - b\|_{C^0}^{n+1} \leq C\, \|\nabla u_{X,\omega}\|_{L^2}\, \|\nabla u_{X,\omega}\|_{C^0}^n.$$

Proof. As in the proof of Lemma 2 of [PSSW1], this follows from a Poincaré-type inequality on Kähler manifolds (M, ω) with ω in $\pi c_1(M)$ (see Theorem 2.4.3 of [F], Lemma 3.1 of [TZ2] or Lemma 2 of [PSSW1]) together with Perelman's non-collapsing result. □

The following is an analogue of the smoothing lemma from [PSSW1] (an adaptation of Bando's smoothing lemma [B] – see also [T] and [CTZ] for related results). It follows from (3.1) and the arguments of [PSSW1].

Proposition 4. *There exist δ, K depending only on n and $C_X = \sup_{t\in[0,\infty)} \|X\|_{C^0}(t)$ so that, for any ε with $0 < \varepsilon \le \delta$ and any $t_0 \ge 0$, if*

$$\|(u_{X,\omega} - b)(t_0)\|_{C^0} \le \varepsilon,$$

then

$$\|\nabla u_{X,\omega}(t_0 + 2)\|_{C^0} + \|(\Delta + X)u_{X,\omega}(t_0 + 2)\|_{C^0} \le K\varepsilon.$$

4. Proof of Theorem 4

We begin by deriving the following analogue for the modified flow of an identity in [PS1],

$$\begin{aligned}\dot{Y}_X = -2\|\bar{\nabla}\bar{\nabla}u\|^2_\theta &+ \int_M (Xu)|\nabla u|^2 e^\theta \omega^n \\ &- \int_M (R_{\bar{k}j} - g_{\bar{k}j} - \nabla_j X_{\bar{k}})\nabla^j u \nabla^{\bar{k}} u e^\theta \omega^n \\ &- \int_M (R - n - \nabla_j X^j)|\nabla u|^2 e^\theta \omega^n.\end{aligned} \tag{4.1}$$

Here $u_{X,\omega}$ and $\theta_{X,\omega}$ have been denoted by just u and θ for simplicity. To establish the above identity, we use (3.1) to obtain

$$\begin{aligned}\partial_t \|\nabla u\|^2_\theta = \int_M (\Delta + X)|\nabla u|^2 e^\theta \omega^n &- \|\nabla\nabla u\|^2_\theta - \|\nabla\bar{\nabla}u\|^2_\theta \\ &+ \int_M |\nabla u|^2 e^\theta \omega^n + \int_M |\nabla u|^2 (Xu) e^\theta \omega^n \\ &- \int_M |\nabla u|^2 (R - n - \nabla_j X^j) e^\theta \omega^n.\end{aligned} \tag{4.2}$$

The first term on the right-hand side of (4.2) actually vanishes since by integration by parts $\int_M e^\theta \omega^n (\Delta + X)\eta = 0$ for any smooth function η. Next, we have a formula of Bochner–Kodaira type, if X^j is a holomorphic vector field and u is a function invariant under Im X,

$$\|\nabla\bar{\nabla}u\|^2_\theta = \|\bar{\nabla}\bar{\nabla}u\|^2_\theta + \int_M R_{\bar{k}j}\nabla^j u \nabla^{\bar{k}} u\, e^\theta \omega^n - \int_M \nabla_j X_{\bar{k}} \nabla^j u \nabla^{\bar{k}} u\, e^\theta \omega^n. \tag{4.3}$$

To establish this, we note that by integration by parts,

$$\begin{aligned}\|\nabla\bar{\nabla}u\|^2_\theta = \|\bar{\nabla}\bar{\nabla}u\|^2_\theta &+ \int_M R_{\bar{k}j}\nabla^j u \nabla^{\bar{k}} u\, e^\theta \omega^n \\ &+ \int_M X_j \nabla_{\bar{p}} \nabla_{\bar{k}} u\, \nabla_q u\, g^{j\bar{p}} g^{q\bar{k}} e^\theta \omega^n \\ &- \int_M X_{\bar{p}} \nabla_j \nabla_{\bar{k}} u\, \nabla_q u\, g^{j\bar{p}} g^{q\bar{k}} e^\theta \omega^n.\end{aligned}$$

Rewrite the integrands of the last two terms in the last line as $X_j \nabla_{\bar{p}} \nabla_{\bar{k}} u \nabla_q u g^{j\bar{p}} g^{q\bar{k}} = \nabla_{\bar{k}}(X^{\bar{p}} \nabla_{\bar{p}} u) \nabla_q u g^{q\bar{k}} - (\nabla_{\bar{k}} X^{\bar{p}}) \nabla_{\bar{p}} u \nabla_q u g^{q\bar{k}}$ and as $X_{\bar{p}} \nabla_j \nabla_{\bar{k}} u \nabla_q u g^{j\bar{p}} g^{q\bar{k}} = \nabla_{\bar{k}}(X^j \nabla_j u) \nabla_q u g^{q\bar{k}}$. But $X^{\bar{p}} \nabla_{\bar{p}} u - X^j \nabla_j u = \bar{X} u - X u = 0$, and thus we are left with the desired formula (4.3). Putting all these identities together gives (4.1).

Once the identity (4.1) is available, the arguments of [PSSW1] apply to give the proof of Theorem 4, with suitable modifications. Write π for the orthogonal projection with respect to the norm $\| \cdot \|_\theta$ of $T^{1,0}$ onto holomorphic vector fields. Then

$$\|\overline{\nabla} \overline{\nabla} u\|_\theta^2 \geq \lambda_X(t) \|\overline{\nabla} u - \pi(\overline{\nabla} u)\|_\theta^2 = \lambda_X(t) \left(\|\overline{\nabla} u\|_\theta^2 - \|\pi(\overline{\nabla} u)\|_\theta^2\right),$$

where $\lambda_X(t)$ is the eigenvalue introduced in §1. Making use of the relations $\|\pi(\overline{\nabla} u)\|_\theta^2 = \int_M \pi(\overline{\nabla} u)^j \partial_j u \, e^\theta \omega^n = -F_X(\pi \overline{\nabla} u)$ we obtain the inequality

$$\begin{aligned} \dot{Y}_X(t) \leq & -2\lambda_X(t) \, Y_X(t) - 2\lambda_X(t) \, F_X(\pi \overline{\nabla} u) + \int_M |\nabla u|^2 (X u) e^\theta \omega^n \\ & - \int_M (R_{\bar{k}j} - g_{\bar{k}j} - \nabla_j X_{\bar{k}}) \nabla^j u \overline{\nabla u^k} \, e^\theta \omega^n \\ & - \int_M (R - n - \nabla_j X^j) |\nabla u|^2 e^\theta \omega^n. \end{aligned} \tag{4.4}$$

We return to the proof of Theorem 4. First, observe that $\|R_{\bar{k}j} - g_{\bar{k}j} - \nabla_j X_{\bar{k}}\|_{L^2} = \|R - n - \nabla_j X^j\|_{L^2}$. This is because one side equals $\|\nabla \overline{\nabla} u\|_{L^2}$ and the other side equals $\|\Delta u\|_{L^2}$, which are readily seen to agree by an integration by parts. Next, we claim that the last three terms on the right-hand side of (4.4) can all be bounded by

$$C \, \|\nabla u\|_{L^2} \, \|(u - b)(t - 2)\|_{C^0}^2.$$

Indeed, since θ is bounded, we can write

$$\begin{aligned} & \left| \int_M (R_{\bar{k}j} - g_{\bar{k}j} - \nabla_j X_{\bar{k}}) \nabla^j u \overline{\nabla^k u} e^\theta \omega^n \right| \\ & \quad \leq C \|\nabla u\|_{C^0} \|\nabla u\|_{L^2} \|R_{\bar{k}j} - g_{\bar{k}j} - \nabla_j X_{\bar{k}}\|_{L^2} \\ & \quad \leq C \, \|\nabla u\|_{L^2} \, \|(u - b)(t - 2)\|_{C^0}^2, \end{aligned}$$

where the last line follows from Proposition 4. Note that if $\|(u - b)(t - 2)\|_{C^0} > \varepsilon$, for ε as in Proposition 4, then we can still obtain the bound

$$\|\nabla u\|_{C^0} \|R - n - \nabla_j X^j\|_{L^2} \leq C \, \|(u - b)(t - 2)\|_{C^0}^2,$$

using the uniform estimates of $\|\nabla u\|_{C^0}$ and $\|\Delta u\|_{C^0}$. Similarly,

$$\left| \int_M (R - n - \nabla_j X^j) |\nabla u|^2 e^\theta \omega^n \right| \leq C \, \|\nabla u\|_{L^2} \, \|(u - b)(t - 2)\|_{C^0}^2,$$

while the estimate for the remaining term, $\left|\int_M |\nabla u|^2 (Xu) e^{\theta} \omega^n\right|$ is even easier, since $|Xu| \leq |X| \cdot |\nabla u| \leq C\, \|(u-b)(t-2)\|_{C^0}$.

Let $0 < \rho := 1/(n+1) < 1$. By Propositions 3 and 4,

$$\|(u-b)(t-2)\|_{C^0}^2 \leq C\, Y_X(t-2)^{\rho} \|(u-b)(t-4)\|_{C^0}^{2(1-\rho)}.$$

Note that the sum of the exponents on either side always match. Iterating,

$$\begin{aligned}\|(u-b)(t-2)\|_{C^0}^2 &\leq C\, Y_X(t-2)^{\rho} Y_X(t-4)^{2(1-\rho)\rho} \|(u-b)(t-6)\|_{C^0}^{2(1-\rho)^2} \\ &\leq C\, Y_X(t-2)^{\frac{\delta_1}{2}} Y_X(t-4)^{\frac{\delta_2}{2}} \\ &\qquad \dots Y_X(t-2N)^{\frac{\delta_N}{2}} \|(u-b)(t-2(N+1))\|_{C^0}^{2(1-\rho)^N},\end{aligned}$$

with $\sum_{j=1}^{N} \delta_j + 2(1-\rho)^N = 2$. Fix N with $2(1-\rho)^N < 1$ and set $\delta_0 = 1$. Since the quantity $\|(u-b)(t-2(N+1))\|_{C^0}$ is bounded by Lemma 2, the statement of Theorem 4 follows.

5. Proof of Theorem 2

The variation of the modified Mabuchi energy along the modified Kähler–Ricci flow is

$$\dot{\mu}_X = -\frac{1}{V} \int_M |\nabla u_{X,\omega}|^2\, e^{\theta_{X,\omega}} \omega^n = -\frac{1}{V} Y_X(t).$$

Integrating in t, we see that condition (A_X) implies: $\int_0^\infty Y_X(t)dt < \infty$. On the other hand, from (4.2) and the uniform bounds of θ, $Xu_{X,\omega}$, R and $\nabla_j X^j$ we obtain $\dot{Y}_X \leq C Y_X$. These inequalities imply (as in Section §2 of [PS1]) that $Y_X(t) \to 0$ as $t \to \infty$. Next, by the uniform bound of $\|\nabla u_{X,\omega}\|_{C^0}$ and Proposition 3 we have $\|u_{X,\omega} - b\|_{C^0} \to 0$ as $t \to \infty$. Then from Proposition 4 we see that $\|\Delta u_{X,\omega}\|_{C^0} \to 0$ as $t \to \infty$. Since $\Delta u_{X,\omega} = R - n - \nabla_j X^j$, the first part of Theorem 2 is established. The L^p integrability of $\|R - n - \nabla_j X^j\|_{C^0}$ on $[0, \infty)$ is established in the same way as part (ii) of Theorem 1 in [PSSW1].

6. Proof of Theorem 1

It is convenient to introduce the following fifth condition:

(o) For each $k = 0, 1, 2, \dots$, there exists a finite constant A_k so that

$$\sup_{t \geq 0} \|\varphi\|_{C^k} \leq A_k.$$

We shall prove (o) $\Leftrightarrow$ (iii), (o) $\Rightarrow$ (iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (i)$\Rightarrow$ (iii).

Proof of (o) $\Leftrightarrow$ (iii)

This is the extension to the modified Kähler–Ricci flow of the fact that a C^0 estimate for the complex Monge–Ampère equation implies C^k estimates to all orders. We note that in [TZ2], a different method is used to obtain higher order estimates, involving a modification of the potential φ along the flow. We give here a direct proof of the higher order estimates, emphasizing only new complications due to $X \neq 0$.

The first step is to show that C^0 estimates for φ imply second order estimates for φ. For ease of notation, we use $\hat{g}_{\bar{k}j}$ to denote the original metric $g^0_{\bar{k}j}$, and $\widehat{\Delta}$ for the Laplacian with respect to this metric. As in the approach of Yau [Y1] and Aubin [A], we apply the maximum principle to $\log(n + \widehat{\Delta}\varphi) - A\varphi$, where A is a large constant to be chosen later, but with the operator $\Delta - \partial_t$ replaced by the operator $\Delta_t + X - \partial_t$. We use the formulas obtained in [PSS] for general flows and introduce the endomorphism

$$h^\alpha{}_\beta = \hat{g}^{\alpha\bar{\gamma}} g_{\bar{\gamma}\beta}.$$

Then $n + \widehat{\Delta}\varphi = \operatorname{Tr} h$, and we have (see e.g. [PSS], eq. (2.27) and subsequent bounds)

$$(\Delta + X - \partial_t) \log \operatorname{Tr} h \geq \frac{1}{\operatorname{Tr} h}\Big(\widehat{\Delta}(\log \frac{\omega^n}{\omega_0^n} - \dot{\varphi}) + X \operatorname{Tr} h\Big) - C_1 \operatorname{Tr} h^{-1}.$$

For the modified Kähler–Ricci flow, we have $\widehat{\Delta}(\log \frac{\omega^n}{\omega_0^n} - \dot{\varphi}) = -\operatorname{Tr} h + n - \widehat{\Delta}\theta - \widehat{\Delta} f(\omega_0)$. The new term compared to the Kähler–Ricci flow is $-\widehat{\Delta}\theta$, which is not yet known to be bounded. This is the reason why the term $X \operatorname{Tr} h$ was introduced, since $X \operatorname{Tr} h = X(\widehat{\Delta}\varphi) = \widehat{\Delta}\theta - \widehat{\Delta}\hat{\theta} + (\widehat{\nabla}_j X^m)\, h^j{}_m - \widehat{\nabla}_m X^m$, where $\hat{\theta} = \theta_{X,\omega_0}$. Thus the term $-\widehat{\Delta}\theta$ cancels out, and we obtain

$$(\Delta + X - \partial_t) \log \operatorname{Tr} h \geq -C_2 - C_3 \operatorname{Tr} h^{-1}.$$

Set $A = C_3 + 1$. Since $\Delta\varphi = n - \operatorname{Tr} h^{-1}$, and $\dot{\varphi},\ X\varphi = \theta - \hat{\theta}$ are bounded, we have

$$(\Delta + X - \partial_t)(\log \operatorname{Tr} h - A\varphi) \geq -C_4 + \operatorname{Tr} h^{-1}.$$

The maximum principle applies now as usual to show that $\operatorname{Tr} h$ is uniformly bounded.

We now give the third order estimates. As in [Y1], set $\varphi_{j\bar{k}m} = \widehat{\nabla}_m \partial_{\bar{k}} \partial_j \varphi$ and $S \equiv g^{j\bar{r}} g^{s\bar{k}} g^{m\bar{t}} \varphi_{j\bar{k}m} \varphi_{\bar{r}s\bar{t}}$. Again, it is convenient to work instead with the connection $\nabla h h^{-1}$,

$$S = g^{m\bar{\gamma}} g_{\bar{\mu}\beta} g^{\ell\bar{\alpha}} (\nabla_m h\, h^{-1})^\beta{}_\ell \overline{(\nabla_\gamma h\, h^{-1})^\mu{}_\alpha} = |\nabla h\, h^{-1}|^2.$$

Applying [PSS] eq. (2.48), to the modified Kähler–Ricci flow gives

$$\begin{aligned}(\Delta - \partial_t) S = {} & |\overline{\nabla}(\nabla h\, h^{-1})|^2 + |\nabla(\nabla h\, h^{-1})|^2 + |\nabla h\, h^{-1}|^2 \\ & + g^{m\bar{\gamma}} \nabla^{\bar{q}} \widehat{R}_{\bar{q}m}{}^\beta{}_\ell \overline{(\nabla_\gamma h\, h^{-1})_{\bar{\beta}}{}^{\bar{\ell}}} + g^{m\bar{\gamma}} (\nabla_m h\, h^{-1})_{\bar{\mu}}{}^{\bar{\alpha}} \overline{\nabla^{\bar{q}} \widehat{R}_{\bar{q}\gamma}{}^\mu{}_\alpha} \\ & + \text{(I)} + \text{(II)} + \text{(III)} + \text{(IV)} + \text{(V)},\end{aligned}$$

where the terms (I)–(V) are due to the modifications arising from the holomorphic vector field X, and given explicitly by

$$\begin{aligned}
\text{(I)} &= \nabla^{\bar\gamma} X^m g_{\bar\mu\beta} g^{\ell\bar\alpha} (\nabla_m h\, h^{-1})^\beta{}_\ell \overline{(\nabla_\gamma h\, h^{-1})^\mu{}_\alpha},\\
\text{(II)} &= -g^{m\bar\gamma} g_{\bar\mu\beta} g^{\ell\bar\alpha} \nabla_m \nabla_\ell X^\beta \overline{(\nabla_\gamma h\, h^{-1})^\mu{}_\alpha},\\
\text{(III)} &= g^{m\bar\gamma} g_{\bar\mu\beta} \nabla^{\bar\alpha} X^\ell (\nabla_m h\, h^{-1})^\beta{}_\ell \overline{(\nabla_\gamma h\, h^{-1})^\mu{}_\alpha},\\
\text{(IV)} &= -g^{m\bar\gamma} g_{\bar\mu\beta} g^{\ell\bar\alpha} (\nabla_m h\, h^{-1})^\beta{}_\ell \overline{\nabla_\gamma \nabla_\alpha X^\mu},\\
\text{(V)} &= -g^{m\bar\gamma} \nabla_\beta X_{\bar\mu} g^{\ell\bar\alpha} (\nabla_m h\, h^{-1})^\beta{}_\ell \overline{(\nabla_\gamma h\, h^{-1})^\mu{}_\alpha}.
\end{aligned}$$

Since the first covariant derivatives of X are of order $O(S^{\frac{1}{2}})$, we have

$$|\text{(I)}| + |\text{(III)}| + |\text{(V)}| \le C_5\, S\, |\nabla X|.$$

The terms (II), (IV) involve the second covariant derivatives of X^m and thus

$$|\text{(II)}| + |\text{(IV)}| \le C_6\, S + \frac{1}{2} |\nabla(\nabla h\, h^{-1})|^2 + S\, |\nabla X| + C_7.$$

Putting this all together, we obtain the following estimate for the flow of S,

$$(\Delta - \partial_t) S \ge \frac{1}{2} |\nabla(\nabla h\, h^{-1})|^2 + |\bar\nabla(\nabla h\, h^{-1})|^2 - C_8\, S\, |\nabla X| - C_9(1+S). \quad (6.1)$$

By the method of [Y1], we can control terms of order $O(S)$ using the evolution equation for $\mathrm{Tr}\, h$. However, we will need an additional argument to deal with the quantity $S|\nabla X|$ which is of the order $O(S^{3/2})$. Since $|X|$ is uniformly bounded along the flow, we have

$$(\Delta - \partial_t)|X|^2 = |\nabla X|^2 - |X|^2 - \partial_i \partial_{\bar j} \theta X^i \overline{X^j} \ge \frac{1}{2} |\nabla X|^2 - C_{10}. \quad (6.2)$$

We define a constant $K = 65 \sup_{M\times[0,\infty)}(|X|^2+1)$ and compute the evolution of the quantity $S/(K-|X|^2)$. Combining (6.1) and (6.2) we have

$$\begin{aligned}
(\Delta - \partial_t)\Big(\frac{S}{K-|X|^2}\Big) \ge{}& \frac{\big(|\nabla(\nabla h\cdot h^{-1})|^2 - |\bar\nabla(\nabla h\cdot h^{-1})|^2\big)}{2(K-|X|^2)} + \frac{S|\nabla X|^2}{2(K-|X|^2)^2}\\
&+ \frac{2\mathrm{Re}(g^{i\bar j}\partial_i S\, \partial_{\bar j}|X|^2)}{(K-|X|^2)^2} + \frac{2S|\nabla|X|^2|^2}{(K-|X|^2)^3}\\
&- \frac{C_8 S|\nabla X|}{K-|X|^2} - C_{11}(1+S).
\end{aligned}$$

We will use the good first and second terms on the right-hand side of this inequality to deal with the bad third and fifth terms. We estimate the third term as follows:

$$\frac{|2g^{i\bar{j}}\partial_i S\,\partial_{\bar{j}}|X|^2|}{(K-|X|^2)^2} \le \frac{S|\nabla X|^2}{4(K-|X|^2)^2} + \frac{\big(|\nabla(\nabla h\cdot h^{-1})|^2 + |\overline{\nabla}(\nabla h\cdot h^{-1})|^2\big)}{2(K-|X|^2)}.$$

For the fifth term, observe that

$$\frac{C_8 S|\nabla X|}{K-|X|^2} \le \frac{S|\nabla X|^2}{4(K-|X|^2)^2} + C_8^2 S.$$

Combining all of the above, we obtain

$$(\Delta-\partial_t)\Big(\frac{S}{K-|X|^2}\Big) \ge -C_{12}(1+S).$$

But from the computation for the second order estimate, we have

$$(\Delta-\partial_t)\mathrm{Tr}\,h \ge \frac{1}{2}S - C_{13},$$

and so applying the maximum principle to the quantity $\big(S/(K-|X|^2) + 3C_{12}\,\mathrm{Tr}\,h\big)$ it follows that $S/(K-|X|^2)$ and hence S is bounded. (An alternative proof is to show that $(\Delta-\partial_t)|T|^2 \ge -B_1 S - B_2$, where $T_j^k = (\nabla_j h\,h^{-1})_l^k X^l$ and B_1, B_2 are constants. Combining this with the evolution of $\mathrm{Tr}\,h$ gives an upper bound of $|T|$ and hence $|\nabla X|$. Thus $S|\nabla X|$ is of order $O(S)$ and one can combine it with (6.1) to bound S.)

In order to apply the standard parabolic estimates to obtain the higher order estimates, we require a derivative bound of $g_{\bar{k}j}$ in the t-direction (cf. §5.5 of [Ch], for example). Given the estimates proved so far, it is sufficient to bound $|\mathrm{Ric}(g)|$. We compute

$$\begin{aligned}(\Delta-\partial_t+X)|\mathrm{Ric}(g)| &= \frac{1}{|\mathrm{Ric}(g)|}\{|\nabla\mathrm{Ric}(g)|^2 - |\nabla|\mathrm{Ric}|\,|^2 + |\mathrm{Ric}(g)|^2\\ &\quad - R_{\bar{k}j}{}^{r\bar{s}}R_{\bar{s}r}R^{j\bar{k}} + \nabla^{\bar{k}}X^p g^{j\bar{q}}R_{\bar{k}j}R_{\bar{q}p} - R_{\bar{k}\ell}\nabla_j X^\ell R^{j\bar{k}}\}\\ &\ge -C_{14}(|\mathrm{Rm}|^2+1).\end{aligned}$$

But from above, there exist constants C_{15}, C_{16} with $C_{15} > 0$ such that

$$(\Delta-\partial_t+X)S \ge C_{15}|\mathrm{Rm}|^2 - C_{16}.$$

We then apply the maximum principle to $|\mathrm{Ric}(g)| + \frac{1}{C_{15}}(C_{14}+1)S$ to bound $|\mathrm{Ric}(g)|$.

We have now established uniform parabolic C^1 estimates for $g_{\bar{k}j}$. The higher order estimates can be obtained in the usual way (see e.g. [L]).

Proof of (o) ⇒ (iv) The remaining implications are all straightforward adaptations of arguments in [PSSW1]. In particular Lemma 3 follows from §5 of [PSSW1], and Lemma 4 from the uniform boundedness of $\theta_{X,\omega}$ and the argument for Lemma 1 of [PSSW2]:

Lemma 3. *Let $W(t)$ be a non-negative C^∞ function of $t \in [0,\infty)$ with $W(t) \leq K_0$ satisfying the difference-differential inequality*

$$\dot W(t) \leq -2\lambda\, W(t) + \lambda \prod_{j=0}^{N} W(t-2j)^{\frac{\nu_j}{2}} \quad \text{for } t \geq K_1 \geq 2N,$$

where λ is a strictly positive constant, and $\nu_j \geq 0$ satisfy $\frac{1}{2}\sum_{j=0}^N \nu_j = 1$. Then there exist constants C, κ with $\kappa > 0$ depending only on K_0, K_1, λ, N, ν_j so that $W(t) \leq C\, e^{-\kappa t}$.

Lemma 4. *There exist constants $c_1, c_2 > 0$ depending only on the complex manifold M and the holomorphic vector field X such that for all $\omega \in \mathcal{K}_X$,*

$$c_1\lambda(\omega) \leq \lambda_X(\omega) \leq c_2\lambda(\omega).$$

Lemma 5. *Let $Y_X(t)$ be given as in (1.3) for the modified Kähler–Ricci flow. Assume that* (a) *$F_X \equiv 0$;* (b) *$Y_X(t) \to 0$ as $t \to \infty$; and* (c) *$\inf_{t\geq 0}\lambda(t) > 0$. Then there exists constants C, κ with $\kappa > 0$ so that $Y_X(t) \leq Ce^{-\kappa t}$.*

Proof of Lemma 5. Theorem 4 and conditions (a)–(c), together with Lemma 4, imply that $Y_X(t)$ satisfies a difference-differential inequality exactly of the type formulated in Lemma 3. The desired inequality follows then from Lemma 3. □

Returning to the proof of (o) ⇒ (iv), assume that (o) holds. Then there exists a sequence $t_m \to \infty$ such that $\varphi(t_m) \to \varphi(\infty)$ in C^∞, for some $\varphi(\infty) \in \mathcal{P}_X(M,\omega_0)$. Since μ_X is decreasing along the modified flow, it follows that for any t, $\mu_X(\varphi(t)) \geq \mu_X(\varphi(\infty))$, and hence μ_X is bounded below along the flow. This implies that the limit metric $g_{\bar k j}(\infty)$ must be a Kähler–Ricci soliton with respect to X (cf. the proof of Theorem 2). By [TZ1], the condition (A_X) is established. Next, we claim that Condition (S) is also satisfied. Otherwise, let $\varphi(t_m)$ be a subsequence with $\lambda(t_m) \to 0$. It contains a subsequence $\varphi(t_\ell)$ such that the corresponding metrics $g_{\bar k j}(t_\ell)$ converge in C^∞ to a Kähler–Ricci soliton $g_{\bar k j}(\infty)$ with respect to X. In [PS1] (see p.162), it was shown that $\lambda(t_\ell) \to \lambda(\infty)$ if $g_{\bar k j}(t_\ell) \to g_{\bar k j}(t_\infty)$ and the dimensions of the holomorphic vector fields of the complex structures for $g_{\bar k j}(t_\ell)$ and $g_{\bar k j}(\infty)$ are the same. In the present case, the complex structures of $g_{\bar k j}(t_\ell)$ and $g_{\bar k j}(\infty)$ are the same, so we do have $\lambda(t_\ell) \to \lambda(\infty)$. Since $\lambda(\infty) > 0$ by definition, we obtain a contradiction. Condition (S) is established.

The existence of a Kähler–Ricci soliton with respect to X implies that (a) in Lemma 5 holds and condition (A_X) gives (b) by Theorem 2. Since (c) in this Lemma is the same as (S), Lemma 5 applies, and (iv) is established.

Proof of (iv) ⇒ (ii) Assume that (iv) is satisfied, and thus $Y_X(t)$ is rapidly decreasing. Then Proposition 3 implies $\|u_{X,\omega} - b\|_{C^0} \leq C\, e^{-\frac{1}{2(n+1)}\kappa t}$. But Proposition 4 implies then that $\|R - n - \nabla_j X^j\|_{C^0} \leq C'\, e^{-\frac{1}{2(n+1)}\kappa t}$ which gives (ii).

Proof of (ii) ⇒ (iii) Assume (ii). Since $\partial_t \log(\omega^n/\omega_0^n) = g^{j\bar{k}} \dot{g}_{\bar{k}j} = -(R - n - \nabla_j X^j)$, we obtain immediately

$$\sup_{t\in[0,\infty)} |\log(\frac{\omega^n}{\omega_0^n})| \leq \int_0^\infty \|R - n - \nabla_j X^j\|_{C^0}\, dt < C.$$

Next, from the modified Kähler–Ricci flow and the uniform bound for $\|\dot{\varphi}\|_{C^0}$ (Lemma 2), it follows that $\|\varphi\|_{C^0} = \|\dot{\varphi} - \log(\frac{\omega^n}{\omega_0^n}) - \theta + f(\omega_0)\|_{C^0} \leq C$.

Proof of (iv) ⇒ (v) Assume (iv). We have already seen that (iv) implies (ii), which implies (iii), which is equivalent to (o). Thus all metrics $g_{\bar{k}j}(t)$ are uniformly equivalent. The same arguments as in [PS1], show that $\|u_{X,\omega}\|_{(s)} \to 0$ exponentially fast for any Sobolev norm s. It follows easily from there that $g_{\bar{k}j}(t)$ converges exponentially fast to a Kähler–Ricci soliton $g_{\bar{k}j}(\infty)$.

All the remaining implications in Theorem 1 are trivial, and the proof is complete.

7. Proof of Theorem 3

If the flow converges to a Kähler–Ricci soliton with respect to X then, by [TZ1], Condition (A_X) is satisfied. Furthermore, as part of the proof of (o) ⇒ (iv), the uniform boundedness of $\|\varphi(t)\|_{C^k}$ for each k implies that Condition (S) is satisfied. Thus it remains only to establish the sufficiency of (A_X) and (S) for the exponential convergence of the flow. By Proposition 1 and Theorem 2, (A_X) implies (a) and (b) of Lemma 5. In addition, (S) gives condition (c). Thus we obtain the exponential decay of $Y_X(t)$, that is, Condition (iv) of Theorem 1 is satisfied. But Theorem 1 implies then the exponential convergence of the modified Kähler–Ricci flow to a Kähler–Ricci soliton.

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References

[A] T. Aubin, Équations du type Monge-Ampère sur les variétés kählériennes compactes. *Bull. Sci. Math.* (2) **102** (1978), no. 1, 63–95. Zbl 0374.53022 MR 0494932

[B] S. Bando, The K-energy map, almost Einstein Kähler metrics and an inequality of the Miyaoka-Yau type. *Tõhuku Math. J.* **39** (1987), 231–235. Zbl 0678.53061 MR 0887939

[C] H.-D. Cao, Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.* **81** (1985), no. 2, 359–372. Zbl 0574.53042 MR 0799272

[CTZ] H.-D. Cao, G. Tian, and X. Zhu, Kähler–Ricci solitons on compact complex manifolds with $c_1(M) > 0$. *Geom. Funct. Anal.* **15** (2005), no. 3, 697–719. Zbl 1084.53035 MR 2221147

[Ch] A. Chau, Convergence of the Kähler–Ricci flow on noncompact Kähler manifolds. *J. Differential Geom.* **66** (2004), 211–232. Zbl 1082.53070 MR 2106124

[CT] X. X. Chen and G. Tian, Ricci flow on Kähler-Einstein surfaces. *Invent. Math.* **147** (2002), 487–544. Zbl 1047.53043 MR 1893004

[DeT] D. DeTurck, Deforming metrics in the direction of their Ricci tensors. *J. Differential Geom.* **18** (1983), 157–162. Zbl 0517.53044 MR 0697987

[D1] S. K. Donaldson, Remarks on gauge theory, complex geometry and 4-manifold topology. In *Fields Medallists' lectures*, ed. by M. F. Atiyah and D. Iagolnitzer, World Scientific Publishing Co., Inc., River Edge, NJ, and Singapore University Press, Singapore 1997, 384–403. Zbl 0913.01020 MR 1622931

[D2] S. K. Donaldson, Scalar curvature and stability of toric varieties. *J. Differential Geom.* **62** (2002), 289–349. Zbl 1074.53059 MR 1988506

[D3] S. K. Donaldson, Lower bounds on the Calabi functional. *J. Differential Geom.* **70** (2005), 453–472. Zbl 1149.53042 MR 2192937

[F] A. Futaki, *Kähler-Einstein metrics and integral invariants*. Lecture Notes in Math. 1314, Springer-Verlag, Berlin 1988. Zbl 0646.53045 MR 0947341

[FM] A. Futaki and T. Mabuchi, Bilinear forms and extremal Kähler vector fields associated with Kähler classes. *Math. Ann.* **301** (1995), 199–210. Zbl 0831.53042 MR 1314584

[H] R. S. Hamilton, Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), 255–306. Zbl 0504.53034 MR 0664497

[L] G. M. Lieberman, *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996. Zbl 0884.35001 MR 1465184

[M] T. Mabuchi, Stability of extremal Kähler manifolds. *Osaka J. Math.* **41** (2004), no. 3, 563–582. Zbl 1076.32017 MR 2107663

[Pa] N. Pali, Characterization of Einstein-Fano manifolds via the Kähler–Ricci flow. *Indiana Univ. Math. J.* **57** (2008), no. 7, 3241–3274. Zbl 1165.53050 MR 2492232

[P] G. Perelman, The entropy formula for the Ricci flow and its geometric applications. Preprint 2002. arXiv:math.DG/0211159 [math.DG] Zbl 1130.53001

[PSS] D. H. Phong, N. Sesum, and J. Sturm, Multiplier ideal sheaves and the Kähler–Ricci flow. *Comm. Anal. Geom.* **15** (2007), no. 3, 613–632. Zbl 1143.53064 MR 2379807

[PS1] D. H. Phong and J. Sturm, On stability and the convergence of the Kähler–Ricci flow. *J. Differential Geom.* **72** (2006), no. 1, 149-168. Zbl 1125.53048 MR 2215459

[PS2] D. H. Phong and J. Sturm, Lectures on stability and constant scalar curvature. In *Current developments in mathematics, 2007*, ed. by D. Jerison, B. Mazur, T. Mrowka, W. Schmid, R. Stanley, and S.-T. Yau, International Press, Somerville, MA, 2009, 101–176. Zbl 05578296 MR 2532997

[PSSW1] D. H. Phong, J. Song, J. Sturm, and B. Weinkove, The Kähler–Ricci flow and the $\overline{\partial}$ operator on vector fields. *J. Differential Geom.* **81** (2009), 631–647. Zbl 1162.32014 MR 2487603

[PSSW2] D. H. Phong, J. Song, J. Sturm, and B. Weinkove, The Kähler–Ricci flow and positive bisectional curvature. *Invent. Math.* **173** (2008), no. 3, 651–665. Zbl 1145.53050 MR 2425138

[ST] N. Sesum and G. Tian, Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman). *J. Inst. Math. Jussieu* **7** (2008), no. 3, 575–587. Zbl 1147.53056 MR 2427424

[Sz1] G. Szekelyhidi, Extremal metrics and K-stability. *Bull. London Math. Soc.* **39** (2007), no. 1, 76–84. Zbl 1111.53057 MR 2303522

[Sz2] G. Szekelyhidi, The Kähler–Ricci flow and K-stability. *Amer. J. Math.* **132** (2010), no. 4, 1077–1090. Zbl 05776008 MR 2663648

[T] G. Tian, Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.* **130** (1997), 1–37. Zbl 0892.53027 MR 1471884

[TZ1] G. Tian and X. H. Zhu, A new holomorphic invariant and uniqueness of Kähler–Ricci solitons. *Comment. Math. Helv.* **77** (2002), 297–325. Zbl 1036.53053 MR 1915043

[TZ2] G. Tian and X. H. Zhu, Convergence of Kähler–Ricci flow. *J. Amer. Math. Soc.* **20** (2007), no. 3, 675–699. Zbl 1185.53078 MR 2291916

[Y1] S. T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I. *Comm. Pure Appl. Math.* **31** (1978), 339–411. Zbl 0369.53059 MR 0480350

[Y2] S. T. Yau, Open problems in geometry. *Proc. Symposia Pure Math.* **54** (1993), 1–28. Zbl 0801.53001 MR 1216573

[Ye] R. Ye, The logarithmic Sobolev inequality along the Ricci flow. Preprint 2007. arXiv:0707.2424v4 [math.DG]

[WZ] X.-J. Wang, and X. H. Zhu, Kähler–Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.* **188** (2004), no. 1, 87–103. Zbl 1086.53067 MR 2084775

[Zha] Q. S. Zhang, A uniform Sobolev inequality under Ricci flow. *Internat. Math. Res. Notices* **2007** (2007), no. 17, Art. ID rnm056; Erratum, *Internat. Math. Res. Notices* **2007** (2007), no. 19, Art. ID rnm096; Addendum, *Internat. Math. Res. Notices* **2008** (2008), no. 1, Art. ID rnm 138. Zbl 1141.53064 Zbl 1142.53050 Zbl 1155.53327 MR 2354801 MR 2359549 MR 2417793

[Zhu1] X. H. Zhu, Kähler–Ricci soliton typed equations on compact complex manifolds with $C_1(M) > 0$. *J. Geom. Anal.* **10** (2000), 759–774. Zbl 1036.53054 MR 1817785

[Zhu2] X. H. Zhu, Kähler-Ricci flow on a toric manifold with positive first Chern class. Preprint 2007. arXiv:math/0703486v1 [math.DG]

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