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On the centralizer of diffeomorphisms of the half-line

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Abstract. Let f be a smooth diffeomorphism of the half-line fixing only the origin and Z^r its centralizer in the group of C^r diffeomorphisms. According to well-known results of Szekeres and Kopell, Z^1 is a one-parameter group. On the other hand, Sergeraert constructed an f whose centralizer Z^r, $2 \leq r \leq \infty$, reduces to the infinite cyclic group generated by f. We show that Z^r can actually be a proper dense and uncountable subgroup of Z^1 and that this phenomenon is not scarce.

Mathematics Subject Classification (2010). 37E05.

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Let f be a smooth diffeomorphism of the closed half-line $\mathbb{R}_+$ with a single fixed point at the origin. In this article, we study the centralizer of f in the group D^r of C^r diffeomorphisms of $\mathbb{R}_+$, $1 \leq r \leq \infty$, that is, the (closed) subgroup Z_f^r of D^r made up of all diffeomorphisms commuting with f. The first things to observe are that Z^r decreases with r, contains the infinite cyclic subgroup generated by f and is quite small. Indeed, for $r = 1$, well-known theorems by G. Szekeres and N. Kopell [7], [4] show that Z_f^1 is always a one-parameter subgroup of D^1 (see also Chapter 4 in [9] and Chapter 4 in [5] for complete proofs and more discussion). For $r \geq 2$, the situation is more subtle, and for instance both of the limit cases permitted by the inclusions

$$\mathbb{Z} \cong \{f^n,\ n \in \mathbb{Z}\} \subset Z_f^r \subset Z_f^1 \cong \mathbb{R}$$

can occur. According to F. Takens' work [8], if f is not infinitely tangent to the identity at 0 then Z_f^1 consists of smooth diffeomorphisms and therefore coincides with Z_f^∞. On the other hand, in [6], F. Sergeraert builds a diffeomorphism f whose centralizer Z_f^2 is strictly contained in Z_f^1, and one can actually check [2] that, in this example, Z_f^2 reduces to the group spanned by f – and is hence as small as possible. The following result says that there exist intermediate situations:

Theorem A. *There exists a smooth diffeomorphism f of $\mathbb{R}_+$ with a single fixed point at the origin, whose centralizer Z_f^r, for $2 \leq r \leq \infty$, is a proper, dense and uncountable subgroup of the one-parameter group Z_f^1.*

This theorem follows from the proposition below, where f is the flow at time one of the vector field ξ coming out:

Proposition 1. *There exists a complete C^1 vector field ξ on $\mathbb{R}_+$, vanishing only at 0, whose flow f^t at time t is not C^2 at 0 for $t = 1/2$ but is smooth on $\mathbb{R}_+$ for all $t \in \mathbb{Z} \oplus \sum_{\tau\in K} \tau\mathbb{Z}$, where $K \subset \mathbb{R} \setminus \mathbb{Q}$ is a Cantor set.*

A natural question to ask now is whether diffeomorphisms f whose centralizer Z^r_f, $r \geq 2$, is neither the one-parameter group generated by f (namely, $Z^1_f \cong \mathbb{R}$) nor the discrete group spanned by f (that is, $\{f^n, n \in \mathbb{Z}\} \cong \mathbb{Z}$), are very peculiar or not. At the end of the paper, Theorem B gives a partial answer to this question: every diffeomorphism of $\mathbb{R}_+$ which satisfies a certain oscillation condition and belongs to a smooth flow (with the usual hypotheses on the unique fixed point) can be approximated in a suitable sense by diffeomorphisms f whose centralizer Z^r_f is as in Theorem A. The proof of this second theorem is very similar to that of the first one but involves more technicalities. For this reason, we discuss the weaker statement in priority.

It would also be interesting to know whether the centralizer Z^r_f, when it is a proper subgroup of $\mathbb{R} \cong Z^1_f$, can contain any Diophantine number. It turns out [2] that the Cantor set we construct in our proof of Proposition 1 contains only Liouville numbers.

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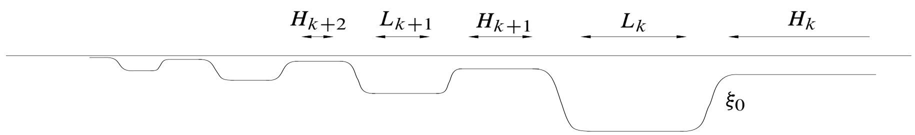
Proof of Proposition 1

1. Overview. The following proof combines the strategy used by F. Sergeraert in [6], Section 4, with the method of approximation by conjugation introduced by D. Anosov and A. Katok in [1] and later developped by many authors (see [3] and references therein). We start with a particular smooth vector field ξ_0 (the same as in [6]) and build ξ as the limit of a sequence of deformations ξ_k where each ξ_k is the pullback

$h_k^*\xi_0$ of ξ_0 by a smooth diffeomorphism h_k. Thus, the flow f_k^t of ξ_k is related to the flow f_0^t of ξ_0 by $f_k^t = h_k^{-1} \circ f_0^t \circ h_k$. The point is to cook up the conjugations h_k so that the diffeomorphisms f_k^t, $k \geq 1$, converge in the C^∞ topology for a dense set of times t but converge only in the C^1 topology for some other time. In particular, the diffeomorphisms h_k diverge in the C^2 topology. Here, the behaviour of the initial vector field plays a crucial role: we take a vector field ξ_0 presenting plateaux which accumulate at the origin and whose heights tend to zero but with wild oscillations. According to a theorem of F. Sergeraert [6], Section 3, these oscillations are necessary if we want to create a non-smooth flow with small perturbations h_k of the identity. Furthermore, Theorem B at the end of this paper states an oscillation condition which is sufficient for our construction to work.

Let us indicate now how these oscillations come into play. First of all, we pick an initial vector field ξ_0 vanishing only at the origin, and contracting: every point is attracted by 0 in the future. Or, in other words, the function ξ_0/∂_x is negative away from 0. The graph of this function can then be depicted as an undersea landscape consisting of a sequence of alternating lowlands L_k and highlands H_k whose respective altitudes $-v_k$ and $-u_k$ (measured from the water surface, so that $0 < u_k < v_k$) go to zero when k grows, but "oscillate wildly" in the sense that the ratios v_k/u_k tend to infinity.



A consequence of this behaviour is that, if an element f_0^t of the flow takes a segment $S \subset H_k$ into L_k for some large k, then its restriction to S is an affine map with big dilation factor v_k/u_k.

In our deformation process, the diffeomorphisms h_k are defined inductively and all coincide with the identity near 0. Each new perturbation is described by the diffeomorphism $g_k = h_k \circ h_{k-1}^{-1}$ and its role is to modify the flow of ξ_0 locally at a specific time $1/q_k$, in a fundamental segment S_k of f_0^{1/q_k} lying in the lowland L_k. In other words, $g_k^{-1} \circ f_0^{1/q_k} \circ g_k$ agrees with f_0^{1/q_k} outside S_k. Furthermore, we take g_k close enough to the identity so that the C^k norms of the maps

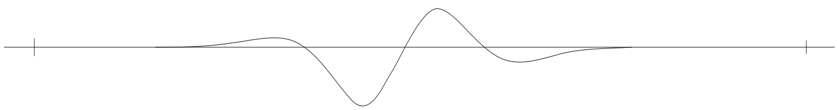
$$g_k^{-1} \circ f_0^t \circ g_k - f_0^t, \quad t \in \tfrac{1}{q_k}\mathbb{Z} \cap [0,1],$$

and also

$$h_k^{-1} \circ f_0^t \circ h_k - h_{k-1}^{-1} \circ f_0^t \circ h_{k-1},$$

are all strictly bounded by 2^{-k}, and we denote by I_k a compact neighbourhood of $\frac{1}{q_k}\mathbb{Z} \cap (0,1)$ such that the non-strict bounds still hold for all $t \in I_k$. With a suitable choice of the sequence q_k, we can arrange that the intersection of the compact sets I_k is a Cantor set K consisting of irrational times t for which the diffeomorphisms $h_k^{-1} \circ f_0^t \circ h_k$, $k \geq 1$, converge in the C^∞ topology. Indeed, it suffices to pick q_k at each step in such a way that $\frac{1}{q_k}\mathbb{Z}$ meets any component of I_{k-1} in at least two points, and also avoids the k^{th} rational number (for an arbitrary enumeration of $\mathbb{Q}$) so that $K = \bigcap I_k$ has no rational point.

Although the action of the perturbation diffeomorphism g_k on the map f_0^{1/q_k} is local, its action on the vector field ξ_0 and on general elements of its flow is not at all. To see this, let us consider the difference $\nu_k = g_k^*\xi_0 - \xi_0$. Since g_k commutes with f_0^{1/q_k} out of S_k and coincides with the identity near 0, it is actually the identity on the whole interval $[0, \min S_k]$. In particular, ν_k vanishes identically there. Inside S_k, our choice of g_k gives ν_k the following shape of a C^k-small wave:



On the other hand, the half-line $[\max S_k, +\infty)$ is tiled by the segments $S_k^p = f_0^{-p/q_k}(S_k)$, $p \geq 1$. The commutation property noted above now implies that, for every $p \geq 1$,

$$\nu_{k\,|S_k^p} = \big(f_0^{p/q_k}\big)^*(\nu_{k\,|S_k}). \tag{1}$$

In other words, the wave $\nu_{k\,|S_k}$ is propagated to the right of S_k by the iterates of f_0^{1/q_k}. Let us look at the wave $\nu_{k\,|S_k^p}$ when S_k^p sits on the highland H_k. As explained before, the restriction of f_0^{p/q_k} to S_k^p for such a p is an affine map of the form

$$x \in S_k^p \mapsto \frac{v_k}{u_k}\,x + c_k \quad \text{for some } c_k \in \mathbb{R}.$$

Then, according to (1),

$$\big(\nu_{k\ |S_k^p}\big)(x) = \frac{(\nu_{k\,|S_k})\big(f_0^{p/q_k}(x)\big)}{Df_0^{p/q_k}(x)} = \left(\frac{v_k}{u_k}\right)^{-1}(\nu_{k\,|S_k})\left(\frac{v_k}{u_k}\,x + c_k\right),$$

and so, for any integer $m \geq 1$,

$$D^m\big(\nu_{k\ |S_k^p}\big)(x) = \left(\frac{v_k}{u_k}\right)^{m-1} D^m(\nu_{k\ |S_k})\left(\frac{v_k}{u_k}\,x + c_k\right).$$

Thus, in the course of the propagation, the wave remains C^1 small but its higher order derivatives are amplified and can become big. As we already said, the difficulty is then to adjust the perturbation diffeomorphisms g_k so that the differences $h_k^*\xi_0 - \xi_0$

(which are essentially the superpositions of the propagated waves ν_l, $l \leq k$) diverge in the C^2 topology while the conjugates $h_k^{-1} \circ f_0^t \circ h_k$, for t in the Cantor set K, still converge in the C^∞ topology. Following Sergeraert, a solution is roughly to take u_k and v_k respectively equal to 2^{-k^4} and 2^{-k^2}, while the size of the wave $\nu_{k\ |S_k}$ is defined as 2^{-k^3}.

2. Notation and toolbox. In this short section, we assume that all necessary conditions are met so that the expressions we write make sense. For any C^k map g defined on an interval $I \subset \mathbb{R}$ (open or closed), we set

$$\|g\|_k = \sup\{|D^m g(x)|,\ 0 \leq m \leq k,\ x \in I\} \in [0, +\infty].$$

If $f : I \to f(I)$ is an orientation-preserving C^2 diffeomorphism, we define Lf to be

$$Lf = D \log Df = \frac{D^2 f}{Df}.$$

The non-linear differential operator L satisfies the following chain rule:

$$L(h \circ g) = Lh \circ g \cdot Dg + Lg. \tag{2}$$

To compute higher order derivatives of compositions, we will also use Faà di Bruno's formula in the form

$$D^m(h \circ g) = \sum_{\pi \in \Pi_m} \big(D^{|\pi|}h\big) \circ g \cdot \prod_{B \in \pi} D^{|B|}g, \tag{3}$$

where Π_m is the set of all partitions π of $\{1, \ldots, m\}$ and $|X|$, for any finite set X, is the number of its elements.

Let η be a vector field on an interval J. Throughout the paper, we will make no difference between η and the function η/∂_x, where x is the underlying coordinate in J, and in particular we will identify ∂_x with 1. If J is both the source of g and the target of h (where g and h are diffeomorphisms), we can define two new vector fields, $g_*\eta$ and $h^*\eta$, which are the pushforward of η by g and its pullback by h, respectively. Viewed as functions, these vector fields are given by

$$g_*\eta = Dg \circ g^{-1} \cdot \eta \circ g^{-1}, \tag{4}$$

$$h^*\eta = \frac{\eta \circ h}{Dh} \tag{5}$$

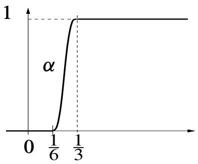
and so we easily get the following expressions for the derivatives:

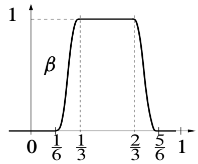
$$D(g_*\eta) = D\eta \circ g^{-1} + Lg \circ g^{-1} \cdot \eta \circ g^{-1}, \tag{6}$$

$$D(h^*\eta) = D\eta \circ h - \frac{D^2 h}{(Dh)^2}\, \eta \circ h. \tag{7}$$

3. The initial vector field. The construction involves two smooth functions α, $\beta\colon \mathbb{R} \to [0,1]$ satisfying the following conditions:

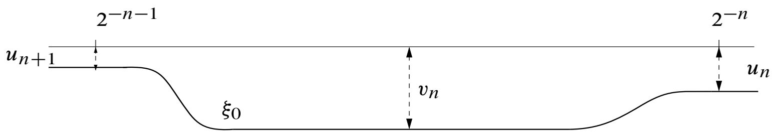
- $\alpha(x)$ equals 0 if $x \le 1/6$ and 1 if $x \ge 1/3$;
- $\beta(x)$ equals 0 if $x \le 1/6$ or $x \ge 5/6$, and 1 if $1/3 \le x \le 2/3$;





Now, setting $u_n = 2^{-n^4}$ and $v_n = 2^{-n^2}$, we define the vector field ξ_0 as in [6] by

$$\xi_0(x) = -u_{n+1} - (u_n - u_{n+1})\,\alpha(2^{n+1}x - 1) - (v_n - u_n)\,\beta(2^{n+1}x - 1)$$
$$\text{for } x \in [2^{-n-1}, 2^{-n}], \quad \xi_0(0) = 0 \quad \text{and} \quad \xi_0(x) = -1 \quad \text{for } x \ge 1.$$



From now on, we denote by $\{f_0^t, t \in \mathbb{R}\}$ the flow of ξ_0 and by $\psi\colon \mathbb{R} \to \mathbb{R}_+^*$ the diffeomorphism given by $\psi(t) = f_0^t(1)$ for all $t \in \mathbb{R}$. Note that, since $D\psi = \xi_0 \circ \psi$,

$$\xi_0 = D\psi \circ \psi^{-1} \quad \text{and} \quad D\xi_0 = L\psi \circ \psi^{-1}. \tag{8}$$

We also fix a forward orbit $\{a_l, l \ge 0\}$ of $f_0 = f_0^1$, where $a_0 = 1$ and $a_l = f_0(a_{l-1}) = \psi(l)$ for all $l \ge 1$.

One easily checks that ξ_0 is smooth, contracting, infinitely flat at the origin and C^1-bounded – with $1 < \|\xi_0\|_1 < +\infty$. Furthermore, ξ_0 equals $-v_n$ identically on the central third of $[2^{-n-1}, 2^{-n}]$, namely $[2^{-n-1} + 2^{-n-1}/3, 2^{-n} - 2^{-n-1}/3]$, and $-u_n$ on $[2^{-n} - 2^{-n-1}/6, 2^{-n} + 2^{-n}/6]$. A simple computation of travel time at constant speed shows that for all $n \ge 4$, there exist integers $i(n)$ and $j(n)$ such that

$$2^{-n} - \tfrac{1}{6}2^{-n-1} \le a_{i(n)+2} < a_{i(n)-1} \le 2^{-n} + \tfrac{1}{6}2^{-n} \tag{9}$$

and

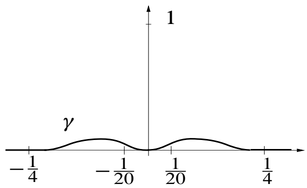
$$2^{-n-1} + \tfrac{1}{3}2^{-n-1} \le a_{j(n)+2} < a_{j(n)-1} \le 2^{-n} - \tfrac{1}{3}2^{-n-1}. \tag{10}$$

Thus ξ_0 equals $-v_n$ on $[a_{j(n)+2}, a_{j(n)-1}]$, and hence f_0^t induces on $[a_{j(n)+1}, a_{j(n)-1}]$ the translation by $-tv_n$ for $0 \le t \le 1$. Similarly, f_0^t induces the translation by $-tu_n$ in a neighbourhood of $a_{i(n)}$.

4. The deformation process. Our goal is now to produce a sequence h_k of smooth diffeomorphisms of $\mathbb{R}_+$ such that the vector fields $\xi_k = h_k^*\xi_0$ converge in the C^1 topology to the vector field ξ of Proposition 1. In order to have regular perturbation patterns (and easier computations), we actually work at time scale, *i.e.* we define h_k as the conjugate $\psi \circ \Phi_k \circ \psi^{-1}$ of a smooth diffeomorphism Φ_k of $\mathbb{R}$ (which coincides with the identity near $+\infty$ so that h_k is also the identity near 0). Conforming to the general scheme of the approximation by conjugation method (see [3]), Φ_k is obtained as a composition

$$\Phi_k = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1$$

where the diffeomorphisms φ_k are manufactured inductively from a fixed function γ and two adjustment integer parameters q_k and n_k. The details of the construction follow.



Let $\gamma\colon \mathbb{R} \to [0,1]$ be a smooth function supported in $[-1/4, 1/4]$ and satisfying $\gamma(t) = t^2/2$ around 0. Given positive integers q, n, set $w_n = 2^{-n^3}$ and denote by $\gamma_{q,n}\colon \mathbb{R} \to [0,1]$ the smooth function defined by

$$\gamma_{q,n}(t) = w_n \gamma\big(q\big(t - j(n)\big)\big) \quad \text{for all } t \in \mathbb{R}. \tag{11}$$

Clearly, $\gamma_{q,n}$ is supported in $\big[j(n) - \frac{1}{4q}, j(n) + \frac{1}{4q}\big]$. Moreover, for every integer $m \geq 1$ and all $t \in \mathbb{R}$,

$$D^m\gamma_{q,n}(t) = w_n q^m D^m \gamma\big(q\big(t - j(n)\big)\big)$$

and hence

$$\|\gamma_{q,n}\|_m = w_n q^m \|\gamma\|_m.$$

In particular, by taking n large compared to q once m is fixed, one can make the C^m norm of $\gamma_{q,n}$ arbitrarily small.

Now let $J_{q,n}$ be the interval $\big[j(n) - \frac{1}{2q}, j(n) + \frac{1}{2q}\big]$ and define $\varphi_{q,n}\colon \mathbb{R} \to \mathbb{R}$ as the map meeting the following properties:

- $\varphi_{q,n}(t) = t$ for $t > j(n) + \frac{1}{2q}$;
- $\varphi_{q,n}(t) = t + \gamma_{q,n}(t)$ for $t \in J_{q,n}$;
- $\varphi_{q,n}$ commutes with the translation by $\frac{1}{q}$ outside $J_{q,n}$, and so

$$\varphi_{q,n}(t) = t + \gamma_{q,n}\Big(t + \frac{p}{q}\Big) \quad \text{if } t \in \Big(J_{q,n} - \frac{p}{q}\Big),\ p \geq 0.$$

In short, we can write

$$\varphi_{q,n}(t) = t + \sum_{p\geq 0} \gamma_{q,n}\Big(t + \frac{p}{q}\Big) \quad \text{for all } t \in \mathbb{R}, \tag{12}$$

and similarly,

$$D\varphi_{q,n}(t) = 1 + \sum_{p\geq 0} D\gamma_{q,n}\Big(t + \frac{p}{q}\Big),$$
$$D^m\varphi_{q,n}(t) = \sum_{p\geq 0} D^m\gamma_{q,n}\Big(t + \frac{p}{q}\Big) \quad \text{for all } m \geq 2.$$

Note that for every $t \in \mathbb{R}$, at most one term in each sum is nonzero since the support of $\gamma_{q,n}$ has length less than $1/q$. These equations imply that

$$\|\varphi_{q,n} - \mathrm{id}\|_m = \|\gamma_{q,n}\|_m \tag{13}$$

and, in particular, $\varphi_{q,n}$ is a diffeomorphism provided $\|\gamma_{q,n}\|_1 < 1$.

The following lemma will be used later (in the proof of Lemma 4) to show that the limit flow coming out of our construction is not smooth at time $1/2$:

Lemma 2. *For all $l \geq 1$, let q_l and n_l be positive integers with q_l odd and $w_{n_l} q_l \|\gamma\|_1 < 1$. Then for every $k \geq 1$ the diffeomorphism Φ_k defined by*

$$\Phi_k = \varphi_k \circ \varphi_{k-1} \circ \cdots \circ \varphi_1, \quad \textit{where } \varphi_l = \varphi_{q_l,n_l},$$

has the following behaviour on $\frac{1}{2}\mathbb{Z}$:

- *Φ_k coincides with the identity in a neighbourhood of $\mathbb{Z} + \frac{1}{2}$;*
- *Φ_k is tangent to the identity on $\mathbb{Z}$ – meaning that $\Phi_k(l) = l$ and $D\Phi_k(l) = 1$ for all $l \in \mathbb{Z}$;*
- *$(L\Phi_k - L\Phi_{k-1})(l)$, for $l \in \mathbb{Z}$, equals $w_{n_k} q_k^2$ if $l \leq j(n_k)$ and 0 otherwise.*

Proof. Since $\gamma_l = \gamma_{q_l,n_l}$ is supported in $\big[-\frac{1}{4q_l}, \frac{1}{4q_l}\big] + j(n_l)$ and

$$\varphi_l = \mathrm{id} + \sum_{p\geq 0} \gamma_l\Big(t + \frac{p}{q_l}\Big),$$

φ_l is the identity on the $\frac{1}{4q_l}$-neighbourhood of $\frac{1}{q_l}\mathbb{Z} + \frac{1}{2q_l}$. But q_l is odd, say $q_l = 2s_l + 1$, so

$$\frac{1}{2} = \frac{q_l}{2q_l} = \frac{2s_l + 1}{2q_l} = \frac{s_l}{q_l} + \frac{1}{2q_l} \in \frac{1}{q_l}\mathbb{Z} + \frac{1}{2q_l},$$

and hence $\frac{1}{q_l}\mathbb{Z} + \frac{1}{2q_l}$ contains $\mathbb{Z} + \frac{1}{2}$ for all $l \geq 1$. Therefore $\Phi_k = \varphi_k \circ \cdots \circ \varphi_1$ is the identity in a neighbourhood of $\mathbb{Z} + \frac{1}{2}$. On the other hand, since $\gamma(0) = D\gamma(0) = 0$, each φ_l is C^1 tangent to the identity on $\frac{1}{q_l}\mathbb{Z} \supset \mathbb{Z}$, so Φ_k is C^1 tangent to the identity on $\mathbb{Z}$.

Now, applying the chain rule (2) for the operator $L = D^2/D$ to $\Phi_k = \varphi_k \circ \Phi_{k-1}$, we get

$$L\Phi_k = L\varphi_k \circ \Phi_{k-1} \cdot D\Phi_{k-1} + L\Phi_{k-1}.$$

For $l \in \mathbb{Z}$, we have seen above that $\Phi_{k-1}(l) = l$ and $D\Phi_{k-1}(l) = 1$, so

$$(L\Phi_k - L\Phi_{k-1})(l) = L\varphi_k(l).$$

If $l > j(n_k)$ then $L\varphi_k(l) = 0$ just because φ_k agrees with the identity on the interval $[j(n_k) + \frac{1}{2q_k}, +\infty)$. If $l \leq j(n_k)$ then (11) and (12) give

$$L\varphi_k(l) = \frac{D^2\varphi_k(l)}{D\varphi_k(l)} = D^2\varphi_k(l) = D^2\gamma_k\big(j(n_k)\big) = w_{n_k} q_k^2,$$

which completes the proof. □

For the next lemma, we fix an enumeration of the rational numbers, $\mathbb{Q} = \{r_k\}_{k\geq 1}$, and set $\Phi_0 = \mathrm{id}$ and $I_0 = [0, 1]$. Moreover, as in Lemma 2, we will henceforth abbreviate φ_{q_l,n_l} as φ_l (and similarly γ_{q_l,n_l} as γ_l and J_{q_l,n_l} as J_l).

Lemma 3. *For suitably chosen increasing sequences of positive integers q_k and n_k, the diffeomorphisms $\Phi_k = \varphi_k \circ \cdots \circ \varphi_1$ and $h_k = \psi \circ \Phi_k \circ \psi^{-1}$, the vector fields $\xi_k = h_k^* \xi_0$ and their flows f_k^t satisfy the following estimates for every $k \geq 1$:*

$$\|\Phi_k - \Phi_{k-1}\|_{k+1} \leq 2^{-k-1}, \qquad (\mathrm{i}_k)$$

$$\|\xi_k - \xi_{k-1}\|_1 \leq 2^{-k}, \qquad (\mathrm{ii}_k)$$

$$\|\big(f_k^t - f_{k-1}^t\big)\,|_{[0,1]}\|_k \leq 2^{-k} \quad \textit{for all} \quad t \in I_k \cup \{1\}, \qquad (\mathrm{iii}_k)$$

where $I_k \subset I_{k-1}$ is a compact set avoiding the k^{th} rational number r_k and consisting of 2^k disjoint segments of nonzero length, two in each component of I_{k-1}.

Proof. Let $k \geq 1$ and assume we already chose q_l and n_l for $1 \leq l \leq k-1$ in such a way that estimates (i_l), (ii_l) and (iii_l) hold. In particular, since $\Phi_0 = \mathrm{id}$ by convention,

$$\|\Phi_{k-1} - \mathrm{id}\|_2 \leq \sum_{l=1}^{k-1} \|\Phi_l - \Phi_{l-1}\|_2 \leq \sum_{l=1}^{k-1} 2^{-l-1} = \frac{1}{2} - 2^{-k} \leq \frac{1}{2}. \qquad (14)$$

Take an odd integer $q_k > q_{k-1}$ such that $\frac{1}{q_k}\mathbb{Z}$ avoids r_k and meets the interior of each component of I_{k-1} in at least two points. Then pick $n_k > n_{k-1}$ such that

$$\|\gamma_k\|_{k+1} \leq \frac{2^{-k-4}\ v_{n_k}^{k-1}}{|\Pi_{k+1}|\ \|D\Phi_{k-1}\|_k^{k+1}\ \|\xi_0\|_1}, \tag{15}$$

i.e.

$$\frac{w_{n_k}}{v_{n_k}^{k-1}} \leq \frac{2^{-k-4}\ q_k^{-k-1}}{|\Pi_{k+1}|\ \|\gamma\|_{k+1}\ \|D\Phi_{k-1}\|_k^{k+1}\ \|\xi_0\|_1},$$

which is possible since

$$\frac{w_n}{v_n^{k-1}} = 2^{-n^3+(k-1)n^2} = \mathrm{o}(1).$$

Note that inequality (15) clearly implies $\|\gamma_k\|_1 < 1$, and so φ_k is a diffeomorphism (remember that $\|\varphi_k - \mathrm{id}\|_m = \|\gamma_k\|_m$).

Let us first prove that this choice of n_k implies (i_k). Since $\Phi_k = \varphi_k \circ \Phi_{k-1}$, Faà di Bruno's formula (3) gives, for $0 \leq m \leq k+1$,

$$D^m(\Phi_k - \Phi_{k-1}) = \sum_{\pi\in\Pi_m} D^{|\pi|}(\varphi_k - \mathrm{id}) \circ \Phi_{k-1} \cdot \prod_{B\in\pi} D^{|B|}\Phi_{k-1}.$$

But for every partition $\pi \in \Pi_m$ with $m \leq k+1$,

$$\|D^{|\pi|}(\varphi_k - \mathrm{id}) \circ \Phi_{k-1}\|_0 = \|\gamma_k\|_{|\pi|} \leq \|\gamma_k\|_{k+1}$$

and

$$\prod_{B\in\pi} |D^{|B|}\Phi_{k-1}| \leq \|D\Phi_{k-1}\|_k^{k+1},$$

and so

$$\|\Phi_k - \Phi_{k-1}\|_{k+1} \leq |\Pi_{k+1}|\ \ \|\gamma_k\|_{k+1}\, \|D\Phi_{k-1}\|_k^{k+1}.$$

Thus, by the choice of n_k in (15),

$$\|\Phi_k - \Phi_{k-1}\|_{k+1} \leq \frac{2^{-k-4}\ \ v_{n_k}^{k-1}}{\|\xi_0\|_1} \leq 2^{-k-1},$$

which is the desired estimate (i_k) (note that $\|\xi_0\|_1 \geq 1$).

To prove (ii_k), let us define

$$\eta_k = \Phi_k^*\partial_t - \Phi_{k-1}^*\partial_t \quad \text{and} \quad \zeta_k = \varphi_k^*\partial_t - \partial_t,$$

so that

$$\eta_k = \Phi_{k-1}^*\zeta_k \quad \text{and} \quad \xi_k - \xi_{k-1} = \psi_*\eta_k.$$

Viewing ζ_k as a function,

$$\zeta_k = \frac{1}{D\varphi_k} - 1 \quad \text{and} \quad D\zeta_k = -\frac{D^2\varphi_k}{(D\varphi_k)^2}.$$

Given the choice of n_k in (15),

$$\|D\varphi_k - 1\|_0 = \|D\gamma_k\|_0 \leq 2^{-k-4}\ \|\xi_0\|_1^{-1} \quad (\text{and so} \left\|\frac{1}{D\varphi_k}\right\|_0 \leq 2),$$

and

$$\|D^2\varphi_k\|_0 = \|D^2\gamma_k\|_0 \leq 2^{-k-4}\ \|\xi_0\|_1^{-1},$$

so

$$|\zeta_k| \leq 2^{-k-3}\ \|\xi_0\|_1^{-1} \quad \text{and} \quad |D\zeta_k| \leq 2^{-k-2}\ \|\xi_0\|_1^{-1}. \tag{16}$$

Next, applying (5) and (7) to $\eta_k = \Phi_{k-1}^*\zeta_k$,

$$\eta_k = \frac{\zeta_k \circ \Phi_{k-1}}{D\Phi_{k-1}} \quad \text{and} \quad D\eta_k = D\zeta_k \circ \Phi_{k-1} - \frac{D^2\Phi_{k-1}}{(D\Phi_{k-1})^2}\ \zeta_k \circ \Phi_{k-1}$$

so, according to (14) and (16),

$$|\eta_k| \leq 2^{-k-2}\ \|\xi_0\|_1^{-1},$$
$$|D\eta_k| \leq 2^{-k-2}\ \|\xi_0\|_1^{-1} + \frac{4}{2}\ 2^{-k-3}\ \|\xi_0\|_1^{-1} = 2^{-k-1}\ \|\xi_0\|_1^{-1}.$$

Now, applying (4), (6) and (8) to $\xi_k - \xi_{k-1} = \psi_*\eta_k$,

$$\begin{aligned} |\xi_k - \xi_{k-1}| &= |\eta_k \circ \psi^{-1} \cdot \xi_0| \leq \|\eta_k\|_0\ \|\xi_0\|_0 \leq 2^{-k-2}, \\ |D(\xi_k - \xi_{k-1})| &= |D\eta_k \circ \psi^{-1} + D\xi_0 \cdot \eta_k \circ \psi^{-1}| \\ &\leq 2\|\eta_k\|_1\ \|\xi_0\|_1 \leq 2^{-k}. \end{aligned}$$

Thus, $\|\xi_k - \xi_{k-1}\|_1 \leq 2^{-k}$ as stated in estimate (ii_k).

Let us finally prove (iii_k). Set $\varphi_0 = \mathrm{id}$ and denote by σ_l^t the flow of $\varphi_l^*\partial_t$ for $0 \leq l \leq k$. Then σ_0^t is just the translation by t and

$$\sigma_k^t = \varphi_k^{-1} \circ \sigma_0^t \circ \varphi_k.$$

Since

$$\xi_k = \psi_*\Phi_k^*\partial_t = \psi_*\Phi_{k-1}^*\varphi_k^*\partial_t \quad \text{and} \quad \xi_{k-1} = \psi_*\Phi_{k-1}^*\partial_t,$$

their flows are given by

$$f_k^t = \psi \circ \Phi_{k-1}^{-1} \circ \sigma_k^t \circ \Phi_{k-1} \circ \psi^{-1} \quad \text{and} \quad f_{k-1}^t = \psi \circ \Phi_{k-1}^{-1} \circ \sigma_0^t \circ \Phi_{k-1} \circ \psi^{-1}.$$

By definition, $\varphi_k = \varphi_{q_k,n_k}$ commutes with the translation σ_0^{1/q_k} outside $J_k = J_{q_k,n_k}$. Consequently, φ_k commutes with any iterate σ_0^{p/q_k}, $p \geq 1$, outside the interval

$$\left[j(n_k) + \frac{1}{2q_k} - \frac{p}{q_k}, j(n_k) + \frac{1}{2q_k}\right] = \bigcup_{q=0}^{p-1}\left(J_k - \frac{q}{q_k}\right).$$

Therefore, σ_k^{p/q_k} equals σ_0^{p/q_k} outside this interval, and in particular, for $0 \leq p \leq q_k$, outside

$$M_k = \left[j(n_k) - 1 + \frac{1}{2q_k}, j(n_k) + \frac{1}{2q_k}\right].$$

On the other hand, for $t \in J_k$,

$$\begin{aligned}
\sigma_k^{1/q_k}(t) &= \varphi_k^{-1}\left(\varphi_k(t) + \frac{1}{q_k}\right) \\
&= \varphi_k^{-1}\left(t + \gamma_k(t) + \frac{1}{q_k}\right) \qquad \text{by definition of } \varphi_k \text{ on } J_k \\
&= t + \frac{1}{q_k} + \gamma_k(t) \qquad \text{because } t + \gamma_k(t) + \frac{1}{q_k} > j(n_k) + \frac{1}{2q_k}, \\
&= \sigma_0^{1/q_k}(t) + \gamma_k(t).
\end{aligned}$$

Thus, $\sigma_k^{1/q_k} - \sigma_0^{1/q_k} = \gamma_k$. Similarly, for any $p \geq 1$,

$$\sigma_k^{p/q_k}(t) - \sigma_0^{p/q_k}(t) = \sum_{q=0}^{p-1} \gamma_k\left(t + \frac{q}{q_k}\right) \quad \text{for all } t \in \mathbb{R}, \tag{17}$$

so

$$\left\|\sigma_k^{p/q_k} - \sigma_0^{p/q_k}\right\|_m = \|\gamma_k\|_m.$$

(again since at most one term of the sum is nonzero in (17)). Now, in the region M_k where σ_k^{p/q_k} and σ_0^{p/q_k} disagree for $0 \leq p \leq q_k$, the diffeomorphism Φ_{k-1} is the identity. Moreover, $\psi(j(n_k)) = a_{j(n_k)}$ and $\psi(M_k) \subset [a_{j(n_k)+1}, a_{j(n_k)-1}]$ so, by (10), ψ restricted to M_k is an affine map with slope $-v_{n_k}$. As a consequence, the derivatives of

$$f_k^{p/q_k} = \psi \circ \Phi_{k-1}^{-1} \circ \sigma_k^{p/q_k} \circ \Phi_{k-1} \circ \psi^{-1}$$

have a simple expression on $\psi(M_k)$:

$$D^m\left(f_k^{p/q_k}\right) = (-v_{n_k})^{1-m}\, D^m\left(\sigma_k^{p/q_k}\right) \circ \psi^{-1}.$$

Similarly, again on $\psi(M_k)$,

$$D^m\left(f_{k-1}^{p/q_k}\right) = (-v_{n_k})^{1-m}\, D^m\left(\sigma_0^{p/q_k}\right) \circ \psi^{-1}.$$

Therefore, for $0 \leq p \leq q_k$ and $0 \leq m \leq k$,

$$\begin{aligned}\big|D^m\big(f_k^{p/q_k} - f_{k-1}^{p/q_k}\big)\big| &\leq v_{n_k}^{1-m}\big\|\sigma_k^{p/q_k} - \sigma_0^{p/q_k}\big\|_m \\ &= v_{n_k}^{1-m}\|\gamma_k\|_m \leq v_{n_k}^{1-k}\|\gamma_k\|_k \leq 2^{-k-4}\end{aligned}$$

according to our choice of n_k in (15), and thus

$$\|f_k^t - f_{k-1}^t\|_k \leq 2^{-k-4} \quad \text{for all } t \in \frac{1}{q_k}\mathbb{Z} \cap [0,1].$$

Now let T_k be a subset of $\frac{1}{q_k}\mathbb{Z} \cap I_{k-1}$ with exactly two points in each of the 2^{k-1} connected components of I_{k-1} (remember that q_k was chosen so that there are at least two points there). Since both vector fields ξ_k and ξ_{k-1} are smooth on $\mathbb{R}_+$, we can find a compact neighbourhood I_k of T_k in $I_{k-1} \setminus \{r_k\}$ consisting of 2^k segments, such that

$$\big\|\big(f_k^t - f_{k-1}^t\big)\,|_{[0,1]}\big\|_k \leq 2^{-k} \quad \text{for all } t \in I_k \cup \{1\}$$

(in fact, the restriction to $[0,1]$ is not essential here: one can prove that the difference $f_k^t - f_{k-1}^t$ is 1-periodic on $[1,\infty)$, and I_k can thus be chosen so that the above bound holds on all of $\mathbb{R}_+$). This completes the proof of (iii$_k$), and thus of Lemma 3. □

5. The limit vector field

Lemma 4. *The vector fields ξ_k, $k \geq 1$, of Lemma 3 converge in the C^1 topology on $\mathbb{R}_+$, and in the C^∞ topology on $\mathbb{R}_+^*$, to a vector field ξ which satisfies all properties stated in Proposition 1 with $K = \bigcap I_k$.*

Proof. The C^1 convergence of the vector fields ξ_k on $\mathbb{R}_+$ follows directly from estimate (ii$_k$) in Lemma 3. Next, estimate (i$_k$) shows that the diffeomorphisms Φ_k converge in the C^∞ topology to a smooth diffeomorphism Φ of $\mathbb{R}$, so the vector fields $\Phi_k^*\partial_t$ converge in the C^∞ topology to $\Phi^*\partial_t$. Now ξ_k equals $\psi_*\Phi_k^*\partial_t$ on $\mathbb{R}_+^*$ and ψ is a smooth diffeomorphism from $\mathbb{R}$ to $\mathbb{R}_+^*$. Given any compact set $A \subset \mathbb{R}_+^*$ and any integer $m \geq 0$, the restriction of ψ to $\psi^{-1}(A)$ is C^m-bounded, and hence the vector fields ξ_k converge C^m uniformly to ξ on A. Therefore, the vector fields ξ_k converge to ξ on $\mathbb{R}_+^*$ in the C^∞ (compact-open) topology.

The convergence of the vector fields ξ_k implies a similar convergence of their flows f_k^t to the flow f^t of ξ. Furthermore, estimate (iii$_k$) in Lemma 3 shows that, for $t \in K \cup \{1\}$, the restrictions $f_k^t\,{}_{|[0,1]}$ converge in the C^m topology on $[0,1]$ for any $m \geq 0$. In the end, the diffeomorphisms f_k^t, $t \in K \cup \{1\}$, converge in the C^∞ topology on $\mathbb{R}_+$, so f^t is smooth for all $t \in K \cup \{1\}$, and hence for all $t \in \mathbb{Z} \oplus \sum_{\tau \in K} \tau\mathbb{Z}$. Note here that each I_k, by construction, is a compact set avoiding the k^{th} rational number and consisting of 2^k segments, two in each component of I_{k-1}, so $K = \bigcap I_k$ is indeed a Cantor set.

The last thing we have to prove is that $f^{1/2}$ is not C^2 at 0 or, equivalently, that $Lf^{1/2} = D^2 f^{1/2}/Df^{1/2}$ is not continuous at 0. Let us compute $Lf^{1/2}$ at a point $a_{i(n_l)}$, as defined in (9), for $l \in \mathbb{N}$. Taking the limit of the maps

$$f_k^{1/2} = \psi \circ \Phi_k^{-1} \circ \left(\mathrm{id} + \frac{1}{2}\right) \circ \Phi_k \circ \psi^{-1},$$

we get

$$f^{1/2} = \psi \circ \Phi^{-1} \circ \left(\mathrm{id} + \frac{1}{2}\right) \circ \Phi \circ \psi^{-1}.$$

Let us set $\sigma = \Phi^{-1} \circ (\mathrm{id} + \frac{1}{2}) \circ \Phi$, so that $f^{1/2} = \psi \circ \sigma \circ \psi^{-1}$. Near $a_{i(n_l)}$, the map ψ^{-1} is affine, with slope $-u_{n_l}^{-1}$, so

$$Lf^{1/2}\big(a_{i(n_l)}\big) = -\frac{1}{u_{n_l}} L\sigma\big(i(n_l)\big).$$

On the other hand, by (2) applied twice,

$$L\sigma\big(i(n_l)\big) = L\Phi^{-1}\left(\Phi\big(i(n_l)\big) + \frac{1}{2}\right) \cdot D\Phi\big(i(n_l)\big) + L\Phi\big(i(n_l)\big).$$

According to Lemma 2, each Φ_k, and hence Φ, is tangent to the identity on $\frac{1}{2}\mathbb{Z}$ provided all integers q_k were chosen odd. Moreover, Φ_k and Φ_k^{-1} coincide with the identity near $\mathbb{Z} + \frac{1}{2}$, so $L\Phi^{-1}\left(i(n_l) + \frac{1}{2}\right) = 0$. Summing up, and using the third property in Lemma 2, we get

$$L\sigma\big(i(n_l)\big) = L\Phi\big(i(n_l)\big) = \sum_{k \geq 1} (L\Phi_k - L\Phi_{k-1})\big(i(n_l)\big) = \sum_{k \geq l} w_{n_k} q_k^2. \tag{18}$$

Therefore,

$$Lf^{1/2}\big(a_{i(n_l)}\big) = -\frac{1}{u_{n_l}} \sum_{k \geq l} w_{n_k} q_k^2 < -\frac{w_{n_l}}{u_{n_l}} \to -\infty,$$

and so $f^{1/2}$ is not C^2 at 0. □

More examples

Let S denote the space of smooth diffeomorphisms of $\mathbb{R}_+$ which are infinitely tangent to the identity at the origin and have no other fixed point. We say that a diffeomorphism f of $\mathbb{R}_+$ is contracting if $f(x) < x$ for all $x > 0$, and we call *Szekeres vector field* of f the unique C^1 vector field generating the one-parameter group Z_f^1 [7], [4].

As mentioned in the introduction, the question we discuss in this section is whether the phenomenon presented in Theorem A is very peculiar or quite general. First of all, because of Takens' work [8], this phenomenon is limited to S. A difficulty then is that there is no obviously relevant topology on S for our problem. In particular, the C^∞ compact-open topology restricted to S is extremely coarse: given any two diffeomorphisms $f, g \in S$, which are both contracting, say, it is easy to construct a sequence of diffeomorphisms $f_k \in S$ which converge to f in the C^∞ topology and whose germs at 0 are all equal to that of g. In other words, the C^∞ topology does not see the germ at 0 while this germ precisely determines the smoothness of the Szekeres vector field and hence the nature of the centralizers in the groups D^r for $r \geq 2$. So we do not claim that the phenomenon described in Theorem A is generic in any way, but the following result shows that it is at least not scarce:

Theorem B. *Let f_0 be a smooth contracting diffeomorphism of $\mathbb{R}_+$ having a smooth and C^1-bounded Szekeres vector field, and satisfying the following oscillation condition*:

$$\limsup_{x\to 0}\left(\sup_{0<y\leq x}\frac{|\log(x-f_0(x))|}{|\log(y-f_0(y))|}\right)=+\infty. \tag{19}$$

Then, for every $k \geq 0$ and every $\varepsilon > 0$, there exists a smooth diffeomorphism f of $\mathbb{R}_+$ which is close to f_0 in the sense that

$$|D^m(f-f_0)(x)| \leq \varepsilon |D^m(f_0-\mathrm{id})(x)| \quad \textit{for all } m \leq k \textit{ and all } x \in \mathbb{R}_+, \tag{20}$$

and whose centralizer Z_f^∞ is a proper, dense and uncountable subgroup of Z_f^1.

Note that the oscillation condition (19) forces f_0 to be infinitely tangent to the identity at 0.

It is interesting to compare this result with Theorem 3.1 in [6]. Indeed, the latter says that, if a smooth contracting diffeomorphism f does not oscillate much in the sense that

$$\sup_{0<y\leq x}\big(y-f(y)\big)=\mathrm{O}\big((x-f(x))^\lambda\big) \quad \text{for some } \lambda>\frac{r-1}{r},$$

then the Szekeres vector field of f is C^r. Theorem B can be thought of as a kind of "partial converse".

Proof. The idea of the proof is the same as for Theorem A: we start with a smooth vector field, here the Szekeres vector field ξ_0 of the given f_0 instead of Sergeraert's vector field, and construct deformations ξ_k of ξ_0 which converge to the Szekeres vector field ξ of the wanted f. We will just hint at how to adapt the arguments in this more general setting. As before, we denote by f_0^t the flow of ξ_0 (so that $f_0 = f_0^1$) and by ψ the diffeomorphism from $\mathbb{R}$ to $\mathbb{R}_+^*$ given by $\psi(t) = f_0^t(1)$ for all $t \in \mathbb{R}$.

We also fix a forward orbit of f_0, namely $\{a_l = f_0^l(1) = \psi(l), l \geq 0\}$, and we set $V_l = [a_{l+2}, a_{l-2}]$ for all $l \geq 0$.

Lemma 5. *There exist two alternating sequences of integers $i(n)$ and $j(n)$, $n \geq 0$, with $i(n) < j(n) < i(n+1) < j(n+1) < \cdots$, such that*

$$\frac{\log u_n}{\log v_n} \xrightarrow[n\to\infty]{} +\infty \tag{21}$$

where $u_n = \sup_{V_{i(n)}} |\xi_0|$ and $v_n = \inf_{V_{j(n)}} |\xi_0|$. In particular, $V_{i(n)}$ and $V_{j(n)}$ are disjoint when n is large enough.

Proof of lemma 5. This proof is rather elementary. Still we give it for the reader's convenience. The oscillation property (19) means that there exist decreasing sequences $(x_n)_n$ and $(y_n)_n$ converging to 0, with $y_n < x_n$, satisfying

$$\lim_{n\to\infty} \frac{\log(x_n - f_0(x_n))}{\log(y_n - f_0(y_n))} = +\infty \tag{22}$$

(the numerator and denominator are negative when n is large enough). We can assume in addition that $x_{n+1} \leq f_0^2(y_n)$ for all n. Let

$$i(n) = \max\{k \in \mathbb{N},\ a_k \geq x_n\},$$
$$j(n) = \min\{k \in \mathbb{N},\ a_k \leq y_n\}.$$

Any fundamental interval $(f_0(x), x] \subset [0, 1]$ of f_0 contains exactly one element of the forward orbit $\{a_i, i \in \mathbb{N}\}$ of $a_0 = 1$, so the definitions of $i(n)$ and $j(n)$ imply

$$x_{n+1} \leq a_{i(n+1)} < f_0^{-1}(x_{n+1}) \leq f_0(y_n) < a_{j(n)} \leq y_n < x_n \leq a_{i(n)},$$

and *a fortiori* $i(n) < j(n) < i(n+1)$ for all n. Let us now prove that for this choice of alternating sequences $i(n)$ and $j(n)$,

$$\frac{\log u_n}{\log v_n} \xrightarrow[n\to\infty]{} +\infty,$$

where $u_n = \sup_{V_{i(n)}} |\xi_0|$ and $v_n = \inf_{V_{j(n)}} |\xi_0|$. By definition of $i(n)$, $j(n)$, $V_{i(n)}$ and $V_{j(n)}$, there exist t_n and s_n in $[-3, 3]$ such that

$$u_n = \left|\xi_0\left(f_0^{t_n}(x_n)\right)\right| \quad \text{and} \quad v_n = \left|\xi_0\left(f_0^{s_n}(y_n)\right)\right|.$$

Now

$$\frac{d}{dt} f_0^t(x) = \xi_0\left(f_0^t(x)\right) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}_+,$$

so

$$\begin{aligned} x_n - f_0(x_n) &= f_0^0(x_n) - f_0^1(x_n) \\ &= -\xi_0\left(f_0^{\tau_n}(x_n)\right) = \left|\xi_0\left(f_0^{\tau_n}(x_n)\right)\right| \quad \text{for some } \tau_n \in [0,1], \end{aligned}$$

and similarly

$$y_n - f_0(y_n) = -\xi_0\left(f_0^{\sigma_n}(y_n)\right) = \left|\xi_0\left(f_0^{\sigma_n}(y_n)\right)\right| \quad \text{for some } \sigma_n \in [0,1].$$

So

$$\frac{\log u_n}{\log v_n} = \frac{\log\left|\xi_0\left(f_0^{t_n}(x_n)\right)\right|}{\log\left|\xi_0\left(f_0^{s_n}(y_n)\right)\right|} = \frac{\log(x_n - f_0(x_n)) + \log\left|\frac{\xi_0(f_0^{t_n}(x_n))}{\xi_0(f_0^{\tau_n}(x_n))}\right|}{\log(y_n - f_0(y_n)) + \log\left|\frac{\xi_0(f_0^{s_n}(y_n))}{\xi_0(f_0^{\sigma_n}(y_n))}\right|}. \tag{23}$$

The flow $\left(f_0^t\right)_{t\in\mathbb{R}}$ of ξ_0 preserves ξ_0, *i.e.*

$$\xi_0\left(f_0^t(x)\right) = Df_0^t(x)\,\xi_0(x) \quad \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}_+.$$

As a result,

$$\frac{\xi_0\left(f_0^{t_n}(x_n)\right)}{\xi_0\left(f_0^{\tau_n}(x_n)\right)} = D\left(f_0^{(t_n-\tau_n)}\right)\left(f_0^{\tau_n}(x_n)\right)$$

and

$$\frac{\xi_0\left(f_0^{s_n}(y_n)\right)}{\xi_0\left(f_0^{\sigma_n}(y_n)\right)} = D\left(f_0^{(s_n-\sigma_n)}\right)\left(f_0^{\sigma_n}(y_n)\right).$$

One easily checks that

$$Df_0^t(0) = e^{tD\xi_0(0)} \quad \text{for all } t \in \mathbb{R},$$

so since $Df_0^1(0) = 1$ ($f_0^1 = f_0$ has to be infinitely tangent to the identity at 0 to satisfy the oscillation condition (19)), $D\xi_0(0) = 0$ and

$$Df_0^t(0) = 1 \quad \text{for all } t \in \mathbb{R}.$$

Since ξ_0 is C^1 on $\mathbb{R}_+$, both $(t,x) \mapsto f_0^t(x)$ and $(t,x) \mapsto Df_0^t(x)$ are uniformly continuous on every compact subset of $\mathbb{R}\times\mathbb{R}_+$. Thus, since $|\tau_n| \le 1$, $|\sigma_n| \le 1$, $|t_n - \tau_n| \le 4$, $|s_n - \sigma_n| \le 4$, and x_n and y_n converge to 0,

$$D\left(f_0^{(t_n-\tau_n)}\right)\left(f_0^{\tau_n}(x_n)\right) \to 1 \quad \text{and} \quad D\left(f_0^{(s_n-\sigma_n)}\right)\left(f_0^{\sigma_n}(y_n)\right) \to 1.$$

This, together with (22), (23), and the fact that

$$\log(x_n - f_0(x_n)) \to -\infty \quad \text{and} \quad \log(y_n - f_0(y_n)) \to -\infty$$

implies that

$$\frac{\log u_n}{\log v_n} \to +\infty,$$

which concludes the proof. □

We now choose a sequence w_n with intermediate decay, *i.e.* satisfying $w_n = \mathrm{o}(v_n^m)$ for all m and $u_n = \mathrm{o}(w_n)$ (one can take for instance $w_n = \sqrt{u_n}$). Then we define the maps $\gamma_{q,n}$ and $\varphi_{q,n}$ by formulae (11) and (12), using the same function γ but the new parameters w_n and $j(n)$. Extending thence all other definitions and notation of Subsection 4, our task is to show that Lemmas 3 and 4 still hold.

Proof of Lemma 3 in the general setting. We only insist here on the points that differ from the proof in Subsection 4. Again, we proceed by induction. At step k, the choice of q_k is just the same, but we need to be more careful about n_k. The reason is that the map ψ is no longer affine on the regions we consider, and hence the computation of higher derivatives of compositions is trickier.

First, using the fact that ξ_0 is smooth and infinitely flat at 0, one can check that, for any fixed $m \geq 1$,

$$\sup\{|D^m\psi(t)|,\ t \in [j(n)-1,\infty)\} \xrightarrow[n\to\infty]{} 0$$

and

$$v_n^{m+1} \sup\left\{|D^m\psi^{-1}(x)|,\ x \in \left[a_{j(n)+1}, a_{j(n)-1}\right]\right\} \xrightarrow[n\to\infty]{} 0.$$

(this is derived from the relations $D\psi = \xi_0 \circ \psi$ and $D\psi^{-1} = 1/\xi_0$).

Then we pick an integer $n_k > n_{k-1}$ meeting the following three conditions:

$$\left\| D\psi_{\ |[j(n_k)-1,\infty)} \right\|_{k-1} < 1, \tag{24}$$

$$\left\| D^m\psi^{-1}_{\ |[a_{j(n_k)+1}, a_{j(n_k)-1}]} \right\|_0 < v_{n_k}^{-m-1} \quad \text{for } 1 \leq m \leq k, \tag{25}$$

and

$$\|\gamma_k\|_{k+1} \leq \frac{2^{-k^2-4}\ v_{n_k}^{2k}}{|\Pi_{k+1}|^2 \|D\Phi_{k-1}\|_k^{k+1}\ \|\xi_0\|_1}. \tag{26}$$

Inequality (26) is stronger than (15) and thus implies (i_k) and (ii_k) of Lemma 3 (the arguments are strictly the same). The proof of (iii_k) is more complicated but we still have (with our former notation)

$$f_k^t = \psi \circ \Phi_{k-1}^{-1} \circ \sigma_k^t \circ \Phi_{k-1} \circ \psi^{-1} \quad \text{and} \quad f_{k-1}^t = \psi \circ \Phi_{k-1}^{-1} \circ \sigma_0^t \circ \Phi_{k-1} \circ \psi^{-1}.$$

For $t = p/q_k$, $0 \leq p \leq q_k$, again $\sigma_k^t = \sigma_0^t$ outside

$$M_k = \left[j(n_k) - 1 + \frac{1}{2q_k}, j(n_k) + \frac{1}{2q_k} \right],$$

so $f_k^t - f_{k-1}^t = 0$ outside $\psi(M_k)$. Furthermore, $\Phi_{k-1} = \mathrm{id}$ on M_k. Thus, on $\psi(M_k)$,

$$f_k^t = \psi \circ \sigma_k^t \circ \psi^{-1} \quad \text{and} \quad f_{k-1}^t = \psi \circ \sigma_0^t \circ \psi^{-1}$$

or, equivalently,

$$f_k^t - f_{k-1}^t = (\psi \circ \sigma_k^t) \circ \psi^{-1} - (\psi \circ \sigma_0^t) \circ \psi^{-1}.$$

For $m \leq k$, Faà di Bruno's formula gives

$$D^m\big(f_k^t - f_{k-1}^t\big) = \sum_{\pi \in \Pi_m} D^{|\pi|}\big(\psi \circ \sigma_k^t - \psi \circ \sigma_0^t\big) \circ \psi^{-1} \cdot \prod_{B \in \pi} D^{|B|}\psi^{-1}. \tag{27}$$

According to inequality (25),

$$\Big|\prod_{B \in \pi} D^{|B|}\psi^{-1}\Big| < v_{n_k}^{-2k} \quad \text{on} \quad \psi(M_k) \subset \big[a_{j(n_k)+1}, a_{j(n_k)-1}\big]. \tag{28}$$

Now write

$$\psi \circ \sigma_k^t - \psi \circ \sigma_0^t = (\psi \circ \sigma_0^t) \circ (\sigma_0^{-t} \circ \sigma_k^t) - (\psi \circ \sigma_0^t)$$

and observe, using (17), that

$$\sigma_0^{-t} \circ \sigma_k^t = \mathrm{id} + \sum_{q=0}^{p-1} \gamma_k\Big(\mathrm{id} + \frac{q}{q_k}\Big) \quad \text{for } t = \frac{p}{q_k}, 0 \leq p \leq q_k.$$

For $l \leq k$, Faà di Bruno's formula gives

$$\begin{aligned} D^l\left(\psi \circ \sigma_k^t - \psi \circ \sigma_0^t\right) &= D^l\left((\psi \circ \sigma_0^t) \circ (\sigma_0^{-t} \circ \sigma_k^t) - (\psi \circ \sigma_0^t)\right) \\ &= \sum_{\substack{\pi \in \Pi_l \\ |\pi| < l}} D^{|\pi|}\left(\psi \circ \sigma_0^t\right) \circ \left(\sigma_0^{-t} \circ \sigma_k^t\right) \cdot \prod_{B \in \pi} D^{|B|}\left(\sigma_0^{-t} \circ \sigma_k^t\right). \end{aligned}$$

Since $\sigma_0^t = \mathrm{id} + t$, it follows from (24) that

$$\big|D^{|\pi|}(\psi \circ \sigma_0^t) \circ (\sigma_0^{-t} \circ \sigma_k^t)\big| < 1 \quad \text{on } M_k.$$

Now for any partition $\pi \in \Pi_l$ with less than l blocks, *i.e.* $|\pi| < l$, one block B of π has at least two elements, so at least one factor in the product

$$\prod_{B \in \pi} D^{|B|}\left(\sigma_0^{-t} \circ \sigma_k^t\right) = \prod_{B \in \pi} D^{|B|}\Big(\mathrm{id} + \sum_{q=0}^{p-1} \gamma_k\Big(\mathrm{id} + \frac{q}{q_k}\Big)\Big)$$

is a derivative of order at least 2, and hence is bounded above by $\|\gamma_k\|_k$, while the others are all less than 2. So the product is bounded above by $2^{l-2}\|\gamma_k\|_k \leq 2^{k-2}\|\gamma_k\|_k$. Therefore,

$$\big|D^l(\psi \circ \sigma_k^t - \psi \circ \sigma_0^t)\big| \leq |\Pi_l|\, 2^{l-2}\|\gamma_k\|_k \leq |\Pi_k|\, 2^{k-2}\|\gamma_k\|_k.$$

In view of (27), (28) and (26) this implies that $\|f_k^t - f_{k-1}^t\|_k \leq 2^{-k-4}$ for all $t = p/q_k$, $0 \leq p \leq q_k$, and one completes the proof of Lemma 3 just as in Subsection 4. □

Proof of Lemma 4 in the general setting. The proof that the vector fields ξ_k converge and that the limit flow f^t is smooth for $t \in \mathbb{Z} \oplus \sum_{\tau \in K} \tau\mathbb{Z}$ is strictly the same as in Subsection 5. Note that if we start our construction at step k_0 instead of step 1, the limit diffeomorphism f satisfies the condition (20) for $l \leq k_0$ and $\varepsilon = 2^{-k_0-1}$, so one can construct f arbitrarily close to f_0 in the sense of Theorem B.

The part of Lemma 4 that needs a little extra effort is the irregularity of $f^{1/2}$. Again,

$$f^{1/2} = \psi \circ \sigma \circ \psi^{-1},$$

with $\sigma = \Phi^{-1} \circ (\mathrm{id} + 1/2) \circ \Phi$. The computation of $L\sigma\big(i(n_l)\big)$ leading to (18) can be integrally transposed here, and yields $L\sigma\big(i(n_l)\big) = \sum_{k \geq l} w_{n_k} q_k^2$ (with the new w_n). However, this time ψ is not affine on the involved region, so the computation of $Lf^{1/2}\big(i(n_l)\big)$ is a bit longer. Formula (2) applied twice gives

$$Lf^{1/2} = \big(L\psi \circ (\sigma \circ \psi^{-1}) \cdot D(\sigma \circ \psi^{-1})\big) + \big(L\sigma \circ \psi^{-1} \cdot D\psi^{-1}\big) + L\psi^{-1},$$

and hence, since $D\psi^{-1} = 1/\xi_0$,

$$Lf^{1/2}(a_{i(n_l)}) = \big[L\psi \circ (\sigma \circ \psi^{-1}) \cdot D(\sigma \circ \psi^{-1}) + L\psi^{-1}\big]\,(a_{i(n_l)}) + \frac{L\sigma\big(i(n_l)\big)}{\xi_0\big(a_{i(n_l)}\big)}.$$

Now, according to Lemma 2 (still valid in our new setting), the limit Φ of the diffeomorphisms Φ_k coincides with the translation by $1/2$ at order one on $\mathbb{Z}$, so the first term of the above sum is equal to

$$\left[L\psi \circ \left(\mathrm{id} + \frac{1}{2}\right) \circ \psi^{-1} \cdot D\left(\left(\mathrm{id} + \frac{1}{2}\right) \circ \psi^{-1}\right) + L\psi^{-1}\right](a_{i(n_l)}) = Lf_0^{1/2}(a_{i(n_l)}).$$

But when l grows, $Lf_0^{1/2}(a_{i(n_l)})$ tends to $Lf_0^{1/2}(0) = 0$. Therefore

$$Lf^{1/2}(a_{i(n_l)}) \sim \frac{\sum_{k \geq l} w_{n_k} q_k^2}{\xi_0\big(a_{i(n_l)}\big)} < -\frac{w_{n_l}}{u_{n_l}} \xrightarrow[l \to \infty]{} -\infty,$$

so $f^{1/2}$ is not C^2 at 0. This concludes the proof of Lemma 4 and of Theorem B. □

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