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Mathematics — *Existence for semilinear parabolic stochastic equations*, by VIOREL BARBU, communicated on 14 May 2010.

ABSTRACT. — The boundary value problem for semilinear parabolic stochastic equations of the form $dX - \Delta X\, dt + \beta(X)\, dt \ni \sqrt{Q}\, dW_t$, where W_t is a Wiener process and β is a maximal monotone graph everywhere defined, is well posed.

KEY WORDS: Wiener process, mild solution, random differential equation.

AMS SUBJECT CLASSIFICATION: 35K58, 35R60.

1. INTRODUCTION

Consider the stochastic differential equation

$$\begin{aligned} &dX - \Delta X\, dt + \beta(X)\, dt \ni \sqrt{Q}\, dW_t && \text{in } (0,T) \times \mathcal{O} = Q_T, \\ &X(0) = x && \text{in } \mathcal{O}, \\ &X = 0 && \text{on } (0,T) \times \partial\mathcal{O} = \Sigma_T. \end{aligned} \tag{1}$$

Here, $\mathcal{O}$ is an open and bounded subset of R^d with smooth boundary $\partial\mathcal{O}$, $d \geq 1$, and W_t is a cylindrical Wiener process in $L^2(\mathcal{O}) = H$ defined by

$$W_t = \sum_{k=1}^{\infty} e_k(\xi)\beta_k(t), \quad \xi \in \mathcal{O},\ t \geq 0,$$

where $\{\beta_k\}_k$ are mutually independent Brownian motions on a probability space $\{\Omega, \mathscr{F}, \mathbb{P}\}$ and $\{e_k\}$ is an orthonormal basis in H. The operator $Q \in L(H,H)$ is self–adjoint, positive and of finite trace.

Finally, $\beta : R \to 2^R$ is a maximal monotone graph (see [1]) everywhere defined on R.

The main result of this note is that, under suitable assumptions on Q (see (H1) below), equation (1) has a unique strong(mild) solution (Theorem 2). A similar result was proven in [2] for the stochastic porous media equation.

Compared with standard existence theory for equation (1) (see [3], [4]), where the main assumption is that β is continuous, monotonically increasing, here β might be multivalued and, therefore, discontinuous. Also, as seen later on, β might be a time dependent function $\beta = \beta(t, \cdot)$ measurable in $t \in [0,T]$.

Moreover, our existence results apply to multivalued graphs β everywhere defined on R. Such a graph (multivalued) arises naturally when in equation (1) the function β is monotonically increasing and discontinuous in $\{r_j\}_{j=1}^{\infty}$. Then, one redefines β by

$$\tilde{\beta}(r) = \beta(r) \quad \text{for } r \neq r_j, \quad \tilde{\beta}(r_j) = [\beta(r_j), \beta(r_j + 0)]$$

and get a maximal monotone graph $\tilde{\beta}$. So, one might say that the existence result established here in Theorem 2 below applies as well to discontinuous monotonically increasing besides continuous functions β.

We shall denote by $C_W([0,T];H)$ the space of all adapted processes $X \in C([0,T];L^2(\Omega,\mathscr{F},\mathbb{P},H))$, $H = L^2(\mathcal{O})$ and by $L^2_W(0,T;H^1_0(\mathcal{O}))$ the space of all adapted processes $X \in L^2(0,T;L^2(\Omega,\mathscr{F},\mathbb{P},H^1_0(\mathcal{O}))$ (see [3]). Here, $H^1_0(\mathcal{O})$ is the standard Sobolev space.

We denote also by W_A the stochastic convolution

$$W_A(t) = \int_0^t e^{-A(t-s)} \sqrt{Q}\, dW_s, \quad t \geq 0,$$

where $A = -\Delta$, $D(A) = H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})$. We recall that $W_A(t)$ is a Gaussian process and $E(|W_A(t)|^2] < \infty$, $\forall t \geq 0$ (see [3], p. 21).

2. The main result

The following hypotheses will be assumed.

(H1) *$W_A(\cdot,\cdot)$ is continuous on $[0,T] \times \overline{\mathcal{O}}$, $\mathbb{P}$-a.s..*
(H2) *$\beta : R \to 2^R$ is a maximal monotone graph such that $D(\beta) = R$.*

Here, $D(\beta) = \{r \in R; \beta(r) \neq \emptyset\}$.

In particular, hypotheses (H2) holds if β is a monotonically nondecreasing and continuous function.

As regards hypotheses (H1), we refer to [3], Theorem 2.13, for sufficient conditions on Q under which it holds.

Definition 1. By strong (or mild) solution to equation (1) we mean a process $X \in C([0,T];H)$ which satisfies

$$X(t) = e^{-At}x - \int_0^t e^{-A(t-s)} \eta(s)\, ds + W_A(t), \quad \mathbb{P}\text{-a.s., } t \in [0,T], \tag{2}$$

where $\eta \in L^1((0,T) \times \mathcal{O} \times \Omega)$ is such that

$$\eta(t,\xi) \in \beta(X(t,\xi)), \quad \text{a.e. } (t,\xi) \in Q_T,\ \mathbb{P}\text{-a.s.} \tag{3}$$

Theorem 2. *Under hypotheses* (H1), (H2)*, for each $x \in H = L^2(\mathcal{O})$ there is a unique strong solution X to equation* (1)*, such that*

$$X \in L^2_W([0,T];H^1_0(\mathcal{O})), \tag{4}$$

$$j(X), j^*(\eta) \in L^1((0,T) \times \mathcal{O} \times \Omega). \tag{5}$$

Here, j is the subpotential associated with β, i.e., $\partial j = \beta$ and j^* is the conjugate of j. (See the notation below.)

3. Proof of Theorem 2

Existence. By using a standard device, we shall reduce equation (1) to the random differential equation

$$(6)\qquad \begin{aligned} & y_t - \Delta y + \beta(y + W_A) \ni 0, && (t,\xi) \in Q_T = (0,T) \times \mathcal{O}, \\ & y(0,\xi) = x(\xi), && \xi \in \mathcal{O}, \\ & y = 0 && \text{on } (0,T) \times \partial\mathcal{O} = \Sigma_T, \end{aligned}$$

where $y = X - W_A$ and $y_t = \dfrac{\partial}{\partial t} y$.

We fix $\omega \in \Omega$ and approximate (6) by

$$(7)\qquad \begin{aligned} & (y_\varepsilon)_t - \Delta y_\varepsilon + \beta_\varepsilon(y_\varepsilon + W_A) \ni 0, && (t,\xi) \in Q_T, \\ & y_\varepsilon(0,\xi) = x(\xi), && \text{in } \mathcal{O}, \\ & y = 0 && \text{on } \Sigma_T, \end{aligned}$$

where $\beta_\varepsilon = \dfrac{1}{\varepsilon}(1 - (1 + \varepsilon\beta)^{-1})$ is the Yosida approximation of β (see, e.g., [1]). Since β_ε is Lipschitzian, equation (7) has a unique solution

$$\begin{gathered} y_\varepsilon \in C([0,T]; L^2(\mathcal{O})) \cap L^2(0,T; H_0^1(\mathcal{O})) \\ \sqrt{t}(y_\varepsilon)_t \in L^2(0,T; L^2(\mathcal{O})), \quad \sqrt{t}y_\varepsilon \in L^2(0,T; H^2(\mathcal{O})). \end{gathered}$$

Denote by $j : R \to R$ the subpotential function corresponding to β, that is $\partial j = \beta$, where ∂j is subdifferential of β (see, e.g., [1], p. 53). Let j^* be the conjugate of j, that is,

$$j^*(p) = \sup\{p \cdot r - j(r); r \in R\}$$

and recall that $p \in \partial\beta(r)$ if and only if

$$(8)\qquad j(r) + j^*(p) = rp.$$

We have also $\beta_\varepsilon = \nabla j_\varepsilon$, where

$$(9)\qquad \begin{aligned} j_\varepsilon(r) &= \inf\left\{\frac{|r-s|^2}{2\varepsilon} + j(s); s \in R\right\} \\ &= \frac{1}{2\varepsilon}|(1+\varepsilon\beta)^{-1}r - r|^2 + j((1+\varepsilon\beta)^{-1}r), \quad \forall r \in R. \end{aligned}$$

Multiplying (7) by y_ε and integrating on $(0,T)\times\mathcal{O}$, we obtain that

$$(10)\qquad \frac{1}{2}\|y_\varepsilon(t)\|^2_{L^2(\mathcal{O})}+\int_0^t\|y_\varepsilon(s)\|^2_{H_0^1(\mathcal{O})}\,ds+\int_0^t\int_{\mathcal{O}} j_\varepsilon(y_\varepsilon+W_A)\,ds\,d\xi \le \frac{1}{2}\|x\|^2_{L^2(\mathcal{O})}+\int_0^t\int_{\mathcal{O}} j_\varepsilon(W_A)\,ds\,d\xi\le C,\quad \forall t\in[0,T].$$

Hence, on a subsequence $\varepsilon\to 0$, we have

$$(11)\qquad y_\varepsilon\to y^*\quad\text{weakly in } L^2(0,T;H_0^1(\mathcal{O}))\text{ and weak-star in } L^\infty(0,T;L^2(\mathcal{O})).$$

Also, by (9)~(10), we see that, for $\varepsilon\to 0$,

$$(12)\qquad (1+\varepsilon\beta)^{-1}(y_\varepsilon+W_A)\to y^*+W_A\quad\text{weak-star in } L^\infty(0,T;L^2(\mathcal{O})).$$

By (8), we have

$$j^*(\beta_\varepsilon(y_\varepsilon+W_A))+j((1+\varepsilon\beta)^{-1}(y_\varepsilon+W_A)) = (\beta_\varepsilon(y_\varepsilon+W_A))(1+\varepsilon\beta)^{-1}(y_\varepsilon+W_A)\le\beta_\varepsilon(y_\varepsilon+W_A)(y_\varepsilon+W_A).$$

This yields

$$(13)\qquad \int_{Q_T} j^*(\beta_\varepsilon(y_\varepsilon+W_A))\,d\xi\,dt\le\int_{Q_T}\beta_\varepsilon(y_\varepsilon+W_A)y_\varepsilon\,d\xi\,dt-\int_{Q_T}\beta_\varepsilon(y_\varepsilon+W_A)W_A\,d\xi\,dt = -\frac{1}{2}\|y_\varepsilon(T)\|^2_{L^2(\mathcal{O})}+\frac{1}{2}\|x\|^2_{L^2(\mathcal{O})}-\|y_\varepsilon\|^2_{L^2(0,T;H_0^1(\mathcal{O}))}-\int_{Q_T}\beta_\varepsilon(y_\varepsilon+W_A)W_A\,d\xi\,dt.$$

Since $D(\beta)=R$, we have that

$$(14)\qquad \lim_{|r|\to\infty}\frac{j^*(r)}{|r|}=+\infty.$$

Then, by (14) we obtain that for each n there is $C_n>0$ such that

$$(15)\qquad j^*(\beta_\varepsilon(y_\varepsilon+W_A))\ge n|\beta_\varepsilon(y_\varepsilon+W_A)|\quad\text{a.e. on } \{(\xi,t);|\beta_\varepsilon(y_\varepsilon+W_A)(\xi,t)|\ge C_n\}.$$

We shall use this to prove that $\{\beta_\varepsilon(y_\varepsilon+W_A)\}_{\varepsilon>0}$ is weakly compact in $L^1(Q_T)$. To this purpose, it suffices to show that

$$(16)\qquad \int_{Q_T}|\beta_\varepsilon(y_\varepsilon+W_A)|\,d\xi\,dt\le C,\quad\forall\varepsilon>0,$$

and that, for each $\delta > 0$, there is C_δ such that for any measurable subset $Q^* \subset Q_T$ with the Lebesgue measure $m(Q^*) \le C_\delta$, we have

$$\int_{Q^*} |\beta_\varepsilon(y_\varepsilon + W_A)|\, d\xi\, dt \le \delta, \quad \forall \varepsilon > 0, \tag{17}$$

(C_δ independent of ε).

Estimate (16) follows by (13) and (15). As regards (17), we start from the inequality

$$\begin{aligned}\int_{Q^*} |\beta_\varepsilon(y_\varepsilon + W_A)|\, d\xi\, dt &\le \int_{Q^* \cap [|\beta_\varepsilon(y_\varepsilon + W_A)| \ge n]} |\beta_\varepsilon(y_\varepsilon + W_A)|\, d\xi\, dt \\ &\quad + nm(Q^*) \le \frac{1}{n} \int_{Q^*} j_\varepsilon^*(\beta_\varepsilon(y_\varepsilon + W_A))\, d\xi\, dt + nm(Q^*) \\ &\le \frac{1}{n} \|W_A\|_{L^\infty(Q_T)} \|\beta_\varepsilon(y_\varepsilon + W_A)\|_{L^1(Q_T)} \le \frac{C}{n} + nm(Q^*).\end{aligned}$$

(Here, we have used (13), (15), (16) and (H1).)

Hence, for $n \ge \dfrac{\delta}{2C}$ and $m(Q^*) \le \dfrac{\delta}{2n}$, we obtain (17), as claimed.

Then, by the Pettis theorem, $\{\beta_\varepsilon(y_\varepsilon + W_A)\}_{\varepsilon > 0}$ is weakly compact in $L^1(Q_T)$ and so, on a subsequence, again denoted ε, we have

$$\beta_\varepsilon(y_\varepsilon + W_A) \to \eta \quad \text{weakly in } L^1(Q_T). \tag{18}$$

Inasmuch as $\{\beta_\varepsilon(y_\varepsilon + W_A)\}$ is bounded in $L^1(Q_T)$, it follows by (7) that $\{y_\varepsilon\}$ is compact in $C([0, T]; L^1(\mathcal{O}))$ and, therefore, for $\varepsilon \to 0$,

$$y_\varepsilon \to y^* \quad \text{strongly in } C([0, T]; L^1(\mathcal{O})) \tag{19}$$

and

$$\begin{aligned} &y_t^* - \Delta y^* + \eta = 0 && \text{in } Q_T, \\ &y^*(0) = x, \quad y^*(t) \in H_0^1(\mathcal{O}), && \text{a.e. } t \in [0, T]. \end{aligned} \tag{20}$$

In order to conclude the proof of existence for equation (6), it remains to be proven that

$$\eta(t, \xi) \in \beta(y^*(t, \xi) + W_A(t, \xi)), \quad \text{a.e. } (t, \xi) \in Q_T. \tag{21}$$

To this end, we start from the inequality

$$\begin{aligned}&\int_{Q_0} \beta_\varepsilon(y_\varepsilon + W_A)(y_\varepsilon + W_A - z)\, d\xi\, dt \\ &\quad \ge \int_{Q_0} j_\varepsilon(y_\varepsilon + W_A)\, d\xi\, dt - \int_{Q_0} j_\varepsilon(z)\, d\xi\, dt, \quad \forall z \in L^\infty(Q_0),\end{aligned} \tag{22}$$

for any measurable subset $Q_0 \subset Q_T$.

On the other hand, by (19), by Egorov Theorem, it follows that for each $\delta > 0$ there is $Q_\delta \subset Q_T$ such that $m(Q_T \backslash Q_\delta) \le \delta$ and $y_\varepsilon \to y^*$ uniformly on Q_δ as $\varepsilon \to 0$. Taking $Q_0 = Q_\delta$ in (22), we obtain

$$\int_{Q_\delta} \eta(y^* + W_A - z)\, d\xi\, dt \ge \int_{Q_\delta} (j(y^* + W_A) - j(z))\, d\xi\, dt, \quad \forall z \in L^\infty(Q_\delta).$$

The latter implies by a standard device the pointwise inequality

$$\eta(y^* + W_A - z) \ge j(y^* + W_A) - j(z), \quad \text{a.e. in } Q_\delta,\ \forall z \in R,$$

and, therefore, $\eta \in \partial j(y^* + W_A) = \beta(y^* + W_A)$, a.e. in Q_δ, and since δ is arbitrary, we obtain (21), as claimed.

Now, it is clearly seen that $X(t) = y(t) + W_A$ is a solution to (1) in the sense made precise in Definition 1. (The fact that the process $X(t) = \lim_{\varepsilon \to 0} y_\varepsilon(t) + W_A(t)$ is adapted is obvious because so is $X_\varepsilon(t) = y_\varepsilon(t) + W_A(t)$.) By (10) and (13), it is also easily seen that $j(X), j^*(\eta) \in L^1((0,T) \times \mathcal{O} \times \Omega)$. This completes the proof of the existence.

Uniqueness. It is immediate, because if X_i, $i = 1, 2$, are solutions to (1) in the above sense, then $y_i = X_i - W_A$, $i = 1, 2$, are $\mathbb{P}$-a.s. solutions to equation (6), which clearly has a unique solution by monotonicity of β.

Remark 3. Theorem 2 remains true for time dependent maximal monotone graphs $\beta = \beta(t, \cdot)$ which satisfy the following assumptions.

(H2)′ *For almost all $t \in (0, T)$, $\beta(t, \cdot) : R \to 2^R$ is maximal monotone, measurable in t and for each $M > 0$ there is C_M independent of t such that*

$$|\beta(t, r)| \le C_M \quad \text{a.e. } t \in (0, T),\ \forall r \in [-M, M]. \tag{23}$$

If β is independent of t, (H2)′ is implied by (H2). The proof is exactly the same as that of Theorem 2.

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References

[1] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer, Berlin. New York, 2010.

[2] V. Barbu - G. Da Prato - M. Roeckner, *Existence of strong solutions for stochastic porous media equation under general monotonicity conditions*, The Annals of Probability, vol. 37, no. 2 (2009), 428–452.

[3] G. Da Prato, *Kolmogorov Equations for Stochastic PDEs*, Birkhäuser Berlag, Basel, 2004.

[4] G. Da Prato - J. Zabzyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, UK, 1992.

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