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Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces

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Abstract. We characterize the possible asymptotic behaviors of the compression associated to a uniform embedding into some L^p-space, with $1 < p < \infty$, for a large class of groups including connected Lie groups with exponential growth and word-hyperbolic finitely generated groups. In particular, the Hilbert compression exponent of these groups is equal to 1. This also provides new and optimal estimates for the compression of a uniform embedding of the infinite 3-regular tree into some L^p-space. The main part of the paper is devoted to the explicit construction of affine isometric actions of amenable connected Lie groups on L^p-spaces whose compressions are asymptotically optimal. These constructions are based on an asymptotic lower bound of the L^p-isoperimetric profile inside balls. We compute the asymptotic behavior of this profile for all amenable connected Lie groups and for all $1 \leq p < \infty$, providing new geometric invariants of these groups. We also relate the Hilbert compression exponent with other asymptotic quantities such as volume growth and probability of return of random walks.

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Contents

1. Introduction

The study of uniform embeddings of locally compact groups into Banach spaces and especially of those associated to proper affine isometric actions plays a crucial role in various fields of mathematics ranging from K-theory to geometric group theory.

Recall that a locally compact group is called a-T-menable if it admits a proper affine action by isometries on a Hilbert space (for short: a proper isometric Hilbert action). An amenable σ-compact locally compact group is always a-T-menable [CCJJV]; but the converse is false since for instance non-amenable free groups are a-T-menable. However, if a locally compact, compactly generated group G admits a proper isometric Hilbert action whose compression ρ satisfies

$$\rho(t) \succ t^{1/2},$$

then G is amenable[1]. On the other hand, in [CTV], we prove that non-virtually abelian polycyclic groups cannot have proper isometric Hilbert actions with linear compression. These results motivate a systematic study of the possible asymptotic behaviors of compression functions, especially for amenable groups.

In this paper, we "characterize" the asymptotic behavior of the L^p-compression, with $1 < p < \infty$, for a large class of groups including all connected Lie groups with exponential growth. Some partial results in this direction for $p = 2$ had been obtained in [GK] and [BrSo] by completely different methods.

1.1. L^p-compression: optimal estimates. Let us recall some basic definitions. Let G be some locally compact compactly generated group. Equip G with the word length function $|\cdot|_S$ associated to a compact symmetric generating subset S and consider a uniform embedding F of G into some Banach space. The compression ρ of F is the nondecreasing function defined by

$$\rho(t) = \inf_{|g^{-1}h|_S \geq t} \|F(g) - F(h)\|.$$

Let $f, g\colon \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ be nondecreasing, nonzero functions. We write respectively $f \preceq g$, $f \prec g$ if there exists $C > 0$ such that $f(t) = O(g(Ct))$, resp. for all $c > 0$, $f(t) = o(g(ct))$ when $t \to \infty$. We write $f \approx g$ if both $f \preceq g$ and $g \preceq f$. The asymptotic behavior of f is its class modulo the equivalence relation $\approx$.

Note that the asymptotic behavior of the compression of a uniform embedding does not depend on the choice of S.

In the sequel, an L^p-space denotes a Banach space of the form $L^p(X, m)$ where (X, m) is a measure space. An L^p-representation of G is a continuous linear G-action on some L^p-space. Let π be an isometric L^p-representation of G and consider a 1-cocycle $b \in Z^1(G, \pi)$, or equivalently an affine isometric action of G with linear part π: see the preliminaries for more details. The compression of b is defined by

$$\rho(t) = \inf_{|g|_S \geq t} \|b(g)\|_p.$$

[1] This was proved for finitely generated groups in [GK]. In [CTV], we give a shorter argument that applies to all locally compact compactly generated groups.

In this paper, we mainly focus our attention on groups in the two following classes. Denote $(\mathcal{L})$ the class of groups including

(1) polycyclic groups and connected amenable Lie groups;

(2) semidirect products $\mathbb{Z}[\frac{1}{mn}] \rtimes_{\frac{m}{n}} \mathbb{Z}$, with m, n co-prime integers with[2] $|mn| \geq 2$ (if $n = 1$ this is the Baumslag–Solitar group $BS(1, m)$); semidirect products $(\mathbb{R} \oplus \bigoplus_{p \in P} \mathbb{Q}_p) \rtimes_{\frac{m}{n}} \mathbb{Z}$ with m, n coprime integers and P a finite family of primes dividing mn;

(3) wreath products $F \wr \mathbb{Z}$ for F a finite group.

Denote $(\mathcal{L}')$ the class of groups including groups in the class $(\mathcal{L})$ and

(1) connected Lie groups and their cocompact lattices;

(2) irreducible lattices in semisimple groups of rank ≥ 2;

(3) hyperbolic finitely generated groups.

Let μ be a left Haar measure on the locally compact group G and write $L^p(G) = L^p(G, \mu)$. The group G acts by isometry on $L^p(G)$ via the left regular representation $\lambda_{G,p}$ defined by

$$\lambda_{G,p}(g)\varphi = \varphi(g^{-1}\cdot).$$

Theorem 1. *Fix some $1 \leq p < \infty$. Let G be a group of the class $(\mathcal{L})$ and let f be an increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\int_1^\infty \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} < \infty. \qquad (C_p)$$

Then there exists a 1-cocycle $b \in Z^1(G, \lambda_{G,p})$ whose compression ρ satisfies

$$\rho \succeq f.$$

Corollary 2. *Fix some $1 \leq p < \infty$. Let G be a group of the class $(\mathcal{L}')$ and let f be an increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying Property (C_q), with $q = \max\{p, 2\}$. Then there exists a uniform embedding of G into some L^p-space whose compression ρ satisfies*

$$\rho \succeq f.$$

Let us sketch the proof of the corollary. First, recall [W], III.A.6, that for $1 \leq p \leq 2$, $L^2([0, 1])$ is isomorphic to a subspace of $L^p([0, 1])$. It is thus enough to prove the theorem for $2 \leq p < \infty$. This is an easy consequence of Theorem 1 since every group of class $(\mathcal{L}')$ quasi-isometrically embeds into a group of $(\mathcal{L})$. Indeed, any connected Lie group admits a closed cocompact connected solvable subgroup.

[2] This condition guaranties that the group is compactly generated.

On the other hand, irreducible lattices in semisimple groups of rank ≥ 2 are quasi-isometrically embedded [LMR]. Finally, any hyperbolic finitely generated group quasi-isometrically embeds into the real hyperbolic space $\mathbf{H}^n$ for n large enough [BoS] which is itself quasi-isometric to $\mathrm{SO}(n,1)$.

The particular case of nonabelian free groups, which are quasi-isometric to 3-regular trees, can also be treated by a more direct method. More generally that method applies to any simplicial[3] tree with possibly infinite degree.

Theorem 3 (see Theorem 7.3). *Let T be a simplicial tree. For every increasing function $f:\mathbb{R}_+\to\mathbb{R}_+$ satisfying*

$$\int_1^\infty \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} < \infty, \qquad (C_p)$$

there exists a uniform embedding F of T into $\ell^p(T)$ with compression $\rho \succeq f$.

Remark 1.1. In [BuSc1], [BuSc2], it is shown that real hyperbolic spaces and word hyperbolic groups quasi-isometrically embed into finite products of (simplicial) trees. Thus the restriction of Corollary 2 to word hyperbolic groups and to simple Lie groups of rank 1 can be deduced from Proposition 7.3. Nevertheless, not every connected Lie group quasi-isometrically embeds into a finite product of trees. Namely, a finite product of trees is a CAT(0) space, and in [Pau] it is proved that a non-abelian simply connected nilpotent Lie group cannot quasi-isometrically embed into any CAT(0) space.

Theorem 4. *Let T_N be the binary rooted tree of depth N. Let ρ be the compression of some 1-Lipschitz map from T_N to some L^p-space for $1<p<\infty$. Then there exists $C<\infty$, depending only on p, such that*

$$\int_1^{2N} \left(\frac{\rho(t)}{t}\right)^q \frac{dt}{t} \leq C,$$

where $q=\max\{p,2\}$.

Although this result is a strengthening (see Corollary 6.3) of Theorem 1 in [Bou], its proof is based on the same arguments. As a consequence, we have

Corollary 5. *Assume that the 3-regular tree quasi-isometrically embeds into some metric space X. Then, the compression ρ of any uniform embedding of X into any L^p-space for $1<p<\infty$ satisfies (C_q) for $q=\max\{p,2\}$.*

[3] By simplicial, we mean that every edge has length 1.

In [BeSc], Theorem 1.5, it is proved that the 3-regular tree quasi-isometrically embeds into any graph with bounded degree and positive Cheeger constant (e.g. any non-amenable finitely generated group). On the other hand, in a work in preparation with Cornulier [CT], we prove that finitely generated linear groups with exponential growth, and finitely generated solvable groups with exponential growth admit quasi-isometrically embedded free non-abelian sub-semigroups. Together with the above corollary, they lead to the optimality of Theorem 1 (resp. Corollary 2) when the group has exponential growth and when $2 \leq p < \infty$ (resp. $1 < p < \infty$).

Corollary 6. *Let G be a finitely generated group with exponential growth which is either virtually solvable or non-amenable. Let φ be a uniform embedding of G into some L^p-space for $1 < p < \infty$. Then its compression ρ satisfies Condition (C_q) for $q = \max\{p, 2\}$.*

Corollary 7. *Let G be a group of class $(\mathcal{L}')$ with exponential growth. Consider an increasing map f and some $1 < p < \infty$; then f satisfies Condition (C_q) with $q = \max\{p, 2\}$ if and only if there exists a uniform embedding of G into some L^p-space whose compression ρ satisfies $\rho \succeq f$.*

Note that the 3-regular tree cannot uniformly embed into a group with subexponential growth. So the question of the optimality of Theorem 1 for non-abelian nilpotent connected Lie groups remains open.

About Condition (C_p). First, note that if $p \leq q$, then (C_p) implies (C_q): this immediately follows from the fact that a nondecreasing function f satisfying (C^p) also satisfies $f(t)/t = O(1)$.

Let us give examples of functions f satisfying Condition (C_p). Clearly, if f and h are two increasing functions such that $f \preceq h$ and h satisfies (C_p), then f satisfies (C_p). The function $f(t) = t^a$ satisfies (C_p) for every $a < 1$ but not for $a = 1$. More precisely, the function

$$f(t) = \frac{t}{(\log t)^{1/p}}$$

does not satisfy (C_p) but

$$f(t) = \frac{t}{((\log t)(\log\log t)^a)^{1/p}}$$

satisfies (C_p) for every $a > 1$. In comparison, in [BrSo], the authors construct a uniform embedding of the free group of rank 2 into a Hilbert space with compression larger than

$$\frac{t}{((\log t)(\log\log t)^2)^{1/2}}.$$

As $t/(\log t)^{1/p}$ does not satisfy (C_p), one may wonder if (C_p) implies

$$\rho(t) \preceq \frac{t}{(\log t)^{1/p}}.$$

The following proposition answers negatively to this question. We say that a function f is sublinear if $f(t)/t \to 0$ when $t \to \infty$.

Proposition 8 (see Proposition 7.5). *For any increasing sublinear function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ and every $1 \leq p < \infty$, there exists a nondecreasing function f satisfying (C_p), a constant $c > 0$ and an increasing sequence of integers (n_i) such that*

$$f(n_i) \geq ch(n_i) \quad \forall i \in \mathbb{N}.$$

In particular, it follows from Theorem 1 that the compression ρ of a uniform embedding of a 3-regular tree in a Hilbert space does not satisfy any *a priori* majoration by any sublinear function.

1.2. Isoperimetry and compression. To prove Theorem 1, we observe a general relation between the L^p-isoperimetry inside balls and the L^p-compression. Let G be a locally compact compactly generated group and consider some compact symmetric generating subset S. For every $g \in G$, write[4]

$$|\widetilde{\nabla}\varphi|(g) = \sup_{s\in S} |\varphi(sg) - \varphi(g)|.$$

Let $2 \leq p < \infty$ and let us call the L^p-isoperimetric profile inside balls the nondecreasing function $J^b_{G,p}$ defined by

$$J^b_{G,p}(t) = \sup_{\varphi} \frac{\|\varphi\|_p}{\|\widetilde{\nabla}\varphi\|_p},$$

where the supremum is taken over all measurable functions in $L^p(G)$ with support in the ball $B(1,t)$. Note that the group G is amenable if and only if $\lim_{t\to\infty} J^b_{G,p}(t) = \infty$. Theorem 1 results from the two following theorems.

Theorem 9 (see Theorem 5.1). *Let G be a group of class $(\mathcal{L})$. Then $J^b_{G,p}(t) \approx t$.*

Theorem 10 (see Corollary 4.6). *Let G be a locally compact compactly generated group and let f be a nondecreasing function satisfying*

$$\int_1^\infty \left(\frac{f(t)}{J^b_{G,p}(t)}\right)^p \frac{dt}{t} < \infty \qquad (CJ_p)$$

[4]We write $\widetilde{\nabla}$ instead of ∇ because this is not a“metric” gradient. The gradient associated to the metric structure would be the right gradient: $|\nabla\varphi|(g) = \sup_{s\in S} |\varphi(gs) - \varphi(g)|$. This distinction is only important when the group is non-unimodular.

for some $1 < p < \infty$. *Then there exists a* 1*-cocycle* $b \in Z^1(G, \lambda_{G,p})$ *whose compression* ρ *satisfies* $\rho \succeq f$.

Theorem 9 may sound as a "functional" property of groups of class ($\mathcal{L}$). Nevertheless, our proof of this result is based on a purely geometric construction. Namely, we prove that these groups admit controlled Følner pairs (see Definition 4.8). In particular, when $p = 1$ we obtain the following corollary of Theorem 9, which has its own interest.

Theorem 11 (see Remark 4.10 and Theorem 5.1). *Let* G *be a group of class* ($\mathcal{L}$) *and let* S *be some compact generating subset of* G. *Then* G *admits a sequence of compact subsets* $(F_n)_{n\in\mathbb{N}}$ *satisfying the two following conditions:*

(i) *there is a constant* $c > 0$ *such that*

$$\mu(sF_n \,\triangle\, F_n) \leq c\mu(F_n)/n \quad \forall s \in S, \forall n \in \mathbb{N};$$

(ii) *for every* $n \in \mathbb{N}$, F_n *is contained*[5] *in* S^n.

In particular, G *admits a controlled Følner sequence in the sense of* [CTV].

This theorem is a strengthening of the well-known construction by Pittet [Pit]. It is stronger first because it does not require the group to be unimodular, second because the control (ii) of the diameter is really a new property that was not satisfied in general by the sequences constructed in [Pit].

1.3. Compression, subexponential growth, and random walks. Let π be an isometric L^p-representation of G. Denote by $B_\pi(G)$ the supremum of all α such that there exists a 1-cocycle $b \in Z^1(G, \pi)$ whose compression ρ satisfies $\rho(t) \succeq t^\alpha$. Denote by $B_p(G)$ the supremum of $B_\pi(G)$ over all isometric L^p-representations π. For $p = 2$, $B_2(G) = B(G)$ has been introduced in [GK] where it was called the equivariant Hilbert compression rate (we suggest that the term *exponent* would be more appropriate here than the term *rate*). On the other hand, define

$$\alpha_{G,p} = \liminf_{t\to\infty} \frac{\log J^b_{G,p}(t)}{\log t}.$$

As a corollary of Theorem 1, we have

Corollary 12. *For every* $1 \leq p < \infty$*, and every group* G *of the class* ($\mathcal{L}$)*, we have* $B_p(G) = 1$.

The following result is a corollary of Theorem 10.

[5] Actually, they also satisfy $S^{[cn]} \subset F_n$ for a constant $c > 0$.

Corollary 13 (see Corollary 4.6). *Let G be a locally compact compactly generated group. For every $0 < p < \infty$, we have*

$$B_{\lambda_{G,p}}(G) \geq \alpha_{G,p}.$$

The interest of this corollary is illustrated by the two following propositions. Recall the volume growth of G is the $\approx$ equivalence class V_G of the function $r \mapsto \mu(B(1,r))$.

Proposition 14 (see Proposition 7.1). *Assume that there exists $\beta < 1$ such that $V_G(r) \preceq e^{r^\beta}$. Then*

$$\alpha_{G,p} \geq 1 - \beta.$$

As an example we obtain that $B(G) \geq 0,19$ for the first Grigorchuk's group (see [Ba] for the best known upper bound of the growth function of this group).

Let G be a finitely generated group and let ν be a symmetric finitely supported probability measure on G. Write $\nu^{(n)} = \nu * \cdots * \nu$ (n times). Recall that $\nu^{(n)}(1)$ is the probability of return of the random walk starting at 1 whose probability transition is given by ν.

Proposition 15 (see Proposition 7.2). *Assume that there exists $\gamma < 1$ such that $\nu^{(n)}(1) \succeq e^{-n^\gamma}$. Then*

$$\alpha_{G,2} \geq (1-\gamma)/2.$$

In [PS], it is proved that if G is a finitely generated extension

$$1 \to K \to G \to N \to 1$$

where K is abelian and N is abelian with $\mathbb{Q}$-rank d. Then

$$\limsup_n \log(-\log(\nu^{(n)}(1))) \leq 1 - 2/(d+2)$$

for any symmetric finitely supported probability on G.

Corollary 16. *Assume that G is a finitely generated extension $1 \to K \to G \to N \to 1$ where K is abelian and N is abelian with $\mathbb{Q}$-rank d. Then*

$$B(G) \geq 1/(d+2).$$

In particular, $B(G) > 0$ for any finitely generated metabelian group G.

1.4. The case of $\mathbb{Z} \wr \mathbb{Z}$. Combining the construction of Theorem 1 for $C_2 \wr \mathbb{Z}$ with the cocycle induced by the morphism of $\mathbb{Z}^{(\mathbb{Z})} \to \ell^p(\mathbb{Z})$, we obtain (see Proposition 7.6 for the details).

Theorem 17. *Fix some $1 \leq p < \infty$. Let $G = \mathbb{Z} \wr \mathbb{Z}$ and let f be an increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\int_1^\infty \left(\frac{f(t)}{t^{p/(2p-1)}}\right)^p \frac{dt}{t} < \infty. \qquad (C_p)$$

Then there exists a 1-cocycle $b \in Z^1(G, \lambda_{G,p})$ whose compression ρ satisfies

$$\rho \succeq f.$$

In particular,

$$B_p(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{p}{2p-1}.$$

In a previous version of this paper, we stated the lower bound $B(\mathbb{Z} \wr \mathbb{Z}) \geq 2/3$, but the proof that we gave relied on a wrong version of Proposition 15 (we stated $\alpha_{G,2} \geq 1-\gamma$, which is wrong as shown by a counter-example in [NP]). The mistake, together with a proof of the full statement $B_p(\mathbb{Z} \wr \mathbb{Z}) \geq \frac{p}{2p-1}$ (see [NP], Lemma 7.8) was communicated to us by Naor and Peres. The proof that we propose here is essentially the same as the one of [NP], but it was actually also known by the author.

1.5. Questions

Question 1.2 (Condition (C_p) for nilpotent connected Lie groups.). Let N be a simply connected non-abelian nilpotent Lie group and let ρ be the compression of a 1-cocycle with values in some L^p-space (resp. of a uniform embedding into some L^p-space) for $2 \leq p < \infty$. Does ρ always satisfies Condition (C_p)?

A positive answer would lead to the optimality of Theorem 1. On the contrary, one should wonder if it is possible, for any increasing sublinear function f, to find a 1-cocycle (resp. a uniform embedding) in L^p with compression $\rho \succeq f$. This would also be optimal since we know [Pau] that N cannot quasi-isometrically embed into any uniformly convex Banach space. Namely, the main theorem in [Pau] states that such a group cannot quasi-isometrically embed into any CAT(0)-space. So this only directly applies to Hilbert spaces, but the key argument, consisting in a comparison between the large scale behavior of geodesics (not exactly in the original spaces but in tangent cones of ultra-products of them) is still valid if the target space is a Banach space with unique geodesics, a property satisfied by uniformly convex Banach spaces.

Question 1.3 (Quasi-isometric embeddings into $\boldsymbol{L^1}$-spaces.). Which connected Lie groups quasi-isometrically embed into some L^1-space?

It is easy to quasi-isometrically embed a simplicial tree T into ℓ^1 (see for instance [GK]). In [BuSc1], [BuSc2], it is proved that every semisimple Lie group of rank 1 quasi-isometrically embeds into a finite product of simplicial trees, hence into a ℓ^1-space. The above question is of particular interest for simply-connected non-abelian nilpotent Lie groups since they do not quasi-isometrically embed into any finite product of trees. Kleiner and Cheeger recently announced a proof that the Heisenberg group cannot quasi-isometrically embed into any L^1-space.

Question 1.4. If G is an amenable group, is it true that

$$B_p(G) = \alpha_{G,p}?$$

We conjecture that this is true for $\mathbb{Z} \wr \mathbb{Z}$, i.e. that $B(\mathbb{Z} \wr \mathbb{Z}) = 2/3$. A first step to prove this is done by Proposition 3.9 which, applied to $G = \mathbb{Z} \wr \mathbb{Z}$ says that

$$B(\mathbb{Z} \wr \mathbb{Z}) = B_{\lambda_{G,2}}(\mathbb{Z} \wr \mathbb{Z}).$$

As a variant of the above question, we may wonder if the weaker equality $B_{\lambda_{G,p}}(G) = \alpha_{G,p}$ holds, in other words if Corollary 13 is optimal for all amenable groups. Possible counterexamples would be wreath products of the form $G = \mathbb{Z} \wr H$ where H has non-linear growth (e.g. $H = \mathbb{Z}^2$).

Question 1.5. Does there exist an amenable group G with $B(G) = 0$?

A candidate would be the wreath product $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ since the probability of return of any non-degenerate random walk in this group satisfies

$$\nu^{(n)}(1) \prec e^{-n^{\gamma}}$$

for every $\gamma < 1$ ([Er], Theorem 2). It is proved in [AGS] that $B(\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})) \leq 1/2$.

Question 1.6. Let G be a compactly generated locally compact group. If G admits an isometric action on some L^p-space, $p \geq 2$, with compression $\rho(t) \succ t^{1/p}$, does it imply that G is amenable?

Recall that this was proved in [GK], [CTV] for $p = 2$. The generalization to every $p \geq 2$ would be of great interest. For instance, this would prove the optimality of a recent result of Yu [Yu] saying that every finitely generated hyperbolic group admits a proper isometric action on some ℓ^p-space for large p enough, with[6] compression $\rho(t) \approx t^{1/p}$.

[6] This is clear in the proof.

Acknowledgments. First, I would like to thank Assaf Naor and Yuval Peres for showing me their paper [NP] where they noticed a mistake in my earlier proof of $B(\mathbb{Z} \wr \mathbb{Z}) \geq 2/3$, and proposed a correct proof of this result (see Section 1.4). Let us also mention that in their paper, Naor and Peres answer to many of our questions, in particular they answer negatively to Question 1.4. On the other hand, in [ANP], Austin, Naor and Peres prove our conjecture that $B(\mathbb{Z} \wr \mathbb{Z}) = 2/3$.

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2. Preliminaries

2.1. Compression. Let us recall some definitions. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $F : X \to Y$ is called a uniform embedding of X into Y if

$$d_X(x, y) \to \infty \iff d_Y(F(x), F(y)) \to \infty.$$

Note that this property only concerns the large-scale geometry. A metric space (X, d) is called *quasi-geodesic* if there exist $\delta > 0$ and $\gamma \geq 1$ such that for all $x, y \in X$, there exists a chain $x = x_0, x_1, \dots, x_n = y$ satisfying:

$$\sum_{k=1}^{n} d(x_{k-1}, x_k) \leq \gamma d(x, y),$$

$$\forall k = 1, \dots, n, \quad d(x_{k-1}, x_k) \leq \delta.$$

If X is quasi-geodesic and if $F : X \to Y$ is a uniform embedding, then it is easy to see that F is large-scale Lipschitz, i.e. there exists $C \geq 1$ such that

$$\forall x, y \in X, \quad d_Y(F(x), F(y)) \leq C d_X(x, y) + C.$$

Nevertheless, such a map is not necessarily large scale bi-Lipschitz (in other words, quasi-isometric).

Definition 2.1. We define the *compression* $\rho : \mathbb{R}_+ \to [0, \infty]$ of a map $F : X \to Y$ by

$$\forall t > 0, \quad \rho(t) = \inf_{d_X(x,y) \geq t} d_Y(F(x), F(y)).$$

Clearly, if F is large-scale Lipschitz, then $\rho(t) \preceq t$.

2.2. Length functions on a group. Now, let G be a group. A length function on G is a function $L\colon G \to \mathbb{R}_+$ satisfying $L(1) = 0$, $L(gh) \leq L(g) + L(h)$, and $L(g) = L(g^{-1})$. If L is a length function, then $d(g,h) = L(g^{-1}h)$ defines a left-invariant pseudo-metric on G. Conversely, if d is a left-invariant pseudo-metric on G, then $L(g) = d(1,g)$ defines a length function on G.

Let G be a locally compact compactly generated group and let S be some compact symmetric generating subset of G. Equip G with a proper, quasi-geodesic length function by

$$|g|_S = \inf\{n \in \mathbb{N} : g \in S^n\}.$$

Denote d_S the associated left-invariant distance. Note that any proper, quasi-geodesic left-invariant metric is quasi-isometric to d_S, and so belongs to the same "asymptotic class".

2.3. Affine isometric actions and first cohomology. Let G be a locally compact group, and π an isometric representation (always assumed continuous) on a Banach space $E = E_\pi$. The space $Z^1(G,\pi)$ is defined as the set of continuous functions $b\colon G \to E$ satisfying, for all g, h in G, the 1-cocycle condition $b(gh) = \pi(g)b(h) + b(g)$. Observe that, given a continuous function $b\colon G \to \mathcal{H}$, the condition $b \in Z^1(G,\pi)$ is equivalent to saying that G acts by affine isometries on $\mathcal{H}$ by $\alpha(g)v = \pi(g)v + b(g)$. The space $Z^1(G,\pi)$ is endowed with the topology of uniform convergence on compact subsets.

The subspace of coboundaries $B^1(G,\pi)$ is the subspace (not necessarily closed) of $Z^1(G,\pi)$ consisting of functions of the form $g \mapsto v - \pi(g)v$ for some $v \in E$. In terms of affine actions, $B^1(G,\pi)$ is the subspace of affine actions fixing a point.

The first cohomology space of π is defined as the quotient space

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi).$$

Note that if $b \in Z^1(G,\pi)$, the map $(g,h) \to \|b(g) - b(h)\|$ defines a left-invariant pseudo-distance on G. Therefore the compression of a 1-cocycle $b\colon (G,d_S) \to E$ is simply given by

$$\rho(t) = \inf_{|g|_S \geq t} \|b(g)\|.$$

The compression of an affine isometric action is defined as the compression of the corresponding 1-cocycle.

Remark 2.2. When the space E is a Hilbert space[7], it is well known [HV], §4.a, that $b \in B^1(G,\pi)$ if and only if b is bounded on G.

[7] The same proof holds for uniformly convex Banach spaces.

3. The maximal L^p-compression functions $M\rho_{G,p}$ and $M\rho_{\lambda_{G,p}}$

3.1. Definitions and general results. Let (G, d_S, μ) be a locally compact compactly generated group, generated by some compact symmetric subset S and equipped with a left Haar measure μ. Denote by $Z^1(G, p)$ the collection of all 1-cocycles with values in any L^p-representation of G. Denote by ρ_b the compression function of a 1-cocycle $b \in Z^1(G, p)$.

Definition 3.1. We call *maximal L^p-compression function* of G the nondecreasing function $M\rho_{G,p}$ defined by

$$M\rho_{G,p}(t) = \sup\Big\{\rho_b(t) : \ b \in Z^1(G, p), \ \sup_{s\in S} \|b(s)\| \leq 1\Big\}.$$

We call maximal *regular* L^p-compression function of G the nondecreasing function $M\rho_{\lambda_{G,p}}$ defined by

$$M\rho_{\lambda_{G,p}} = \sup\Big\{\rho_b(t) : \ b \in Z^1(G, \lambda_{G,p}), \ \sup_{s\in S} \|b(s)\| \leq 1\Big\}.$$

Note that the asymptotic behaviors of both $M\rho_{G,p}$ and $M\rho_{\lambda_{G,p}}$ do not depend on the choice of the compact generating set S. Moreover, we have

$$M\rho_{\lambda_{G,p}}(t) \leq M\rho_{G,p}(t) \preceq t.$$

Let φ be a measurable function on G such that $\varphi - \lambda(s)\varphi \in L^p(G)$ for every $s \in S$. For every $t > 0$, define

$$\mathrm{Var}_p(\varphi, t) = \inf_{|g|_S \geq t} \|\varphi - \lambda(g)\varphi\|_p.$$

The function φ and p being fixed, the map $t \mapsto \mathrm{Var}_p(\varphi, t)$ is nondecreasing.

Proposition 3.2. *We have*

$$M\rho_{\lambda_{G,p}}(t) = \sup_{\|\tilde{\nabla}\varphi\|_p \leq 1} \mathrm{Var}_p(\varphi, t).$$

Proof. We trivially have

$$M\rho_{\lambda_{G,p}}(t) \geq \sup_{\|\tilde{\nabla}\varphi\|_p \leq 1} \mathrm{Var}_p(\varphi, t).$$

Let b be an element of $Z^1(G, \lambda_{G,p})$. By convoluting $b(g)$, for every g, on the right by a Dirac approximation, one can approximate b by a cocycle b' such that $x \to b'(g)(x)$ is continuous for every g in G. Hence, we can assume that $b(g)$ is continuous for every g in G. Now, setting $\varphi(g) = b(g)(g)$, we define a measurable function satisfying

$$b(g) = \varphi - \lambda(g)\varphi.$$

So we have

$$\rho(t) = \mathrm{Var}_p(\varphi, t) \leq M\rho_{\lambda_{G,p}}(t)$$

where ρ is the compression of b. □

Remark 3.3. It is not difficult to prove that the asymptotic behavior of $M\rho_{\lambda_{G,p}}$ is invariant under quasi-isometry between finitely generated groups.

Proposition 3.4. *The group G admits a proper*[8] *1-cocycle with values in some L^p-representation if and only if $M\rho_{G,p}(t)$ goes to infinity as $t \to \infty$.*

Proof. The "only if" part is trivial. Assume that $M\rho_{G,p}(t)$ goes to infinity. Let (t_k) be an increasing sequence growing fast enough so that

$$\sum_{k\in\mathbb{N}} \frac{1}{t_k^p} < \infty.$$

For every $k \in \mathbb{N}$, choose some $b_k \in Z^1(G, p)$ whose compression ρ_k satisfies

$$\rho_k(t_k) \geq \frac{M\rho_{G,p}(t_k)}{2}$$

and such that

$$\sup_{s\in S} \|b_k(s)\| \leq 1.$$

Clearly, we can define a 1-cocycle $b \in Z^1(G, p)$ by

$$b = \bigoplus\nolimits_k^{\ell^p} \tfrac{1}{t_k} b_k.$$

That is, if for every k, b_k takes values in the representation π_k, then b takes values in the direct sum $\oplus_k^{\ell^p} \pi_k$. Now, observe that for $|g| \geq t_k$ and $j \leq k$, we have $\|b_j(g)\| \geq 1/2$, so that

$$\|b(g)\|^p \geq k/2^p.$$

Thus the cocycle b is proper. □

The following proposition, which is a quantitative version of the previous one, plays a crucial role in the sequel.

[8]For $p = 2$, this means that G is a-T-menable if and only if $M\rho_{G,2}$ goes to infinity. It should be compared to the role played by the H-metric (see § 2.6 in [C], and § 7.4) for Property (T).

Proposition 3.5. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing map satisfying*

$$\int_1^\infty \left(\frac{f(t)}{M\rho_{G,p}(t)}\right)^p \frac{dt}{t} < \infty, \qquad (CM_p)$$

Then,

(1) *there exists a* 1*-cocycle $b \in Z^1(G, p)$ such that*

$$\rho \succeq f;$$

(2) *if one replaces $M\rho_{G,p}$ by $M\rho_{\lambda_{G,p}}$ in Condition (CM_p), then b can be chosen in $Z^1(G, \lambda_{G,p})$.*

Proof. (1): For every $k \in \mathbb{N}$, choose some $b_k \in Z^1(G, p)$ (for (2), we take $b_k \in Z^1(G, \lambda_{G,p})$) whose compression ρ_k satisfies

$$\rho_k(2^{k+1}) \geq \frac{M\rho_{G,p}(2^{k+1})}{2}$$

and such that

$$\sup_{s\in S} \|b_k(s)\| \leq 1.$$

Then define another sequence of cocycles $\tilde{b}_k \in Z^1(G, p)$ by

$$\tilde{b}_k = \frac{f(2^k)}{M\rho_{G,p}(2^{k+1})} b_k.$$

Since $M\rho_{G,p}$ and f are nondecreasing, for any $2^k \leq t \leq 2^{k+1}$, we have

$$\frac{f(2^k)}{M\rho_{G,p}(2^{k+1})} \leq \frac{f(t)}{M\rho_{G,p}(t)}.$$

Hence, for $s \in S$,

$$\begin{aligned}\sum_k \|\tilde{b}_k(s)\|_p^p &\leq \sum_k \left(\frac{f(2^k)}{M\rho_{G,p}(2^{k+1})}\right)^p \\ &\leq 2\int_1^\infty \left(\frac{f(t)}{M\rho_{G,p}(t)}\right)^p \frac{dt}{t} < \infty\end{aligned}$$

So we can define a 1-cocycle on $b \in Z^1(G, p)$ by

$$b = \bigoplus_k \tilde{b}_k. \qquad (3.1)$$

On the other hand, if $|g|_S \geq 2^{k+1}$, then

$$\begin{aligned}\|b(g)\|_p &\geq \|\tilde{b}_k(g)\|_p \\ &\geq \frac{f(2^k)}{M\rho_{\lambda_{G,p}}(2^{k+1})}\rho_k(2^{k+1}) \\ &\geq f(2^k).\end{aligned}$$

So if ρ is the compression of the 1-cocycle b, we have $\rho \succeq f$.

(2): We keep the previous notation. Assume that f satisfies

$$\int_1^{\infty}\left(\frac{f(t)}{M\rho_{\lambda_{G,p}}(t)}\right)^p\frac{dt}{t} < \infty.$$

The cocycle b provided by the proof of (1) has the expected compression but it takes values in an infinite direct sum of regular representation $\lambda_{G,p}$. Now, we would like to replace the direct sum $b = \oplus_k b_k$ by a mere sum, in order to obtain a cocycle in $Z^1(G, \lambda_{G,p})$. Since G is not assumed unimodular, the measure μ is not necessarily right-invariant. However, one can define an isometric representation $r_{G,p}$ on $L^p(G)$, called the right regular representation by

$$r_{G,p}(g)\varphi = \Delta(g)^{-1}\varphi(\cdot g) \quad \forall \varphi \in L^p(G),$$

where Δ is the modular function of G. We will use the following well-known property of the representation $r_{G,p}$, for $p > 1$. To simplify, let us write $r(g)$ instead of $r_{G,p}(g)$. For every $(\varphi, \psi) \in L^p(G) \times L^p(G)$, we have

$$\lim_{|g|\to\infty} \|r(g)\varphi + \psi\|_p^p = \|\varphi\|_p^p + \|\psi\|_p^p. \tag{3.2}$$

Moreover, this limit is uniform on compact subsets of $(L^p(G))^2$. As $r_{G,p}$ and $\lambda_{G,p}$ commute, $r_{G,p}$ acts by isometries on $Z^1(G, \lambda_{G,p})$.

Lemma 3.6. *There exists a sequence (g_k) of elements of G such that $b' = \sum r(g_k)b_k$ defines a cocycle in $Z^1(G, \lambda_{G,p})$ and such that*

$$\left|\|b'(g)\|_p^p - \left\|\sum_{j=0}^{k-1} r(g_j)b_j(g)\right\|_p^p - \sum_{j\geq k}\|b_j(g)\|_p^p\right| \leq 1 \tag{3.3}$$

for any k large enough and every $g \in B(1, 2^{k+2})$.

Proof of Lemma 3.6. By an immediate induction, using (3.2), we construct a sequence $(g_k) \in G^{\mathbb{N}}$ satisfying, for every $K \geq 0$, $s \in S$,

$$\left\|\sum_{k=0}^{K} r(g_k)b_k(s)\right\|_p^p \leq \sum_{k=0}^{K}\|b_k(s)\|_p^p + \sum_{k=0}^{K} 2^{-k-1} \leq 1,$$

which implies that b' is a well-defined 1-cocycle in $Z^1(G, \lambda_{G,p})$. Similarly, one can choose (g_k) satisfying the additional property that, for every $k \in \mathbb{N}$, $|g| \le 2^{k+2}$,

$$\left| \left\| \sum_{j=0}^{k} r(g_j) b_j(g) \right\|_p^p - \left\| \sum_{j=0}^{k-1} r(g_j) b_j(g) \right\|_p^p - \|b_k(g)\|_p^p \right| \le 2^{-k-1}.$$

Fixing $k \in \mathbb{N}$, an immediate induction over K shows that for every $|g| \le 2^{k+2}$ and every $K \ge k$,

$$\left| \left\| \sum_{j=0}^{K} r(g_j) b_j(g) \right\|_p^p - \left\| \sum_{j=0}^{k-1} r(g_j) b_j(g) \right\|_p^p - \sum_{j=k}^{K} \|b_j(g)\|_p^p \right| \le \sum_{j=k}^{K} 2^{-j-1}.$$

This proves (3.3). □

By the lemma, for $|g| \le 2^{k+2}$,

$$\|b'(g)\|_p^p \ge \|b_k(g)\|_p^p - 1.$$

Then, for $2^{k+1} \le |g| \le 2^{k+2}$, we have

$$\|b'(g)\|_p^p \ge f(2^k) - 1$$

Therefore, the compression ρ' of b' satisfies

$$\rho' \succeq f$$

and we are done. □

We have the following immediate consequence.

Corollary 3.7. *For every* $1 \le p < \infty$,

$$B(G, p) = \liminf_{t \to \infty} \frac{\log M\rho_{G,p}(t)}{\log t}.$$

Example 3.8. Let F_r be the free group of rank $r \ge 2$ and let $A(F_r)$ be the set of edges of the Cayley graph of F_r associated to the standard set of generators. The standard isometric affine action of F_r on $\ell^p(A(F_r))$, whose linear part is isomorphic to a direct sum $\lambda_{G,p} \oplus_{\ell^p} \cdots \oplus_{\ell^p} \lambda_{G,p}$ of r copies of $\lambda_{G,p}$ has compression $\approx t$. This shows that $M\rho_{\lambda_{F_r,p}}(t) \succeq t^{1/p}$.

3.2. Reduction to the regular representation for $p = 2$. In the Hilbert case, we prove that if a group admits a 1-cocycle with large enough compression, then $M\rho_{G,2} = M\rho_{\lambda_{G,2}}$. This result is mainly motivated by Question 1.4 since it implies that

$$B(\mathbb{Z} \wr \mathbb{Z}) = B_{\lambda_{G,2}}(\mathbb{Z} \wr \mathbb{Z}).$$

Proposition 3.9. *Let π be a unitary representation of the group G on a Hilbert space $\mathcal{H}$ and let $b \in Z^1(G,\pi)$ be a cocycle whose compression ρ satisfies*

$$\rho(t) \succ t^{1/2}.$$

Then[9]

$$\rho \preceq M\rho_{\lambda_{G,2}}.$$

In particular,

$$M\rho_2 = M\rho_{\lambda_{G,2}}.$$

combining with Proposition 3.5, we obtain

Corollary 3.10. *With the same hypotheses, we have*

$$B(G) = B(G,\lambda_{G,2}) = \liminf_{t\to\infty} \frac{\log M\rho_{\lambda_{G,2}}(t)}{\log t}.$$

Proof of Proposition 3.9. For every $t > 0$, define

$$\varphi_t(g) = e^{-\|b(g)\|^2/t^2}.$$

By Schoenberg's Theorem (Appendix C in [BHV]), φ_t is positive definite. It is easy to prove that φ_t is square-summable (see [CTV], Theorem 4.1). By [Dix], Théorème 13.8.6, it follows that there exists a positive definite, square-summable function ψ_t on G such that $\varphi_t = \psi_t * \psi_t$, where $*$ denotes the convolution product. In other words, $\varphi_t = \langle\lambda(g)\psi_t, \psi_t\rangle$. In particular,

$$\varphi_t(1) = 1 = \|\psi_t\|_2^2$$

and for every $s \in S$,

$$\begin{aligned}\|\psi_t - \lambda(s)\psi_t\|_2^2 &= 2(\|\psi_t\|_2^2 - \langle\lambda(s)\psi_t, \psi_t\rangle)\\ &= 2(1-\varphi_t(s))\\ &= 2(1-e^{-\|b(s)\|^2/t^2})\\ &\preceq 1/t^2\end{aligned}$$

[9] Note that the hypotheses of the proposition also imply that G is amenable [CTV] (Theorem 4.1), [GK].

On the other hand, for g such that $\rho(|g|_S) \geq t$, we have

$$\begin{aligned}\|\psi_t - \lambda(g)\psi_t\|_2^2 &= 2(1 - e^{-\|b(g)\|^2/t^2}) \\ &\geq 2(1 - e^{-\rho(|g|_S)^2/t^2}) \\ &\geq 2(1 - 1/e).\end{aligned}$$

So, we have

$$\frac{\|\psi_t - \lambda(g)\psi_t\|_2}{\|\tilde{\nabla}\psi_t\|_2} \geq ct$$

where c is a constant. In other words,

$$\mathrm{Var}_2(\psi_t, \rho^{-1}(t)) \geq ct.$$

It follows from the definitions that $M\rho_{\lambda_{G,2}} \succeq \rho$. □

4. L^p-isoperimetry inside balls

4.1. Comparing $J^b_{G,p}$ and $M\rho_{\lambda_{G,p}}$. Let G be a locally compact compactly generated group and let S be a compact symmetric generating subset of G. Let A be a subset of the group G. One defines the L^p-isoperimetric profile inside A by

$$J_p(A) = \sup_{\varphi} \frac{\|\varphi\|_p}{\|\tilde{\nabla}\varphi\|_p}$$

where the supremum is taken over nonzero functions in $L^p(G)$ with support included in A.

Definition 4.1. The L^p-*isoperimetric profile* inside balls is the nondecreasing function $J^b_{G,p}$ defined by

$$J^b_{G,p}(t) = J_p(B(1,t)).$$

Remark 4.2. The usual L^p-isoperimetric profile of G (see for example [Cou]) is defined by

$$j_{G,p}(t) = \sup_{\mu(A)=t} J_p(A).$$

Note that our notion of isoperimetric profile depends on the diameter of the subsets instead of their measure.

Remark 4.3. The asymptotic behavior of $J^b_{p,G}$ is invariant under quasi-isometry between compactly generated groups [T]. In particular, it is also invariant under passing to a cocompact lattice [CS].

Remark 4.4. Using basic L^p-calculus, one can easily prove [Cou] that if $p \le q$, then

$$(J^b_{G,p})^{p/q} \preceq J^b_{G,q} \preceq J^b_{G,p}.$$

Now let us compare $J^b_{p,G}$ and $M\rho_{\lambda_{G,p}}$ introduced in § 3.

Proposition 4.5. *For every $2 \le p < \infty$, we have*

$$M\rho_{\lambda_{G,p}} \succeq J^b_{G,p}.$$

Proof. Fix some $t > 0$ and choose some $\varphi \in L^p(X)$ whose support lies in $B(1,t)$ such that

$$\frac{\|\varphi\|_p}{\|\tilde{\nabla}\varphi\|_p} \ge J^b_{G,p}(t)/2.$$

Take $g \in G$ satisfying $|g|_S \ge 3t$. Note that $B(1,t) \cap \lambda(g)B(1,t) = \emptyset$. So φ and $\lambda(g)\varphi$ have disjoint supports. In particular,

$$\|\varphi - \lambda(g)\varphi\|_p \ge \|\varphi\|_p$$

and

$$\|\tilde{\nabla}(\varphi - \lambda(g)\varphi)\|_p = 2^{1/p}\|\tilde{\nabla}\varphi\|_p.$$

This clearly implies the proposition. □

Combining with Proposition 3.5, we obtain

Corollary 4.6. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing map be satisfying*

$$\int_1^\infty \left(\frac{f(t)}{J^b_{G,p}(t)}\right)^p \frac{dt}{t} < \infty \qquad (CJ_p)$$

for some $1 \le p < \infty$. Then there exists a 1-cocycle b in $Z^1(G, \lambda_{G,p})$ such that

$$\rho \succeq f.$$

Question 4.7. For which groups G do we have $M\rho_{\lambda_{G,p}} \approx J^b_{G,p}$?

We show that the question has positive answer for groups of class $(\mathcal{L})$. On the contrary, note that the group G is nonamenable if and only if $J^b_{G,p}$ is bounded. But we have seen in the previous section that for a free group of rank ≥ 2, $M\rho_{\lambda_{G,p}}(t) \succeq t^{1/p}$. More generally, the answer to Question 4.7 is no for every nonamenable group admitting a proper 1-cocycle with values in the regular representation. This question is therefore only interesting for amenable groups.

4.2. Sequences of controlled Følner pairs. In this section, we give a method, adapted[10] from [CGP] to estimate J_p^b.

Definition 4.8. Let G be a compactly generated, locally compact group equipped with a left invariant Haar measure μ. Let $\alpha = (\alpha_n)$ be a nondecreasing sequence of integers. A sequence of *α-controlled Følner pairs* of G is a family (H_n, H_n') where H_n and H_n' are nonempty compact subsets of G satisfying for some constant $C > 0$ the following conditions:

(1) $S^{\alpha_n} H_n \subset H_n'$

(2) $\mu(H_n') \leq C\mu(H_n)$;

(3) $H_n' \in B(1, Cn)$

If $\alpha_n \approx n$, we call (H_n, H_n') a controlled sequence of Følner pairs.

Proposition 4.9. *Assume that G admits a sequence of α-controlled Følner pairs. Then*

$$J_{G,p}^b \succeq \alpha.$$

Proof. For every $n \in \mathbb{N}$, consider the function $\varphi_n : G \to \mathbb{R}_+$ defined by

$$\varphi_n(g) = \min\{k \in \mathbb{N} : g \in S^k (H_n')^c\}$$

where $A^c = G \setminus A$. Clearly, φ_n is supported in H_n'. It is easy to check that

$$\|\tilde{\nabla}\varphi_n\|_p \leq (\mu(H_n'))^{1/p}$$

and that

$$\|\varphi_n\|_p \geq \alpha_n (\mu(H_n))^{1/p}.$$

Hence by (2),

$$\|\varphi_n\|_p \geq C^{-1/p}\alpha_n \|\tilde{\nabla}\varphi_n\|_p,$$

so we are done. □

Remark 4.10. Note that if H and H' are subsets of G such that $S^k H \subset H'$ and $\mu(H') \leq C\mu(H)$, then there exists by pigeonhole principle an integer $0 \leq j \leq k-1$ such that

$$\mu(\partial S^j H) = \mu(S^{j+1} H \setminus S^j H) \leq \frac{C}{k}\mu(S^j H).$$

So in particular if (H_n, H_n') is a α-controlled sequence of Følner pairs, then there exists a Følner sequence (K_n) such that $H_n \subset K_n \subset H_n'$ and

$$\frac{\mu(\partial K_n)}{\mu(K_n)} \leq C/\alpha_n.$$

Moreover, if $\alpha_n \approx n$, then one obtains a controlled Følner sequence in the sense of [CTV], Definition 4.8.

[10] In [CGP], the authors are interested in estimating the L^2-isoperimetric profile of a group.

5. Isoperimetry in balls for groups of class ($\mathcal{L}$)

The purpose of this section is to prove the following theorem.

Theorem 5.1. *Let G be a group belonging to the class ($\mathcal{L}$). Then, G admits a controlled sequence of Følner pairs. In particular, $J^b_{G,p}(t) \approx t$.*

Note that Theorem 1 follows from Theorem 5.1 and Corollary 4.6.

5.1. Wreath products $F \wr \mathbb{Z}$. Let F be a finite group. Consider the wreath product $G = F \wr \mathbb{Z} = \mathbb{Z} \ltimes F^{(\mathbb{Z})}$, the group law being defined as $(n, f)(m, g) = (n + m, \tau_m f + g)$ where $\tau_m f(x) = f(m+x)$. As a set, G is a Cartesian product $\mathbb{Z} \times U$ where U is the direct sum $F^{(\mathbb{Z})} = \bigoplus_{n\in\mathbb{Z}} F_n$ of copies F_n of F. The set $S = S_F \cup S_{\mathbb{Z}}$, where $S_F = F_0$ and $S_{\mathbb{Z}} = \{-1, 0, 1\}$ is clearly a symmetric generating set for G. Define

$$H_n = I_n \times U_n$$

and

$$H'_n = I_{2n} \times U_n$$

where $U_n = F^{[-2n,2n]}$ and $I_n = [-n, n]$.

Let us prove that $(H_n, H'_n)_n$ is a sequence of controlled Følner pairs. We therefore have to show that

(1) $S^n H_n \subset H'_n$

(2) $|H'_n| \leq 2|H_n|$;

(3) there exists $C > 0$ such that $H'_n \subset B(1, Cn)$

Property (2) is trivial. To prove (1) and (3), recall that the length of an element of $g = (k, u)$ of G equals $L(\gamma) + \sum_{h\in\mathbb{Z}} |u(h)|_F$ where $L(\gamma)$ is the length of a shortest path γ from 0 to k in $\mathbb{Z}$ passing through every element of the support of u (see [Par], Theorem 1.2). In particular,

$$|(u, k)|_S \leq 2L(\gamma).$$

Thus, if $g \in H_n$, then $L(\gamma) \leq 30n$. So (3) follows. On the other hand, if $g = (k, u) \in S^n$, then

$$|k|_{\mathbb{Z}} \leq L(\gamma) \leq n$$

and

$$\mathrm{Supp}(u) \subset I_n.$$

So $H_n g \subset H'_n$. □

Remark 5.2. Note that the proof still works replacing $\mathbb{Z}$ by any group with linear growth. On the other hand, replacing it by a group of polynomial growth of degree d yields a sequence of $n^{1/d}$-controlled Følner pairs. For instance, as a corollary, we obtain that $B(F \wr \mathbb{Z}^d) \geq 1/d$.

5.2. Semidirect products $(\mathbb{R} \oplus \bigoplus_{p\in P} \mathbb{Q}_p) \rtimes_{\frac{m}{n}} \mathbb{Z}$. Note that discrete groups of type (2) of the class ($\mathcal{L}$) are cocompact lattices of a group of the form

$$G = \mathbb{Z} \ltimes_{\frac{m}{n}} \Big(\mathbb{R} \oplus \bigoplus_{p\in P} \mathbb{Q}_p\Big)$$

with m, n coprime integers and P a finite set of primes (possibly infinite) dividing mn. To simplify notation, we will only consider the case when $P = \{p\}$ is reduced to one single prime, the generalization presenting no difficulty. The case where $p = \infty$ will result from the case of connected Lie groups (see next section) since $\mathbb{Z} \ltimes_{\frac{m}{n}} \mathbb{R}$ embeds as a closed cocompact subgroup of the group of positive affine transformations $\mathbb{R} \ltimes \mathbb{R}$.

So consider the group $G = \mathbb{Z} \ltimes_{1/p} \mathbb{Q}_p$. Define a compact symmetric generating set by $S = S_{\mathbb{Q}_p} \cup S_{\mathbb{Z}}$ where $S_{\mathbb{Q}_p} = \mathbb{Z}_p$ and $S_Z = \{-1, 0, 1\}$. Define (H_k, H_k') by

$$H_k = I_k \times p^{-2k}\mathbb{Z}_p$$

and

$$H_k' = I_{2k} \times p^{-2k}\mathbb{Z}_p,$$

where $I_k = [-k, k]$. Using the same kind of arguments as previously for $F \wr \mathbb{Z}$, one can prove easily that (H_k, H_k') is a controlled sequence of Følner pairs. $\square$

5.3. Amenable connected Lie groups. Let G be a solvable simply connected Lie group. Let S be a compact symmetric generating subset. In [Gu] (see also [O]), it is proved that G admits a maximal normal connected subgroup such that the quotient of G by this subgroup has polynomial growth. This subgroup is called the exponential radical and is denoted $\mathrm{Exp}(G)$. We have $\mathrm{Exp}(G) \subset N$, where N is the maximal nilpotent normal subgroup of G. Let T be a compact symmetric generating subset of $\mathrm{Exp}(G)$. An element $g \in G$ is called strictly exponentially distorted if the S-length of g^n grows as $\log|n|$. The subset of strictly exponentially distorted elements of G coincides with $\mathrm{Exp}(G)$. That is,

$$\mathrm{Exp}(G) = \{g \in G : \ |g^n|_S \approx \log|n|\} \cup \{1\}.$$

Moreover, $\mathrm{Exp}(G)$ is strictly exponentially distorted in G in the sense that there exists $\beta \geq 1$ such that for every $h \in \mathrm{Exp}(G) \setminus \{1\}$,

$$\beta^{-1}\log(|h|_T + 1) - \beta \leq |h|_S \leq \beta\log(|h|_T + 1) + \beta \tag{5.1}$$

where T is a compact symmetric generating subset of $\mathrm{Exp}(G)$.

We will need the following two lemmas.

Lemma 5.3. *Let G be a locally compact group. Let H be a closed normal subgroup. Let λ and ν be respectively left Haar measures of H and G/H. Let i be a measurable* *left-section of the projection* $\pi\colon\ G \to G/H$, *i.e.* $G = \sqcup_{x\in G/H} i(x)H$. *Identify* G *with the cartesian product* $G/H \times H$ *via the map* $(x,h) \mapsto i(x)h$. *Then the product measure* $\nu \otimes \lambda$ *is a left Haar measure on* G.

Proof. We have to prove that $\nu \otimes \lambda$ is left-invariant on G. Fix g in G. Define a measurable map σ_g from G/H to H by

$$\sigma_g(x) = (i(\pi(g)x)^{-1} g i(x).$$

In other words, $\sigma_g(x)$ is the unique element of H such that

$$g i(x) = i(\pi(g)x)\sigma_g(x).$$

Let $\varphi\colon G \to \mathbb{R}$ be a continuous, compactly supported function. We have

$$\int_{G/H\times H} \varphi[g i(x)h] d\nu(x) d\lambda(h) = \int_{G/H\times H} \varphi[i(\pi(g)x)\sigma_g(x)h] d\nu(x) d\lambda(h).$$

As ν and λ are respectively left Haar measures on G/H and H, the Jacobian of the transformation $(x,h) \mapsto (\pi(g)x, \sigma_g(x)h)$ is equal to 1. Hence,

$$\int_{G/H\times H} \varphi[i(\pi(g)x)\sigma_g(x)h] d\nu(x) d\lambda(h) = \int_{G/H\times H} \varphi[i(x)h] d\nu(x) d\lambda(h).$$

Thus $\nu \otimes \lambda$ is left-invariant. □

Lemma 5.4. *Let* G *be a connected Lie group and* H *be a normal subgroup. Consider the projection* $\pi\colon G \to G/H$. *There exists a compact generating set* S *of* G *and a* σ*-compact cross-section* σ *of* G/H *inside* G *such that* $\sigma(\pi(S)^n) \subset S^{n+1}$.

Proof. Since π is a submersion, there exists a compact neighborhood S of 1 in G such that $\pi(S)$ admits a continuous cross-section σ_1 in S. Now, let X be a minimal (discrete) subset of G/H satisfying $G/H = \bigcup_{x\in X} x\pi(S)$. Since this covering is locally finite and $\pi(S)$ is compact, one can construct by induction a partition $(A_x)_{x\in X}$ of G/H such that every A_x is a constructible, and therefore σ-compact subset of $x\pi(S)$. Let $\sigma_2\colon X \to G$ be a cross-section of X satisfying $\sigma_2(X \cap \pi(S)^n) \subset S^n$. Now, for every $z \in A_x$, define

$$\sigma(z) = \sigma_2(x)\sigma_1(x^{-1}z).$$

Clearly, σ satisfies to the hypotheses of the lemma. □

Equip the group $P = G/\mathrm{Exp}(G)$ with a Haar measure ν and with the symmetric generating subset $\pi(S)$, where π is the projection on P. Assume that S satisfies to the hypotheses of Lemma 5.4 and let σ be a σ-compact cross-section of P inside G

such that $\sigma(\pi(S)^n) \subset S^{n+1}$. For every $n \in \mathbb{N}$, write $F_n = \sigma(\pi(S)^n)$. Let α be some large enough positive number that we will determine later. Denote by $\lfloor x \rfloor$ the integer part of a real number x. Define, for every $n \in \mathbb{N}$,

$$H_n = S^n T^{\lfloor \exp(\alpha n) \rfloor}$$

and

$$H'_n = S^{2n} T^{\lfloor \exp(\alpha n) \rfloor}.$$

Note that $H'_n = S^n H_n$. On the other hand, since $\mathrm{Exp}(G)$ is strictly exponentially distorted, there exists $a \geq 1$ only depending on α and β such that, for every $n \in \mathbb{N}$,

$$S^n T^{\lfloor \exp(\alpha n) \rfloor} \subset S^{an}.$$

Hence, to prove that (H_n, H'_n) is a sequence of controlled Følner pairs, it suffices to show that $\mu(H'_n) \leq C\mu(H_n)$. Consider another sequence (A_n, A'_n) defined by, for every $n \in \mathbb{N}^*$,

$$A_n = F_{n-1} T^{\lfloor \exp(\alpha n) \rfloor}$$

and

$$A'_n = F_{2n} T^{2\lfloor \exp(\alpha n) \rfloor}.$$

As F_n is σ-compact, A_n and A'_n are measurable. To compute the measures of A_n and A'_n, we choose a normalization of the Haar measure λ on $\mathrm{Exp}(G)$ such that the measure μ disintegrates over λ and the pull-back measure of ν on $\sigma(P)$ as in Lemma 5.3. We therefore obtain

$$\mu(A_n) = \nu(\pi(S)^{n-1})\lambda(T^{\lfloor \exp(\alpha n) \rfloor})$$

and

$$\mu(A'_n) = \nu(\pi(S)^{2n})\lambda(T^{2\lfloor \exp(\alpha n) \rfloor}).$$

Since P and $\mathrm{Exp}(G)$ have both polynomial growth, there is a constant C such that, for every $n \in \mathbb{N}^*$,

$$\mu(A'_n) \leq C\mu(A_n).$$

So now, it suffices to prove that

$$A_n \subset H_n \subset H'_n \subset A'_n,$$

where the only nontrivial inclusion is $H'_n \subset A'_n$. Let $g \in S^{2n}$; let $f \in F_{2n}$ be such that $\pi(g) = \pi(f)$. Since $F_{2n} \subset S^{2n+2} \subset S^{3n}$,

$$gf^{-1} \in S^{6n} \cap \mathrm{Exp}(G).$$

On the other hand, by (5.1),

$$S^{6n} \cap \mathrm{Exp}(G) \subset T^{2\lfloor \exp(6\beta n) \rfloor}.$$

Therefore, for every $n \in \mathbb{N}^*$,

$$H'_n \subset F_{2n} T^{2\lfloor \exp(6\beta n)\rfloor} T^{\lfloor \exp(\alpha n)\rfloor} = F_{2n} T^{2\lfloor \exp(6\beta n)\rfloor + \lfloor \exp(\alpha n)\rfloor}.$$

Hence, choosing $\alpha \geq 6\beta + \log 2$, we have

$$H'_n \subset F_{2n} T^{2\lfloor \exp(\alpha n)\rfloor} = A'_n,$$

and we are done. □

6. On embedding of finite trees into uniformly convex Banach spaces

Definition 6.1. A Banach space X is called *q-uniformly convex* ($q > 0$) if there is a constant $a > 0$ such that for any two points x, y in the unit sphere satisfying $\|x - y\| \geq \varepsilon$, we have

$$\left\| \frac{x + y}{2} \right\| \leq 1 - a\varepsilon^q.$$

Note that by a theorem of Pisier [Pis], every uniformly convex Banach space is isomorphic to some q-uniformly convex Banach space.

In this section, we prove that the compression of a Lipschitz embedding of a finite binary rooted tree into a q-uniformly convex space X always satisfies condition (C_q). Theorem 4 follows from the fact that an L^p-space is $\max\{p, 2\}$-uniformly convex.

Theorem 6.2. *Let T_J be the binary rooted tree of depth J and let $1 < q < \infty$. Let F be a 1-Lipschitz map from T_J to some q-uniformly convex Banach space X and let ρ be the compression of F. Then there exists $C = C(q) < \infty$ such that*

$$\int_1^{2J} \left(\frac{\rho(t)}{t} \right)^q \frac{dt}{t} \leq C. \tag{6.1}$$

Corollary 6.3. *Let F be any uniform embedding of the 3-regular tree T into some q-uniformly convex Banach space. Then the compression ρ of F satisfies Condition (C_q).* □

As a corollary, we also reobtain the theorem of Bourgain.

Corollary 6.4 ([Bou], Theorem 1). *With the notation of Theorem 6.2, there exists at least two vertices x and y in T_J such that*

$$\frac{\|F(x) - F(y)\|}{d(x, y)} \leq \left(\frac{C}{\log J} \right)^{1/q}.$$

Proof. For every $1 \le t \le 2J$, there exist $z, z' \in T_J$, $d(z,z') \ge t$ such that:

$$\frac{\rho(t)}{t} = \frac{\|F(z)-F(z')\|}{t} \ge \frac{\|F(z)-F(z')\|}{d(z,z')}.$$

Therefore

$$\min_{z \ne z' \in T_J} \frac{\|F(z)-F(z')\|}{d(z,z')} \le \min_{1 \le u \le 2J} \frac{\rho(u)}{u}.$$

But by (6.1)

$$\Big(\min_{1\le u\le 2J} \frac{\rho(u)}{u}\Big)^q \int_1^{2J} \frac{1}{t} dt \le \int_1^{2J} \Big(\frac{\rho(t)}{t}\Big)^q \frac{dt}{t} \le C.$$

We then have

$$\min_{z \ne z' \in T_J} \frac{\|F(z)-F(z')\|}{d(z,z')} \le \Big(\frac{C}{\log J}\Big)^{1/q}.$$

□

Proof of Theorem 6.2. Since the proof follows closely the proof of Theorem 1 in [Bou], we keep the same notation to allow the reader to compare them. For $j = 1, 2, \dots$, denote $\Omega_j = \{-1,1\}^j$ and $T_j = \bigcup_{j' \le j} \Omega_{j'}$. Thus T_j is the finite tree with depth j. Denote d the tree-distance on T_j.

Lemma 6.5 ([Pis], Proposition 2.4). *There exists $C = C(q) < \infty$ such that if $(\xi_s)_{s\in\mathbb{N}}$ is an X-valued martingale on some probability space Ω, then*

$$\sum_s \|\xi_{s+1} - \xi_s\|_q^q \le C \sup_s \|\xi_s\|_q^q \tag{6.2}$$

where $\| \ \|_q$ stands for the norm in $L^q_X(\Omega)$.

Lemma 6.5 is used to prove

Lemma 6.6. *If $x_1, \dots, x_J$, with $J = 2^r$, is a finite system of vectors in X, then*

$$\sum_{s=1}^r 2^{-qs} \min_{2^s < j \le J-2^s} \|2x_j - x_{j-2^s} - x_{j+2^s}\|^q \le C \sup_{1\le j\le J-1} \|x_{j+1} - x_j\|^q. \tag{6.3}$$

Denote $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \dots \subset \mathcal{D}_r$ the algebras of intervals on $[0,1]$ obtained by successive dyadic refinements. Define the X-valued function

$$\xi = \sum_{1\le j\le J-1} 1_{[\frac{j}{J}, \frac{j+1}{J}[}(x_{j+1} - x_j)$$

and consider expectations $\xi_s = \mathbf{E}[\xi|\mathcal{D}_s]$ for $s = 1, \dots, r$. Since ξ_s form a martingale ranging in X, it satisfies inequality (6.2). On the other hand

$$\begin{aligned}\|\xi_{s+1} - \xi_s\|_q^q &= 2^{-r+s} 2^{qs} \sum_{1<t\le 2^{r-s}}^{r} 2^{-qs} \|2x_{t2^s} - x_{(t-1)2^s} - x_{(t+1)2^s}\|^q \\ &\le 2^{-qs} \min_{2^s<j\le J-2^s} \|2x_j - x_{j-2^s} - x_{j+2^s}\|^q.\end{aligned}$$

So (6.3) follows from the fact that

$$\|\xi_s\|_q^q \le \|\xi_{s+1} - \xi_s\|_\infty^q = \sup_j \|x_{j+1} - x_j\|^q. \qquad \square$$

Lemma 6.7. *If $f_1, \dots, f_J$, with $J = 2^r$, is a finite system of functions in $L_X^\infty(\Omega)$. Then*

$$\sum_{s=1}^{r} 2^{-qs} \min_{2^s<j\le J-2^s} \|2f_j - f_{j-2^s} - f_{j+2^s}\|^q \le C \sup_{1\le j\le J-1} \|f_{j+1} - f_j\|_\infty^q. \tag{6.4}$$

Proof. Replace X by $L_X^q(\Omega)$, for which (6.2) remains valid, and use (6.3). $\square$

Lemma 6.8. *Let $f_1, \dots, f_J$, with $J = 2^r$, be a sequence of functions on $\{1,-1\}^J$ where f_j only depends on $\varepsilon_1, \dots, \varepsilon_j$. Then*

$$\begin{aligned}\sum_{s=1}^{r} 2^{-qs} \min_{2^s<j\le J-2^s} \left(\int_{\Omega_j\times\Omega_{2^s}\times\Omega_{2^s}} \|f_{j+2^s}(\varepsilon,\delta) - f_{j+2^s}(\varepsilon,\delta')\|^q \, d\varepsilon\, d\delta\, d\delta' \right) \\ \le 2^q C \sup_{1\le j\le J-1} \|f_{j+1} - f_j\|_\infty^q.\end{aligned}$$

Proof. For every $d < j \le J - d$, using the triangle inequality, we obtain

$$\begin{aligned}\|2f_j - f_{j-d} - f_{j+d}\|_q^q &= \int_{\Omega_j\times\Omega_d} \|2f_j - f_{j-d} - f_{j+d}\|^q \, d\varepsilon\, d\delta \\ &\ge 2^{-q} \int_{\Omega_j\times\Omega_d\times\Omega_d} \|f_{j+2^s}(\varepsilon,\delta) - f_{j+2^s}(\varepsilon,\delta')\|^q \, d\varepsilon\, d\delta\, d\delta'.\end{aligned}$$

The lemma then follows from (6.4). $\square$

Now, let us prove Theorem 6.2. Fix J and consider a 1-Lipschitz map $F : T_J \to X$. Apply Lemma 6.8 to the functions $f_1, \dots, f_J$ defined by

$$\forall \alpha \in \Omega_j, \quad f_j(\alpha) = F(\alpha).$$

By definition of the compression, we have

$$\rho\left(d\left((\varepsilon,\delta),(\varepsilon,\delta')\right)\right) \le \| f_{j+2^s}(\varepsilon,\delta) - f_{j+2^s}(\varepsilon,\delta')\| \tag{6.5}$$

where $\varepsilon \in \Omega_j$ and $\delta, \delta' \in \Omega_{2^s}$.

But, on the other hand, with probability $1/2$, we have

$$d\left((\varepsilon,\delta),(\varepsilon,\delta')\right) = 2.2^s.$$

So combining this with Lemma 6.8, (6.5) and with the fact that F is 1-Lipschitz, we obtain

$$\sum_{s=1}^{r} 2^{-qs}\rho(2^s)^q \le 2^{q+1}C$$

But since ρ is decreasing, we have

$$2^{-qs}\rho(2^s)^q \ge 2^{-q-1}\int_{2^{s-1}+1}^{2^s} \frac{1}{t}\left(\frac{\rho(t)}{t}\right)^q dt.$$

So (6.1) follows. □

7. Applications and further results

7.1. Hilbert compression, volume growth and random walks. Let G be a locally compact group generated by a symmetric compact subset S containing 1. Let us denote $V(n) = \mu(S^n)$ and $S(n) = V(n+1) - V(n) = \mu(S^{n+1} \smallsetminus S^n)$. Extend V as a piecewise linear function on $\mathbb{R}_+$ such that $V'(t) = S(n)$ for $t \in]n, n+1[$.

Proposition 7.1. *Let G be a compactly generated locally compact group. For any* $2 \le p < \infty$,

$$J_{G,p}(t) \preceq \frac{t}{\log V(t)}.$$

Proof. For every $n \in \mathbb{N}$, define

$$k(n) = \sup\{k,\ V(n-k) \ge V(n)/2\}$$

and

$$j(n) = \sup_{1\le j\le n} k(j).$$

For every positive integer $l \le n/j(n)$,

$$V(n) \ge 2^l V(n - lj(n)).$$

Hence, as $V(0) = 1$,
$$V(n) \geq 2^{n/(j(n)+1)}.$$
Thus, there is a constant $c > 0$ such that
$$j(n) \geq \frac{cn}{\log V(n)}.$$
Let $q_n \leq n$ be such that $j(n) = k(q_n)$. Now define
$$\varphi_n = \sum_{k=1}^{q_n-1} 1_{B(1,k)}.$$
Note that the subsets $SB(1,k) \vartriangle B(1,k) = B(1,k+1) \smallsetminus B(1,k)$, for $k \in \mathbb{N}$, are piecewise disjoint. Thus, an easy computation shows that
$$\|\tilde{\nabla}\varphi_n\|_p \leq V(q_n)^{1/p}.$$
On the other hand
$$\|\varphi_n\|_p \geq j(n)V(q_n - j(n))^{1/p} \geq \frac{cn}{\log V(n)}(V(q_n)/2)^{1/p}.$$
Since $J^b_{G,p}(n) \geq \|\varphi_n\|_p/\|\tilde{\nabla}\varphi_n\|_p$, we conclude that $J^b_{G,p}(n) \succeq n/\log V(n)$. □

Now, consider a symmetric probability measure ν on a finitely generated group G, supported by a finite generating subset S. Given an element φ of $\ell^2(G)$, a simple calculation shows that
$$\frac{1}{2}\iint |\varphi(sx) - \varphi(x)|^2 d\nu^{(2)}(s)d\mu(x) = \int(\varphi - \nu^{(2)} * \varphi)\varphi d\mu = \|\varphi\|_2^2 - \|\nu * \varphi\|_2^2$$
where μ denotes the counting measure on G. Let us introduce a (left) gradient on G associated to ν. Let φ be a function on G; define
$$|\tilde{\nabla}\varphi|_2^2(g) = \int |\varphi(sg) - \varphi(g)|^2 d\nu^{(2)}(s).$$
This gradient satisfies
$$\||\tilde{\nabla}\varphi|_2\|_2^2 = 2(\|\varphi\|_2^2 - \|\nu * \varphi\|_2^2).$$
We have
$$\mu(S)^{-1/2}|\tilde{\nabla}\varphi|_2 \leq |\tilde{\nabla}\varphi| \leq |\tilde{\nabla}\varphi|_2.$$

Proposition 7.2. *Assume that $\nu^{(n)}(1) \succeq e^{-Cn^b}$ for some $b < 1$. Then*
$$J^b_{G,2}(t) \succeq Ct^{1-b}.$$

Proof. Let us prove that there exists a constant $C' < \infty$ such that for every $n \in \mathbb{N}$, there exists $n \leq k \leq 2n$ such that

$$\frac{\| \, |\tilde{\nabla} \nu^{(2k)}|_2 \, \|_2^2}{\| \, \nu^{(2k)} \, \|_2^2} \leq C' n^{b-1}.$$

Since $\nu^{(2k)}$ is supported in $S^{2k} \subset S^{4n}$, this will prove the proposition. Let C_n be such that for every $n \leq q \leq 2n$,

$$\frac{\| \, |\tilde{\nabla} \nu^{(2q)}|_2 \, \|_2^2}{\| \, \nu^{(2q)} \, \|_2^2} \geq C_n n^{b-1}.$$

Since the function defined by $\psi(q) = \| \, \nu^{(2q)} \, \|_2^2$ satisfies

$$\psi(q+1) - \psi(q) = -\frac{1}{2} \, \| \, |\tilde{\nabla} \nu^{(2q)}|_2 \, \|_2^2,$$

we can extend ψ as a piecewise linear function on $\mathbb{R}_+$ such that

$$\psi'(t) = \frac{1}{2} \, \| \, |\tilde{\nabla} \nu^{(2q)}|_2 \, \|_2^2$$

for every $t \in [q, q+1[$. Then, for every $n \leq t \leq 2n$ we have

$$-\frac{\psi'(t)}{\psi(t)} \geq C_n n^{b-1}$$

which integrates in

$$-\log\left(\frac{\psi(2n)}{\psi(n)}\right) \geq C_n n^b.$$

Since $\psi(n) < 1$, this implies

$$\psi(2n) \leq e^{-C_n n^b}.$$

But on the other hand,

$$\psi(2n) \geq \| \, \nu^{(4n)} \, \|_2^2 \geq \nu^{(8n)}(1) \geq e^{-8Cn^b}.$$

So $C_n \leq 8C$. □

7.2. A direct construction to embed trees. Here, we propose to show that the method used in [Bou], [GK], [BrSo] to embed trees in L^p-spaces can also be exploited to obtain optimal estimates (i.e. a converse to Theorem 6.2). Moreover, no hypothesis of local finitude is required for this construction.

Theorem 7.3. *Let T be a simplicial tree. For every increasing function $f:\mathbb{R}_+\to\mathbb{R}_+$ satisfying, for $1\le p<\infty$*

$$\int_1^\infty \left(\frac{f(t)}{t}\right)^p \frac{dt}{t} < \infty, \tag{C_p}$$

there exists a uniform embedding F of T into $\ell^p(T)$ with compression $\rho \succeq f$.

Proof. Let us start with a lemma.

Lemma 7.4. *For every nonnegative sequence (ξ_n) such that*

$$\sum_n |\xi_{n+1}-\xi_n|^p < \infty,$$

there exists a Lipschitz map $F: T\to \ell^p(T)$ whose compression ρ satisfies

$$\forall n\in\mathbb{N},\quad \rho(n)\ge \Big(\sum_{j=0}^n \xi_j^p\Big)^{1/p}.$$

Proof. The following construction is a generalization of those carried out in [GK] and [BrSo]. Fix a vertex o. For every $y\in T$, denote δ_y the element of $\ell^p(T)$ that takes value 1 on y and 0 elsewhere. Let x be a vertex of T and let $x_0=x, x_1,\dots,x_l=o$ be the minimal path joining x to o. Define

$$F(x)=\sum_{i=1}^l \xi_i\delta_{x_i}.$$

To prove that F is Lipschitz, it suffices to prove that $\|F(x)-F(y)\|_p$ is bounded for neighbor vertices in T. So let x and y be neighbor vertices in T such that $d(o,y)=d(x,o)+1=l+1$. We have

$$\|F(y)-F(x)\|_p^p \le \xi_0^p + \sum_{j=0}^l |\xi_{n+1}-\xi_n|^p.$$

On the other hand, let x and y be two vertices in T. Let z be the last common vertex of the two geodesic paths joining o to x and y. We have

$$d(x,y)=d(x,z)+d(z,y)$$

and

$$\begin{aligned}\|F(x)-F(y)\|_p^p &= \|F(x)-F(z)\|_p^p+\|F(z)-F(y)\|_p^p\\ &\ge \max\{\|F(x)-F(z)\|_p^p\,,\ \|F(z)-F(y)\|_p^p\}.\end{aligned}$$

Let $k = d(z,x)$; we have

$$\|F(x) - F(z)\|_p^p \geq \sum_{j=0}^{k} \xi_j^p,$$

which proves the lemma. □

Now, let us prove the theorem. Define (ξ_j) by

$$\xi_0 = \xi_1 = 0;$$
$$\forall j \geq 1, \quad \xi_{j+1} - \xi_j = \frac{1}{j^p}\frac{f(j)}{j}$$

and consider the associated Lipschitz map F from T to $\ell^p(T)$. Clearly, we have

$$\sum |\xi_{n+1} - \xi_n|^p < \infty$$

and

$$\sum_{j=0}^{n} \xi_j^p \geq \sum_{j=[n/2]}^{n} \Big(\sum_{k=0}^{j-1} |\xi_{k+1} - \xi_k|\Big)^p \geq n/2\Big(\sum_{k=0}^{[n/2]-1} |\xi_{k+1} - \xi_k|\Big)^p \geq cf([n/2])$$

using the fact that f is nondecreasing. So the theorem now follows from the lemma. □

7.3. Cocycles with lacunar compression

Proposition 7.5. *For any increasing sublinear function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ and every $2 \leq p < \infty$, there exists a function f satisfying (C_p), a constant $c > 0$ and an increasing sequence of integers (n_i) such that*

$$\forall i \in \mathbb{N}, \quad f(n_i) \geq ch(n_i).$$

Proof. Choose a sequence (n_i) such that

$$\sum_{i\in\mathbb{N}} \left(\frac{h(n_i)}{n_i}\right)^p < \infty$$

Define

$$\forall i \in \mathbb{N},\ n_i \leq t < n_{i+1}, \quad f(t) = h(n_i)$$

We have

$$\int_1^{\infty} \frac{1}{t}\Big(\frac{f(t)}{t}\Big)^p dt \leq \sum_i (h(n_i))^p \int_{n_i}^{n_{i+1}} \frac{dt}{t^{p+1}} \leq (p+1)\sum_i \left(\frac{h(n_i)}{n_i}\right)^p < \infty$$

So we are done. □

7.4. The case of $\mathbb{Z} \wr \mathbb{Z}$. The proof of Theorem 17 follows from Proposition 3.5 and from the following observation.

Proposition 7.6. *For all $1 \leq p < \infty$, the maximal ℓ^p-compression function of the group $G = \mathbb{Z} \wr \mathbb{Z}$ satisfies*

$$M\rho_{G,p}(t) \succeq t^{p/(2p-1)}.$$

Proof. Denote by θ the projection $\mathbb{Z} \wr \mathbb{Z} \to C_2 \wr \mathbb{Z}$. Fix two word lengths on $\mathbb{Z} \wr \mathbb{Z}$ and $C_2 \wr \mathbb{Z}$, which for simplicity, we will both denote by $|g|$.

Consider the unique cocycle $b\colon \mathbb{Z}\wr\mathbb{Z} \to \ell^p(\mathbb{Z})$ which extends the natural injective morphism $\mathbb{Z}^{(\mathbb{Z})} \to \ell^p(\mathbb{Z})$. For any $g = (k,u) \in \mathbb{Z} \wr \mathbb{Z} = \mathbb{Z} \ltimes \mathbb{Z}^{(\mathbb{Z})}$, we therefore have $\|b(g)\| = \|u\|_p$. Taking the ℓ^p-direct sum of this cocycle with every cocycle of $\mathbb{Z} \wr \mathbb{Z}$ factorizing through θ, and since $M\rho_{C_2\wr\mathbb{Z},p}(t) \approx t$, we obtain

$$M\rho_{\mathbb{Z}\wr\mathbb{Z},p}(t) \succeq \inf_{g\in\mathbb{Z}\wr\mathbb{Z},\ |g|\geq t} \max\{|p(g)|, \|b(g)\|\}. \tag{7.1}$$

Up to multiplicative constants, (see [Par], Theorem 1.2), the word length of an element $g = (k,u) \in \mathbb{Z} \wr \mathbb{Z}$ is given by

$$L(\gamma) + \sum_{h\in\mathbb{Z}} |u(h)| = L(\gamma) + \|u\|_1,$$

where $L(\gamma)$ is the length of a shortest path γ from 0 to k passing through every element of the support of u. Similarly, $|p(g)| \approx L(\gamma) + |\mathrm{Supp}(u)|$. Hence by (7.1), we can assume that $L(\gamma) \leq |g|/2$, so that $\|u\|_1 \geq |g|/2$. By Hölder's inequality, we have $\|u\|_1 \leq \|u\|_p |\mathrm{Supp}(u)|^{1-1/p}$, which is less than a constant times $\|b(g)\||p(g)|^{1-1/p}$. Therefore

$$M\rho_{\mathbb{Z}\wr\mathbb{Z},p}(t) \succeq \inf_{g\in\mathbb{Z}\wr\mathbb{Z},\ |g|\geq t} \max\left\{|p(g)|, |g|/|p(g)|^{1-1/p}\right\},$$

which immediately implies the proposition. □

7.5. H-metric. Let G be a locally compact, compactly generated group and let S be a compact symmetric generating set. A Hilbert length function is a length function associated to some Hilbert 1-cocycle b, i.e. $L(g) = \|b(g)\|$. Consider the supremum of all Hilbert length functions on G, bounded by 1 on S: it defines a length function on G which in general is no longer a Hilbert length function. This length function has been introduced by Cornulier [C], § 2.6, who called the corresponding metric "H-metric". Observe that if the group G satisfies $M\rho_{G,2}(t) \approx t$, then the H-metric of G is quasi-isometric to the word length. As a consequence of Theorem 5.1 and Proposition 4.5, we get

Proposition 7.7. *For every group in the class ($\mathcal{L}$), the H-metric is quasi-isometric to the word length.*

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