Comment. Math. Helv. 86 (2011), 675–706
DOI 10.4171/CMH/238

Commentarii Mathematici Helvetici

The Khovanov width of twisted links and closed 3-braids

Adam Lowrance

Abstract. Khovanov homology is a bigraded $\mathbb{Z}$-module that categorifies the Jones polynomial. The support of Khovanov homology lies on a finite number of slope two lines with respect to the bigrading. The Khovanov width is essentially the largest horizontal distance between two such lines. We show that it is possible to generate infinite families of links with the same Khovanov width from link diagrams satisfying certain conditions. Consequently, we compute the Khovanov width for all closed 3-braids.

Mathematics Subject Classification (2010). 57M25, 57M27.

Keywords. Khovanov homology, homological width, Turaev genus, Turaev surface, quasi-alternating.

1. Introduction

Let $L \subset S^3$ be an oriented link. The Khovanov homology of L, denoted $\mathrm{Kh}(L)$, was introduced by Mikhail Khovanov in [13], and is a bigraded $\mathbb{Z}$-module with homological grading i and polynomial (or Jones) grading j so that $\mathrm{Kh}(L) = \bigoplus_{i,j} \mathrm{Kh}^{i,j}(L)$. The graded Euler characteristic of $\mathrm{Kh}(L)$ is the unnormalized Jones polynomial:

$$(q + q^{-1}) V_L(q^2) = \sum_{i,j} (-1)^i \operatorname{rank} \mathrm{Kh}^{i,j}(L) q^j.$$

The support of $\mathrm{Kh}(L)$ lies on a finite number of slope 2 lines with respect to the bigrading. Therefore, it is convenient to define the δ-grading by $\delta = j - 2i$ so that $\mathrm{Kh}(L) = \bigoplus_\delta \mathrm{Kh}^\delta(L)$. Also, either all the δ-gradings of $\mathrm{Kh}(L)$ are odd, or they all are even. Let $\delta_{\min}$ be the minimum δ-grading where $\mathrm{Kh}(L)$ is nontrivial and $\delta_{\max}$ be the maximum δ-grading where $\mathrm{Kh}(L)$ is nontrivial. Then $\mathrm{Kh}(L)$ is said to be $[\delta_{\min}, \delta_{\max}]$-*thick*, and the *Khovanov width of* L is defined as

$$w_{\mathrm{Kh}}(L) = \frac{1}{2}(\delta_{\max} - \delta_{\min}) + 1.$$

In this paper, we show the following:

- If a crossing in a link diagram is width-preserving (defined in Section 3), then it can be replaced with an alternating rational tangle and the Khovanov width does not change (Theorem 3.4).
- We compute the Khovanov width of all closed 3-braids (Theorem 4.10).
- We determine the Turaev genus of all closed 3-braids, up to an additive error of at most 1.
- We show that for closed 3-braids the Khovanov width and odd Khovanov width are equal (Corollary 5.5).

The paper is organized as follows. In Section 2, we review some properties of Khovanov homology. In Section 3, we describe the behavior of Khovanov width when a crossing is replaced by an alternating rational tangle. In Section 4, the Khovanov width of any closed 3-braid is computed. Finally, in Section 5 we show that Khovanov width and odd Khovanov width for closed 3-braids are equal.

Acknowledgements. I wish to thank Scott Baldridge, Oliver Dasbach, Mikhail Khovanov, and Peter Ozsváth for many helpful conversations. A portion of the work for this article was done while the author was visiting Columbia University in the Fall of 2008. He thanks the department of mathematics for their hospitality.

2. Khovanov homology background

In this section, we give background material on Khovanov homology. If D is a diagram for L, then denote the Khovanov homology of L by either $\mathrm{Kh}(L)$ or $\mathrm{Kh}(D)$. Similarly, let $w_{\mathrm{Kh}}(L)$ and $w_{\mathrm{Kh}}(D)$ equivalently denote the Khovanov width of L. If $\mathbb{F}$ is a field, then let $\mathrm{Kh}(L;\mathbb{F})$ denote Khovanov homology with coefficients in the field $\mathbb{F}$, and $w_{\mathrm{Kh}}(L;\mathbb{F})$ denote the width of $\mathrm{Kh}(L;\mathbb{F})$.

Let L_1 and L_2 be oriented links, and let C be a component of L_1. Denote by l the linking number of C with its complement $L_1 - C$. Let L_1' be the link L_1 with the orientation of C reversed. Denote the mirror image of L_1 by $\overline{L_1}$ and the disjoint union of L_1 and L_2 by $L_1 \sqcup L_2$. The following proposition was proved by Khovanov in [13].

Proposition 2.1 (Khovanov). *For $i, j \in \mathbb{Z}$ there are isomorphisms*

$$\begin{aligned} \mathrm{Kh}^{i,j}(L_1') &\cong \mathrm{Kh}^{i+2l,j+2l}(L_1), \\ \mathrm{Kh}^{i,j}(\overline{L_1};\mathbb{Q}) &\cong \mathrm{Kh}^{-i,-j}(L_1;\mathbb{Q}) \\ \mathrm{Tor}(\mathrm{Kh}^{i,j}(\overline{L_1})) &\cong \mathrm{Tor}(\mathrm{Kh}^{1-i,-j}(L_1)), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Kh}^{i,j}(L_1 \sqcup L_2) \cong {} & \textstyle\bigoplus_{k,m\in\mathbb{Z}} (\mathrm{Kh}^{k,m}(L_1) \otimes \mathrm{Kh}^{i-k,j-m}(L_2)) \\ & \textstyle\bigoplus_{k,m\in\mathbb{Z}} \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{Kh}^{k,m}(L_1), \mathrm{Kh}^{i-k+1,j-m}(L_2)). \end{aligned}$$

Let D be a diagram for L_1 and D' be the diagram D with the component C reversed. Denote the number of negative crossings in D by $\operatorname{neg}(D)$, where the sign of a crossing is as in Figure 1. Set $s = \operatorname{neg}(D) - \operatorname{neg}(D')$. Then Proposition 2.1 implies

$$\mathrm{Kh}^{\delta}(D') \cong \mathrm{Kh}^{\delta+s}(D),$$

and

$$\mathrm{Kh}^{\delta}(\overline{L_1}; \mathbb{Q}) \cong \mathrm{Kh}^{-\delta}(L_1; \mathbb{Q}).$$

In [14], Khovanov introduced the reduced Khovanov homology. For a knot K, this theory is denoted $\widetilde{\mathrm{Kh}}(K)$. For links of more than one component, the reduced Khovanov homology depends on a choice of a marked component, and hence is denoted $\widetilde{\mathrm{Kh}}(L, C)$, where C is the marked component of L. Similar to the unreduced version, $\widetilde{\mathrm{Kh}}(L, C)$ is a bigraded $\mathbb{Z}$-module with homological grading i and Jones grading j so that $\widetilde{\mathrm{Kh}}(L, C) = \bigoplus_{i,j} \widetilde{\mathrm{Kh}}^{i,j}(L, C)$. The graded Euler characteristic of $\widetilde{\mathrm{Kh}}(L, C)$ is the ordinary Jones polynomial:

$$V_L(q^2) = \sum_{i,j} (-1)^i \operatorname{rank} \widetilde{\mathrm{Kh}}^{i,j}(L, C) q^j.$$

As with Khovanov homology, if $\tilde{\delta}_{\min}$ is the minimum δ-grading where $\widetilde{\mathrm{Kh}}(L, C)$ is nontrivial and $\tilde{\delta}_{\max}$ is the maximum δ-grading where $\widetilde{\mathrm{Kh}}(L, C)$ is nontrivial, then we say that $\widetilde{\mathrm{Kh}}(L, C)$ is $[\tilde{\delta}_{\min}, \tilde{\delta}_{\max}]$-thick. The reduced Khovanov width is defined as $w_{\widetilde{\mathrm{Kh}}}(L) = \frac{1}{2}(\tilde{\delta}_{\max} - \tilde{\delta}_{\min}) + 1$.

Asaeda and Przytycki [2] show that there is a long exact sequence relating reduced and unreduced Khovanov homology.

Theorem 2.2 (Asaeda–Przytycki). *There is a long exact sequence relating the reduced and unreduced versions of Khovanov homology:*

$$\cdots \to \widetilde{\mathrm{Kh}}^{i,j+1}(L, C) \to \mathrm{Kh}^{i,j}(L) \to \widetilde{\mathrm{Kh}}^{i,j-1}(L, C) \to \widetilde{\mathrm{Kh}}^{i+1,j+1}(L, C) \to \cdots$$

Corollary 2.3. *Let L be a link with marked component C. Then $\mathrm{Kh}(L)$ is $[\delta_{\min}, \delta_{\max}]$-thick if and only if $\widetilde{\mathrm{Kh}}(L, C)$ is $[\delta_{\min}+1, \delta_{\max}-1]$-thick. Hence $w_{\mathrm{Kh}}(L)-1 = w_{\widetilde{\mathrm{Kh}}}(L)$.*

Proof. The long exact sequence of Theorem 2.2 can be rewritten with respect to the δ-grading as

$$\cdots \to \widetilde{\mathrm{Kh}}^{\delta+1}(L, C) \to \mathrm{Kh}^{\delta}(L) \to \widetilde{\mathrm{Kh}}^{\delta-1}(L, C) \to \widetilde{\mathrm{Kh}}^{\delta-1}(L, C) \to \cdots.$$

Suppose $\mathrm{Kh}(L)$ is $[\delta_{\min}, \delta_{\max}]$-thick. Therefore $\widetilde{\mathrm{Kh}}^{\delta}(L, C) = 0$ for $\delta > \delta_{\max} + 1$ and for $\delta < \delta_{\min} - 1$.

Suppose $\widetilde{\mathrm{Kh}}^{\delta_{\max}+1}(L,C)$ is nontrivial. Then for some i and j where $j-2i=\delta_{\max}+1$, the group $\widetilde{\mathrm{Kh}}^{i,j}(L,C)$ is nontrivial. By repeatedly applying the long exact sequence of Theorem 2.2, one sees that $\widetilde{\mathrm{Kh}}^{i+k,j+2k}(L,C)$ is nontrivial for all $k \geq 0$. However, the group $\widetilde{\mathrm{Kh}}^{\delta_{\max}+1}(L,C)$ is finitely generated. Hence $\widetilde{\mathrm{Kh}}^{\delta_{\max}+1}(L,C)$ is trivial. Similarly, one can show that $\widetilde{\mathrm{Kh}}^{\delta_{\min}-1}(L,C)$ is also trivial.

The long exact sequence also implies that $\widetilde{\mathrm{Kh}}^{\delta_{\max}-1}(L,C)$ and $\widetilde{\mathrm{Kh}}^{\delta_{\min}+1}(L,C)$ are nontrivial. Thus $\widetilde{\mathrm{Kh}}(L,C)$ is $[\delta_{\min}+1,\delta_{\max}-1]$-thick.

Suppose $\widetilde{\mathrm{Kh}}(L,C)$ is $[\delta_{\min}+1,\delta_{\max}-1]$-thick. Similar to the case above, if either $\mathrm{Kh}^{\delta_{\min}}(L)$ or $\mathrm{Kh}^{\delta_{\max}}(L)$ are trivial, then one can show that $\widetilde{\mathrm{Kh}}^{\delta_{\min}+1}(L,C)$ or $\widetilde{\mathrm{Kh}}^{\delta_{\max}-1}(L,C)$ respectively are infinitely generated. Hence $\mathrm{Kh}(L)$ is $[\delta_{\min},\delta_{\max}]$-thick. □

Corollary 2.3 implies that if C and C' are two components of L, then $\widetilde{\mathrm{Kh}}(L,C)$ is $[\tilde{\delta}_{\min},\tilde{\delta}_{\max}]$-thick if and only if $\widetilde{\mathrm{Kh}}(L,C')$ is $[\tilde{\delta}_{\min},\tilde{\delta}_{\max}]$-thick. Hence, the notation $w_{\widetilde{\mathrm{Kh}}}(L)$ is unambiguous.

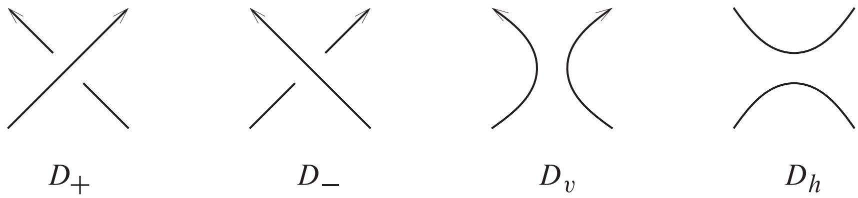

Figure 1. The links in an oriented resolution. D_+ is a positive crossing, and D_- is a negative crossing.

Let D_+, D_-, D_v and D_h be planar diagrams of links that agree outside a neighborhood of a distinguished crossing x as in Figure 1. Define $e=\mathrm{neg}(D_h)-\mathrm{neg}(D_+)$. There are long exact sequences relating the Khovanov homology of each of these links. Khovanov [13] implicitly describes these sequences, and Viro [26] explicitly states both sequences. The graded versions are taken from Rasmussen [20] and Manolescu–Ozsváth [16].

Theorem 2.4 (Khovanov). *There are long exact sequences*

$$\cdots \longrightarrow \mathrm{Kh}^{i-e-1,j-3e-2}(D_h) \longrightarrow \mathrm{Kh}^{i,j}(D_+) \longrightarrow \mathrm{Kh}^{i,j-1}(D_v) \longrightarrow \mathrm{Kh}^{i-e,j-3e-2}(D_h) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathrm{Kh}^{i,j+1}(D_v) \longrightarrow \mathrm{Kh}^{i,j}(D_-) \longrightarrow \mathrm{Kh}^{i-e+1,j-3e+2}(D_h) \longrightarrow \mathrm{Kh}^{i+1,j+1}(D_v) \longrightarrow \cdots .$$

When only the $\delta = j - 2i$ grading is considered, the long exact sequences become

$$\cdots \longrightarrow \mathrm{Kh}^{\delta-e}(D_h) \xrightarrow{f_+^{\delta-e}} \mathrm{Kh}^{\delta}(D_+) \xrightarrow{g_+^{\delta}} \mathrm{Kh}^{\delta-1}(D_v) \xrightarrow{h_+^{\delta-1}} \mathrm{Kh}^{\delta-e-2}(D_h) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathrm{Kh}^{\delta+1}(D_v) \xrightarrow{f_-^{\delta+1}} \mathrm{Kh}^{\delta}(D_-) \xrightarrow{g_-^{\delta}} \mathrm{Kh}^{\delta-e}(D_h) \xrightarrow{h_-^{\delta-e}} \mathrm{Kh}^{\delta-1}(D_v) \longrightarrow \cdots .$$

There are versions of these long exact sequences where Khovanov homology is replaced with reduced Khovanov homology. In the reduced sequences, the gradings are identical to the unreduced sequences.

Let D, D_0 and D_1 be link diagrams differing only in a neighborhood of a crossing x of D (as in Figure 2) with associated link types L, L_0 and L_1 respectively. The set $\mathcal{Q}$ of *quasi-alternating links* is the smallest set of links such that

- The unknot is in $\mathcal{Q}$.
- If the link L has a diagram with a crossing x such that
  (1) both of the links, L_0 and L_1 are in $\mathcal{Q}$,
  (2) $\det(L) = \det(L_0) + \det(L_1)$,

then L is in $\mathcal{Q}$. We will say that D is *quasi-alternating at* x.

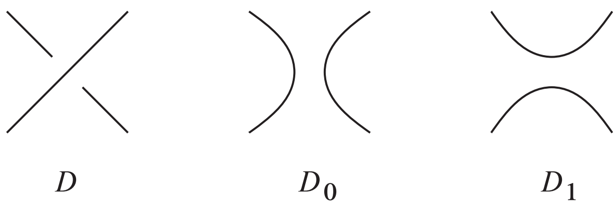

Figure 2. The links in an unoriented resolution.

In [15], Lee showed that alternating links have reduced Khovanov width 1. The set of alternating links is a proper subset of the set of quasi-alternating links. Manolescu and Ozsváth [16] use the long exact sequences for reduced Khovanov homology to show that the same result holds for quasi-alternating links.

Theorem 2.5 (Manolescu–Ozsváth). *Let L be a quasi-alternating link. Then $\widetilde{\mathrm{Kh}}(L)$ is supported entirely in δ-grading $-\sigma(L)$, where $\sigma(L)$ denotes the signature of the link.*

Theorem 2.5 together with Corollary 2.3 imply that if L is quasi-alternating, then $\mathrm{Kh}(L)$ is $[-\sigma(L)-1, -\sigma(L)+1]$-thick and $w_{\mathrm{Kh}}(L) = 2$.

Theorem 2.4 directly implies the following corollary:

Corollary 2.6. *Let D_+, D_-, D_v and D_h be as in Figure 1. Suppose $\mathrm{Kh}(D_v)$ is $[v_{\min}, v_{\max}]$-thick and $\mathrm{Kh}(D_h)$ is $[h_{\min}, h_{\max}]$-thick. Then $\mathrm{Kh}(D_+)$ is $[\delta^+_{\min}, \delta^+_{\max}]$-thick, and $\mathrm{Kh}(D_-)$ is $[\delta^-_{\min}, \delta^-_{\max}]$-thick , where*

$$\delta^+_{\min} = \begin{cases} \min\{v_{\min}+1, h_{\min}+e\} & \text{if } v_{\min} \neq h_{\min}+e+1, \\ v_{\min}+1 & \text{if } v_{\min} = h_{\min}+e+1 \text{ and } h_+^{v_{\min}} \text{ is surjective,} \\ v_{\min}-1 & \text{if } v_{\min} = h_{\min}+e+1 \text{ and } h_+^{v_{\min}} \text{ is not surjective,} \end{cases}$$

$$\delta^+_{\max} = \begin{cases} \max\{v_{\max}+1, h_{\max}+e\} & \text{if } v_{\min} \neq h_{\max}+e+1, \\ v_{\max}-1 & \text{if } v_{\max} = h_{\max}+e+1 \text{ and } h_+^{v_{\max}} \text{ is injective,} \\ v_{\max}+1 & \text{if } v_{\max} = h_{\max}+e+1 \text{ and } h_+^{v_{\max}} \text{ is not injective,} \end{cases}$$

$$\delta^-_{\min} = \begin{cases} \min\{v_{\min}-1, h_{\min}+e\} & \text{if } v_{\min} \neq h_{\min}+e-1, \\ v_{\min}+1 & \text{if } v_{\min} = h_{\min}+e-1 \text{ and } h_-^{v_{\min}} \text{ is surjective,} \\ v_{\min}-1 & \text{if } v_{\min} = h_{\min}+e-1 \text{ and } h_-^{v_{\min}} \text{ is not surjective,} \end{cases}$$

and

$$\delta^-_{\max} = \begin{cases} \max\{v_{\max}-1, h_{\max}+e\} & \text{if } v_{\max} \neq h_{\max}+e-1, \\ v_{\max}-1 & \text{if } v_{\max} = h_{\max}+e-1 \text{ and } h_+^{v_{\max}} \text{ is injective,} \\ v_{\max}+1 & \text{if } v_{\max} = h_{\max}+e-1 \text{ and } h_+^{v_{\max}} \text{ is not injective.} \end{cases}$$

3. Twisted links

3.1. Khovanov width of twisted links. Let $\tau = C(a_1, \dots, a_m)$ be a rational tangle, and let D be a link diagram with a distinguished crossing x. Suppose that near the crossing x, the arcs of D forming x are line segments in the plane whose slopes are either 1 or -1. Define *D twisted at x by τ* to be the diagram obtained by removing x and inserting τ such that a neighborhood of the rightmost crossing or topmost crossing of τ in D_τ looks exactly like a neighborhood of x in D. The resulting link diagram is denoted D_τ. See Figure 3.

The main result of this section, Theorem 3.4, is a generalization of a proposition proved by Champanerkar and Kofman in [8].

Proposition 3.1 (Champanerkar–Kofman). *Let D be a link diagram with crossing x, and let τ be an alternating rational tangle such that D is twisted at x by τ. If D is quasi-alternating at x, then D_τ is quasi-alternating at each crossing of τ.*

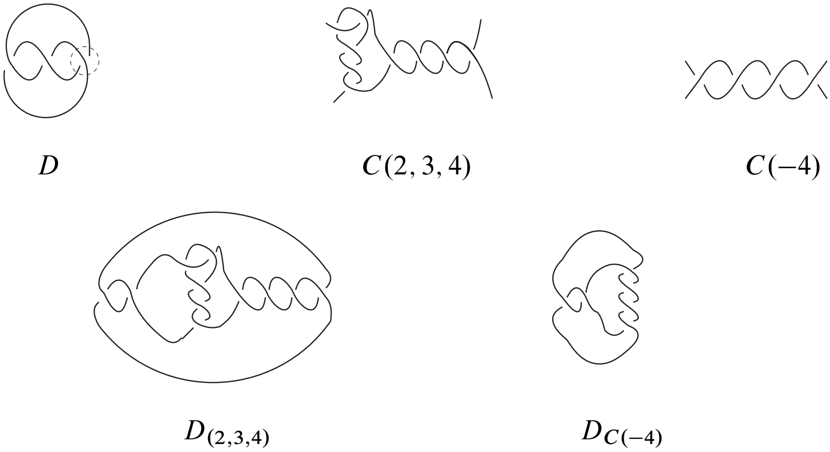

Figure 3. The diagram D twisted by $C(2,3,4)$ and $C(-4)$.

Let D be a diagram with crossing x. Resolve D at the crossing x to obtain diagrams D_v and D_h. Suppose $\mathrm{Kh}(D_v)$ is $[v_{\min}, v_{\max}]$-thick and $\mathrm{Kh}(D_h)$ is $[h_{\min}, h_{\max}]$-thick. As before, set $e = \mathrm{neg}(D_h) - \mathrm{neg}(D_+)$, where D_+ is the same diagram as D except if the crossing x in D is negative, then it is changed to positive in D_+. The diagram D is said to be width-preserving at x if either of the following conditions hold.

- If x is a positive crossing in D, then both $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} \neq h_{\max} + e + 1$.
- If x is a negative crossing in D, then both $v_{\min} \neq h_{\min} + e - 1$ and $v_{\max} \neq h_{\max} + e - 1$.

Proposition 3.2. *Let D be a link diagram with crossing x. If D is quasi-alternating at x, then D is width-preserving at x.*

Proof. Suppose D is quasi-alternating at x. Let D_v and D_h be the two resolutions of D at x. Since D is quasi-alternating at x, it follows that D_v and D_h are also quasi-alternating. Theorem 2.5 implies that $\widetilde{\mathrm{Kh}}(D)$, $\widetilde{\mathrm{Kh}}(D_v)$ and $\widetilde{\mathrm{Kh}}(D_h)$ are each supported entirely in one δ-grading. Suppose $\widetilde{\mathrm{Kh}}(D_v)$ is supported in δ-grading v and $\widetilde{\mathrm{Kh}}(D_h)$ is supported in δ-grading h. Corollary 2.2 implies that $\mathrm{Kh}(D_v)$ is $[v-1, v+1]$-thick and $\mathrm{Kh}(D_h)$ is $[h-1, h+1]$-thick. Let $e = \mathrm{neg}(D_h) - \mathrm{neg}(D_+)$ where D_+ is the same diagram as D except if x is negative in D, then it is changed to positive in D_+. Since $\det(D) = \det(D_v) + \det(D_h)$, it follows that the nontrivial parts of $\widetilde{\mathrm{Kh}}(D)$, $\widetilde{\mathrm{Kh}}(D_v)$ and $\widetilde{\mathrm{Kh}}(D_h)$ lie in three consecutive spots in the long exact sequence of Theorem 2.4 such that $\widetilde{\mathrm{Kh}}(D_v)$ and $\widetilde{\mathrm{Kh}}(D_h)$ are not adjacent. Therefore, if x is positive, then $v = h + e - 1$, and if x is negative, then $v = h + e + 1$. The result follows directly. □

Lemma 3.3. *Let D be an oriented link diagram with crossing x, and let τ be an alternating rational tangle with exactly two crossings x_0 and x_1. Let D_τ be D twisted at x by τ. If D is width-preserving at x, then for any orientation, D_τ is width-preserving at x_0 and x_1. Moreover, $w_{\mathrm{Kh}}(D) = w_{\mathrm{Kh}}(D_\tau)$.*

Proof. There are two ways to twist D at c, either horizontally or vertically. Let $\tau_1 = C(2)$ and $\tau_2 = C(-2)$. For each case, it is only necessary to prove the result for one choice of orientations on D and D_τ. Proposition 2.1 implies the result for all other choices of orientations on D and D_τ.

Let D_v and D_h be the diagrams obtained by resolving D at x, and let D_v^i and D_h^i be the diagrams obtained by resolving D_τ at the crossing x_i for $i = 0, 1$. Suppose $\mathrm{Kh}(D_v)$ and $\mathrm{Kh}(D_h)$ are $[v_{\min}, v_{\max}]$-thick and $[h_{\min}, h_{\max}]$-thick respectively. Let $e = \mathrm{neg}(D_h) - \mathrm{neg}(D_+)$ where D_+ is the same diagram as D except if the crossing x is negative in D, then it is changed to positive in D_+. Similarly set $e_i = \mathrm{neg}(D_h^i) - \mathrm{neg}(D_+^i)$ where D_+^i is the same diagram as D_τ except if the crossing x_i is negative in D_τ, then it is changed to positive in D_+.

Suppose x is positive. Choose the orientation on D_{τ_1} given in Figure 4. Also, Figure 4 shows the resolutions D_v^0 and D_h^0.

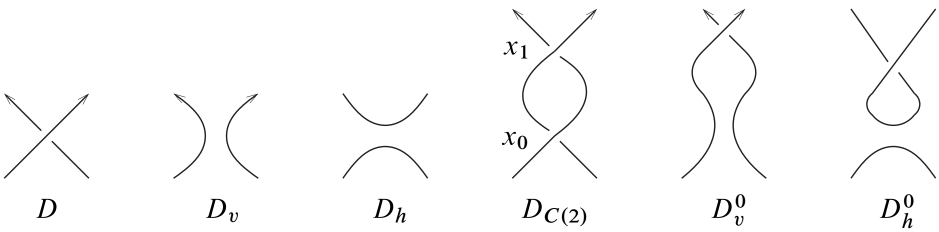

Figure 4. The resolutions for x positive and $\tau = C(2)$.

Observe that x_i is positive in D_{τ_1} for $i = 0, 1$. Corollary 2.6 implies that $\mathrm{Kh}(D)$ is $[\alpha, \beta]$-thick where $\alpha = \min\{v_{\min}+1, h_{\min}+e\}$ and $\beta = \max\{v_{\max}+1, h_{\max}+e\}$. The diagrams D_v^i and D represent the same link, and the diagrams D_h and D_h^i represent the same link. Therefore, $\mathrm{Kh}(D_v^i)$ is $[\alpha, \beta]$-thick and $\mathrm{Kh}(D_h^i)$ is $[h_{\min}, h_{\max}]$-thick. The diagram D_h^i is the same as the diagram D_h except D_h^i has one additional negative Reidemeister I twist, and hence $\mathrm{neg}(D_h^i) = \mathrm{neg}(D_h) + 1$. Since the diagrams D and D_v^i are identical, $\mathrm{neg}(D) = \mathrm{neg}(D_v^i)$. Thus $e_i = e + 1$. Since D is width-preserving, it follows that $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} \neq h_{\max} + e + 1$. Therefore,

$$h_{\min} + e_i + 1 = h_{\min} + e + 2 \neq \alpha,$$

and

$$h_{\max} + e_i + 1 = h_{\max} + e + 2 \neq \beta.$$

Hence D_{τ_1} is width-preserving at x_i. Also, Corollary 2.6 implies that $\mathrm{Kh}(D_{\tau_1})$ is $[\alpha+1, \beta+1]$-thick, and thus $w_{\mathrm{Kh}}(D) = w_{\mathrm{Kh}}(D_{\tau_1})$.

The possible orientations of D_{τ_2} depend on whether the strands forming the crossing x are in the same component of D or different components of D. Suppose they are in the same component. Choose the orientation on D_{τ_2} given in Figure 5. Also, Figure 5 shows the resolutions D_v^0 and D_h^0.

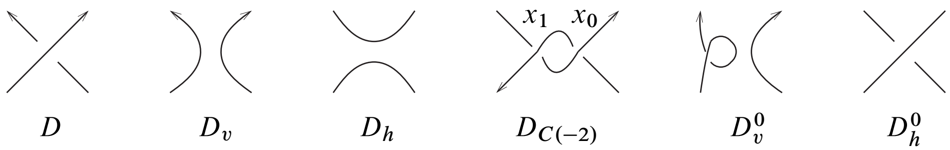

Figure 5. The resolutions for x positive, $\tau = C(-2)$, and with the depicted strands of D in the same component.

Observe that x_i is positive in D_{τ_2} for $i = 0, 1$. With suitably chosen orientations, we have

$$\mathrm{neg}(D_v) = \mathrm{neg}(D) = \mathrm{neg}(D_h^i), \tag{3.1}$$

and

$$\mathrm{neg}(D_+^i) = \mathrm{neg}(D_h). \tag{3.2}$$

The diagram D_v^i is the same as D_v except D_v^i has one component reversed and an additional positive Reidemeister I twist. Therefore, Proposition 2.1 implies that $\mathrm{Kh}(D_v^i)$ is $[v_{\min} - e, v_{\max} - e]$-thick. Also, equations 3.1 and 3.2 imply that $e_i = -e$. The diagram D_h^i is identical to D. Therefore, $\mathrm{Kh}(D_h^i)$ is $[\alpha, \beta]$-thick where $\alpha = \min\{v_{\min} + 1, h_{\min} + e\}$ and $\beta = \max\{v_{\max} + 1, h_{\max} + e\}$. Since D is width-preserving at x, we have $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} \neq h_{\max} + e + 1$. Therefore,

$$\alpha + e_i + 1 = \min\{v_{\min}+1, h_{\min}+e\} - e + 1 = \min\{v_{\min} - e + 2, h_{\min}+1\} \neq v_{\min} - e,$$

and

$$\beta + e_i + 1 = \max\{v_{\max}+1, h_{\max}+e\} - e + 1 = \max\{v_{\max} - e + 2, h_{\max}+1\} \neq v_{\max} - e.$$

Thus D_{τ_2} is width-preserving at x_i. Moreover, Corollary 2.6 implies that $\mathrm{Kh}(D_{\tau_2})$ is $[\alpha - e, \beta - e]$-thick, and hence $w_{\mathrm{Kh}}(D) = w_{\mathrm{Kh}}(D_{\tau_2})$. Suppose the strands forming the crossing x are in different components of the link. Choose the orientation on D_{τ_2} given in Figure 6. Also, Figure 6 shows the resolutions D_v^0 and D_h^0. Observe that x_i is a negative crossing in D_{τ_2} for $i = 0, 1$. Orient D_h^i so that it represents the same oriented link as D_v. With a suitably chosen orientation on D_h, we have

$$\mathrm{neg}(D) = \mathrm{neg}(D_v) = \mathrm{neg}(D_h^i), \tag{3.3}$$

and

$$\mathrm{neg}(D_h) + 1 = \mathrm{neg}(D_+^i) = \mathrm{neg}(D_v^i). \tag{3.4}$$

Equations 3.3 and 3.4 imply that $e_i = -e-1$. The diagram D_v^i is the same as D except

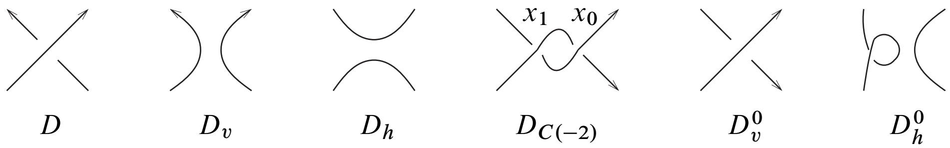

Figure 6. The resolutions for x positive, $\tau = C(-2)$, and with the depicted strands of D in different components.

D_v^i has one component reversed. Equations 3.3 and 3.4 along with Proposition 2.1 imply that $\mathrm{Kh}(D_v^i)$ is $[\alpha - e - 1, \beta - e - 1]$-thick where $\alpha = \min\{v_{\min} + 1, h_{\min} + e\}$ and $\beta = \max\{v_{\max} + 1, h_{\max} + e\}$. Since D_h^i and D_v represent the same oriented link, it follows that $\mathrm{Kh}(D_h^i)$ is $[v_{\min}, v_{\max}]$-thick. Since D is width-preserving at x, we have $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} \neq h_{\max} + e + 1$. Therefore,

$$\alpha - e - 1 = \min\{v_{\min} - e, h_{\min} - 1\} \neq v_{\min} - e - 2 = v_{\min} + e_i - 1,$$

and

$$\beta - e - 1 = \max\{v_{\max} - e, h_{\max} - 1\} \neq v_{\max} - e - 2 = v_{\max} + e_i - 1.$$

Thus D_{τ_2} is width-preserving at x_i. Moreover, Corollary 2.6 implies that $\mathrm{Kh}(D_{\tau_2})$ is $[\alpha - e - 1, \beta - e - 1]$-thick, and hence $w_{\mathrm{Kh}}(D) = w_{\mathrm{Kh}}(D_{\tau_2})$,

The case where x is a negative crossing in D is proved similarly. □

Theorem 3.4. *Let D be a link diagram with crossing x, τ be an alternating rational tangle, and D_τ be the diagram D twisted at x by τ. If D is width-preserving at x, then $w_{\mathrm{Kh}}(D) = w_{\mathrm{Kh}}(D_\tau)$.*

Proof. Let $\tau = C(a_1, \dots, a_m)$. Since τ is alternating, either $a_i > 0$ for all i or $a_i < 0$ for all i. Suppose $a_i > 0$ for all i. Beginning with the diagram D and the crossing x, one can alternate twisting the diagram by $C(2)$ and $C(-2)$. Replacing the appropriate crossings m times results in the diagram $D_{\tau'}$ where $\tau' = C(2, 1, \dots, 1)$. Lemma 3.3 implies that each crossing in $D_{\tau'}$ is width-preserving, and $w_{\mathrm{Kh}}(D) = w_{\mathrm{Kh}}(D_{\tau'})$.

Replace crossings corresponding to the m-th term in τ' by $C(2)$ until the resulting diagram is obtained by twisting D by $C(2, 1, \dots, 1, a_m)$ at x. Next, replace crossings corresponding to the $(m-1)$-st term in $C(2, 1, \dots, 1, a_m)$ with $C(-2)$ until the resulting diagram is obtained by twisting D by $C(2, 1, \dots, 1, a_{m-1}, a_m)$ at x. Continue replacing crossings in the tangle by either $C(2)$ or $C(-2)$ until the resulting diagram is obtained by twisting D by $C(a_1, \dots, a_m)$ at x. Since at each step, the only tangles used are $C(2)$ and $C(-2)$, Lemma 3.3 implies that $w_{\mathrm{Kh}}(D) = w_{\mathrm{Kh}}(D_\tau)$. The case where each $a_i < 0$ is proved similarly. □

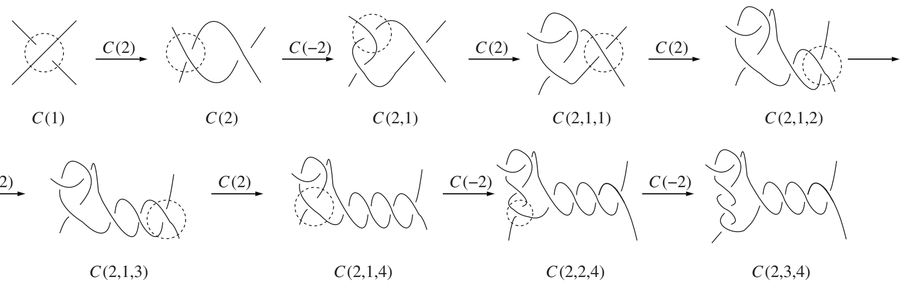

Figure 7. The inductive process of Theorem 3.4. At each step, the circled crossing is replaced with either $C(2)$ or $C(-2)$.

Remark 3.5. Watson [27] proves that $w_{\mathrm{Kh}}(D_\tau)$ is bounded by $w_{\mathrm{Kh}}(D_v)$ and $w_{\mathrm{Kh}}(D_h)$. By assuming that D is width-preserving at x, we are able to strengthen the result and calculate $w_{\mathrm{Kh}}(D_\tau)$.

Suppose D is an oriented diagram with crossing x. If D is twisted at x by $\tau_n = C(n)$ as in Figure 8, then the assumptions of Theorem 3.4 can be relaxed and a slightly stronger result holds. The following technical result is needed to compute the Khovanov width of closed 3-braids.

Proposition 3.6. *Suppose D is an oriented diagram with crossing x. Suppose D is twisted at x by $\tau_n = C(n)$ as in Figure 8. Let D_v and D_h be the two resolutions of D at x. Suppose $\mathrm{Kh}(D_v)$ is $[v_{\min}, v_{\max}]$-thick and $\mathrm{Kh}(D_h)$ is $[h_{\min}, h_{\max}]$-thick. Let $\alpha_\pm = \min\{v_{\min} \pm 1, h_{\min} + e\}$ and $\beta_\pm = \max\{v_{\max} \pm 1, h_{\max} + e\}$.*

(1) *Let $n > 0$. Suppose that $v_{\min} \neq h_{\min} + e + 1$. If $v_{\max} = h_{\max} + e + 1$, then suppose that there exist integers i and j such that $j - 2i = v_{\max}$, $\mathrm{Kh}^{i,j}(D_v)$ is nontrivial, and $\mathrm{Kh}^{k,l}(D_h)$ is trivial for all k whenever $l \leq j - 3e - 1$. Then $\mathrm{Kh}(D_{\tau_n})$ is $[n + \alpha_+, n + \beta_+]$-thick.*

(2) *Let* $n < 0$. *Suppose that* $v_{\max} \neq h_{\max} + e - 1$. *If* $v_{\min} = h_{\min} + e - 1$, *then suppose that there exist integers* i *and* j *such that* $j - 2i = v_{\min}$, $\mathrm{Kh}^{i,j}(D_v)$ *is nontrivial, and* $\mathrm{Kh}^{k,l}(D_h)$ *is trivial for all* k *whenever* $l \geq j - 3e - 1$. *Then* $\mathrm{Kh}(D_{\tau_n})$ *is* $[n + \alpha_-, n + \beta_-]$*-thick.*

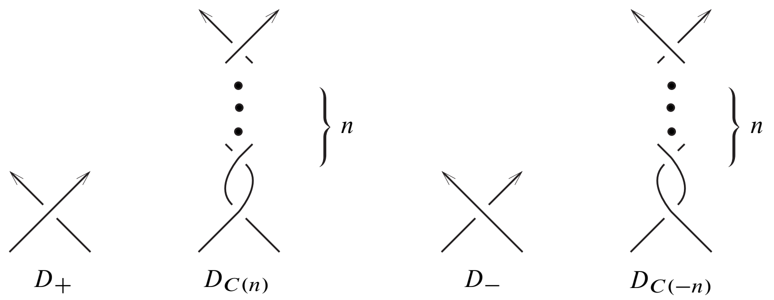

Figure 8. For $n > 0$, twist D_+ by $C(n)$ and twist D_- by $C(-n)$. Then choose the above orientations for $D_{C(n)}$ and $D_{C(-n)}$.

Proof. Let $n > 0$. Since D is twisted at x by τ_n as in Figure 8, it follows that x is a positive crossing. If both $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} \neq h_{\max} + e + 1$, then D is width-preserving at x. It follows from the proof of Theorem 3.4 that $\mathrm{Kh}(D_{\tau_n})$ is $[n + \alpha_+, n + \beta_+]$-thick.

Suppose $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} = h_{\max} + e + 1$. Thus there exist integers i and j such that $j - 2i = v_{\max}$, $\mathrm{Kh}^{i,j}(D_v)$ is nontrivial, and $\mathrm{Kh}^{k,l}(D_h)$ is trivial for all k and for all $l \leq j - 3e - 1$. Since $v_{\min} \neq h_{\min} + e + 1$, it follows that the minimum δ-grading where $\mathrm{Kh}(D_{\tau_n})$ is nontrivial is $n + \alpha_+$. We show, by induction on n, that $\mathrm{Kh}^{i,j+n}(D_{\tau_n}) \cong \mathrm{Kh}^{i,j}(D_v)$. This implies that the maximum δ-grading supporting $\mathrm{Kh}(D_{\tau_n})$ is $n + \beta_+$.

If $n = 1$, then the long exact sequence of Theorem 2.4 looks like

$$0 \to \mathrm{Kh}^{i,j+1}(D) \to \mathrm{Kh}^{i,j}(D_v) \to \mathrm{Kh}^{i-e,j-3e-1}(D_h) \to \cdots .$$

By hypothesis, $\mathrm{Kh}^{i-e,j-3e-1}(D_h)$ is trivial, and hence $\mathrm{Kh}^{i,j+1}(D) \cong \mathrm{Kh}^{i,j}(D_v)$.

Suppose, by way of induction, that $\mathrm{Kh}^{i,j+n}(D_{\tau_n}) \cong \mathrm{Kh}^{i,j}(D_v)$. Resolve $D_{\tau_{n+1}}$ at any crossing in τ_{n+1} to obtain diagrams D_v' and D_h'. Let $e_{n+1} = \mathrm{neg}(D_h') - \mathrm{neg}(D_{\tau_{n+1}})$. Since $\mathrm{neg}(D_h') = \mathrm{neg}(D_h) + n$ and $\mathrm{neg}(D_{\tau_{n+1}}) = \mathrm{neg}(D)$, it follows that $e_{n+1} = e + n$. Observe that D_v' and D_{τ_n} are the same diagram, and D_h' and D_h are diagrams for the same link. Hence the long exact sequence of Theorem 2.4 looks like

$$0 \to \mathrm{Kh}^{i,j+n+1}(D_{\tau_{n+1}}) \to \mathrm{Kh}^{i,j+n}(D_{\tau_n}) \to \mathrm{Kh}^{i-e-n,j-3e-3n-1}(D_h) \to \cdots .$$

Since $j-3e-3n-1 \leq j-3e-1$, it follows that $\mathrm{Kh}^{i-e-n,j-3e-3n-1}(D_h)$ is trivial. Thus $\mathrm{Kh}^{i,j+n+1}(D_{\tau_{n+1}}) \cong \mathrm{Kh}^{i,j+n}(D_{\tau_n}) \cong \mathrm{Kh}^{i,j}(D_v)$. Therefore $\mathrm{Kh}(D_{\tau_n})$ is $[n+\alpha_+, n+\beta_+]$-thick.

The case where $n < 0$ is proved in a similar fashion using the second sequence from Theorem 2.4. □

3.2. The Turaev genus of twisted links. Each link diagram D has an associated Turaev surface Σ_D. Let Γ be the plane graph associated to D. Regard Γ as embedded in $\mathbb{R}^2$ sitting inside $\mathbb{R}^3$. Outside the neighborhoods of the vertices of Γ is a collection of arcs in the plane. Replace each arc by a band that is perpendicular to the plane. In the neighborhoods of the vertices, place a saddle so that the circles obtained from choosing a 0-resolution at each crossing lie above the plane and so that the circles obtained from choosing a 1-resolution at each crossing lie below the plane (see Figure 9). The

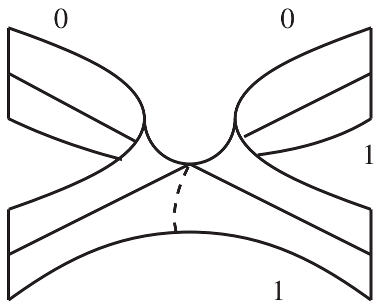

Figure 9. In a neighborhood of each crossing, insert a saddle so that the boundary above the plane corresponds to the 0 resolution and the boundary below the plane corresponds to the 1 resolution.

resulting surface has a boundary of disjoint circles, with circles corresponding to the all 0-resolution above the plane and circles corresponding to the all 1-resolution below the plane. For each boundary circle, insert a disk to obtain a closed surface Σ_D known as the *Turaev surface* (cf. [24]). The genus of this surface is denoted $g(\Sigma_D)$, and can be calculated by the formula

$$g(\Sigma_D) = \frac{2 - s_0(D) - s_1(D) + c(D)}{2},$$

where $c(D)$ is the number of crossings in D and $s_0(D)$ and $s_1(D)$ are the number of circles appearing in the all 0 and all 1 resolutions of D respectively. The *Turaev genus* of a link is defined as

$$g_T(L) = \min\{g(\Sigma_D) \mid D \text{ is a diagram for } L\}.$$

The Turaev genus of a link L is a measure of how far L is away from being alternating. Specifically, Dasbach et. al. [10] prove the following proposition.

Proposition 3.7 (Dasbach–Futer–Kalfagianni–Lin–Stoltzfus). *A link has Turaev genus* 0 *if and only if it is alternating.*

Also, the Turaev genus of L gives a bound on the Khovanov width of L. Manturov [17] and Champanerkar–Kofman–Stoltzfus [9] prove the following inequality.

Proposition 3.8 (Manturov, Champanerkar–Kofman–Stoltzfus). *Let L be a link. Then*
$$w_{\mathrm{Kh}}(L) - 2 \leq g_T(L).$$

The following proposition is implicit in Champanerkar and Kofman [8], but not explicitly proven.

Proposition 3.9. *Let D be a link diagram with crossing x, and let τ be an alternating rational tangle such that D is twisted by τ at x. Then $g(\Sigma_{D_\tau}) = g(\Sigma_D)$.*

Proof. Suppose $\tau = C(a_1, \dots, a_m)$, where $\operatorname{sign}(a_i) = \operatorname{sign}(a_j)$ for all i and j. Let $a = \sum_{i=1}^m |a_i|$. The all 0-resolution of D is the same as the all 0-resolution of D_τ, except D_τ has an additional k circles. Similarly, the all 1-resolution of D is the same as the all 1-resolution of D_τ, except D_τ has an additional l circles. Since τ is alternating, it follows that $k + l = a - 1$. Also, $c(D_\tau) = c(D) + a - 1$. Therefore,
$$\begin{aligned} g(\Sigma_D) &= \frac{2 - s_0(D) - s_1(D) + c(D)}{2} \\ &= \frac{2 - (s_0(D_\tau) + s_1(D_\tau) - (a-1)) + c(D_\tau) - (a-1)}{2} \\ &= \frac{2 - s_0(D_\tau) - s_1(D_\tau) + c(D_\tau)}{2} \\ &= g(\Sigma_{D_\tau}). \end{aligned}$$
□

In the case where D is the closure of a braid, there is a particularly nice version of Proposition 3.9. Let $w = w(\sigma_1, \sigma_1^{-1}, \dots, \sigma_{n-1}, \sigma_{n-1}^{-1}) \in B_n$ be a word in the braid group, and let D be the link diagram obtained from taking the closure of w. Suppose w' is word in B_n obtained by replacing σ_i in w with σ_i^k where $k > 0$ or by replacing σ_i^{-1} in w with σ_i^k where $k < 0$. Let D' be the link diagram obtained by taking the braid closure of w'.

Corollary 3.10. *Let D and D' be link diagrams obtained from the closures of the braids w and w' respectively. Then $g(\Sigma_D) = g(\Sigma_{D'})$.*

4. Applications to 3-braids

Closed 3-braids are a rich class of links in which computation of invariants are possible. In [3], Birman and Menasco classify the link types of closed 3-braids. Several papers (Schreier [21], Murasugi [18], and Garside [11]) give algorithms to determine when two 3-braids are conjugate in B_3. In this paper, we will be interested in Murasugi's solution to the conjugacy problem.

4.1. Torus links. Let $T(p,q)$ denote the (p,q) torus link. In this subsection, we will determine the Turaev genus and Khovanov width of $T(3,q)$. Turner [25] and Stošić [23] give formulas for the rational Khovanov homology of $T(3,q)$. The following theorem specifies the support of $\mathrm{Kh}(T(3,q);\mathbb{Q})$ for $q \geq 3$. If $q \leq -3$, one can deduce the support from this theorem and the fact that $T(3,-q)$ is the mirror of $T(3,q)$.

Theorem 4.1 (Stošić, Turner). *Suppose* $n \geq 1$.

(1) *The group* $\mathrm{Kh}(T(3,3n);\mathbb{Q})$ *is* $[4n-3, 6n-1]$*-thick. Thus*

$$w_{\mathrm{Kh}}(T(3,3n);\mathbb{Q}) = n+2.$$

(2) *The group* $\mathrm{Kh}(T(3,3n+1);\mathbb{Q})$ *is* $[4n-1, 6n+1]$*-thick. Thus*

$$w_{\mathrm{Kh}}(T(3,3n+1);\mathbb{Q}) = n+2.$$

(3) *The group* $\mathrm{Kh}(T(3,3n+2);\mathbb{Q})$ *is* $[4n+1, 6n+3]$*-thick. Thus*

$$w_{\mathrm{Kh}}(T(3,3n+2);\mathbb{Q}) = n+2.$$

The following lemma gives several normal forms for braids in B_3 whose closures are torus links. We will use these normal forms to compute the Turaev genus of a $(3,q)$ torus link as well as the Turaev genus of many closed 3-braids.

Lemma 4.2. *Let B_3 be the braid group on three strands. Then for any $n > 1$, we have*

$$\begin{aligned}
(\sigma_1\sigma_2)^3 &= \sigma_1^2\sigma_2\sigma_1^2\sigma_2,\\
(\sigma_1\sigma_2)^4 &= \sigma_1^2\sigma_2\sigma_1^3\sigma_2\sigma_1,\\
(\sigma_1\sigma_2)^5 &= \sigma_1^3\sigma_2\sigma_1^3\sigma_2\sigma_1^2,\\
(\sigma_1\sigma_2)^{3n} &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+1}\sigma_2,\\
(\sigma_1\sigma_2)^{3n+1} &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+2}\sigma_2\sigma_1,
\end{aligned}$$

and

$$(\sigma_1\sigma_2)^{3n+2} = \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-1}\,\sigma_1^3\sigma_2\sigma_1^{n+1}.$$

Proof. Observe

$$\begin{aligned}(\sigma_1\sigma_2)^3 &= \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\\ &= \sigma_1^2\sigma_2\sigma_1^2\sigma_2,\\ (\sigma_1\sigma_2)^4 &= \sigma_1^2\sigma_2\sigma_1^2\sigma_2\sigma_1\sigma_2\\ &= \sigma_1\sigma_2^2\sigma_1\sigma_2^3\sigma_1,\end{aligned}$$

and

$$\begin{aligned}(\sigma_1\sigma_2)^5 &= \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\\ &= \sigma_1\sigma_2\sigma_1\sigma_2^2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\\ &= \sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1^2\\ &= \sigma_1^3\sigma_2\sigma_1^3\sigma_2\sigma_1^2.\end{aligned}$$

The braid relation directly implies the following two relations:

$$\sigma_1^k\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^k,$$

and

$$\sigma_1\sigma_2\sigma_1^k = \sigma_2^k\sigma_1\sigma_2,$$

for $k > 0$. These relations will be used to prove the last three equations in the lemma.

For $n > 1$, we prove that

$$(\sigma_1\sigma_2)^{3n} = \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+1}\sigma_2$$

by induction. Let $n = 2$. Then

$$\begin{aligned}(\sigma_1\sigma_2)^6 &= \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\\ &= \sigma_1^2\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1^2\sigma_2\\ &= \sigma_1^3\sigma_2\sigma_1^3\sigma_2\sigma_1^3\sigma_2.\end{aligned}$$

Suppose, by way of induction, that

$$(\sigma_1\sigma_2)^{3n} = \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+1}\sigma_2.$$

Then

$$\begin{aligned}(\sigma_1\sigma_2)^{3(n+1)} &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+1}\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+1}\sigma_2\sigma_1\sigma_2^2\sigma_1\sigma_2^2\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+3}\sigma_2\sigma_1^2\sigma_2^2\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{n+2}\sigma_1\sigma_2^2\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1^{n+2}\sigma_2\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^4\sigma_2\sigma_1^3\sigma_2\sigma_1^{n+2}\sigma_2.\end{aligned}$$

Hence, for all $n > 1$,

$$(\sigma_1\sigma_2)^{3n} = \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+1}\sigma_2. \tag{4.1}$$

Equation 4.1 implies

$$\begin{aligned}(\sigma_1\sigma_2)^{3n+1} &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+1}\sigma_2\sigma_1\sigma_2\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+2}\sigma_2\sigma_1.\end{aligned}$$

Furthermore,

$$\begin{aligned}(\sigma_1\sigma_2)^{3n+2} &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1^{n+2}\sigma_2\sigma_1\sigma_1\sigma_2\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2^{n+1}\sigma_1\sigma_2\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^3\sigma_2\sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1^{n+1}\\ &= \sigma_1^3\sigma_2\,\underbrace{\sigma_1^4\sigma_2\ldots\sigma_1^4\sigma_2}_{n-2}\,\sigma_1^4\sigma_2\sigma_1^3\sigma_2\sigma_1^{n+1}.\end{aligned}$$

□

Abe and Kishimoto [1] have independently calculated the Turaev genus for the $(3, q)$ torus links. We give diagrams in closed braid form that minimize Turaev genus, while they have a different approach.

Proposition 4.3. *Suppose $q > 0$. The Turaev genus of $T(3, q)$ and $T(3, -q)$ is $\lfloor q/3 \rfloor$.*

Proof. Let D be the diagram obtained by taking the closure of the normal form for $(\sigma_1\sigma_2)^q$ given in Lemma 4.2. Thus D is a diagram for $T(3, q)$ and is the closure of a braid in the form

$$\sigma_1^{a_1}\sigma_2^{b_1}\ldots\sigma_1^{a_s}\sigma_2^{b_s}\sigma_1^{a_{s+1}},$$

where $s = \lfloor q/3 \rfloor + 1$, both $a_i > 0$ and $b_i > 0$ for all $1 \le i \le s$, and $a_{s+1} \ge 0$. Let D' be the diagram obtained by taking the closure of the braid $(\sigma_1\sigma_2)^s$. Corollary 3.10 implies that $g(\Sigma_D) = g(\Sigma_{D'})$. Since $c(D') = 2\lfloor q/3 \rfloor + 2$, $s_0(D') = 3$ and $s_1(D') = 1$, it follows that $g(\Sigma_{D'}) = \lfloor q/3 \rfloor$. Proposition 3.8 and Theorem 4.1 imply that the Turaev genus of $T(3, q)$ is greater than or equal to $\lfloor q/3 \rfloor$. Therefore, $g_T(T(3, q)) = \lfloor q/3 \rfloor$. The genera of the Turaev surfaces for a diagram and its mirror are equal, and hence $g_T(T(3, -q)) = \lfloor q/3 \rfloor$. □

The next corollary follows directly from Theorem 4.1, Proposition 4.3, and Proposition 3.8.

Corollary 4.4. *Suppose $n \ge 1$.*

(1) *The group* $\mathrm{Kh}(T(3, 3n))$ *is* $[4n - 3, 6n - 1]$*-thick and the group* $\mathrm{Kh}(T(3, -3n))$ *is* $[-6n + 1, -4n + 3]$*-thick. Therefore*

$$w_{\mathrm{Kh}}(T(3, 3n)) = w_{\mathrm{Kh}}(T(3, -3n)) = n + 2.$$

(2) *The group* $\mathrm{Kh}(T(3, 3n + 1))$ *is* $[4n - 1, 6n + 1]$*-thick and the group* $\mathrm{Kh}(T(3, -3n - 1)$ *is* $[-6n - 1, -4n + 1]$*-thick. Therefore*

$$w_{\mathrm{Kh}}(T(3, 3n + 1)) = w_{\mathrm{Kh}}(T(3, -3n - 1) = n + 2.$$

(3) *The group* $\mathrm{Kh}(T(3, 3n + 2))$ *is* $[4n + 1, 6n + 3]$*-thick and the group* $\mathrm{Kh}(T(3, -3n - 2))$ *is* $[-6n - 3, -4n - 1]$*-thick. Therefore*

$$w_{\mathrm{Kh}}(T(3, 3n + 2)) = w_{\mathrm{Kh}}(T(3, -3n - 2)) = n + 2.$$

4.2. Khovanov width of 3-braids. In this subsection, we determine the Khovanov width of closed 3-braids based upon Murasugi's classification of closed 3-braids up to conjugation. In [18], Murasugi proves the following:

Theorem 4.5 (Murasugi). *Let $w \in B_3$ be a braid on three strands, and let $h = (\sigma_1\sigma_2)^3$ be a full twist. Let $n \in \mathbb{Z}$. Then w is conjugate to exactly one of the following:*

(1) $h^n\sigma_1^{p_1}\sigma_2^{-q_1}\dots\sigma_1^{p_s}\sigma_2^{-q_s}$ *where* p_i, q_i *and* s *are positive integers.*

(2) $h^n\sigma_2^m$ *where* $m\in\mathbb{Z}$.

(3) $h^n\sigma_1^m\sigma_2^{-1}$, *where* $m\in\{-1,-2,-3\}$.

Let L be a closed 3-braid. Theorem 4.5 says, in effect, that L is the closure of a braid of the form h^nA. For $n\neq 0$, we say that L has *cancellation* if the braid word for A contains a σ_i^{ε} for $i=1$ or 2, where $\operatorname{sign}(\varepsilon)\neq\operatorname{sign}(n)$. Besides two infinite family of braids, we prove that $w_{\mathrm{Kh}}(L)=|n|+2$ if there is no cancellation and $w_{\mathrm{Kh}}(L)=|n|+1$ if there is cancellation.

The following several propositions establish the support of $\mathrm{Kh}(L)$. The proofs require the computation of Khovanov homology for a few specific links. We represent the rational Khovanov homology as a Poincaré polynomial $P(L)$, a Laurent polynomial in the variables q and t such that the coefficient of q^it^j is the rank of $\mathrm{Kh}^{i,j}(L;\mathbb{Q})$. One can find these computations in KnotInfo [6]. Additionally, if L is a torus $(3,q)$ torus link, then Turner [25] computes $P(L)$. The only link used in a proof below that is not a $(3,q)$ torus link is $L(6,n,1)$ in Thistlethwaite's link table. The Khovanov homology of $L(6,n,1)$ can be determined from Lee's result on the Khovanov homology of alternating links [15] and the long exact sequences of Theorem 2.4.

Proposition 4.6. *Suppose* $n>0$ *and* $k\geq 0$. *Let* D *be the closure of the braid* $(\sigma_1\sigma_2)^{3n}\sigma_1^k\sigma_2^{-1}$, *and let* D' *be the closure of* $(\sigma_1\sigma_2)^{-3n}\sigma_1\sigma_2^{-k}$. *Then* $\mathrm{Kh}(D)$ *is* $[4n+k-2,6n+k-2]$*-thick and* $\mathrm{Kh}(D')$ *is* $[-6n-k+2,-4n-k+2]$*-thick.*

Proof. Observe that $(\sigma_1\sigma_2)^{3n}\sigma_1^k\sigma_2^{-1}=(\sigma_1\sigma_2)^{3n-1}\sigma_1^{k+1}$ for $n>0$. Let D_+ be the closure of the braid $(\sigma_1\sigma_2)^{3n-1}\sigma_1$. Resolve the crossing given by the last σ_1 to obtain two link diagrams D_v and D_h. Then D_v is a diagram for $T(3,3n-1)$, and D_h is a diagram for the unknot. By Corollary 4.4, $\mathrm{Kh}(D_v)$ is $[4n-3,6n-3]$-thick. Since D_h is the unknot, $\mathrm{Kh}(D_h)$ is $[-1,1]$-thick. Recall that $e=\operatorname{neg}(D_h)-\operatorname{neg}(D_+)$. The diagram D_h has $4n-1$ negative crossings, while the diagram D_+ has no negative crossings. Thus $e=4n-1$.

If $n\neq 2$, then D_+ is width-preserving. If $n=2$, then the Poincaré polynomial of $D_+=T(3,5)$ is

$$P(T(3,5))=q^7+q^9+q^{11}t^2+q^{15}t^3+q^{13}t^4+q^{15}t^4+q^{17}t^5+q^{17}t^6+q^{19}t^5+q^{21}t^7.$$

Therefore, $\mathrm{Kh}^{0,9}(D_v)$ is nontrivial. Moreover, $\mathrm{Kh}^{i,j}(D_h)=0$ for all i if $j\leq 9-3e-1=-13$. Therefore, for $n>0$, Proposition 3.6 implies that $\mathrm{Kh}(D)$ is $[4n+k-2,6n+k-2]$-thick. The proof for D' is similar. □

Proposition 4.7. *Let* D *be the closure of the braid* $(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{-b_1}\dots\sigma_1^{a_k}\sigma_2^{-b_k}$, *where each* $a_i,b_i>0$. *Let* $a=\sum_{i=1}^k a_i$ *and* $b=\sum_{i=1}^k b_i$. *If* $n>0$, *then* $\mathrm{Kh}(D)$

is $[4n+a-b-1, 6n+a-b-1]$*-thick. If* $n<0$*, then* $\mathrm{Kh}(D)$ *is* $[6n+a-b+1, 4n+a-b+1]$*-thick. Hence, if* $n \neq 0$*, then* $w_{\mathrm{Kh}}(D) = |n|+1$.

Proof. Suppose $n>0$. We proceed by induction on b. Suppose $b=1$. Let D_1 be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_1^a\sigma_2^{-1}$. Proposition 4.6 states that $\mathrm{Kh}(D_1)$ is supported in the band $[4n+a-2, 6n+a-2]$. Since $(\sigma_1\sigma_2)^3$ is in the center of B_3, it follows that D_1 represents the same link as D_1', the closure of $(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{-1}\sigma_1^{a-a_1}$.

If D_b is the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_1^{p_1}\sigma_2^{-q_1}\dots\sigma_1^{p_j}\sigma_2^{-q_j}$ where each $p_i, q_i > 0$, $\sum_{i=1}^j p_i = a$ and $\sum_{i=1}^j q_i = b$, then by way of induction, suppose $\mathrm{Kh}(D_b)$ is $[4n+a-b-1, 6n+a-b-1]$-thick. Let D_{b+1} be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_1^{p_1'}\sigma_2^{-q_1'}\dots\sigma_1^{p_l'}\sigma_2^{-q_l'}$, where $p_i', q_i' > 0$, $\sum_{i=1}^l p_i' = a$, and $\sum_{i=1}^l q_i' = b+1$.

Resolve D_{b+1} at the crossing corresponding to the last σ_2^{-1} to obtain diagrams D_v and D_h. The diagram D_{b+1} is the closure of a 3-braid consisting of n full twists, followed by an alternating 3-braid. There is a simple closed curve in the plane whose interior is the alternating 3-braid piece of D_{b+1}. After resolving D_{b+1}, the interior of the simple closed curve contains pieces of D_v and D_h. Call the pieces of D_{b+1}, D_v, and D_h contained in the interior of this simple closed curve the alternating parts of D_{b+1}, D_v, and D_h respectively.

By the inductive hypothesis, $\mathrm{Kh}(D_v)$ is $[4n+a-b-1, 6n+a-b-1]$-thick. Let m be the number of negative crossings in the alternating part of D_h, which has a total of $a+b$ crossings. Also, D_h is a non-alternating diagram for an alternating link L. Hence, Theorem 2.5 implies that $\mathrm{Kh}(L)$ is $[-\sigma(L)-1, -\sigma(L)+1]$-thick. One can calculate the signature of an alternating link from any alternating diagram by a result of Gordon and Litherland [12]. Color the regions of the alternating diagram in a checkerboard fashion so that near each crossing it looks like Figure 10.

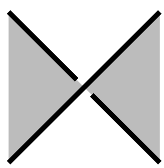

Figure 10. Color the alternating diagram in a checkerboard fashion such that a neighborhood of each crossing appears as above.

Then the signature is given by

$$\sigma(L) = \#(\text{black regions}) - \#(\text{positive crossings}) - 1.$$

There is another diagram representing L that has $b+2$ black regions and $a+b-m$ positive crossings (see Figure 11). Therefore, $\sigma(L) = m-a+3$, and hence $\mathrm{Kh}(D_h')$ is

$[a-m-4, a-m-2]$-thick. Since there are $4n$ negative crossing in the full twist part of D_h and m negative crossings in the alternating part of D_h, it follows that $\mathrm{neg}(D_h) = 4n + m$. Let D_+ be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_1^{p'_1}\sigma_2^{-d'_1}\dots\sigma_1^{p'_l}\sigma_2^{-d'_l+1}\sigma_2$. Then $\mathrm{neg}(D_+) = b$, and thus $e = \mathrm{neg}(D_h) - \mathrm{neg}(D_+) = 4n + m - b$. For $n > 0$,

$$4n + a - b - 1 \neq (a - m - 4) + (4n + m - b) - 1$$

and

$$6n + a - b - 1 \neq (a - m - 2) + (4n + m - b) - 1.$$

Therefore, Theorem 2.6 implies that $\mathrm{Kh}(D_{b+1})$ is $[4n+a-b-2, 6n+a-b-2]$-thick. The proof for $n < 0$ is similar. □

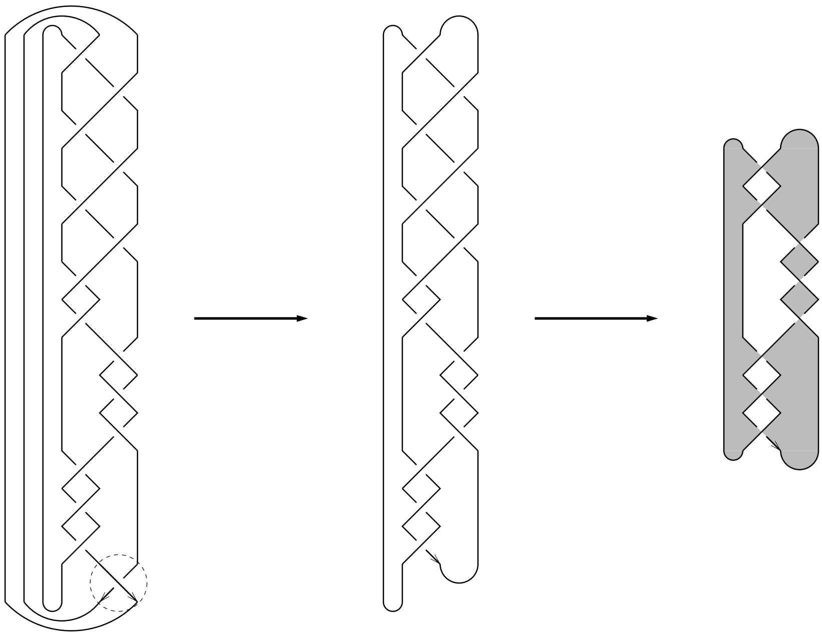

Figure 11. The closure of the braid $(\sigma_1\sigma_2)^3\sigma_1^2\sigma_2^{-3}\sigma_1^3\sigma_2^{-1}$ together with its resolution and an alternating diagram of its resolution. There are 5 black regions and 2 negative crossings in the alternating diagram.

Proposition 4.8. *Let D be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_2^m$.*

(1) *If $n > 0$ and $m \geq 0$, then* $\mathrm{Kh}(D)$ *is* $[4n + m - 3, 6n + m - 1]$*-thick and* $w_{\mathrm{Kh}}(D) = n + 2$.

(2) *If $n < 0$ and $m \leq 0$, then* $\mathrm{Kh}(D)$ *is* $[6n + m + 1, 4n + m + 3]$*-thick and* $w_{\mathrm{Kh}}(D) = -n + 2$.

(3) *If* $n = 1$ *and* $m < -3$*, then* $\mathrm{Kh}(D)$ *is* $[m+3, m+7]$*-thick and* $w_{\mathrm{Kh}}(D) = 3$.

(4) *If* $n = -1$ *and* $m > 3$*, then* $\mathrm{Kh}(D)$ *is* $[m-7, m-3]$*-thick and* $w_{\mathrm{Kh}}(D) = 3$.

(5) *If both* $n = 1$ *and* $-3 \leq m < 0$ *or both* $n > 1$ *and* $m < 0$*, then* $\mathrm{Kh}(D)$ *is* $[4n+m-1, 6n+m-1]$*-thick and* $w_{\mathrm{Kh}}(D) = n+1$.

(6) *If both* $n = -1$ *and* $0 < m \leq 3$ *or both* $n < -1$ *and* $m > 0$*, then* $\mathrm{Kh}(D)$ *is* $[6n+m+1, 4n+m+1]$*-thick and* $w_{\mathrm{Kh}}(D) = -n+1$.

Proof. We prove statements (1), (3), and (5). Statements (2), (4), and (6) are proved similarly.

(1) Suppose $n > 0$ and $m \geq 0$. Let D_+ be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_2$. Resolve D_+ at the crossing corresponding to the last σ_2 to obtain diagrams D_v and D_h. Then D_v is a diagram for $T(3,3n)$. By Corollary 4.4, $\mathrm{Kh}(D_v)$ is $[4n-3, 6n-1]$-thick. Also, D_h is the two component unlink, and hence $\mathrm{Kh}(D_h)$ is $[-2,2]$-thick. The diagram D_h has $4n$ negative crossings, and the diagram D_+ has no negative crossings. Thus $e = 4n$.

Observe that $4n-3 \neq -2+e+1$ and $6n-1 = 2+e+1$ when $n = 2$. If $n = 2$, then D_v is $T(3,6)$, and

$$\begin{aligned}P(T(3,6)) = {} & q^9 + q^{11} + q^{13}t^2 + q^{17}t^3 + q^{15}t^4 + q^{17}t^4 \\ & + q^{19}t^5 + q^{19}t^6 + q^{21}t^7 + q^{21}t^8 + q^{23}t^7 + 3q^{23}t^8 + 2q^{25}t^8.\end{aligned}$$

Therefore $\mathrm{Kh}^{0,11}(D_v)$ is nontrivial. Also, $\mathrm{Kh}^{i,j}(D_h) = 0$ for all i if $j \leq 11 - 3e - 1 = -14$. Hence, Theorem 3.6 implies that $\mathrm{Kh}(D)$ is $[4n+m-3, 6n+m-1]$-thick.

(3) Suppose $n = 1$ and $m < -3$. Let D_- be the closure of $(\sigma_1\sigma_2)^3\sigma_2^{-5}$. Resolve D_- at the crossing corresponding to the last σ_2^{-1} to obtain diagrams D_v and D_h. The diagram D_h is a diagram for the two component unlink, and hence $\mathrm{Kh}(D_h)$ is $[-2,2]$-thick. The diagram D_v is a diagram for the link $L(6,n,1)$ in Thistlethwaite's link table (see Figure 12).

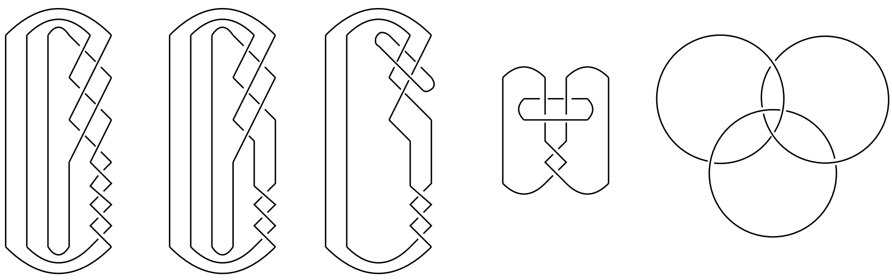

Figure 12. A transformation of the closure of $(\sigma_1\sigma_2)^3\sigma_2^{-4}$ into $L(6,n,1)$.

The Poincaré polynomial for $L(6,n,1)$ is given by

$$P(L(6,n,1)) = 2q^{-1} + 3q + q^3 + qt + q^5t^2 + q^7t^4 + q^9t^4.$$

Therefore $\mathrm{Kh}(D_v)$ is $[-1,3]$-thick. Both of the diagrams D_h and D_+ have 4 negative crossings. Thus $e = 0$. Since $-1 \neq -2+e-1$ and $3 \neq 2+e-1$, Theorem 3.6 implies that $\mathrm{Kh}(D)$ is $[m+3, m+7]$-thick.

(5) If $n = 1$ and $-3 \leq m < 0$, then Baldwin [4] has shown that D is quasi-alternating. Therefore, Theorem 2.5 implies that $\mathrm{Kh}(D)$ is $[-\sigma(L)-1, -\sigma(L)+1]$-thick, where L is the link type of D. A straightforward calculation of signature gives the desired result.

Suppose $n > 1$ and $m < 0$. Observe that $(\sigma_1\sigma_2)^{3n}\sigma_2^{-1} = (\sigma_1\sigma_2)^{3n-1}\sigma_1$. Let D_+ be the closure of the braid $(\sigma_1\sigma_2)^{3n-1}\sigma_1$. Resolve D_+ at the crossing corresponding to the last σ_1 to obtain diagrams D_v and D_h. Since D_v is a diagram for $T(3, 3n-1)$, it follows that $\mathrm{Kh}(D_v)$ is $[4n-3, 6n-3]$-thick. Since D_h is a diagram for the unknot, it follows that $\mathrm{Kh}(D_h)$ is $[-1,1]$-thick. The diagram D_h has $4n-1$ negative crossings, and D_v has no negative crossings. Thus $e = 4n-1$.

If $n = 2$, then $6n-3 = 1+e+1$, and the long exact sequence of Theorem 2.4 looks like

$$0 \to \mathrm{Kh}^{0,10}(D_+) \to \mathrm{Kh}^{0,9}(D_v) \to \mathrm{Kh}^{-7,-13}(D_h) \to \cdots .$$

Since $\mathrm{Kh}^{-7,-13}(D_h) = 0$ and $\mathrm{Kh}^{0,9}(D_v)$ is nontrivial, it follows that $\mathrm{Kh}^{0,10}(D_+)$ is nontrivial. Since $4n-3 \neq -1+e+1$ and $6n-3 \neq 1+e+1$ for $n > 2$, Corollary 2.6 implies that $\mathrm{Kh}(D_+)$ is $[4n-2, 6n-2]$-thick.

Let D_- be the closure of $(\sigma_1\sigma_2)^{3n-1}\sigma_1\sigma_2^{-1}$. Resolve D_- at the crossing given by the last σ_2^{-1} to obtain diagrams D_v and D_h. The diagram D_v is the closure of the braid $(\sigma_1\sigma_2)^{3n-1}\sigma_1$, and hence $\mathrm{Kh}(D_v)$ is $[4n-2, 6n-2]$-thick. The link D_h is a diagram for the two component unlink, and thus $\mathrm{Kh}(D_h)$ is $[-2,2]$-thick. The diagram D_h has $4n-1$ negative crossings, and D_+ has no negative crossings. Thus $e = 4n-1$.

For $n > 1$, we have $4n-2 \neq -2+e-1$ and $6n-2 \neq 2+e-1$. Therefore, Theorem 3.6 implies that $\mathrm{Kh}(D)$ is $[4n+m-1, 6n+m-1]$-thick. □

Proposition 4.9. *Let D be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_1^m\sigma_2^{-1}$, where $m \in \{-1,-2,-3\}$.*

(1) *If $n > 0$, then $\mathrm{Kh}(D)$ is $[4n+m-2, 6n+m-2]$-thick, and $w_{\mathrm{Kh}}(D) = n+1$.*

(2) *If $n < 0$, then $\mathrm{Kh}(D)$ is $[6n+m, 4n+m+2]$-thick, and $w_{\mathrm{Kh}}(D) = -n+2$.*

Proof. (1) Suppose $n > 0$. If $m = -1$, then D is a diagram for $T(3, 3n-1)$, and the result follows.

Let $m = -2$. Then, up to conjugation in B_3, we have

$$\begin{aligned}(\sigma_1\sigma_2)^{3n}\sigma_1^{-2}\sigma_2^{-1} &= (\sigma_1\sigma_2)^{3n-1}\sigma_1^{-1}\\ &= (\sigma_2\sigma_1)^{3n-2}\sigma_2\\ &= (\sigma_1\sigma_2)^{3n-2}\sigma_1.\end{aligned}$$

If D' is the closure of the braid $(\sigma_1\sigma_2)^{3n-2}\sigma_1$, then D and D' represent the same link. Resolve D' at the crossing corresponding to the final σ_1 to obtain diagrams D_v and D_h. Then D_v is a diagram for $T(3, 3n-2)$, and D_h is a diagram for the unknot. Hence $\mathrm{Kh}(D_v)$ is $[4n-5, 6n-5]$-thick, and $\mathrm{Kh}(D_h)$ is $[-1, 1]$-thick. The diagram D_h has $4n-3$ negative crossings, and the diagram D' has none. Thus $e = 4n-3$.

Observe that $4n-5 \neq -1+e+1$, and $6n-5 = 1+e+1$ when $n = 2$. If $n = 2$, the long exact sequence of Theorem 2.4 looks like

$$0 \to \mathrm{Kh}^{0,8}(D') \to \mathrm{Kh}^{0,7}(D_v) \to \mathrm{Kh}^{-5,-9}(D_h) \to \cdots$$

Since $\mathrm{Kh}^{-5,-9}(D_h) = 0$ and $\mathrm{Kh}^{0,7}(L_v)$ is nontrivial, it follows that $\mathrm{Kh}^{0,8}(L)$ is nontrivial. Hence Theorem 2.6 implies that $\mathrm{Kh}(D')$ is $[4n-4, 6n-4]$-thick.

Let $m = -3$. Then, up to conjugation in B_3, we have

$$\begin{aligned}(\sigma_1\sigma_2)^{3n}\sigma_1^{-3}\sigma_2^{-1} &= (\sigma_1\sigma_2)^{3n-1}\sigma_1^{-2}\\ &= (\sigma_2\sigma_1)^{3n-3}\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\\ &= (\sigma_1\sigma_2)^{3n-2}.\end{aligned}$$

Hence D is a diagram for $T(3, 3n-2)$, and the result follows.

(2) Let $n < 0$. If $m = -1$, then D is a diagram for $T(3, 3n-1)$, and the result follows.

Let $m = -2$. Then D is the closure of $(\sigma_1\sigma_2)^{3n-1}\sigma_1^{-1}$. Resolve D at the crossing corresponding to the last σ_1^{-1} to obtain diagrams D_v and D_h. Then D_v is a diagram for $T(3, 3n-1)$, and hence $\mathrm{Kh}(D_v)$ is $[6n-1, 4n+1]$-thick. Also, D_h is a diagram for the unknot, and hence $\mathrm{Kh}(D_h)$ is $[-1, 1]$-thick. The diagram D_h has $-2n-1$ negative crossings, and the diagram D_+ has $-6n+2$ negative crossings. Thus $e = 4n-1$.

Observe that $4n+1 \neq 1+e-1$, and $6n-1 = -1+e-1$ if $n = -1$. If $n = -1$, the long exact sequence of Theorem 2.4 looks like

$$\cdots \to \mathrm{Kh}^{5,9}(D_h) \to \mathrm{Kh}^{0,-7}(D_v) \to \mathrm{Kh}^{0,-8}(D) \to 0.$$

Since $\mathrm{Kh}^{5,9}(D_h) = 0$ and $\mathrm{Kh}^{0,-7}(D_v)$ is nontrivial, it follows that $\mathrm{Kh}^{0,-8}(D)$ is nontrivial. Thus $\mathrm{Kh}(D)$ is $[6n-2, 4n]$-thick.

Let $m = -3$. Then, up to conjugacy in B_3, we have

$$\begin{aligned}(\sigma_1\sigma_2)^{3n}\sigma_1^{-3}\sigma_2^{-1} &= (\sigma_1\sigma_2)^{3n}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-2}\\ &= (\sigma_1\sigma_2)^{3n-2}.\end{aligned}$$

In this case D is a diagram for $T(3, 3n-2)$, and the result follows. □

We collect the results of Propositions 4.7, 4.8 and 4.9 into one theorem giving the Khovanov width of closed 3-braids.

Theorem 4.10. *Let L be a closed 3-braid of the form $h^n A$, as in Theorem 4.5, where $h = (\sigma_1\sigma_2)^3$ and $n \neq 0$. Then*

$$w_{\mathrm{Kh}}(L) = \begin{cases} |n| + 2 & \text{if } L \text{ has no cancellation or if } L \text{ is the} \\ & \text{closure of } h^{\pm 1}\sigma_2^{\mp m} \text{ where } m > 3, \\ |n| + 1 & \text{otherwise.} \end{cases}$$

Remark 4.11. If $n = 0$, then L is a (possibly split) alternating link, and thus $w_{\mathrm{Kh}}(L)$ can be deduced from Theorem 2.5 and Proposition 2.1.

In [4], Baldwin classifies quasi-alternating closed 3-braids.

Proposition 4.12. *Let L be a closed 3-braid and let $h = (\sigma_1\sigma_2)^3$.*

- *If L is the closure of the braid $h^n\sigma_1^{a_1}\sigma_2^{-b_2}\dots\sigma_1^{a_k}\sigma_2^{-b_k}$, where each $a_i, b_i > 0$, then L is quasi-alternating if and only if $n \in \{-1, 0, 1\}$.*
- *If L is the closure of the braid $h^n\sigma_2^m$, then L is quasi-alternating if and only if either $n = 1$ and $m \in \{-1, -2, -3\}$ or $n = -1$ and $m \in \{1, 2, 3\}$.*
- *If L is the closure of the braid $h^n\sigma_1^m\sigma_2^{-1}$, where $m \in \{-1, -2, -3\}$, then L is quasi-alternating if and only if $n \in \{0, 1\}$.*

Using the spectral sequence from reduced Khovanov homology of a link to the Heegaard Floer homology of the branched double cover of that link, Baldwin [4] shows the following corollary. This corollary is also a consequence of Theorem 4.10 and Proposition 4.12.

Corollary 4.13 (Baldwin). *Let L be a closed 3-braid. Then L is quasi-alternating if and only if $w_{\widetilde{\mathrm{Kh}}}(L) = 1$.*

Remark 4.14. Shumakovitch has used his computer package KhoHo [22] to show that the 9_{46} and 10_{140} knots (both closed 4-braids) have reduced Khovanov width 1, but they are not quasi-alternating. One can use either of these knots to generate infinite families of counterexamples to Corollary 4.13 for braids with index greater than 3.

4.3. Turaev genus of closed 3-braids.

Combining Lemma 4.2 with Corollary 3.10 gives a useful tool to compute the Turaev genus of closed 3-braids. By using the lower bound given by Proposition 3.8, the Turaev genus of closed 3-braids can be calculated up to a maximum additive error of at most 1.

Proposition 4.15. *Let L be the link type closure of $(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{-b_1}\dots\sigma_1^{a_k}\sigma_2^{-b_k}$, where each $a_i, b_i > 0$ and $n \neq 0$. Then $|n| - 1 \leq g_T(L) \leq |n|$.*

Proof. Suppose $n > 0$. We have

$$(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{-b_1}\dots\sigma_1^{a_k}\sigma_2^{-b_k} = (\sigma_1\sigma_2)^{3n-1}\sigma_1^{a_1+1}\sigma_2^{-b_1}\dots\sigma_1^{a_k}\sigma_2^{-b_k+1}.$$

If $b_k > 1$, let D be the closure of the braid $(\sigma_1\sigma_2)^n(\sigma_1\sigma_2^{-1})^k$ and if $b_k = 1$, let D be the closure of the braid $(\sigma_1\sigma_2)^n(\sigma_1\sigma_2^{-1})^{k-1}$. By applying the normal form of Lemma 4.2 to $(\sigma_1\sigma_2)^{3n-1}$ and then using Corollary 3.10, it follows that $g_T(L) \leq g_T(D)$. A straightforward calculation shows that $g_T(D) = n$. Since $w_{\text{Kh}}(L) = n+1$ and $w_{\text{Kh}}(L)-2 \leq g_T(L)$, we have $n-1 \leq g_T(L)$. The case where $n < 0$ is similar. □

Proposition 4.16. *Let L be the link type of the closure of $(\sigma_1\sigma_2)^{3n}\sigma_2^m$, where $n \neq 0$.*

1. *If L has no cancellation, then $g_T(L) = |n|$.*
2. *If L has cancellation and $|n| > 1$, then $|n|-1 \leq g_T(L) \leq |n|$.*
3. *If either both $n = 1$ and $-3 \leq m < 0$ or both $n = -1$ and $0 < m \leq 3$, then $g_T(L) = 0$.*
4. *If either both $n = 1$ and $m < -3$ or both $n = -1$ and $m > 3$. Then $g_T(L) = 1$.*

Proof. (1) If L has no cancellation, then either both $n > 0$ and $m \geq 0$ or $n < 0$ and $m \leq 0$. Corollary 3.10 implies that $g_T(L) \leq g_T(T(3,3n)) = |n|$. Since $w_{\text{Kh}}(L) = |n|+2$, it follows that $g_T(L) = |n|$.

(2) Suppose that L has cancellation and $n > 1$. Then $m < 0$ and

$$(\sigma_1\sigma_2)^{3n}\sigma_2^m = (\sigma_1\sigma_2)^{3n-1}\sigma_1\sigma_2^{(m+1)}.$$

If $m < -1$, let D be the closure of $(\sigma_1\sigma_2)^n\sigma_1\sigma_2^{-1}$, and if $m = -1$, let D be the closure $(\sigma_1\sigma_2)^n$. Lemma 4.2 and Corollary 3.10 imply that $g_T(L) \leq g(\Sigma_D)$. A straightforward calculation shows that $g(\Sigma_D) = n$. Since $w_{\text{Kh}}(L) = n+1$, it follows that $n-1 \leq g_T(L)$. The case where $n < -1$ and $m > 0$ is similar.

(3) Suppose $n = 1$ and $-3 \leq m < 0$. As noted in Baldwin's paper [4], we have

$$(\sigma_1\sigma_2)^3\sigma_2^m = \sigma_1\sigma_2^2\sigma_1\sigma_2^2\sigma_2^m.$$

By canceling the σ_2^m with the final σ_2^2, one obtains a diagram for L with 5 crossings or less. Therefore L is alternating and $g_T(L) = 0$. The case where $n = -1$ and $0 < m \leq 3$ is similar.

(4) Suppose $n = 1$ and $m < -3$. Then L can be represented by the closure of $\sigma_1\sigma_2^2\sigma_1\sigma_2^{m+2}$. Let D be the closure of the braid $\sigma_1\sigma_2\sigma_1\sigma_2^{-1}$. By Corollary 3.10, we have $g_T(L) \leq g(\Sigma_D)$, and a straightforward calculation shows that $g(\Sigma_D) = 1$. Since $w_{\text{Kh}}(L) = 3$, it follows that $g_T(L) = 1$. The case where $n = -1$ and $m > 3$ is similar. □

Proposition 4.17. *Let L be the link type of the closure of $(\sigma_1\sigma_2)^{3n}\sigma_1^m\sigma_2^{-1}$ where $m \in \{-1,-2,-3\}$. If $n > 0$, then $g_T(L) = n-1$ and if $n < 0$, then $g_T(L) = |n|$.*

Proof. Let $n > 0$. Using the forms in the proof of Proposition 4.9 and the reductions of Lemma 4.2 and Corollary 3.10, one sees that $g_T(L) \leq g(\Sigma_D)$ where D is the closure of $(\sigma_1\sigma_2)^n$. A straightforward calculation shows that $g(\Sigma_D) = n-1$. Since $w_{\mathrm{Kh}}(L) = n+1$, it follows that $g_T(L) = n-1$.

Let $n < 0$. Using the forms in the proof of Proposition 4.9 and the reductions of Lemma 4.2 and Corollary 3.10, one sees that $g_T(L) \leq g(\Sigma_{D'})$ where D' is the closure of $(\sigma_1\sigma_2)^{n+1}$. A straightforward calculation shows that $g(\Sigma_{D'}) = |n|$. Since $w_{\mathrm{Kh}}(L) = |n|+2$, it follows that $g_T(L) = |n|$. □

The previous results of this section are summarized in the following corollary.

Corollary 4.18. *Let L be a closed 3-braid. Then*

$$0 \leq g_T(L) - (w_{\mathrm{Kh}}(L) - 2) \leq 1.$$

Remark 4.19. Both the lower bound and upper bound of the above inequality are achieved by closed 3-braids. For example, the links in Proposition 4.17 achieve the lower bound while the links in Proposition 4.16 part (4) achieve the upper bound. There are also closed 3-braids (see Proposition 4.15) where it is unknown whether the lower bound or upper bound is achieved.

5. Applications to odd Khovanov homology

In [19], Ozsváth, Rasmussen and Szabó introduced odd Khovanov homology, a knot homology that is closely related to Khovanov homology. Odd Khovanov homology, denoted $\mathrm{Kh}_{\mathrm{odd}}(L)$, is a bigraded $\mathbb{Z}$-module whose graded Euler characteristic is the unnormalized Jones polynomial.

5.1. A spanning tree model for odd Khovanov homology. Champanerkar and Kofman [7] and independently Wehrli [28] developed a spanning tree model for Khovanov homology. In this subsection, we show that the similarities between Khovanov homology and odd Khovanov homology imply that odd Khovanov homology also has a spanning tree model.

Let D be a link diagram and let $\mathcal{X}$ be the set of crossings of D. Suppose $(C(D), \partial)$ is the hypercube of resolutions complex from [13] and [5] that generates Khovanov homology, and suppose $(C_{\mathrm{odd}}(D), \partial_{\mathrm{odd}})$ is the hypercube of resolutions complex from [19] that generates odd Khovanov homology. A vertex in the hypercube is a function $I : \mathcal{X} \to \{0,1\}$. For each vertex I, one obtains a one-manifold D_I be smoothing each crossing of D according to I. Both chain complexes are constructed by associating certain $\mathbb{Z}$-modules to each of the one-manifolds D_I.

Number the crossings of D from 1 to $|\mathcal{X}|$ arbitrarily. One can obtain the vertices of the hypercube as the leaves of a binary tree. The root of this tree is the diagram D. The children of a vertex v at level i are obtained by smoothing the ith crossing of v into either a 0-resolution or a 1-resolution. See Figure 13.

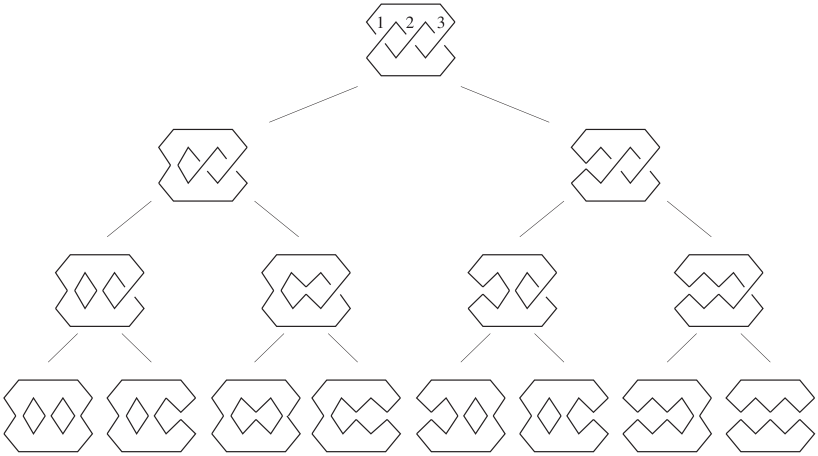

Figure 13. Binary tree whose leaves are the vertices of the hypercube.

Modify the binary tree as follows. If either of the children of a vertex v is disconnected, then the vertex v becomes a leaf and all its descendants are deleted. See Figure 14. The leaves of the modified binary trees are *twisted unknots*, i.e. they are unknots that can be trivialized using only Reidemeister I moves. Also, the leaves are in one-to-one correspondence with the spanning trees of either checkerboard graph associated to D. The details of this correspondence are described in Champarnerkar–Kofman [7] and Wehrli [28]. Denote the set of spanning trees by $\mathcal{T}(D)$, and the diagram associated to a tree $T \in \mathcal{T}(D)$ by D_T.

Let U denote diagram of the unknot with no crossings. The Khovanov complex of the disjoint union of k copies of U is given by

$$C(U^k) = \mathcal{A}^{\otimes k},$$

where $\mathcal{A}$ is the bigraded module defined by $\mathcal{A}^{0,-1} = \mathcal{A}^{0,1} = \mathbb{Z}$ and $\mathcal{A}^{i,j} = 0$ for $(i,j) \neq (0, \pm 1)$. For any bigraded object M, define grading shifts $[m]$ and $\{n\}$ by $(M[n]\{n\})^{i,j} := M^{i-m,j-n}$.

In [28], Wehrli gives the following *spanning tree model* for Khovanov homology. Champanerkar and Kofman prove an analogous result in [7].

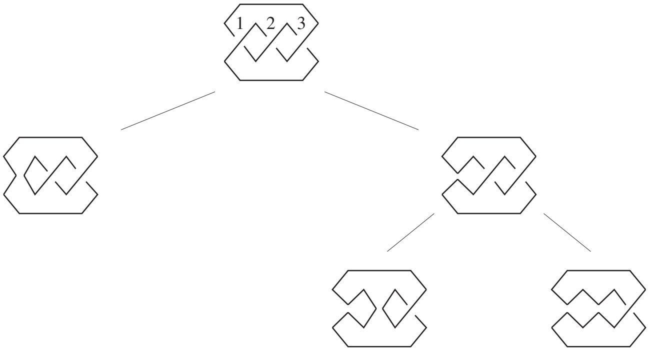

Figure 14. The modified binary tree whose leaves are in correspondence to spanning trees of the checkerboard graph of D.

Proposition 5.1 (Wehrli). *Let D be a connected link diagram. Then there is a decomposition $C(D) = A \oplus B$, where B is contractible and A as a module is given by*

$$A = \bigoplus_{T \in \mathcal{T}(D)} \mathcal{A}[f(D, D_T)]\{g(D, D_T)\},$$

for functions f and g depending on D and D_T.

Let D, D_0 and D_1 be as in Figure 2. The spanning tree model for Khovanov homology is a consequence of

(1) the bigraded $\mathbb{Z}$-module structure of $C(D)$,

(2) the fact that $C(D)$ is isomorphic to the mapping cone of $w : C(D_0) \to C(D_1)$, for some map w, and

(3) the structure of the complex under Reidemeister I moves, which is specified by

$$C(\diagup\!\!\circ) \cong C(\,)\,)\{-1\} \oplus B_1, \quad C(\diagdown\!\!\circ) \cong C(\,)\,)[1]\{2\} \oplus B_2,$$

for contractible complexes B_1 and B_2.

As bigraded $\mathbb{Z}$-modules $C(D)$ and $C_{\text{odd}}(D)$ are isomorphic. Furthermore, from the proof of invariance under Reidemeister I moves in [19], one can see that (2) and (3) also hold for odd Khovanov homology. Therefore odd Khovanov homology also has a spanning tree model.

Proposition 5.2. *Let D be a connected link diagram. Then there is a decomposition $C_{\text{odd}}(D) = A \oplus B$, where B is contractible and A as a module is given by*

$$A = \bigoplus_{T \in \mathcal{T}(D)} \mathcal{A}[f(D, D_T)]\{g(D, D_T)\},$$

for functions f and g, which are the same as in Proposition 5.1.

Proposition 3.8 is a consequence of the bigraded $\mathbb{Z}$-module structure of the spanning tree complex for Khovanov homology. Since odd Khovanov homology has a spanning tree complex with the same bigraded $\mathbb{Z}$-module structure, there is an analogous Turaev genus bound on the odd Khovanov width of a link L, denoted $w_{\mathrm{Kh}_{\mathrm{odd}}}(L)$.

Proposition 5.3. *Let L be a link. Then*

$$w_{\mathrm{Kh}_{\mathrm{odd}}}(L) - 2 \leq g_T(L).$$

5.2. The odd Khovanov width of closed 3-braids. There is a close relationship between Khovanov homology and odd Khovanov homology. Ozsváth, Rasmussen, and Szabó [19] have shown that

$$\mathrm{Kh}(L; \mathbb{Z}_2) \cong \mathrm{Kh}_{\mathrm{odd}}(L; \mathbb{Z}_2),$$

and that odd Khovanov homology satisfies long exact sequences identical to the sequences in Theorem 2.4. These similarities, along with the Turaev genus bound given in Proposition 5.3, imply the following result.

Theorem 5.4. *Let L be a closed* 3*-braid. Then* $\mathrm{Kh}(L)$ *is* $[\delta_{\min}, \delta_{\max}]$*-thick if and only if* $\mathrm{Kh}_{\mathrm{odd}}(L)$ *is* $[\delta_{\min}, \delta_{\max}]$*-thick.*

Proof. Let L' be a link that is the base case for one of the inductions in Propositions 4.6 through 4.9. Then L' is either a $(3, q)$ torus link or the link $L(6, n, 1)$, and

$$w_{\mathrm{Kh}}(L'; \mathbb{Q}) = w_{\mathrm{Kh}_{\mathrm{odd}}}(L') = g_T(L') + 2. \tag{5.1}$$

Suppose $\mathrm{Kh}^{\delta}(L'; \mathbb{Q})$ is nontrivial. Then $\mathrm{Kh}^{\delta}(L'; \mathbb{Z}_2)$ is also nontrivial. Since $\mathrm{Kh}(L; \mathbb{Z}_2) \cong \mathrm{Kh}_{\mathrm{odd}}(L; \mathbb{Z}_2)$, it follows that $\mathrm{Kh}^{\delta}_{\mathrm{odd}}(L'; \mathbb{Z}_2)$ is nontrivial. Therefore, $\mathrm{Kh}^{\delta}_{\mathrm{odd}}(L')$ is nontrivial. Then Equation 5.1 implies

$$g_T(L') + 2 = w_{\mathrm{Kh}}(L'; \mathbb{Q}) \leq w_{\mathrm{Kh}_{\mathrm{odd}}}(L') \leq g_T(L') + 2.$$

Thus $\mathrm{Kh}(L')$ is $[\delta'_{\min}, \delta'_{\max}]$-thick if and only if $\mathrm{Kh}'(L')$ is $[\delta'_{\min}, \delta'_{\max}]$-thick.

The proofs of Propositions 4.6 through 4.9 rely only on the Khovanov homology of the base case and the long exact sequences of Theorem 2.4. Therefore, Propositions 4.6 through 4.9 hold for odd Khovanov homology, and this implies the result. □

Corollary 5.5. *Let L be a closed* 3*-braid. Then*

$$w_{\mathrm{Kh}}(L) = w_{\mathrm{Kh}_{\mathrm{odd}}}(L).$$

Note that Corollary 5.5 is not true for the closure of n-braids where $n > 3$. The examples from Remark 4.14 have $w_{\mathrm{Kh}_{\mathrm{odd}}}(L) > 2$. These examples can be used to generate infinite families of examples of closed n-braids where odd Khovanov width and Khovanov width are different for $n > 3$.

References

[1] T. Abe and K. Kishimoto, The dealternating number and the alternation number of a closed 3-braid. *J. Knot Theory Ramifications* **19** (2010), no. 9, 1157–1181. Zbl 1209.57005 MR 2726563 691

[2] M. Asaeda and J. Przytycki, Khovanov homology: torsion and thickness. In *Advances in topological quantum field theory*, NATO Sci. Ser. II Math. Phys. Chem. 179, Kluwer Acad. Publishers, Dordrecht 2004, 135–166. Zbl 1088.57012 MR 2147419 677

[3] J. Birman and W. Menasco, Studying links via closed braids III: classifying links which are closed 3-braids. *Pacific J. Math.* **161** (1993), 25–115. Zbl 0813.57010 MR 1237139 689

[4] J. Baldwin, Heegaard Floer homology and genus one, one boundary component open books. *J. Topology* **1** (2008), no. 4, 962–992. Zbl 1160.57009 MR 2461862 697, 699, 700

[5] D. Bar Natan, On Khovanov's categorification of the Jones polynomial. *Algebr. Geom. Topol.* **2** (2002), 337–370. Zbl 0998.57016 MR 1917056 701

[6] J. C. Cha and C. Livingston, KnotInfo: Table of knot invariants. http://www.indiana.edu/ knotinfo 693

[7] A. Champanerkar and I. Kofman, Spanning trees and Khovanov homology. *Proc. Amer. Math. Soc.* **137** (2009), no. 6, 2157–2167. Zbl 1183.57011 MR 2480298 701, 702

[8] A. Champanerkar and I. Kofman, Twisting quasi-alternating links. *Proc. Amer. Math. Soc.* **137** (2009), no. 7, 2451–2458. Zbl MR 680, 688

[9] A. Champanerkar, I. Kofman, and N. Stoltzfus, Graphs on surfaces and Khovanov homology. *Algebr. Geom. Topol.* **7** (2007), 1531–1540. Zbl 1169.57004 MR 2366169 688

[10] O. T. Dasbach, D. Futer, E. Kalfagianni, X. Lin, and N. Stoltzfus, The Jones polynomial and graphs on surfaces. *J. Combin. Theory Ser. B* **98** (2008), no. 2, 384–399. Zbl 1135.05015 MR 2389605 688

[11] F. Garside, The braid groups and other groups. *Quart. J. Math. Oxford Ser.* (2) **20** (1969), 235–254. Zbl 0194.03303 MR 0248801 689

[12] C. Gordon and R Litherland, On the signature of a link. *Invent. Math.* **47** (1978) 53–69. Zbl 0391.57004 MR 0500905 694

[13] M. Khovanov, A categorification of the Jones polynomial. *Duke Math J.* **101** (2000), no. 3, 359–426. Zbl 0960.57005 MR 1740682 675, 676, 678, 701

[14] M. Khovanov, Patterns in knot cohomology I. *Experiment. Math.* **12** (2003), no. 3, 365–374. Zbl 1073.57007 MR 2034399 677

[15] E. S. Lee, The support of Khovanov's invariants for alternating knots. Preprint 2002. arXiv:math.GT/0201105 [math.GT]. 679, 693

[16] C. Manolescu and P. Ozsváth, On the Khovanov and knot Floer homologies of quasi-alternating links. In *Proceedings of Gökova Geometry-Topology Conference 2007*, Gökova Geometry/Topology Conference (GGT), Gökova 2008, 60–81. Zbl 05663535 MR 2509750 678, 679

[17] V. Manturov, Minimal diagrams of classical and virtual links. Preprint 2005. arXiv:math.GT/0501393v1 [math.GT]. 688

[18] K. Murasugi, On closed 3-braids. *Mem. Amer. Math. Soc.* **151**, Amer. Math. Soc., Providence, RI, 1974. Zbl 0327.55001 MR 0356023 689, 692

[19] P. Ozsváth, J. Rasmussen, and S. Szabó, Odd Khovanov homology. Preprint 2007. arXiv:0710.4300v1 [math.QA]. 701, 703, 704

[20] J. Rasmussen, Knot polynomials and knot homologies. In *Geometry and topology of manifolds*, Fields Inst. Commun. 47, Amer. Math. Soc., Providence, RI, 2005, 261–280. Zbl 1095.57016 MR 2189938 678

[21] O. Schreier, Über die Gruppen $A^a B^b = 1$. *Abh. Math. Sem. Univ. Hamburg* **3** (1924), 167–169. Zbl 50.0070.01 MR 689

[22] A. Shumakovitch, KhoHo pari package. 699

[23] M. Stošić, Khovanov homology of torus links. *Topology Appl.* **156** (2009), no. 3, 533–541. Zbl 1169.57014 MR 2492301 689

[24] V. G. Turaev, A simple proof of the Murasugi and Kauffman theorems on alternating links. *Enseign. Math.* (2) **33** (1987), no. 3–4, 203–225. Zbl 0668.57009 MR 0925987 687

[25] P. Turner, A spectral sequence for Khovanov homology with an application to $(3, q)$-torus links. *Algebr. Geom. Topol.* **8** (2008), no. 2, 869–884. Zbl 1158.57016 MR 2443099 689, 693

[26] O. Viro, Khovanov homology, its definitions and ramifications. *Fund. Math.* **184** (2004), 317–342. Zbl 1078.57013 MR 2128056 678

[27] L. Watson, Surgery obstructions from Khovanov homology. Preprint 2009. arXiv:0807.1341v3 [math.GT]. 685

[28] S. Wehrli, A spanning tree model for Khovanov homology. *J. Knot Theory Ramifications* **17** (2008), no. 12, 1561–1574. Zbl 1167.57006 MR 2477595 701, 702

Received March 20, 2009

Adam Lowrance, Department of Mathematics, University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, U.S.A.

E-mail: alowrance@math.uiowa.edu