A rigidity property of some negatively curved solvable Lie groups

Nageswari Shanmugalingam and Xiangdong Xie

Abstract. We show that for some negatively curved solvable Lie groups, all self quasiisometries are almost isometries. We prove this by showing that all self quasisymmetric maps of the ideal boundary (of the solvable Lie groups) are biLipschitz with respect to the visual metric.

1. Introduction

In recent years, there have been a lot of interest in the large scale geometry of solvable Lie groups and finitely generated solvable groups ([D], [EFW1], [EFW2], [FM1], [FM2], [FM3], [Pe]). In particular, Eskin, Fisher and Whyte ([EFW1], [EFW2]) proved the quasiisometric rigidity of the 3-dimensional solvable Lie group Sol. In this paper, we use quasiconformal analysis to prove a rigidity property of some negatively curved solvable Lie groups.

Let A be an $n \times n$ diagonal matrix with real eigenvalues α_i with $\alpha_{i+1} > \alpha_i > 0$:

$$A = \begin{pmatrix} \alpha_1 I_{n_1} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \alpha_2 I_{n_2} & \ldots & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \ldots & \alpha_r I_{n_r} \end{pmatrix},$$

where I_{n_i} is the $n_i \times n_i$ identity matrix and the $\mathbf{0}$'s are zero matrices (of various sizes). Let $\mathbb{R}$ act on $\mathbb{R}^n$ by the linear transformations e^{tA} ($t \in \mathbb{R}$) and we can form the semidirect product $G_A = \mathbb{R}^n \rtimes_A \mathbb{R}$. That is, $G_A = \mathbb{R}^n \times \mathbb{R}$ as a smooth manifold, and the group operation is given for all $(x, t), (y, s) \in \mathbb{R}^n \times \mathbb{R}$ by

$$(x, t) \cdot (y, s) = (x + e^{tA}y, t + s).$$

The group G_A is a simply connected solvable Lie group and is the subject of study in this paper.

We endow G_A with the left invariant metric determined by taking the standard Euclidean metric at the identity of $G_A \approx \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. With this metric G_A has sectional curvature $-\alpha_r^2 \le K \le -\alpha_1^2$ (and so is Gromov hyperbolic). Hence G_A has a well defined ideal boundary ∂G_A. There is a so-called cone topology on $\overline{G_A} = G_A \cup \partial G_A$, in which ∂G_A is homeomorphic to the n-dimensional sphere and $\overline{G_A}$ is homeomorphic to the closed $(n + 1)$-ball in the Euclidean space. For each $x \in \mathbb{R}^n$, the map $\gamma_x \colon \mathbb{R} \to G_A, \gamma_x(t) = (x, t)$ is a geodesic. We call such a geodesic a vertical geodesic. It can be checked that all vertical geodesics are asymptotic as $t \to +\infty$. Hence they define a point ξ_0 in the ideal boundary ∂G_A.

Since G_A is Gromov hyperbolic, there is a family of visual metrics on ∂G_A. For each $\xi \in \partial G_A$, there is also the so-called parabolic visual metric on $\partial G_A \setminus \{\xi\}$. The relation between visual metrics and parabolic visual metrics is analogous to the relation between spherical metric (on the sphere) and the Euclidean metric (on the one point complement of the sphere). We next recall the parabolic visual metric D on ∂G_A viewed from ξ_0.

The set $\partial G_A \setminus \{\xi_0\}$ can be naturally identified with $\mathbb{R}^n$ (see Section 2). Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$, where $\mathbb{R}^{n_i}$ is the eigenspace associated to the eigenvalue α_i of A. Each point $x \in \mathbb{R}^n$ can be written as $x = (x_1, \ldots, x_r)$ with $x_i \in \mathbb{R}^{n_i}$. The parabolic visual metric D on $\partial G_A \setminus \{\xi_0\} \approx \mathbb{R}^n$ is defined by

$$D(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|^{\alpha_1/\alpha_2}, \ldots, |x_r - y_r|^{\alpha_1/\alpha_r}\},$$

for all $x = (x_1, \ldots, x_r), y = (y_1, \ldots, y_r) \in \mathbb{R}^n$.

Let $\eta \colon [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding of metric spaces $f \colon X \to Y$ is an η-*quasisymmetric embedding* if for all distinct triples $x, y, z \in X$, we have

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \le \eta\left(\frac{d(x, y)}{d(x, z)}\right).$$

If f is further assumed to be a homeomorphism, we say it is η-*quasisymmetric*. A map $f \colon X \to Y$ is quasisymmetric if it is η-quasisymmetric for some η.

When $r \ge 2$, Bruce Kleiner has proved that ([K]) every self quasisymmetry of ∂G_A (equipped with a visual metric) preserves the horizontal foliation (see Section 3) and fixes the point ξ_0. This is one of the main ingredients in the proof of our main result. Since Kleiner's proof is unpublished, we include a proof here for completeness. Notice that Kleiner's result implies that a self quasisymmetry of ∂G_A induces a self map of $(\mathbb{R}^n, D)$.

The following is the main result of this paper.

Theorem 1.1. *Let G_A and $\xi_0 \in \partial G_A$ be as above. If $r \ge 2$, then every self quasisymmetry of ∂G_A (equipped with a visual metric) is biLipschitz on $\partial G_A \setminus \{\xi_0\}$ with respect to the parabolic visual metric D.*

One should compare this with quasiconformal maps on Euclidean spaces ([GV]) and Heisenberg groups ([B]), where there are non-biLipschitz quasiconformal maps. On the other hand, the conclusion of Theorem 1.1 is not as strong as in the cases of quaternionic hyperbolic spaces, Cayley plane ([P]) and Fuchsian buildings ([BP], [X]), where every quasisymmetric map of the ideal boundary is actually a conformal map. In our case, there are many non-conformal quasisymmetric maps of the ideal boundary of G_A. We also remark that in Section 15 of [T2] Tyson has previously classified (quasi)metric spaces of the form $(\mathbb{R}^n, D)$ up to quasisymmetry.

We list three consequences of Theorem 1.1.

Let $L \geq 1$ and $C \geq 0$. A (not necessarily continuous) map $f : X \to Y$ between two metric spaces is an (L, A)-*quasiisometry* if

(1) $d(x_1, x_2)/L - C \leq d(f(x_1), f(x_2)) \leq L\, d(x_1, x_2) + C$ for all $x_1, x_2 \in X$;

(2) for any $y \in Y$, there is some $x \in X$ with $d(f(x), y) \leq C$.

In the case $L = 1$, we call f an *almost isometry*.

Corollary 1.2. *Assume that $r \geq 2$. Then every self quasiisometry of G_A is an almost isometry.*

Notice that an almost isometry is not necessarily a finite distance away from an isometry.

The following result was previously obtained by B. Kleiner [K].

Corollary 1.3. *If $r \geq 2$, then G_A is not quasiisometric to any finitely generated group.*

In the identification of G_A with $\mathbb{R}^n \times \mathbb{R}$, we view the map $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $h(x, t) = t$ as the height function. A quasiisometry φ of G_A is height-respecting if $|h(\varphi(x, t)) - t|$ is bounded independent of x, t.

Corollary 1.4. *Assume that $r \geq 2$. Then all self quasiisometries of G_A are height-respecting.*

The question of whether a quasiisometry of G_A is height-respecting is important for the following three reasons. First, Farb and Mosher ([FM1]) have classified a large class of solvable Lie groups (including groups of type G_A) up to height-respecting quasiisometries. Second, there is no known examples of non-height-respecting quasiisometries except for rank one symmetric spaces of noncompact type. Finally, showing a quasiisometry is height-respecting is a key step in the proof of the quasiisometric rigidity of Sol ([EFW1], [EFW2]).

When $r = 1$, the group G_A is isometric to a rescaling of the real hyperbolic space. In this case, all the above results fail.

This paper is structured as follows. In Section 2 we review some basics about the group G_A. In Section 3 we prove that quasisymmetric self-maps of $\partial G_A \setminus \{\xi_0\}$ equipped with the parabolic visual metric preserve horizontal foliations, and in Section 4 we will prove that such maps are biLipschitz with respect to this metric. In Section 5 we provide the proofs of Theorem 1.1 and the corollaries stated in Section 1.

2. The solvable Lie groups G_A

In this section we review some basic facts about the group G_A and define several parabolic visual (quasi)metrics on the ideal boundary.

Let A and G_A be as in the Introduction. We endow G_A with the left invariant metric determined by taking the standard Euclidean metric at the identity of $G_A \approx \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. At a point $(x, t) \in \mathbb{R}^n \times \mathbb{R} \approx G_A$, the tangent space is identified with $\mathbb{R}^n \times \mathbb{R}$, and the Riemannian metric is given by the symmetric matrix

$$\begin{pmatrix} e^{-2tA} & 0 \\ 0 & 1 \end{pmatrix}.$$

With this metric G_A has sectional curvature $-\alpha_r^2 \leq K \leq -\alpha_1^2$. Hence G_A has a well defined ideal boundary ∂G_A. All vertical geodesics γ_x ($x \in \mathbb{R}^n$) are asymptotic as $t \to +\infty$. Hence they define a point ξ_0 in the ideal boundary ∂G_A.

The sets $\mathbb{R}^n \times \{t\}$ ($t \in \mathbb{R}$) are horospheres centered at ξ_0. For each $t \in \mathbb{R}$, the induced metric on the horosphere $\mathbb{R}^n \times \{t\} \subset G_A$ is determined by the quadratic form e^{-2tA}. This metric has distance formula $d_{\mathbb{R}^n \times \{t\}}((x, t), (y, t)) = |e^{-tA}(x - y)|$. Here $|\cdot|$ denotes the Euclidean norm. The distance between two horospheres, corresponding to $t = t_1$ and $t = t_2$, is $|t_1 - t_2|$. It follows that for $(x_1, t_1), (x_2, t_2) \in G_A = \mathbb{R}^n \times \mathbb{R}$,

$$d((x_1, t_1), (x_2, t_2)) \geq |t_1 - t_2|. \tag{2.1}$$

Each geodesic ray in G_A is asymptotic to either an upward oriented vertical geodesic or a downward oriented vertical geodesic. The upward oriented geodesics are asymptotic to ξ_0 and the downward oriented vertical geodesics are in 1-to-1 correspondence with $\mathbb{R}^n$. Hence $\partial G_A \setminus \{\xi_0\}$ can be naturally identified with $\mathbb{R}^n$.

Recall that a quasimetric on a set A is a function $\rho \colon A \times A \to [0, \infty)$ satisfying: (1) $\rho(x, y) = \rho(y, x)$ for all $x, y \in A$; (2) $\rho(x, y) = 0$ only when $x = y$; (3) there is a constant $L \geq 1$ such that $\rho(x, z) \leq L(\rho(x, y) + \rho(y, z))$ for all $x, y, z \in A$.

Given $x, y \in \mathbb{R}^n \approx \partial G_A \setminus \{\xi_0\}$, the parabolic visual quasimetric $D_e(x, y)$ is defined as follows: $D_e(x, y) = e^t$, where t is the unique real number such that at height t the two vertical geodesics γ_x and γ_y are at distance one apart in the horosphere; that is, $d_{\mathbb{R}^n \times \{t\}}((x, t), (y, t)) = |e^{-tA}(x - y)| = 1$. Here the subscript e

in D_e means it corresponds to the Euclidean norm. This definition of parabolic visual quasimetric is very natural, but D_e does not have a simple formula. Next we describe another parabolic visual quasimetric which is biLipschitz equivalent with D_e and admits a simple formula.

In addition to the Euclidean norm, there is another norm on $\mathbb{R}^n$ that is naturally associated to G_A. Write $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$, where $\mathbb{R}^{n_i}$ is the eigenspace associated to the eigenvalue α_i of A. Each point $x \in \mathbb{R}^n$ can be written as $x = (x_1, \ldots, x_r)$ with $x_i \in \mathbb{R}^{n_i}$. The block supernorm is given by $|x|_s = \max\{|x_1|, \ldots, |x_r|\}$ for $x = (x_1, \ldots, x_r)$. Using this norm one can define another parabolic visual quasimetric on $\partial G_A \backslash \{\xi_0\}$ as follows: $D_s(x, y) = e^t$, where t is the unique real number such that at height t the two vertical geodesics γ_x and γ_y are at distance one apart with respect to the norm $|\cdot|_s$; that is, $|e^{-tA}(x - y)|_s = 1$. Here the subscript s in D_s means it corresponds to the block supernorm $|\cdot|_s$. Then D_s is given by Lemma 7 of [D]:

$$D_s(x, y) = \max\{|x_1 - y_1|^{\frac{1}{\alpha_1}}, \ldots, |x_r - y_r|^{\frac{1}{\alpha_r}}\},$$

for all $x = (x_1, \ldots, x_r)$, $y = (y_1, \ldots, y_r) \in \mathbb{R}^n$.

Notice that $|x|_s \leq |x| \leq \sqrt{r}\,|x|_s$ for all $x \in \mathbb{R}^n$. Using this, one can verify the following lemma, whose proof is left to the reader.

Lemma 2.1. *For all $x, y \in \mathbb{R}^n$ we have $D_s(x, y) \leq D_e(x, y) \leq r^{1/2\alpha_1} D_s(x, y)$.*

In general, D_s does not satisfy the triangle inequality. However, for each $0 < \epsilon \leq \alpha_1$, the function D_s^ϵ is always a metric, called a parabolic visual metric. In this paper we consider the following parabolic visual metric

$$D(x, y) = D_s^{\alpha_1}(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|^{\alpha_1/\alpha_2}, \ldots, |x_r - y_r|^{\alpha_1/\alpha_r}\}.$$

With respect to this metric the rectifiable curves in $\mathbb{R}^n \approx \partial G_A \setminus \{\xi_0\}$ are necessarily curves of the form $\gamma: I \to \mathbb{R}^n$ with $\gamma(t) = (\gamma_1(t), c_2, \ldots, c_r)$ where $c_i \in \mathbb{R}^{n_i}$, $2 \leq i \leq r$, are constant vectors. This follows from the fact that the directions corresponding to $\mathbb{R}^{n_i}$, $i \geq 2$, have their Euclidean distance components "snowflaked" by the power $\alpha_1/\alpha_i < 1$.

3. Quasisymmetric maps preserve horizontal foliations

In this section we show that every self-quasisymmetry of $(\mathbb{R}^n, D)$ preserves a natural foliation on $\mathbb{R}^n$. We remark that this fact has been observed previously by both B. Kleiner [K] and J. Tyson (Section 15, [T2]). We include a proof here for completeness.

Recall that a metric space X endowed with a Borel measure μ is an *Ahlfors regular space of dimension Q* (for short, a Q-regular space) if there exists a constant $C_0 \geq 1$

so that

$$C_0^{-1} r^Q \le \mu(B_r) \le C_0 r^Q$$

for every ball B_r with radius $r < \operatorname{diam}(X)$.

We need the following result; see [T1] for the definition of the modulus Mod_Q of a family of curves.

Theorem 3.1 ([T1], Theorem 1.4). *Let X and Y be locally compact, connected, Q-regular metric spaces $(Q > 1)$ and let $f : X \to Y$ be an η-quasisymmetric home-omorphism. Then there is a constant C depending only on η, Q and the regularity constants of X and Y so that*

$$\frac{1}{C} \operatorname{Mod}_Q \Gamma \le \operatorname{Mod}_Q f(\Gamma) \le C \operatorname{Mod}_Q \Gamma$$

for every curve family Γ in X.

Recall that we write $\mathbb{R}^n$ as $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$. Set $Y = \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ and write $\mathbb{R}^n = \mathbb{R}^{n_1} \times Y$. Since we assume $r \ge 2$, the set Y is nontrivial. The subsets $\{\mathbb{R}^{n_1} \times \{y\} : y \in Y\}$ form a foliation of $\mathbb{R}^n$. We call this foliation the horizontal foliation and each leaf $\mathbb{R}^{n_1} \times \{y\}$ a horizontal leaf. Since $\frac{\alpha_1}{\alpha_i} < 1$ for all $2 \le i \le r$, we notice that a curve in $(\mathbb{R}^n, D)$ is not rectifiable if it is not contained in a horizontal leaf.

Observe that $(\mathbb{R}^{n_i}, |\cdot|^{\alpha_1/\alpha_i})$ with the Hausdorff measure (which is comparable to the n_i-dimensional Lebesgue measure) is $n_i \alpha_i / \alpha_1$-regular. Let μ be the product of the Hausdorff measures on the factors $(\mathbb{R}^{n_i}, |\cdot|^{\alpha_1/\alpha_i})$. Then it is easy to see that $(\mathbb{R}^n, D)$ with the measure μ is Q-regular with $Q = \Sigma_{i=1}^r n_i \frac{\alpha_i}{\alpha_1}$. It follows that Theorem 3.1 applies to the metric space $(\mathbb{R}^n, D)$. We also point out here that the Hausdorff measure μ is comparable to the canonical n-dimensional Lebesgue measure on $\mathbb{R}^n$.

Theorem 3.2. *If $r \ge 2$, then every quasisymmetry $F : (\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$ preserves the horizontal foliation on $\mathbb{R}^n$.*

Proof. Suppose F does not preserve the horizontal foliation. Then there are two points p and q in some $\mathbb{R}^{n_1} \times \{y\}$ such that $f(p)$ and $f(q)$ are not in the same horizontal leaf. Let γ be the Euclidean line segment from p to q and Γ be the family of straight segments parallel to γ in $\mathbb{R}^n$ whose union is an n-dimensional circular cylinder with γ as the central axis. The curves in Γ are rectifiable with respect to the metric D. Since f is a homeomorphism, by choosing the radius of the circular cylinder to be sufficiently small (by a compactness argument) we may assume that no curve in Γ is mapped into a horizontal leaf. It follows that $f(\Gamma)$ has no locally rectifiable curve and so $\operatorname{Mod}_Q f(\Gamma) = 0$. On the other hand, [V1], 7.2 (page 21), shows that $\operatorname{Mod}_Q \Gamma > 0$ (the Euclidean length element on each $\beta \in \Gamma$ is the same

as the length element on β obtained from the metric D). Since $Q = \Sigma_{i=1}^{r} n_i \frac{\alpha_i}{\alpha_1} > 1$, this contradicts Theorem 3.1. Hence each horizontal leaf is mapped to a horizontal leaf. $\qquad\square$

4. Quasisymmetry implies biLipschitz

In this section we show that each self quasisymmetry of $(\mathbb{R}^n, D)$ is actually a biLipschitz map. One should contrast this with the case of Euclidean spaces and Heisenberg groups, where there are non-biLipschitz quasisymmetric maps ([GV], [B]). On the other hand, $(\mathbb{R}^n, D)$ is not as rigid as the ideal boundary of a quaternionic hyperbolic space or a Cayley plane ([P]) or a Fuchsian building ([BP], [X]), where each self quasisymmetry is a conformal map.

Let $K \geq 1$ and $C > 0$. A bijection $F: X_1 \to X_2$ between two metric spaces is called a K-quasisimilarity (with constant C) if

$$\frac{C}{K} d(x, y) \leq d(F(x), F(y)) \leq C K d(x, y)$$

for all $x, y \in X_1$. It is clear that a map is a quasisimilarity if and only if it is a biLipschitz map. The point of using the notion of quasisimilarity is that sometimes there is control on K but not on C.

Theorem 4.1. *Let $F: (\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$ be an η-quasisymmetry. Then F is a K-quasisimilarity with $K = (\eta(1)/\eta^{-1}(1))^{2r+2}$.*

In this section, we first develop some intermediate results, and then use these results to provide a proof of this theorem. We first recall some definitions.

Let $g: X_1 \to X_2$ be a homeomorphism between two metric spaces. We define for every $x \in X_1$ and $r > 0$,

$$L_g(x, r) = \sup\{d(g(x), g(x')) : d(x, x') \leq r\},$$
$$l_g(x, r) = \inf\{d(g(x), g(x')) : d(x, x') \geq r\}.$$

Notice that if X_1 is connected and $X_1 \setminus B(x, r)$ is non-empty, then $l_g(x, r) \leq L_g(x, r)$. In this paper, we only consider connected metric spaces. Set

$$L_g(x) = \limsup_{r \to 0} \frac{L_g(x, r)}{r}, \quad l_g(x) = \liminf_{r \to 0} \frac{l_g(x, r)}{r}.$$

Then

$$L_{g^{-1}}(g(x)) = \frac{1}{l_g(x)} \quad \text{and} \quad l_{g^{-1}}(g(x)) = \frac{1}{L_g(x)}$$

for any $x \in X_1$. If g is an η-quasisymmetry, then $L_g(x, r) \leq \eta(1) l_g(x, r)$ for all $x \in X_1$ and $r > 0$. Hence if in addition

$$\lim_{r \to 0} \frac{L_g(x, r)}{r} \quad \text{or} \quad \lim_{r \to 0} \frac{l_g(x, r)}{r}$$

exist, then

$$0 \leq l_g(x) \leq L_g(x) \leq \eta(1) l_g(x) \leq \infty.$$

Recall the decomposition $\mathbb{R}^n = \mathbb{R}^{n_1} \times Y$. Given points $y = (x_2, \ldots, x_r)$ and $y' = (x'_2, \ldots, x'_r) \in Y$ with $x_i, x'_i \in \mathbb{R}^{n_i}$, set

$$D_Y(y, y') = \max\{|x_2 - x'_2|^{\frac{\alpha_1}{\alpha_2}}, |x_3 - x'_3|^{\frac{\alpha_1}{\alpha_3}}, \ldots, |x_r - x'_r|^{\frac{\alpha_1}{\alpha_r}}\}.$$

For $p = (x_1, y)$, $p' = (x'_1, y') \in \mathbb{R}^{n_1} \times Y$, we have $D(p, p') = \max\{|x_1 - x'_1|, D_Y(y, y')\}$. We notice that for every $y_1, y_2 \in Y$, the distance in the metric D of the two horizontal leaves,

$$D(\mathbb{R}^{n_1} \times \{y_1\}, \mathbb{R}^{n_1} \times \{y_2\}) = D_Y(y_1, y_2). \tag{4.1}$$

Also, for any $p = (x_1, y_1) \in \mathbb{R}^{n_1} \times Y$ and any $y_2 \in Y$,

$$D(p, \mathbb{R}^{n_1} \times \{y_2\}) = D_Y(y_1, y_2). \tag{4.2}$$

By Theorem 3.2 the quasisymmetry F preserves the horizontal foliation. Hence it induces a map $G \colon Y \to Y$ such that for any $y \in Y$, $F(\mathbb{R}^{n_1} \times \{y\}) = \mathbb{R}^{n_1} \times \{G(y)\}$. For each $y \in Y$, let $H(\cdot, y) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ be the map such that $F(x, y) = (H(x, y), G(y))$ for all $x \in \mathbb{R}^{n_1}$. Because $F \colon (\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$ is an η-quasisymmetry, it follows that for each fixed $y \in Y$, the map $H(\cdot, y) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ is an η-quasisymmetry with respect to the Euclidean metric on $\mathbb{R}^{n_1}$. The following lemma together with equations (4.1) and (4.2) imply that $G \colon (Y, D_Y) \to (Y, D_Y)$ is also an η-quasisymmetry.

Lemma 4.2 ([T2], Lemma 15.9). *Let $g \colon X_1 \to X_2$ be an η-quasisymmetry and $A, B, C \subset X_1$. If $d(A, B) \leq t\, d(A, C)$ for some $t \geq 0$, then there is some $a \in A$ such that*

$$d(g(A), g(B)) \leq \eta(t) d(g(a), g(C)).$$

We recall that if $g \colon X_1 \to X_2$ is an η-quasisymmetry, then $g^{-1} \colon X_2 \to X_1$ is an η_2-quasisymmetry, where $\eta_2(t) = (\eta^{-1}(t^{-1}))^{-1}$, see [V2], Theorem 6.3. Note that $\eta_2(1) = 1/\eta^{-1}(1)$ and $\eta_2^{-1}(1) = 1/\eta(1)$.

In the proofs of the following lemmas, the quantities $l_G, L_G, l_{G^{-1}}, L_{G^{-1}}$ are defined with respect to the metric D_Y. Similarly, $l_{H(\cdot, y)}, L_{H(\cdot, y)}, l_{I_y}$ and L_{I_y} are defined with respect to the Euclidean metric on $\mathbb{R}^{n_1}$, where $I_y := H(\cdot, y)^{-1} \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$. Lemmas 4.6 and 4.7 together verify Theorem 4.1 for the case $r = 2$. At the end of this section we will use induction to then complete the proof of Theorem 4.1 for the general case $r \geq 2$.

Lemma 4.3. *The following holds for all $y \in Y$ and $x \in \mathbb{R}^{n_1}$:*

(1) $L_G(y, r) \leq \eta(1) \, l_{H(\cdot, y)}(x, r)$ *for $r > 0$;*
(2) $\eta^{-1}(1) \, l_{H(\cdot, y)}(x) \leq l_G(y) \leq \eta(1) \, l_{H(\cdot, y)}(x)$;
(3) $\eta^{-1}(1) \, L_{H(\cdot, y)}(x) \leq L_G(y) \leq \eta(1) \, L_{H(\cdot, y)}(x)$.

Proof. To prove (1), let $y \in Y$, $x \in \mathbb{R}^{n_1}$ and $r > 0$. Let $y' \in Y$ be an arbitrary point with $D_Y(y, y') \leq r$ and $x' \in \mathbb{R}^{n_1}$ an arbitrary point with $|x - x'| \geq r$. Then $D((x, y), (x, y')) \leq r \leq D((x, y), (x', y))$. Since F is η-quasisymmetric, we have

$$D_Y(G(y), G(y')) \leq D(F(x, y), F(x, y')) \leq \eta(1) \, D(F(x, y), F(x', y))$$
$$= \eta(1) \, |H(x, y) - H(x', y)|.$$

Since y' and x' are chosen arbitrarily and are independent of each other, the inequality follows.

Next we prove (2) and (3). Since Y is connected, we have $l_G(y, r) \leq L_G(y, r)$. Now the second inequality of (2) follows from (1). Similarly the second inequality of (3) follows from (1) and the fact that $l_{H(\cdot, y)}(x, r) \leq L_{H(\cdot, y)}(x, r)$.

To prove the first inequalities in (2) and (3), observe that the inverse map

$$F^{-1} \colon (\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$$

is an η_2-quasisymmetry, with

$$F^{-1}(x, y) = (H(\cdot, G^{-1}(y))^{-1}(x), G^{-1}(y)) = (I_{G^{-1}(y)}(x), G^{-1}(y)).$$

Applying the second inequality of (2) proven above to I_y and G^{-1}, we obtain

$$\frac{1}{L_G(y)} = l_{G^{-1}}(G(y)) \leq \eta_2(1) \cdot l_{I_y}(H(x, y)) = \frac{1}{\eta^{-1}(1)} \cdot \frac{1}{L_{H(\cdot, y)}(x)},$$

hence $L_G(y) \geq \eta^{-1}(1) L_{H(\cdot, y)}(x)$, which is the first inequality of (3). Similarly, using the second inequality of (3) we obtain the first inequality of (2). $\square$

When $r = 2$, we have $Y = \mathbb{R}^{n_2}$ and $D_Y = |\cdot|^{\frac{\alpha_1}{\alpha_2}}$.

Lemma 4.4. *Assume that $r = 2$. Then $0 < l_G(y) \leq L_G(y) \leq \eta(1) l_G(y) < \infty$ for a.e. $y \in Y$ with respect to the Lebesgue measure on $Y = \mathbb{R}^{n_2}$.*

Proof. Observe in this case that $D_Y(y, y') = |y - y'|^{\alpha_1 / \alpha_2}$ for $y, y' \in Y = \mathbb{R}^{n_2}$. Because G is an η-quasisymmetry with respect to the metric D_Y, it is η_1-quasisymmetric with respect to the Euclidean metric, where $\eta_1(t) = (\eta(t^{\alpha_1 / \alpha_2}))^{\alpha_2 / \alpha_1}$. Hence the map $G \colon (\mathbb{R}^{n_2}, |\cdot|) \to (\mathbb{R}^{n_2}, |\cdot|)$ is differentiable a.e. with respect to the Lebesgue measure. With L_G^e, l_G^e the distortion quantities of the map G with respect to the Euclidean metric, the differentiability property of G shows that $\lim_{r \to 0} \frac{L_G^e(y, r)}{r}$ and $\lim_{r \to 0} \frac{l_G^e(y, r)}{r}$

exist. Since $L_G(y, r) = L_G^e(y, r^{\alpha_2/\alpha_1})^{\alpha_1/\alpha_2}$ and $l_G(y, r) = l_G^e(y, r^{\alpha_2/\alpha_1})^{\alpha_1/\alpha_2}$, this implies that both $\lim_{r \to 0} \frac{L_G(y,r)}{r}$ and $\lim_{r \to 0} \frac{l_G(y,r)}{r}$ exist for a.e. $y \in Y$. It follows that

$$0 \le l_G(y) \le L_G(y) \le \eta(1) l_G(y) \le \infty.$$

Fix $y \in Y$ such that both $\lim_{r \to 0} \frac{L_G(y,r)}{r}$ and $\lim_{r \to 0} \frac{l_G(y,r)}{r}$ exist. We next prove that $L_G(y) \ne 0, \infty$. Suppose that $L_G(y) = \infty$. Then $l_G(y) = \infty$ and so by Lemma 4.3 (2), $l_{H(\cdot,y)}(x) = \infty$ for all $x \in \mathbb{R}^{n_1}$. Hence $I_y = H(\cdot, y)^{-1} \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ has the property that $L_{I_y}(x) = 0$ for all $x \in \mathbb{R}^{n_1}$. This implies that I_y is a constant map, contradicting the fact that it is a homeomorphism. Similarly we use Lemma 4.3 (3) to show that $L_G(y) \ne 0$. $\qquad\square$

In the next two lemmas we use the fact that $\eta(1) \ge 1$ and $0 < \eta^{-1}(1) \le 1$.

Lemma 4.5. *Suppose that $r = 2$. Then, for a.e. $y \in Y$, the map $H(\cdot, y) \colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ is an $\eta(1)/\eta^{-1}(1)$-quasisimilarity with constant $l_G(y) > 0$.*

Proof. By Lemma 4.3 (2) we have $l_{H(\cdot,y)}(x) \ge l_G(y)/\eta(1)$. Lemma 4.3 (3) and Lemma 4.4 imply that, for a.e. $y \in Y$, we have $l_G(y) > 0$ and

$$L_{H(\cdot,y)}(x) \le L_G(y)/\eta^{-1}(1) \le (\eta(1)/\eta^{-1}(1)) \, l_G(y)$$

for all $x \in \mathbb{R}^{n_1}$. Because $\mathbb{R}^{n_1}$ is a geodesic space, for a.e. $y \in Y$ the map $H(\cdot, y)$ is an $\eta(1)/\eta^{-1}(1)$-quasisimilarity with constant $l_G(y)$. $\qquad\square$

Lemma 4.6. *If $r = 2$, then there exists a constant $C > 0$ with the following properties:*

(1) *For each $y \in Y$, $H(\cdot, y)$ is an $(\eta(1)/\eta^{-1}(1))^4$-quasisimilarity with constant C;*

(2) *$G \colon (Y, D_Y) \to (Y, D_Y)$ is an $(\eta(1)/\eta^{-1}(1))^5$-quasisimilarity with constant C.*

Proof. (1) Fix any $y_0 \in Y$ that satisfies both Lemma 4.4 and Lemma 4.5. Set $C = l_G(y_0)$. Let $y \in Y$ be an arbitrary point satisfying both Lemma 4.4 and Lemma 4.5. Fix $x_0 \in \mathbb{R}^{n_1}$ and choose $x \in \mathbb{R}^{n_1}$ such that $|x - x_0| \ge D_Y(y, y_0)$. Then

$$D((x, y_0), (x_0, y)) = D((x, y), (x_0, y)) = |x - x_0|.$$

By choosing x so that in addition $|H(x, y_0) - H(x_0, y)| > D_Y(G(y_0), G(y))$, by the η-quasisymmetry of F we have

$$\begin{aligned}
|H(x, y_0) - H(x_0, y)| &= D(F(x, y_0), F(x_0, y)) \\
&\le \eta(1) D(F(x, y), F(x_0, y)) = \eta(1) |H(x, y) - H(x_0, y)|.
\end{aligned}$$

By the choice of y and Lemma 4.5, we have

$$|H(x, y) - H(x_0, y)| \le (\eta(1)/\eta^{-1}(1)) l_G(y) |x - x_0|.$$

On the other hand,

$$|H(x, y_0) - H(x_0, y)| \geq |H(x, y_0) - H(x_0, y_0)| - |H(x_0, y_0) - H(x_0, y)|$$
$$\geq \frac{l_G(y_0)}{\eta(1)/\eta^{-1}(1)}|x - x_0| - |H(x_0, y_0) - H(x_0, y)|.$$

Combining the above inequalities and letting $|x - x_0| \to \infty$, we obtain

$$l_G(y) \geq \frac{1}{(\eta(1))^3(\eta^{-1}(1))^{-2}}l_G(y_0) = \frac{C}{(\eta(1))^3(\eta^{-1}(1))^{-2}}. \tag{4.3}$$

Switching the roles of y_0 and y, we obtain $l_G(y) \leq (\eta(1))^3(\eta^{-1}(1))^{-2}l_G(y_0)$. By Lemma 4.4, we have

$$L_G(y) \leq \eta(1)l_G(y) \leq (\eta(1))^4(\eta^{-1}(1))^{-2}C. \tag{4.4}$$

Because $\mathbb{R}^{n_1}$ is a geodesic space, to show that $H(\cdot, y)$ is a quasisimilarity it suffices to gain control over $l_{H(\cdot,y)}$ and $L_{H(\cdot,y)}$. By (4.4) and Lemma 4.3 (3),

$$L_{H(\cdot,y)}(x) \leq L_G(y)/\eta^{-1}(1) \leq C(\eta(1))^4(\eta^{-1}(1))^{-3}$$

for all $x \in \mathbb{R}^{n_1}$, and by (4.3) and Lemma 4.3 (2),

$$l_{H(\cdot,y)}(x) \geq \frac{1}{\eta(1)}l_G(y) \geq \frac{C}{(\eta(1))^4(\eta^{-1}(1))^{-2}}.$$

for all $x \in \mathbb{R}^{n_1}$. Hence for a.e. y, $H(\cdot, y)$ is an $(\eta(1)/\eta^{-1}(1))^4$-quasisimilarity with constant C. A limiting argument shows this is true for all y. Hence (1) holds.

(2) Recall that when $r = 2$ we have $Y = \mathbb{R}^{n_2}$ and $D_Y = |\cdot|^{\alpha_1/\alpha_2}$. Hence to prove (2) it suffices to show that $G: (\mathbb{R}^{n_2}, |\cdot|) \to (\mathbb{R}^{n_2}, |\cdot|)$ is a K-quasisimilarity with $K = (\eta(1)/\eta^{-1}(1))^{5\alpha_2/\alpha_1}$. As observed before, G is η_1-quasisymmetric with respect to the Euclidean metric, where $\eta_1(t) = (\eta(t^{\alpha_1/\alpha_2}))^{\alpha_2/\alpha_1}$. Because $\mathbb{R}^{n_2}$ is a geodesic space, it suffices to gain control over l_G^e and L_G^e, where l_G^e and L_G^e are similar to l_G and L_G, but with the Euclidean metric instead of the metric D_Y. Because $l_G^e(p) = l_G(p)^{\alpha_2/\alpha_1}$ and $L_G^e(p) = L_G(p)^{\alpha_2/\alpha_1}$, it suffices to gain control over the quantities l_G and L_G in terms of $(\eta(1)/\eta^{-1}(1))^5$.

Notice that (1) implies

$$\frac{C}{(\eta(1)/\eta^{-1}(1))^4} \leq l_{H(\cdot,y)}(x) \leq L_{H(\cdot,y)}(x) \leq C(\eta(1)/\eta^{-1}(1))^4$$

for all $x \in \mathbb{R}^{n_1}$ and all $y \in Y$. By Lemma 4.3, for all $y \in Y$ we have

$$\frac{C}{(\eta(1)/\eta^{-1}(1))^5} \leq l_G(y) \leq L_G(y) \leq C(\eta(1)/\eta^{-1}(1))^5.$$

Hence (2) holds. $\square$

Lemma 4.7. *Suppose that $r \geq 2$ and there are constants $K \geq 1$ and $C > 0$ with the following properties:*

(1) *$G : (Y, D_Y) \to (Y, D_Y)$ is a K-quasisimilarity with constant C;*

(2) *For each $y \in Y$, $H(\cdot, y)$ is a K-quasisimilarity with constant C.*

Then F is an $(\eta(1)/\eta^{-1}(1))K$-quasisimilarity with constant C.

Proof. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^{n_1} \times Y$. We shall first establish a lower bound for $D(F(x_1, y_1), F(x_2, y_2))$. If $|x_1 - x_2| \leq D_Y(y_1, y_2)$, then $D((x_1, y_1), (x_2, y_2)) = D_Y(y_1, y_2)$ and by (1),

$$D(F(x_1, y_1), F(x_2, y_2)) \geq D_Y(G(y_1), G(y_2))$$
$$\geq \frac{C}{K} D_Y(y_1, y_2) = \frac{C}{K} D((x_1, y_1), (x_2, y_2)).$$

If $|x_1 - x_2| > D_Y(y_1, y_2)$, then

$$D((x_1, y_1), (x_2, y_2)) = D((x_1, y_2), (x_2, y_2)) = |x_1 - x_2|,$$

and since F is an η-quasisymmetry, by using (2),

$$D(F(x_1, y_1), F(x_2, y_2)) \geq \frac{1}{\eta(1)} D(F(x_1, y_2), F(x_2, y_2))$$
$$= \frac{1}{\eta(1)} |H(x_1, y_2) - H(x_2, y_2)|$$
$$\geq \frac{C}{\eta(1)K} |x_1 - x_2|$$
$$= \frac{C}{\eta(1)K} D((x_1, y_1), (x_2, y_2)).$$

Hence we have a lower bound for $D(F(x_1, y_1), F(x_2, y_2))$.

By (1), $G^{-1} : (Y, D_Y) \to (Y, D_Y)$ is a K-quasisimilarity with constant C^{-1}. Similarly, (2) implies that for each $y \in Y$, $(H(\cdot, y))^{-1}$ is a K-quasisimilarity with constant C^{-1}. Also recall that F^{-1} is an η_2-quasisymmetry and F is an η-quasisymmetry. Now the argument in the previous paragraph applied to F^{-1} implies

$$D(F^{-1}(x_1, y_1), F^{-1}(x_2, y_2)) \geq \frac{1}{CK\eta_2(1)} D((x_1, y_1), (x_2, y_2)).$$

It follows that

$$D(F(x_1, y_1), F(x_2, y_2)) \leq CK\eta_2(1) D((x_1, y_1), (x_2, y_2))$$
$$= \frac{CK}{\eta^{-1}(1)} D((x_1, y_1), (x_2, y_2))$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^n$, completing the proof. $\square$

Proof of Theorem 4.1. We induct on r. Lemmas 4.6 and 4.7 yield the desired result in the case $r = 2$. Now we assume that $r \geq 3$ and that the Theorem is true for $r - 1$. By Lemma 4.2, F induces an η-quasisymmetry $G \colon (Y, D_Y) \to (Y, D_Y)$. It follows that G is η_1-quasisymmetric with respect to the metric $D_Y^{\alpha_2/\alpha_1}$ (and it is easy to verify that this is indeed a metric), where $\eta_1(t) = [\eta(t^{\alpha_1/\alpha_2})]^{\alpha_2/\alpha_1}$. We point out here that for $(x_2, \ldots, x_r), (x_2', \ldots, x_r') \in Y$,

$$\begin{aligned} &D_Y((x_2, \ldots, x_r), (x_2', \ldots, x_r'))^{\alpha_2/\alpha_1} \\ &= \max\{|x_2 - x_2'|, |x_3 - x_3'|^{\alpha_2/\alpha_3}, \ldots, |x_r - x_r'|^{\alpha_2/\alpha_r}\}. \end{aligned}$$

Hence the induction hypothesis applied to $G \colon (Y, D_Y^{\alpha_2/\alpha_1}) \to (Y, D_Y^{\alpha_2/\alpha_1})$ shows that G is an $(\eta_1(1)/\eta_1^{-1}(1))^{2r}$-quasisimilarity with constant C. Therefore $G \colon (Y, D_Y) \to (Y, D_Y)$ is a K_1-quasisimilarity with constant C^{α_1/α_2}, where

$$K_1 = \left(\frac{\eta_1(1)}{\eta_1^{-1}(1)} \right)^{\frac{2r\alpha_1}{\alpha_2}} = \left(\frac{\eta(1)}{\eta^{-1}(1)} \right)^{2r}. \tag{4.5}$$

This implies that $C^{\alpha_1/\alpha_2}/K_1 \leq l_G(y) \leq L_G(y) \leq C^{\alpha_1/\alpha_2} K_1$ for all $y \in Y$. Now Lemma 4.3 yields

$$C^{\alpha_1/\alpha_2} \frac{1}{K_1 \eta(1)} \leq l_{H(\cdot, y)}(x) \leq L_{H(\cdot, y)}(x) \leq C^{\alpha_1/\alpha_2} \frac{K_1}{\eta^{-1}(1)}$$

for all $y \in Y$ and all $x \in \mathbb{R}^{n_1}$. Since $\mathbb{R}^{n_1}$ is a geodesic space, for each $y \in Y$ the map $H(\cdot, y)$ is a $K_1 \frac{\eta(1)}{\eta^{-1}(1)}$-quasisimilarity with constant C^{α_1/α_2}. By Lemma 4.7, the map F is a $K_1 (\frac{\eta(1)}{\eta^{-1}(1)})^2$-quasisimilarity with constant C^{α_1/α_2}. Here K_1 is as in (4.5). $\qquad \square$

5. Proof of the main results

In this section we will prove Theorem 1.1 and the corollaries from the introduction.

Let X be a CAT(-1) space. Bourdon [Bo] defined a visual metric d_x on ∂X for each $x \in X$. For each $\omega \in \partial X$, there is also a so-called parabolic visual metric on $\partial X \setminus \{\omega\}$, whose definition is recalled below. Let $\omega \in \partial X$, and $b \colon X \to \mathbb{R}$ a Busemann function associated to ω. Define the Gromov product with respect to this Busemann function by

$$(x|x')_b = \frac{1}{2}(b(x) + b(x') - d(x, x')),$$

which extends to points at infinity. The corresponding Hamenstädt metric [H] ρ_b is then defined as $\rho_b(\xi, \eta) = e^{-(\xi|\eta)_b}$ for $\xi, \eta \in \partial X \setminus \{\omega\}$. If $x \in X$ is such that

$b(x) = 0$, then

$$\rho_b(\xi, \eta) = \frac{d_x(\xi, \eta)}{d_x(\xi, \omega) d_x(\eta, \omega)} \tag{5.1}$$

for $\xi, \eta \in \partial X \setminus \{\omega\}$, see [FS], Section 2.3.

We remark that parabolic visual metrics have also been considered by Hersonsky–Paulin [HP] and Bonk–Kleiner [BK].

Let A and G_A be as in the introduction, and g_0 the Riemannian metric on G_A. Let $g = \alpha_1^2 g_0$. Since (G_A, g_0) has sectional curvature $-\alpha_r^2 \leq K \leq -\alpha_1^2$, we conclude that (G_A, g) has sectional curvature $-\alpha_r^2/\alpha_1^2 \leq K \leq -1$ and hence is CAT(-1). Notice that $b := -\alpha_1 h$ is a Busemann function on (G_A, g) associated to ξ_0, where $h \colon G_A \to \mathbb{R}$, $h(x, t) = t$ is the height function. It is not difficulty to see that for $X = (G_A, g)$, the metric ρ_b on $\partial G_A \setminus \{\xi_0\}$ is biLipschitz equivalent to D.

Proof of Theorem 1.1. Let $F \colon (\partial G_A, d_x) \to (\partial G_A, d_x)$ be a quasisymmetric map, where d_x is the Bourdon metric associated to some base point $x \in G_A$ in the CAT(-1) space $(G_A, \alpha_1^2 g_0)$. We first prove that $F(\xi_0) = \xi_0$. Let D be the metric on $\partial G_A \setminus \{\xi_0\} = \mathbb{R}^n$ considered in the previous sections. We have observed that D is biLipschitz equivalent with the Hamenstädt metric ρ_b, where $b = -\alpha_1 h$. Let μ be the product of the Hausdorff measures on the factors $(\mathbb{R}^{n_i}, |\cdot|^{\frac{\alpha_1}{\alpha_i}})$ of $\mathbb{R}^n$. Then the metric measure space $(\mathbb{R}^n, D, \mu)$ is Q-regular with $Q = \Sigma_{i=1}^r n_i \frac{\alpha_i}{\alpha_1}$. On ∂X we define a measure μ' as follows: $\mu'(\{\xi_0\}) = 0$, and on $\partial G_A \setminus \{\xi_0\} = \mathbb{R}^n$, μ' is absolutely continuous with respect to μ with Radon–Nikodym derivative

$$\frac{d\mu'}{d\mu}(\xi) = \frac{1}{(1 + D(o, \xi))^{2Q}}$$

for $\xi \in \partial G_A$. It can be shown that $(\partial G_A, d_x)$ with μ' is also Q-regular. Hence Theorem 3.1 applies to the map $F \colon (\partial G_A, d_x) \to (\partial G_A, d_x)$ and the measure μ'.

Suppose $F(\xi_0) \neq \xi_0$. Then $F^{-1}(\xi_0)$ lies in exactly one horizontal leaf. Fix some $y \in Y$ such that $\mathbb{R}^{n_1} \times \{y\}$ does not contain $F^{-1}(\xi_0)$. Notice that the subset $(\mathbb{R}^{n_1} \times \{y\}) \cup \{\xi_0\}$ of ∂G_A is an n_1-dimensional topological sphere. So $F(\mathbb{R}^{n_1} \times \{y\} \cup \{\xi_0\})$ is an n_1-dimensional topological sphere in $\mathbb{R}^n$. Since each horizontal leaf is an n_1-dimensional Euclidean space, the set $F(\mathbb{R}^{n_1} \times \{y\} \cup \{\xi_0\})$ is not contained in any horizontal leaf. It follows that as a dense subset of $F(\mathbb{R}^{n_1} \times \{y\} \cup \{\xi_0\})$, the set $F(\mathbb{R}^{n_1} \times \{y\})$ is also not contained in any horizontal leaf. Hence there are two points p and q in $\mathbb{R}^{n_1} \times \{y\}$ such that $F(p)$ and $F(q)$ are not in the same horizontal leaf.

Let γ be the Euclidean line segment from p to q and Γ be the family of straight segments parallel to γ in $\mathbb{R}^n$ whose union is an n-dimensional circular cylinder C with γ as the central axis. The curves in Γ are rectifiable with respect to the metric D. Since F is a homeomorphism, by choosing the radius of the circular cylinder to be sufficiently small (by a compactness argument) we may assume that no curve in

Γ is mapped into a horizontal leaf and that $F^{-1}(\xi_0)$ is not in this cylinder. It follows that $F(\Gamma)$ has no locally rectifiable curve with respect to D. Now notice that $F(C)$ is a compact subset of $\partial G_A \backslash \{\xi_0\}$. By (5.1) the two metrics d_x and ρ_b are biLipschitz equivalent on $F(C)$. As D and ρ_b are biLipschitz equivalent, the metrics d_x and D are biLipschitz equivalent on $F(C)$. It follows that $F(\Gamma)$ has no locally rectifiable curve with respect to d_x. Hence $\text{Mod}_Q F(\Gamma) = 0$ in the metric measure space $(\partial G_A, d_x, \mu')$. Theorem 3.1 then implies that $\text{Mod}_Q \Gamma = 0$ in the metric measure space $(\partial G_A, d_x, \mu')$. On the other hand, $\text{Mod}_Q \Gamma > 0$ in the metric measure space $(\mathbb{R}^n, D, \mu)$ (see the proof of Theorem 3.2). Since D and d_x are biLipschitz equivalent on C, and μ and μ' are also comparable on C, we have $\text{Mod}_Q \Gamma > 0$ in the metric measure space $(\partial G_A, d_x, \mu')$, a contradiction.

Next we prove that F is biLipschitz with respect to the metric D. Since $F(\xi_0) = \xi_0$, the map F induces a map $F : (\partial G_A \backslash \{\xi_0\}, d_x) \to (\partial G_A \backslash \{\xi_0\}, d_x)$. Then (5.1) implies that $F : (\partial G_A \backslash \{\xi_0\}, \rho_b) \to (\partial G_A \backslash \{\xi_0\}, \rho_b)$ is quasisymmetric. Since ρ_b and D are biLipschitz equivalent, $F : (\partial G_A \backslash \{\xi_0\}, D) \to (\partial G_A \backslash \{\xi_0\}, D)$ is also quasisymmetric. Now the result follows from Theorem 4.1. $\square$

We note that because G_A has sectional curvature $-\alpha_r^2 \le K \le -\alpha_1^2$, G_A is a proper geodesic δ-hyperbolic space with δ depending only on α_1.

Proof of Corollary 1.3. Suppose there is a quasiisometry $f : G_A \to G$ from G_A to a finitely generated group G, where G is equipped with a fixed word metric. Since G_A is Gromov hyperbolic, it follows that G is Gromov hyperbolic and f induces a quasisymmetric map $\partial f : \partial G_A \to \partial G$. The left translation of G on itself induces an action of G on the Gromov boundary ∂G by quasisymmetric maps. By conjugating this action with ∂f we obtain an action of G on ∂G_A by quasisymmetric maps. By Theorem 1.1, this action has a global fixed point. It follows that the action of G on ∂G has a global fixed point. This can happen only when G is virtually infinite cyclic, in which case the Gromov boundary ∂G consists of only two points. This contradicts the fact that ∂G_A is a sphere of dimension $n \ge 2$ (since $r \ge 2$). $\square$

The proofs of Corollaries 1.4 and 1.2 require some preparation.

Let X be a proper geodesic δ-hyperbolic space and $\xi_1, \xi_2, \xi_3 \in X \cup \partial X$ be three distinct points. For any constant $C \ge 0$, a point $x \in X$ is called a *C-quasi-center* of the three points ξ_1, ξ_2, ξ_3 if for each $i = 1, 2, 3$, there is a geodesic σ_i joining ξ_i and ξ_{i+1} ($\xi_4 := \xi_1$) such that the distance from x to σ_i is at most C. For any $C \ge 0$, there is a constant C' that depends only on δ and C such that the distance between any two C-quasi-centers of ξ_1, ξ_2, ξ_3 is at most C'.

Notice that, if X is a CAT(-1) space, $\omega, \xi, \eta \in \partial X$ are three distinct points, $x \in X$ is a C-quasi-center of ω, ξ, η for some $C \ge 0$, and b is a Busemann function associated to ω, then $|(\xi|\eta)_b - b(x)| \le C'$ for some constant C' depending only on C. It follows that

$$e^{-C'} e^{-b(x)} \le \rho_b(\xi, \eta) \le e^{C'} e^{-b(x)}.$$

Corollaries 1.2 and 1.4 follow from Theorem 1.1 and the following lemma. Recall that, the metric ρ_b is biLipschitz equivalent to D, and by Theorem 1.1, for any quasi-isometry $f : G_A \to G_A$, the boundary map ∂f fixes ξ_0 and restricts to a biLipschitz map of $(\partial G_A \setminus \{\xi_0\}, D)$, which we still denote by ∂f.

Let M be a CAT(-1) space, $\xi_0 \in \partial M$, and $x_0 \in M$ a base point. Let γ be the geodesic ray with $\gamma(0) = x_0$ and $\gamma(\infty) = \xi_0$, and B_γ the Busemann function associated with γ. Let $\rho_M := \rho_{B_\gamma}$ be the Hamenstädt metric associated with the Busemann function B_γ.

Let N be another CAT(-1) space, and $f : M \to N$ an (L_0, C_0)-quasiisometry. For any $\xi \in \partial M$ and $x \in M$, we set $\xi' = \partial f(\xi)$ and $x' = f(x)$. Let γ' be the geodesic ray with $\gamma'(0) = x_0'$ and $\gamma'(\infty) = \xi_0'$, and $B_{\gamma'}$ the Busemann function associated with γ'. Let $\rho_N := \rho_{\gamma'}$ be the Hamenstädt metric associated with the Busemann function $B_{\gamma'}$.

Lemma 5.1. *Let $s > 0$. Suppose that there exists a constant $C_1 \geq 0$ such that every point $x \in M$ is a C_1-quasi-center of some distinct triple (ξ_0, ξ, η) in ∂M. Then the following three conditions are equivalent:*

(i) *there is a constant $C \geq 0$ such that $|B_{\gamma'}(f(x)) - s \cdot B_\gamma(x)| \leq C$ for all $x \in M$;*

(ii) *the boundary map $\partial f : (\partial M \setminus \{\xi_0\}, \rho_M^s) \to (\partial N \setminus \{\xi_0'\}, \rho_N)$ is biLipschitz;*

(iii) *there exists a constant $C \geq 0$ such that $s \cdot d(x, y) - C \leq d(f(x), f(y)) \leq s \cdot d(x, y) + C$ for all $x, y \in M$.*

Proof. $(2) \implies (1)$. Suppose that the boundary map

$$\partial f : (\partial M \setminus \{\xi_0\}, \rho_M^s) \to (\partial N \setminus \{\xi_0'\}, \rho_N)$$

is L-biLipschitz for some $L \geq 1$. Let $x \in M$. By assumption, x is a C_1-quasi-center of some distinct triple (ξ_0, ξ, η) in ∂M. By the property of Hamenstädt metric we have

$$\frac{1}{C_2} e^{-B_\gamma(x)} \leq \rho_M(\xi, \eta) \leq C_2 e^{-B_\gamma(x)}, \tag{5.2}$$

where C_2 depends only on C_1. Since $f : M \to N$ is an (L_0, C_0)-quasiisometry, the point $f(x)$ is a C_3-quasi-center of (ξ_0', ξ', η') with $C_3 = C_3(L_0, C_0, C_1)$. It follows that

$$\frac{1}{C_4} e^{-B_{\gamma'}(f(x))} \leq \rho_N(\xi', \eta') \leq C_4 e^{-B_{\gamma'}(f(x))}, \tag{5.3}$$

where C_4 depends only on C_3. Since $\partial f : (\partial M \setminus \{\xi_0\}, \rho_M^s) \to (\partial N \setminus \{\xi_0'\}, \rho_N)$ is L-biLipschitz, (5.2) and (5.3) imply $|B_{\gamma'}(f(x)) - s \cdot B_\gamma(x)| \leq \ln(L C_4 C_2^s)$.

$(1) \implies (2)$. Suppose there is a constant C such that $|B_{\gamma'}(f(x)) - s \cdot B_\gamma(x)| \leq C$ for all $x \in M$. Let $\xi \neq \eta \in \partial M \setminus \{\xi_0\}$. Then the triple (ξ_0, ξ, η) has a C_2-quasi-center $x \in M$, where C_2 is a universal constant. Since $f : M \to N$ is

an (L_0, C_0)-quasiisometry, the point $f(x)$ is a C_3-quasi-center of (ξ_0', ξ', η') with $C_3 = C_3(L_0, C_0)$. Then we have

$$\frac{1}{C_4} e^{-B_\gamma(x)} \leq \rho_M(\xi, \eta) \leq C_4 e^{-B_\gamma(x)}$$

and

$$\frac{1}{C_5} e^{-B_{\gamma'}(f(x))} \leq \rho_N(\xi', \eta') \leq C_5 e^{-B_{\gamma'}(f(x))},$$

where C_4 depends only on C_2 and C_5 depends only on C_3. Now it follows from $|B_{\gamma'}(f(x)) - s \cdot B_\gamma(x)| \leq C$ that $\partial f : (\partial M \backslash \{\xi_0\}, \rho_M^s) \rightarrow (\partial N \backslash \{\xi_0'\}, \rho_N)$ is L-biLipschitz, where $L = C_5 e^C C_4^s$.

(1) $\Longrightarrow$ (3). Suppose there is a constant $C \geq 0$ such that $|B_{\gamma'}(f(x)) - s \cdot B_\gamma(x)| \leq C$ for all $x \in M$. Let $x, y \in M$. Let $p \in M$ be a C_2-quasi-center of (x, y, ξ_0), where C_2 is a universal constant. Since $f : M \rightarrow N$ is an (L_0, A_0)-quasiisometry, $f(p)$ is a C_3-quasi-center of $(f(x), f(y), \xi_0')$ with $C_3 = C_3(L_0, C_0, C_2)$. Then

$$|B_\gamma(x) - B_\gamma(p) - d(x, p)| \leq 2C_2,$$

$$|B_\gamma(y) - B_\gamma(p) - d(y, p)| \leq 2C_2,$$

$$|B_{\gamma'}(f(y)) - B_{\gamma'}(f(p)) - d(f(y), f(p))| \leq 2C_3$$

and

$$|B_{\gamma'}(f(x)) - B_{\gamma'}(f(p)) - d(f(x), f(p))| \leq 2C_3.$$

Also notice that $|d(x, y) - d(p, x) - d(p, y)| \leq 2C_2$ and

$$|d(f(x), f(y)) - d(f(p), f(x)) - d(f(p), f(y))| \leq 2C_3.$$

These inequalities together with assumption (1) imply

$$|d(f(x), f(y)) - s \cdot d(x, y)| \leq 6C_3 + 4C + 6sC_2.$$

(3) $\Longrightarrow$ (1): Suppose (3) holds. Fix some $\xi \in \partial M \backslash \{\xi_0\}$. The image $f(\xi \xi_0)$ is an (L_0, C_0)-quasi-geodesic from ξ' to ξ_0'. By the stability of quasi-geodesics, the Hausdorff distance $HD(f(\xi \xi_0), \xi' \xi_0') \leq C_2$ for some constant $C_2 = C_2(L_0, C_0)$. It follows from this and (3) that

$$|B_{\gamma'}(f(p)) - B_{\gamma'}(f(q)) - s[B_\gamma(p) - B_\gamma(q)]| \leq C + 4C_2$$

for all $p, q \in \xi \xi_0$. Fixing $q \in \xi \xi_0$ we obtain

$$|B_{\gamma'}(f(p)) - sB_\gamma(p)| \leq C_3 := C + 4C_2 + |B_{\gamma'}(f(q)) - sB_\gamma(q)| \qquad (5.4)$$

for all $p \in \xi \xi_0$. Now let $x \in M$ be arbitrary. Let $p \in \xi \xi_0$ be the projection of x onto $\xi \xi_0$. Then p is a C_4-quasi-center of (x, ξ, ξ_0) for some universal constant C_4. It

follows that $f(p)$ is a C_5-quasi-center of $(f(x), \xi', \xi_0')$ with $C_5 = C_5(L_0, C_0, C_4)$. Hence

$$|B_\gamma(x) - B_\gamma(p) - d(x, p)| \leq 2C_4 \qquad (5.5)$$

and

$$|B_{\gamma'}(f(x)) - B_{\gamma'}(f(p)) - d(f(x)), f(p))| \leq 2C_5. \qquad (5.6)$$

Now (5.4), (5.5), (5.6) and assumption (3) imply

$$|B_{\gamma'}(f(x)) - sB_\gamma(x)| \leq C + 2C_5 + C_3 + 2sC_4$$

for all $x \in M$. □

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Nageswari Shanmugalingam, Department of Mathematical Sciences, P.O.Box 210025, University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.

E-mail: shanmun@uc.edu

Xiangdong Xie, Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403-0221, U.S.A.

E-mail: xiex@bgsu.edu