

## Equivariant classes of matrix matroid varieties

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*Dedicated to the memory of T. Brylawski (1944–2007)*

**Abstract.** To each subset  $I$  of  $\{1, \dots, k\}$  associate an integer  $r(I)$ . Denote by  $X$  the collection of those  $n \times k$  matrices for which the rank of a union of columns corresponding to a subset  $I$  is  $r(I)$ , for all  $I$ . We study the equivariant cohomology class represented by the Zariski closure  $Y = \bar{X}$ . This class is an invariant of the underlying matroid structure. Its calculation incorporates challenges similar to the calculation of the ideal of  $Y$ , namely, the determination of the geometric theorems for the matroid. This class also gives information on the degenerations and hierarchy of matroids. New developments in the theory of Thom polynomials of contact singularities (namely, a recently found stability property) help us to calculate these classes and present their basic properties. We also show that the coefficients of this class are solutions to problems in enumerative geometry, which are natural generalization of the linear Gromov–Witten invariants of projective spaces.

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### 1. Introduction

**1.1. Matroid representation varieties.** To each subset  $I$  of  $\{1, \dots, k\}$  associate an integer  $r(I)$ . Denote by  $X$  the collection of those  $n \times k$  matrices for which the rank of a union of columns corresponding to a subset  $I$  is  $r(I)$ , for all  $I$ . Our main object of study in this paper is the Zariski closure  $Y$  of this set. This is a version of matroid representation varieties. Other versions (e.g. contained in Grassmannians, instead of the affine space of matrices) are also known, and they are closely related to ours. A dual point of view is considering the hyperplanes determined by the column vectors of the matrices. From this point of view  $Y$  is the parameter space of certain hyperplane arrangements.

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Matroid representation varieties are universal objects in algebraic geometry in the sense that any complication of varieties can be modeled on them. The precise statement of this universality theorem is called Mnëv's theorem, see [Mnë88], [RG95], or a recent account in [Vak06]. Hence one does not hope that any reasonable question on these varieties has an easy answer. One manifestation of this phenomenon is the determination of the ideal of these varieties. In Section 3 below we will explain with examples how the generators of the ideal encode projective geometry theorems.

The problem we will consider about matroid varieties is an enumerative geometry problem, a generalization of the linear Gromov–Witten invariants of projective spaces. Suppose a matroid variety is given, as above. Consider  $k$  generic linear subspaces  $V_i$  in  $\mathbb{C}^n$ . The question is, how many  $n \times k$  matrices exist that belong to our matroid variety such that the  $i$ 'th column vector is in  $V_i$ . For example, after projectivizing, we can ask the following question: given 8 generic straight lines and a generic point in the projective plane, how many Pappus configurations exist with 8 points of the Pappus configuration belonging to the 8 lines, and the 9'th point coinciding with the given point. The precise definitions, and the answer are given below. In the special case, when the matroid variety is the variety of rank  $\leq 2$  matrices of size  $n \times k$ , this enumerative question is equivalent to the determination of  $k$ -point linear Gromov–Witten invariants in projective spaces. For general matroid varieties, however, no classical geometric or Gromov–Witten-type methods are known to compute the generalized Gromov–Witten invariants.

The nature of the matroid Gromov–Witten invariants in  $\mathbb{P}^2$  can be visualized by pictures. Some interactive presentations, created with the Interactive Geometry Software Cinderella [RGK] can be found at [www.unc.edu/~rimanyi/matroid\\_show](http://www.unc.edu/~rimanyi/matroid_show).

We will show in Theorem 5.3 that our matroid versions of linear Gromov–Witten invariants can be computed through the equivariant classes  $[Y]$ . These are cohomology classes that the varieties  $Y$  represent in the  $\mathrm{GL}(n) \times \mathrm{GL}(1)^k$ -equivariant cohomology ring of the vector space of  $n \times k$  matrices.

## 1.2. Equivariant classes represented by invariant varieties of a representation.

Let the group  $G$  act on the complex vector space  $V$ , and let  $Y \subset V$  be an invariant variety of complex codimension  $c$ . Then  $Y$  represents a cohomology class  $[Y] \in H_G^{2c}(V)$  in equivariant cohomology. There are various definitions and names for this class, e.g. equivariant Poincaré dual, Thom polynomial, multidegree. We will call it the equivariant class of  $Y$ . Since  $H_G^*(V)$  is naturally isomorphic to the ring  $H^*(BG)$  of  $G$ -characteristic classes, the equivariant class  $[Y]$  is simply a  $G$ -characteristic class of degree  $2c$ .

The equivariant class of the variety  $Y$  encodes a lot of geometric information on  $Y$ ; let us just allude to the effectiveness of Schubert calculus (the Giambelli formula is such an equivariant class) or the generalization involving classes of quiver loci. Other applications include the enumerative geometry results coming from Thom

polynomials of singularities, see e.g. [Kle76] for a classic review or [MR07] for a recent addition.

Let us remark that the equivariant class of a matroid representation variety can be interpreted as a class of a quiver locus for the “star quiver” (based on a star shaped graph). However, the equivariant properties of quiver representations are only well understood for quivers of Dynkin type ADE, see [BF99], [FR02], [KMS06], [KS06], [Buc08] and references therein. It would be interesting to compare our results with quiver coefficients defined in [Buc08] for non-Dynkin quivers.

The usual tools to calculate equivariant classes represented by invariant subvarieties involve equivariant resolution, equivariant degeneration, or equivariant localization techniques. These techniques require the understanding of the ideal or the singularities of the variety in question. For matroid varieties we lack this essential information.

Another main approach to calculate equivariant classes, effective for equivariant classes of contact singularities as well, is an interpolation method described in [FR04]. Below we will study an improvement of this interpolation method. In essence, we will describe certain constraints that a particular  $[Y]$  must satisfy. Some of these constraints originate from the topological arguments of [FR04], some others from the enumerative interpretation of some coefficients.

Finally, a key advance of this paper, a third set of constraints, follow from the analogue of a stabilization property recently proved for contact singularities [FR07]. This property stems from understanding how an equivariant class changes at non-transversal intersection. In turn, one obtains a bound on the number of factors in each term of such and equivariant class.

Overwhelming experience shows that these three sets of constraints are sufficient to determine the equivariant classes  $[Y]$ , providing several enumerative applications. However, at the moment, we have no theorem claiming that for a particular matroid a certain set of constraints is sufficient.

In Section 7 we show a certain stabilization property connecting the equivariant classes of matroid varieties in different dimensions  $n$ . As a corollary we prove a vanishing theorem on certain coefficients. In Section 8 we outline the method of calculating these invariants.

A particularly interesting question, subject to future study, is whether the matroid Gromov–Witten invariants can be organized as structure constants of an algebraic object with some kind of associativity property – mimicking the construction of the (big) quantum cohomology ring.

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## 2. Matrix matroid varieties

We will denote the set of natural numbers  $\{0, 1, \dots\}$  by  $\mathbb{N}$ , and the set  $\{1, 2, \dots, k\}$  by  $[k]$ . For a set  $X$  let  $2^X$  denote its power set, i.e. the set of subsets of  $X$ . We will identify the vector spaces  $(\mathbb{C}^n)^k$  and  $\mathbb{C}^{n \times k}$  by the rule

$$(v_1, \dots, v_k) \leftrightarrow \left( \left( v_1 \right) \dots \left( v_k \right) \right) \quad (1)$$

Elements in  $(\mathbb{C}^n)^k$  will be referred to as (ordered) *vector configurations* in  $\mathbb{C}^n$ . For a matrix  $M \in \mathbb{C}^{n \times k}$  and subsets  $U \subset [n]$ ,  $V \subset [k]$ , let  $M_V^U$  denote the submatrix consisting of the  $(i, j)$ -entries of  $M$  for  $i \in U$ ,  $j \in V$ . Let  $M_V = M_V^{[n]}$ .

The vector configuration  $\mathcal{C} = (v_1, v_2, \dots, v_k) \in (\mathbb{C}^n)^k$  defines the rank function  $r_{\mathcal{C}} : 2^{[k]} \rightarrow \mathbb{N}$ ,

$$r_{\mathcal{C}}(V) = \dim \operatorname{span}\{v_i\}_{i \in V}.$$

**Definition 2.1.** For a configuration  $\mathcal{C}$  we define

$$X_{\mathcal{C}} = \{M \in \mathbb{C}^{n \times k} : \operatorname{rank}(M_V) = r_{\mathcal{C}}(V) \text{ for all } V \subset [k]\}.$$

The Zariski closure  $\overline{X_{\mathcal{C}}} \subset \mathbb{C}^{n \times k}$  will be called the *matrix matroid variety* associated with  $\mathcal{C}$ , and will be denoted by  $Y_{\mathcal{C}}$ .

If we identify  $n \times k$  matrices with  $k$ -tuples of  $n$ -vectors as in (1), then  $X_{\mathcal{C}}$  consists of those configurations whose rank function is the same as that of  $\mathcal{C}$ . For example,  $\mathcal{C}$  itself belongs to  $X_{\mathcal{C}}$ . The matrix matroid variety  $Y_{\mathcal{C}}$  consists of those configurations that are limits (degenerations) of elements in  $X_{\mathcal{C}}$ .

Observe that  $X_{\mathcal{C}}$  and  $Y_{\mathcal{C}}$  do not change if we re-scale, i.e. multiply, any vector  $v_i$  in  $\mathcal{C}$  by any non-zero complex number. Hence  $X_{\mathcal{C}}$  and  $Y_{\mathcal{C}}$  are determined by the list of points  $P_i := [v_i] \in \mathbb{P}^{n-1}$  for  $v_i \neq 0$ , and the list of those  $v_i$  which are 0. By abusing language, such a list will also be called a configuration.

**Example 2.2.** Let  $n = 2$ ,  $k = 6$ , and consider the following configuration:

$$P_1 = P_2 = P_3 = (0 : 1) \in \mathbb{P}^1, \quad P_4 = P_5 = (1 : 1) \in \mathbb{P}^1, \quad v_6 = 0.$$

This configuration is illustrated in Figure 1. Matrices in  $X_{\mathcal{C}}$  are those  $2 \times 6$  matrices whose

- first three columns are proportional *non-zero* vectors,
- the fourth and fifth columns are proportional *non-zero* vectors,
- *the first and the fourth columns are non-proportional*,
- the sixth column is the zero vector.

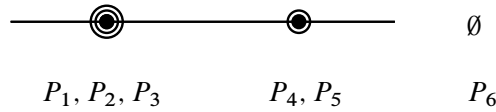


Figure 1

It is true that  $Y_{\mathcal{E}}$  consists of matrices satisfying the “closed” conditions above, but not necessarily the “open” ones. That is,  $Y_{\mathcal{E}}$  consists of matrices whose first three columns are proportional, fourth and fifth columns are proportional, and sixth column is 0. However, the easy procedure of dropping the open conditions will *not* specify  $Y_{\mathcal{E}}$  in general.

**Example 2.3.** *Matrix Schubert varieties.* Consider a complete flag of linear spaces

$$L^0 \subset L^1 \subset \dots \subset L^{n-1} \subset L^n$$

in  $\mathbb{C}^n$ . Let  $\ell = (l_0, l_1, \dots, l_n) \in \mathbb{N}^{n+1}$  with  $\sum l_i = k$ . Choose  $l_i$  generic points  $v_1^{(i)}, \dots, v_{l_i}^{(i)}$  in  $L^i$ . The matrix matroid variety corresponding to the configuration

$$\mathcal{C}_{\ell} = (v_1^{(0)}, \dots, v_{l_0}^{(0)}, v_1^{(1)}, \dots, v_{l_1}^{(1)}, \dots, v_1^{(n)}, \dots, v_{l_n}^{(n)})$$

is studied in the papers [Ful92], [FR03, Section 5], [KM05], and is called the matrix Schubert variety corresponding to Grassmannian permutation.

For the problems to be considered later in this paper, matrix Schubert varieties will be the simple case. Products of matrix Schubert varieties will also be considered simple. Note that Example 2.2 is such a product of matrix Schubert varieties (after identifying  $\mathbb{C}^{n \times k_1} \times \mathbb{C}^{n \times k_2}$  with  $\mathbb{C}^{n \times (k_1 + k_2)}$ ), namely,

$$Y_{\mathcal{E}} = Y_{\mathcal{E}_{(0,3,0)}} \times Y_{\mathcal{E}_{(1,2,0)}} = Y_{\mathcal{E}_{(0,3,0)}} \times Y_{\mathcal{E}_{(0,2,0)}} \times Y_{\mathcal{E}_{(1,0,0)}}. \tag{2}$$

Examples of matrix matroid varieties which are not products of matrix Schubert varieties will be given below.

**Remark 2.4.** Other candidate names for matrix matroid varieties would be “matroid variety” or “matroid representation variety”. We chose the name “matrix matroid variety”, because of the analogy with matrix Schubert varieties.

### 3. The ideal of matrix matroid varieties

The rank of a matrix is  $r$  if its  $(r+1) \times (r+1)$  minors vanish, and at least one  $r \times r$  minor does not vanish. Hence the algebraic description of  $X_{\mathcal{E}} \subset \mathbb{C}^{n \times k}$  is a collection of

equations (several minors vanish), together with a collection of conditions expressing that certain polynomials (some other minors) do not vanish together (cf. the open and closed conditions of Example 2.2):

$$X_{\mathcal{E}} = \{M = (m_{i,j}) \in \mathbb{C}^{n \times k} : p_u(m_{i,j}) = 0 \text{ for } u = 1, 2, \dots; \text{ for } v = 1, 2, \dots, q_1^{(v)}(m_{i,j}), \dots, q_{w_v}^{(v)}(m_{i,j}) \text{ do not vanish together}\}.$$

It follows that

$$Y_{\mathcal{E}} \subset \{M = (m_{i,j}) \in \mathbb{C}^{n \times k} : p_u(m_{i,j}) = 0 \text{ for } u = 1, 2, \dots\}. \tag{3}$$

Although it is tempting to think that we have equality in formula (3), in general, this is not the case. First we give an intuitive reason for this.

**3.1. Motivation: the Menelaus configuration.** Consider the Menelaus configuration  $\mathcal{C}_M$  of Figure 2, with  $n = 3, k = 6$ .

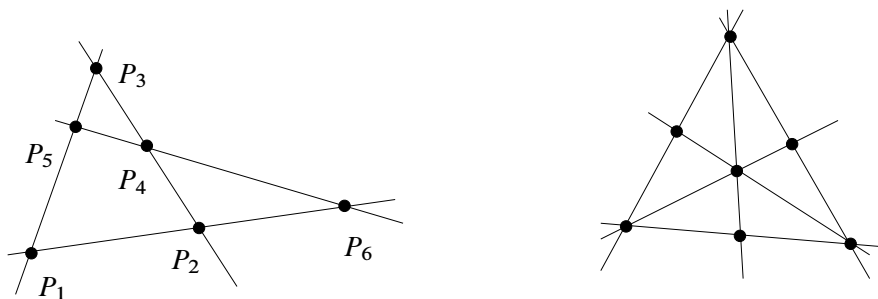


Figure 2. The Menelaus configuration  $\mathcal{C}_M$ , and the Ceva configuration  $\mathcal{C}_C$ .

The equations  $p_u$  of formula (3) are the four  $3 \times 3$  minors of the  $3 \times 6$  matrix  $(m_{i,j})$  corresponding to the following triples of column-indices: 126, 135, 234, 456. The right hand side of formula (3) hence contains all  $3 \times 6$  matrices for which these four minors vanish. We claim that there is a matrix for which these minors vanish, but it is *not* in  $Y_{\mathcal{E}_M}$ , i.e. it is not a limit of matrices from  $X_{\mathcal{E}_M}$ . Indeed, consider an affine chart of  $\mathbb{P}^2$ , a line  $l$  in it, and the configuration  $\mathcal{C}'$  of six generic points on  $l$ . If this configuration were in the closure of  $X_{\mathcal{E}_M}$  then there would be a family of configurations belonging to  $X_{\mathcal{E}_M}$ , all in the affine chart, converging to  $\mathcal{C}'$ . For all these configurations Menelaus' theorem [Men00] holds, which we recall now.

**Theorem 3.1** (Complex affine version of Menelaus's theorem). *Consider the configuration  $\mathcal{C}_M$  of points in  $\mathbb{C}^2$ . Choose an identification of the  $P_1P_3P_5$  line with  $\mathbb{C}$ . Observe that the complex number  $(P_5 - P_1)/(P_3 - P_5)$  does not depend on the*

choice; denote this ratio by  $P_1 P_5 / P_5 P_3$ . Then (using similar notations for the other straight lines) we have

$$\frac{P_1 P_5}{P_5 P_3} \cdot \frac{P_3 P_4}{P_4 P_2} \cdot \frac{P_2 P_6}{P_6 P_1} = -1. \tag{4}$$

Our reasoning is finished by observing that for  $\mathcal{C}'$  the Menelaus identity (4) does not hold; this proves that for  $\mathcal{C}_M$  the two sides of Formula (3) are not equal.

One may wonder if there is a complex projective version of Menelaus’s theorem, which would eliminate the need for the affine chart in the geometric proof above. The answer is given in the next section.

What we learned from the Menelaus example is that

- the “naive” equations  $p_u$  are not enough to cut out  $Y_{\mathcal{C}}$  even set-theoretically from  $\mathbb{C}^{n \times k}$ ;
- the extra equations needed (besides the naive ones) encode the not-so-obvious geometric theorems of the configuration.

**3.2. The ideal of  $X_{\mathcal{C}}$ , examples.** Let  $I_{\mathcal{C}}$  denote the ideal of the variety  $Y_{\mathcal{C}}$ , i.e. the homogeneous ideal of polynomials vanishing on  $Y_{\mathcal{C}}$ .

**Example 3.2.** In Schubert calculus the following statement is well known [Fu192]: For the matrix Schubert variety  $\mathcal{C}_\ell$  of Example 2.3 the “naive” equations generate  $I_{\mathcal{C}_\ell}$ :

$$I_{\mathcal{C}_\ell} = \left( \det \left( M_{\substack{\{i_1, \dots, i_s\} \\ \{j_1, \dots, j_s\}}} \right) = 0 \right)$$

for  $i_1 < \dots < i_s, j_1 < \dots < j_s, j_s \leq l_0 + \dots + l_{s-1}, s = 1, \dots, n$ .

Now consider the Menelaus configuration of Section 3.1, and consider the variety corresponding to the naive equations

$$Y_{\text{naive}} = \{M \in \mathbb{C}^{3 \times 6} : \det M_{126} = 0, \det M_{135} = 0, \det M_{234} = 0, \det M_{456} = 0\}.$$

Computer algebra packages [GPS01] can be used to find that this variety is the union of two irreducible varieties

$$Y_{\text{naive}} = Y_{\mathcal{C}_M} \cup \{M \in \mathbb{C}^{3 \times 6} : \text{rank}(M) \leq 2\}. \tag{5}$$

This decomposition sheds light on the intuitive reasoning of Section 3.1. As a byproduct, the computer algebra package finds generators of the ideal  $I_{\mathcal{C}_M}$ . It turns out that  $I_{\mathcal{C}_M}$  can be minimally generated by polynomials of degrees

$$3, 3, 3, 3, 5, 5, 5, \underbrace{6, \dots, 6}_{12}.$$

The four degree 3 polynomials can be chosen to be the four naive equations. As a consequence, the collection of the other equations can be considered as the *extra*, non-trivial complex projective identities holding for Menelaus configurations. We might as well call this set of polynomials the “complex projective Menelaus’ theorem”.

For completeness let us show how to generate degree 6 and degree 5 polynomials in  $I_{\mathcal{E}_M}$  knowing only the usual version of Menelaus’ theorem (Theorem 3.1). In the projective plane  $(x, y, z)$  we can choose the  $y$  coordinate to be at infinity. In the remaining affine plane we can identify ratios of complex numbers by appropriate projections. Hence, for example, from (4) we can obtain

$$\frac{x_5/y_5 - x_1/y_1}{x_3/y_3 - x_5/y_5} \cdot \frac{z_4/y_4 - z_3/y_3}{z_2/y_2 - z_4/y_4} \cdot \frac{z_6/y_6 - z_2/y_2}{z_1/y_1 - z_6/y_6} = -1. \quad (6)$$

Rearranging this equality we obtain a degree 6 polynomial. Making other choices we may obtain several other degree 6 polynomials. Getting degree 5 ones is more delicate. Consider the degree 6 polynomial obtained from (6), and the ones obtained from the next three similar equalities

$$\begin{aligned} \frac{z_5/y_5 - z_1/y_1}{z_3/y_3 - z_5/y_5} \cdot \frac{z_4/y_4 - z_3/y_3}{z_2/y_2 - z_4/y_4} \cdot \frac{x_6/y_6 - x_2/y_2}{x_1/y_1 - x_6/y_6} &= -1, \\ \frac{x_1/y_1 - x_2/y_2}{x_2/y_2 - x_6/y_6} \cdot \frac{z_6/y_6 - z_4/y_4}{z_4/y_4 - z_5/y_5} \cdot \frac{z_5/y_5 - z_3/y_3}{z_3/y_3 - z_1/y_1} &= -1, \\ \frac{z_1/y_1 - z_2/y_2}{z_2/y_2 - z_6/y_6} \cdot \frac{z_6/y_6 - z_4/y_4}{z_4/y_4 - z_5/y_5} \cdot \frac{x_5/y_5 - x_3/y_3}{x_3/y_3 - x_1/y_1} &= -1. \end{aligned}$$

It turns out that the sum of these four degree 6 polynomials is  $y_4$  times

$$\begin{aligned} &-x_5y_1z_3z_6y_2 + x_5y_1z_3z_2y_6 + x_3y_5z_2z_1y_6 - x_5y_3z_2z_1y_6 \\ &+ z_1y_5z_3x_6y_2 - z_1y_5z_3x_2y_6 - z_3y_5z_2x_6y_1 + z_5y_3z_2x_6y_1z_5y_3 \\ &+ x_2y_1z_6z_3y_5 - x_2y_1z_6 - x_6y_2z_5z_1y_3 + x_2y_6z_5z_1y_3 \\ &- z_1y_2z_6x_3y_5 + z_1y_2z_6x_5y_3 + z_6y_2z_5x_3y_1 - z_2y_6z_5x_3y_1. \end{aligned} \quad (7)$$

This latter is one of the degree 5 generators of  $I_{\mathcal{E}_M}$ . The other two can be obtained by similar calculations, or appropriate changes of variables in (7).

For an arbitrary configuration  $\mathcal{C}$  determining  $I_{\mathcal{C}}$  seems to be a hopelessly difficult problem. One might start with the ideal generated by the naive equations, and try to get rid of the “fake” components (just like the determinantal variety in (5)) by primary decomposition or by dividing (or saturating) with ideals of extra components. In practice, none of these strategies is feasible in reasonable time for configurations even a little more complicated than the Menelaus configuration.

Below, in Section 4 we will study another invariant of matrix matroid varieties, namely, their equivariant classes, which will be much better computable than  $I_{\mathcal{C}}$ , and which can answer various questions about these varieties without determining their ideals.



**3.3. The codimension of matrix matroid varieties.** A consequence of the universality theorem mentioned in the Introduction, is that matrix matroid varieties in general can have multiple components, possibly of different dimensions. Let us mention that it seems a difficult problem to show an example for this phenomenon. Nevertheless, we do not expect any easy general procedure which would give the (co)dimension of a matrix matroid variety corresponding to a given matroid. In practice, however, one may often determine the codimension by imagining the matroid builded up step by step, and keeping track of the degrees of freedom. Consider, for example, the Menelaus configuration of Figure 2. The points  $P_1, P_2, P_4, P_5$  are projectively free. Adding  $P_3$  however means a restriction: it can not be anywhere in the plane, it has to be in the intersection of  $P_1P_5$  and  $P_2P_4$ . This is a 2-codimensional restriction. Similarly, adding the point  $P_6$  is another 2 codimensional restriction. Hence the codimension of  $Y_{\mathcal{C}_M}$  is  $2 + 2 = 4$ . Now consider the Pappus configuration in Figure 3. The subset  $P_1, P_2, P_4, P_5$  is free. The points  $P_3$  and  $P_6$  are on the straight lines  $P_1P_2$  and  $P_4P_5$  respectively. Hence each represents a codimension 1 restriction. These 6 points determine the remaining 3, which hence represent 2-codimensional restrictions each. Therefore the codimension of this matrix matroid variety is  $1 + 1 + 2 + 2 + 2 = 8$ .

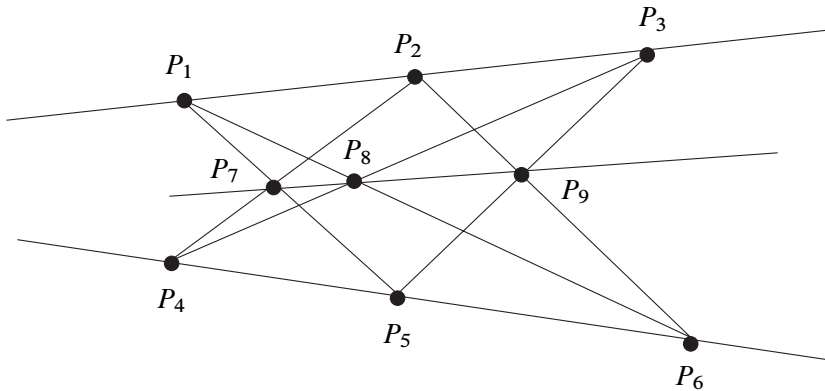


Figure 3. The Pappus configuration  $\mathcal{C}_P$ .

**4. Equivariant classes of matrix matroid varieties.**

We will work in the complex algebraic category; cohomology will be meant with integer coefficients; and  $GL(n)$  will denote the general linear group  $GL(n, \mathbb{C})$ .

**4.1. Equivariant classes in general.** If  $Y$  is a complex codimension  $c$  subvariety in a compact complex manifold  $M$ , then  $Y$  represents a cohomology class  $[Y]$  in  $H^{2c}(M)$ . The following equivariant version of this notion is more delicate to define; see e.g. [Kaz97], [EG98], [FR04], [MS04, 8.5], [Ful07].

Let the group  $G$  act on the complex vector space  $V$ , and let  $Y \subset V$  be an invariant variety of complex codimension  $c$ . Then  $Y$  represents a cohomology class  $[Y] \in H_G^{2c}(V)$  in equivariant cohomology. Since  $H_G^*(V)$  is naturally isomorphic to the ring  $H^*(BG)$  of  $G$ -characteristic classes, the equivariant class  $[Y]$  is simply a  $G$ -characteristic class of degree  $2c$ .

**4.2. Equivariant classes of matrix matroid varieties.** Let  $D(k)$  be the group of diagonal matrices of size  $k$ . Consider the action of  $G_{n,k} = \mathrm{GL}(n) \times D(k)$  on the vector space  $\mathbb{C}^{n \times k}$  of  $n \times k$  matrices by

$$(A, B) \cdot M = AMB^{-1}, \quad A \in \mathrm{GL}(n), \quad B \in D(k), \quad M \in \mathbb{C}^{n \times k}.$$

Viewing elements of  $\mathbb{C}^{n \times k}$  as vector configurations as in (1), the action of  $(A, B) \in G_{n,k}$  reparametrizes  $\mathbb{C}^n$  (the action of  $A$ ) and rescales the vectors one by one (the action of  $B$ ). Therefore, the spaces  $X_{\mathcal{E}}$  and hence the varieties  $Y_{\mathcal{E}}$  are  $G_{n,k}$ -invariant.

In the rest of the paper the main concept of interest will be the equivariant class

$$[Y_{\mathcal{E}}] \in H_{G_{n,k}}^*(\mathbb{C}^{n \times k}) = H^*(BG_{n,k}) = \mathbb{Z}[c_1, \dots, c_n, d_1, \dots, d_k], \quad (8)$$

where  $c_i$  are the Chern classes of  $\mathrm{GL}(n)$ , and  $d_i$  are the first Chern classes of the  $\mathrm{GL}(1)$  components of  $D^k = \mathrm{GL}(1)^k$ . We have  $\deg c_i = 2i$ ,  $\deg d_i = 2$ .

**4.3. Examples.** In Sections 5–9 we will show how to calculate the classes  $[Y_{\mathcal{E}}]$ , and discuss their geometric meaning. Before that, however, we show some examples.

**Example 4.1.** Consider the configuration  $\mathcal{C}$  of Example 2.2. We have

$$\begin{aligned} [Y_{\mathcal{E}}] = & (d_1 d_2 + d_1 d_3 + d_2 d_3 - c_1(d_1 + d_2 + d_3) + c_1^2 - c_2) \\ & \cdot (c_1 - d_4 - d_5)(d_6^2 - c_1 d_6 + c_2). \end{aligned}$$

Let us now consider matrix Schubert varieties  $\mathcal{C}_{\ell}$  of Example 2.3. That is, we have  $\ell = (l_0, \dots, l_n)$ ,  $\sum l_i = k$ . We may assume without loss of generality that there is an  $r$  such that  $l_1, \dots, l_r$  are all non-zero, while  $l_{r+1} = \dots = l_n = 0$ . (Indeed, observe for example that  $Y_{\mathcal{C}_{2,0,2}} = Y_{\mathcal{C}_{2,1,1}}$ , by changing the complete flag.) Define

$$\mu_i = \begin{cases} \sum_{j=0}^{i-1} l_j + 1, & i \leq r \\ k + i - r & i > r, \end{cases} \quad \lambda_{n+1-i} = \mu_i - i, \text{ for } i = 1, \dots, n.$$

Let  $\beta_j^{(i)}$  be degree  $j$  polynomials in the ring (8), defined by

$$1 + \beta_1^{(i)}t + \beta_2^{(i)}t^2 + \dots = \frac{\prod_{j < \mu_i} (1 + d_j t)}{1 + c_1 t + \dots + c_n t^n}.$$

**Theorem 4.2.** *Using the notation above, the matrix Schubert variety  $\mathcal{E}_\ell \subset \mathbb{C}^{n \times k}$  has complex codimension  $|\lambda| = \sum \lambda_i$ , and we have*

$$[Y_{\mathcal{E}_\ell}] = (-1)^{|\lambda|} \det \left( \beta_{\lambda_i + j - i}^{(n+1-i)} \right)_{i,j=1,\dots,n}. \tag{9}$$

*Proof.* This is, in fact, not a new theorem. Observe, that  $Y_{\mathcal{E}_\ell}$  is not only invariant under the action of  $GL(n) \times D(k)$ , but under the same action of  $GL(n) \times B(k)$ , where  $B(k)$  is the Borel group of upper triangular  $k \times k$  matrices associated with the complete flag. The varieties  $Y_{\mathcal{E}_\ell}$  are, in fact, the orbit closures of this extended action. The equivariant classes of these orbit closures are calculated in Theorem 5.1 of [FR03] to be the double Schur polynomials of (9), see also [KM05]. Since the inclusion  $D(k) \subset B(k)$  is a homotopy equivalence, the  $G_{n,k}$ -equivariant classes are the same as the  $GL(n) \times B(k)$ -equivariant classes. This observation also shows that expression (9) is symmetric in the  $d_i$  variables.  $\square$

Equivariant classes of products of varieties multiply in the obvious sense. For instance, the result of Example 4.1 can be recovered from the factorization (2) and the application of Theorem 4.2 to the three factors. Namely, we have

$$\begin{aligned} [Y_{\mathcal{E}_{0,3,0}}] &= \det \begin{pmatrix} \frac{\prod_{i=1}^3 (1+d_i t)}{1+c_1 t+c_2 t^2} | 2 & \frac{\prod_{i=1}^3 (1+d_i t)}{1+c_1 t+c_2 t^2} | 3 \\ 0 & \frac{\prod_{i=1}^0 (1+d_i t)}{1+c_1 t+c_2 t^2} | 0 \end{pmatrix} \\ &= d_1 d_2 + d_1 d_3 + d_2 d_3 - c_1 (d_1 + d_2 + d_3) + c_1^2 - c_2. \end{aligned}$$

Here, and later,  $f(t)|_i$  means the  $i$ 'th coefficient of the Taylor series  $f(t)$  in the formal variable  $t$ . Similarly,

$$[Y_{\mathcal{E}_{0,2,0}}] = -(d_4 + d_5 - c_1), \quad [Y_{\mathcal{E}_{(1,0,0)}}] = d_6^2 - c_1 d_6 + c_2,$$

after appropriate shifting of indices.

It is rather difficult to present equivariant classes of matrix matroid varieties which are not products of matrix Schubert varieties. For example the class  $[Y_{\mathcal{E}_M}]$  for the Menelaus configuration of Section 3.1 is a degree 8 polynomial with 173 terms (in  $c$ - $d$ -monomials). To indicate how it looks we show this polynomial after we substitute 0 for all the  $d_i$  variables:  $[Y_{\mathcal{E}}]^* = [Y_{\mathcal{E}}]_{d_i=0 \forall i}$ . Observe that the  $[Y_{\mathcal{E}}]^*$  class is the  $GL(n)$  equivariant class represented by  $Y_{\mathcal{E}}$ . We have

- $[Y_{\mathcal{E}_M}]^* = 3c_1^2 c_2 - 2c_1 c_3 - c_2^2 = 3\Delta_{(211)} + 2\Delta_{(22)} + 3\Delta_{(31)}.$

Here the Schur polynomials  $\Delta_\lambda$  corresponding to a partition  $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r)$  are defined by  $\Delta_\lambda = \det(c_{\lambda_i + j - i})_{i,j=1,\dots,r}$ . The significance of the Schur basis is presented in Sections 6, 7. The expression in Theorem 4.2 can also be interpreted as a Schur polynomial. Here is a list of similar specializations of  $[Y_{\mathcal{E}}]$  for the Ceva, Pappus, and Desargues configurations.

- $[Y_{\mathcal{E}_C}]^* = 6\Delta_{(2211)} + 4\Delta_{(222)} + 3\Delta_{(3111)} + 8\Delta_{(321)} + \Delta_{(33)}$ ;
- $[Y_{\mathcal{E}_P}]^* = 11\Delta_{(221111)} + 16\Delta_{(22211)} + 8\Delta_{(2222)} + 12\Delta_{(311111)} + 28\Delta_{(32111)} + 28\Delta_{(3221)} + 17\Delta_{(3311)} + 15\Delta_{(332)}$ ;
- $[Y_{\mathcal{E}_D}]^* = 15\Delta_{(222111)} + 20\Delta_{(22221)} + 20\Delta_{(321111)} + 50\Delta_{(32211)} + 30\Delta_{(3222)} + 30\Delta_{(33111)} + 45\Delta_{(3321)} + 10\Delta_{(333)}$ .

The coefficients of these classes in terms of Schur polynomials are all non-negative. We will prove this property for general matroids in Theorem 6.6.

The equivariant classes are rather meaningless formulas unless we find geometric applications. We finish this section with a rather simple application; more delicate geometric meaning will be discussed in Sections 5 and 6.

**4.4. The degree of  $\mathbb{P}(Y_{\mathcal{E}})$ .** Suppose the group  $G$  acts on the vector space  $V$ , and  $Y$  is an invariant cone. Then the degree of the projective variety  $\mathbb{P}(Y)$  can be recovered from the equivariant class  $[Y] \in H_G^*(V)$  by the following procedure.

Let  $T^n$  be a maximal torus of  $G$  with corresponding Chern roots  $\alpha_i$ . If  $w_1, \dots, w_n, w$  are integers with the property that for any  $z \in \mathbb{C}$ ,  $|z| = 1$  we have

$$(z^{w_1}, \dots, z^{w_n}) \cdot v = z^w v, \quad (z^{w_1}, \dots, z^{w_n}) \in T^n, \quad v \in V,$$

then

$$\deg \mathbb{P}(Y) = [Y](\alpha_i = \frac{w_i}{w}).$$

On the right hand side we have the equivariant class, with the number  $w_i/w$  substituted into the Chern root corresponding to the  $i$ 'th factor of  $T^n$ . This theorem easily follows from the study of the change of equivariant classes when the torus action is pulled back to another torus action; see e.g. [MS04, Exercise 8.14, 8.15], [FNR05, 6.4].

For matrix matroid varieties we have two natural choices for the substitution. Either we substitute

$$c_i = \binom{n}{i}, \quad d_i = 0, \quad \text{or} \quad c_i = 0, \quad d_i = -1.$$

Observe that the first substitutions can be carried out for the specialized classes above ( $d_i = 0$  for all  $i$ ), hence the following theorem can be checked from the classes presented above.

**Theorem 4.3.** *For the Menelaus, Ceva, Pappus, and Desargues configurations (see Figures 2, 3, 4) we have*

$$\begin{aligned} \deg \mathbb{P}(Y_{\mathcal{E}_M}) &= 66, & \deg \mathbb{P}(Y_{\mathcal{E}_C}) &= 297, \\ \deg \mathbb{P}(Y_{\mathcal{E}_P}) &= 2943, & \deg \mathbb{P}(Y_{\mathcal{E}_D}) &= 4680. \end{aligned}$$

**Remark 4.4.** The degree of  $\mathbb{P}(Y_{\mathcal{E}_M})$  can also be recovered from the decomposition in (5). Indeed, in this decomposition all three varieties are 4 codimensional;  $Y_{\text{naive}}$  has degree  $3^4$  because of Bézout’s theorem; the determinantal variety  $\{M \in \mathbb{C}^{3 \times 6} \mid \text{rank } M \leq 2\}$  has degree  $\binom{6}{4}$  (see [Ful84, 14.4.14]). Hence  $\deg Y_{\mathcal{E}_M} = 81 - 15 = 66$ . The same argument shows that

$$\begin{aligned} [Y_{\mathcal{E}_M}] &= (c_1 - d_1 - d_2 - d_6)(c_1 - d_1 - d_3 - d_5)(c_1 - d_2 - d_3 - d_4) \\ &\quad (c_1 - d_4 - d_5 - d_6) - \frac{\prod_{j=1}^6 (1 + d_j t)}{1 + c_1 t + c_2 t^2 + c_3 t^3} |4. \end{aligned}$$

For the other varieties in the theorem we know no other way of determining the degree, but to calculate the equivariant class as in Section 8, then carry out the described substitution.

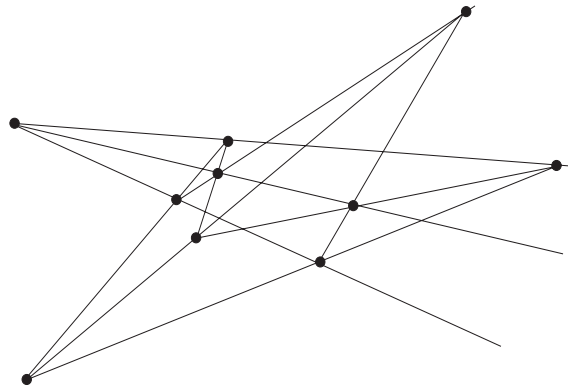


Figure 4. The Desargues configuration  $\mathcal{C}_D$ .

### 5. Matroid versions of linear Gromov–Witten invariants

In the rest of the paper, for simplicity, we will assume that  $\mathcal{C}$  is a configuration of  $k$  non-zero vectors in  $\mathbb{C}^n$ , and that  $Y_{\mathcal{C}}$  is pure dimensional.

For non-negative integers  $q_1, \dots, q_k$  with  $\sum q_i = \text{codim}_{\mathbb{C}}(Y_{\mathcal{C}} \subset \mathbb{C}^{n \times k})$  we define

$$N(\mathcal{C}; q_1, \dots, q_k) = \#\{([v_1], \dots, [v_k]) \in (\mathbb{P} \mathbb{C}^n)^k : (v_1, \dots, v_k) \in Y_{\mathcal{C}}, v_i \in V_i\},$$

where  $V_1, \dots, V_k$  is a generic collection of linear spaces with  $\dim V_i = q_i + 1$ .

More generally, instead of generic linear spaces  $V_i$ , we could have considered varieties of different dimensions and degree. These generalized enumerative problems can be reduced to the linear version above.

**Example 5.1.** Consider the configuration  $\mathcal{C}$  in Figure 5 (a).

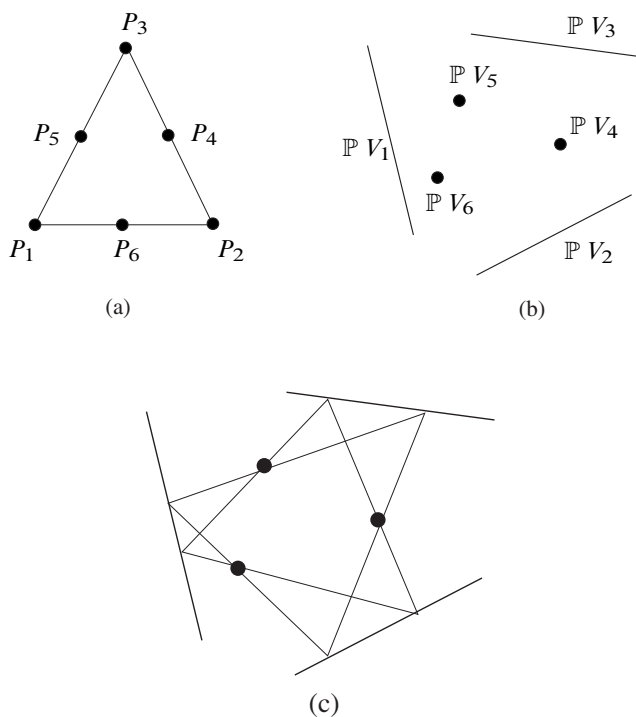


Figure 5. (a)  $\mathcal{C}$ ; (b) the enumerative problem; (c) the solution is 2.

The number  $N(\mathcal{C}; 1, 1, 1, 0, 0, 0)$  is the number of solutions to the following problem: given 3 points and 3 straight lines (generically) on the plane  $\mathbb{P}^2$  (Figure 5 (b)). How many triangles exist, whose vertices are on the straight lines, and whose sides pass through the given points? The solution is 2 (Figure 5 (c)) due to the following well-known argument: Choosing a point  $X$  on  $\mathbb{P} V_1$  we can project it through  $\mathbb{P} V_6$  to  $\mathbb{P} V_2$ , then project further through  $\mathbb{P} V_4$  to  $\mathbb{P} V_3$ , then further through  $\mathbb{P} V_5$  back to

$\mathbb{P} V_1$ , obtaining a point  $X'$ . The map  $X \mapsto X'$  is a projective transformation of the projective line  $\mathbb{P} V_1$ , whose number of fixed points is the question. Since projective transformations have the form  $x \mapsto (ax + b)/(cx + d)$  (in affine coordinate), the number of fixed points is 2. (The construction of the fixed points using a compass and straightedge is the famous Steiner construction, see e.g. [PT01, Addendum 3, p. 167].)

**Example 5.2.** Consider the number  $N(\mathcal{C}_M; 1, 1, 1, 1, 0, 0)$  for the Menelaus configuration of Figure 2. One may try to follow the argument of Example 5.1: choose a point  $X$  on  $\mathbb{P} V_1$ , projecting it through  $\mathbb{P} V_6$  to  $\mathbb{P} V_2$ , then further project through the intersection of  $\mathbb{P} V_4$  and the line  $\overline{\mathbb{P} V_5, \mathbb{P} V_6}$  to  $\mathbb{P} V_3$ , then even further through the point  $\mathbb{P} V_5$ , back to  $\mathbb{P} V_1$ , obtaining  $X'$ . The transformation  $X \mapsto X'$  is a projective transformation of  $\mathbb{P} V_1$ , hence it has 2 fixed points, suggesting that  $N(\mathcal{C}_M; 1, 1, 1, 1, 0, 0) = 2$ . However, this is wrong, the correct number is  $N(\mathcal{C}_M; 1, 1, 1, 1, 0, 0) = 1$ . One of the two fixed points of the transformation  $X \mapsto X'$  is on the “other” component of  $Y_{\text{naive}}$  in (5), not on  $Y_{\mathcal{C}_M}$ . Geometrically, one of the fixed points of the transformation corresponds to all the points lying on the line of  $\mathbb{P} V_5$  and  $\mathbb{P} V_6$ , which configuration does not belong to  $Y_{\mathcal{C}_M}$ . There are configurations (e.g. the Ceva configuration) for which some “extra” components in  $Y_{\text{naive}} \setminus Y_{\mathcal{C}}$  have bigger dimensions than  $Y_{\mathcal{C}}$ . For these, arguments similar to that in Example 5.1 suggest the incorrect  $N(\mathcal{C}; q_1, \dots, q_k) = \infty$ .

If the ideal  $I_{\mathcal{C}}$  is known, determining the numbers  $N(\mathcal{C}; q_1, \dots, q_k)$  reduces to algebraic calculations, which are, at least theoretically, doable. However, as we mentioned, the ideal  $I_{\mathcal{C}}$  is not known in general. The equivariant class defined in Section 4.2 provides an answer.

**Theorem 5.3.** *The coefficient of  $d_1^{q_1} d_2^{q_2} \dots d_k^{q_k}$  in  $[Y_{\mathcal{C}}]$  is*

$$(-1)^{\text{codim } \mathcal{C}} N(\mathcal{C}; q_1, q_2, \dots, q_k).$$

This follows from a more or less standard intersection theoretic argument, which we show in detail in the next section.

**Example 5.4.** The matrix matroid variety corresponding to the configuration of Example 5.1 is a product of matrix Schubert varieties. Hence, its equivariant class is computed by Theorem 4.2 to be

$$(c_1 - d_1 - d_3 - d_5)(c_1 - d_2 - d_3 - d_4)(c_1 - d_1 - d_2 - d_6).$$

The coefficient of  $d_1 d_2 d_3$  is  $-2$ , reproducing the result of Example 5.1.

**Example 5.5.** Special cases of  $N(\mathcal{C}; \mathbf{q})$ 's are solutions to certain so-called *Schubert problems*. We illustrate this with the prototype of Schubert problems: how

many straight lines intersect 4 generic lines in  $\mathbb{P}^3$ ? In our language the answer is  $N(\mathcal{C}_{(0,1,3,0,0)}; 1, 1, 1, 1)$ . According to Theorems 5.3 and 4.2 we have

$$N(\mathcal{C}_{(0,1,3,0,0)}; 1, 1, 1, 1) = \text{coefficient of } d_1d_2d_3d_4$$

in

$$\det \begin{pmatrix} d_1d_2 + \dots + d_4d_5 & d_1d_2d_3 + \dots + d_3d_4d_5 \\ d_1 + \dots + d_4 & d_1d_2 + \dots + d_3d_4 \end{pmatrix},$$

which is clearly  $\binom{4}{2} - \binom{4}{1} = 2$ .

**Remark 5.6.** Certain  $N(\mathcal{C}; \mathbf{q})$  invariants are 0 for obvious reasons. For example, if  $P_1, P_2$ , and  $P_3$  are on one line in the configuration  $\mathcal{C}$ , then  $N(\mathcal{C}; 0, 0, 0, q_4, \dots, q_k) = 0$ . Indeed,  $P_1, P_2$ , and  $P_3$ , being on one line, can not be three *generic* points. Similarly, if there is a subset  $I = \{i_1, \dots, i_s\} \subset [k]$  such that  $\sum(q_{i_j} + 1) > r_{\mathcal{C}}(\{i_1, \dots, i_s\})$ , then obviously  $N(\mathcal{C}; \mathbf{q}) = 0$ . It is easy to see that the existence of such an  $I$  is the only reason for vanishing  $N(\mathcal{C}; \mathbf{q})$ .

The method in Section 8 to calculate  $[Y_{\mathcal{C}}]$  for the configurations  $\mathcal{C}_M$  (Menelaus),  $\mathcal{C}_C$  (Ceva),  $\mathcal{C}_P$  (Pappus), and  $\mathcal{C}_D$  (Desargues) (just like any other configuration we tried) works, leading to the knowledge of all the Gromov–Witten invariants of these configurations. Below we present some information on these invariants.

**Menelaus.** All non-zero coefficients of the pure  $d$  monomials of  $[Y_{\mathcal{C}_M}]$  are 1 – for example the one studied in Example 5.2.

**Ceva.** The same holds for the Ceva configuration: the range of invariants is  $\{0, 1\}$ .

**Pappus.** The range of the  $N(\mathcal{C}_P; \mathbf{q})$  invariants is  $\{0, 1, 2, 3, 4, 5\}$ . Here are some sample results.

- $N(\mathcal{C}_P; 1, 1, 1, 1, 1, 1, 1, 0) = 5$ , that is, the number of Pappus configurations on the plane whose  $i$ 'th vertex is on a pre-described generic line  $l_i$  for  $i = 1, \dots, 8$ , and whose 9'th point is a pre-described generic point, is 5. We know no other way of finding this number.
- $N(\mathcal{C}_P; 2, 0, 0, 1, 1, 1, 1, 1) = 4$ . The argument in Example 5.1 would suggest the wrong answer 5.
- $N(\mathcal{C}_P; 1, 1, 0, 2, 1, 1, 0, 0, 2) = N(\mathcal{C}_P; 1, 1, 1, 1, 1, 0, 0, 1, 2) = 3$ .

**Desargues.** Again, the range of the invariants  $N(\mathcal{C}_D; \mathbf{q})$  is  $\{0, 1\}$ .

It would be interesting to find a geometric interpretation of the property of a configuration, which is equivalent to the condition that the range of  $N(\mathcal{C}; \mathbf{q})$  is  $\{0, 1\}$ .



**6. Proof of Theorem 5.3**

We are going to present a proof of Theorem 5.3 which also proves positivity and enumerative properties of other coefficients.

Let  $\mathcal{C}$  be a configuration of  $k$  vectors in  $\mathbb{C}^n$ , such that  $Y_{\mathcal{C}}$  is a codimension  $l$  subvariety of  $\text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ .

Let  $\tau_n$  be the universal tautological bundle over the Grassmannian  $\text{Gr}_n \mathbb{C}^\infty$  (universal subbundle) which we will approximate with the finite Grassmannian  $\text{Gr}_n \mathbb{C}^N$  for a large  $N \gg n, k$ . Below we will refer to certain numbers as ‘large’; by this we mean that those numbers tend to infinity as  $N \rightarrow \infty$ . The cohomology of the finite Grassmannian is a factor of the cohomology  $H^*(\text{Gr}_n \mathbb{C}^\infty) = \mathbb{Z}[c_1, \dots, c_n]$  by an ideal with large degree generators. In our notations we will ignore this ideal, and identify the two cohomology rings.

Let  $\phi: \text{Gr}_n \mathbb{C}^N \rightarrow \text{Gr}_n \mathbb{C}^N$  be the (necessarily non-holomorphic) classifying map of the dual vector bundle  $\tau_n^*$ , and consider the induced diagram

$$\begin{array}{ccccc}
 \widehat{\Sigma}_{\mathcal{C}} & \hookrightarrow & \text{Hom}(\tau_1^k, \tau_n^*) & \xrightarrow{\psi} & \text{Hom}(\tau_1^k, \tau_n) & \longleftarrow & \Sigma_{\mathcal{C}} \\
 & & \downarrow \xi & & \downarrow & & \\
 & & \text{Gr}_n \mathbb{C}^N \times (\mathbb{P}^{N-1})^k & \xrightarrow{\phi \times \text{id}} & \text{Gr}_n \mathbb{C}^N \times (\mathbb{P}^{N-1})^k & & 
 \end{array}$$

Here  $\psi$  is the map induced by  $\phi \times \text{id}$ , and  $\Sigma_{\mathcal{C}}$  is the collection of the copies of  $Y_{\mathcal{C}}$  in each fiber of the bundle  $\text{Hom}(\tau_1^k, \tau_n)$ . That is, by definition, the cohomology class represented by  $\Sigma_{\mathcal{C}}$  in the cohomology  $H^*(\text{Hom}(\tau_1^k, \tau_n)) = H^*(\text{Gr}_n \mathbb{C}^N \times (\mathbb{P}^{N-1})^k)$  is the equivariant class  $[Y_{\mathcal{C}}]$ . We set  $\widehat{\Sigma}_{\mathcal{C}} = \psi^{-1}(\Sigma_{\mathcal{C}})$ .

The homomorphism

$$(\phi \times \text{id})^*: \mathbb{Z}[c_1, \dots, c_n, d_1, \dots, d_k] \rightarrow \mathbb{Z}[c_1, \dots, c_n, d_1, \dots, d_k]$$

induced by the map  $\phi \times \text{id}$  maps  $c_i$  to  $(-1)^i c_i$ , and  $d_i$  to  $d_i$ . Hence we have that

$$[\widehat{\Sigma}_{\mathcal{C}} \subset \text{Hom}(\tau_1^k, \tau_n^*)] = (\phi \times \text{id})^* [\Sigma_{\mathcal{C}} \subset \text{Hom}(\tau_1^k, \tau_n)] = [Y_{\mathcal{C}}]_{|_{c_i \mapsto (-1)^i c_i}}.$$

Observe that the bundle  $\xi = \text{Hom}(\tau_1^k, \tau_n^*)$  has a large dimensional space of sections. Indeed, let  $\alpha_i: \mathbb{C}^N \rightarrow (\mathbb{C}^N)^*$  be linear maps for  $i = 1, \dots, k$ . Then the map

$$s_{\alpha}: (L^n \leq \mathbb{C}^N, l_1^1, \dots, l_k^1 \leq \mathbb{C}^N) \mapsto \sum_{i=1}^k \iota_L^* \alpha_i \iota_{l_i} \tag{10}$$

is a section of  $\xi$ ; where  $\iota_L: L \rightarrow \mathbb{C}^N$ ,  $\iota_{l_i}: l_i \rightarrow \mathbb{C}^N$  are the canonical inclusions, and  $\iota_L^*: (\mathbb{C}^N)^* \rightarrow L^*$  is the adjoint map.

We want to show that for an appropriate choice of  $\alpha = (\alpha_1, \dots, \alpha_k)$  the section  $s_\alpha$  of  $\xi$  is transversal to  $\widehat{\Sigma}_\mathcal{E}$ . We will use the following straightforward generalization of the classical Bertini theorem.

**Proposition 6.1.** *Let  $E \rightarrow X$  be a smooth vector bundle,  $B$  a vector space and  $\varphi: B \rightarrow \Gamma(E)$  is a linear family of smooth sections. Suppose that  $E$  is generated by the sections  $\varphi(b), b \in B$ , i.e.  $\Phi(b, x) := \varphi(b)(x): B \times X \rightarrow E$  is surjective, and  $Y$  is a smooth submanifold of the total space  $E$ . Then there is a  $b \in B$  such that  $\varphi(b)$  is transversal to  $Y$ .*

For the proof we will use two lemmas.

**Lemma 6.2** ([GG73], II.4.6). *Let  $B, X, Y$  be smooth manifolds,  $\Phi: B \times X \rightarrow Y$  smooth and transversal to the submanifold  $W \subset Y$ . The map  $\Phi$  encodes a family of maps  $\Phi_b(x) = \Phi(b, x): X \rightarrow Y$  for  $b \in B$ . Then*

$$\{b \in B : \Phi_b \pitchfork W\}$$

is dense in  $B$ . □

**Lemma 6.3.** *Let  $E \rightarrow X$  be a smooth vector bundle,  $B$  a vector space, and  $\varphi: B \rightarrow \Gamma(E)$  a linear family of smooth sections. Suppose that  $E$  is generated by the sections  $\varphi(b), b \in B$ , i.e.*

$$\Phi(b, x) := \varphi(b)(x): B \times X \rightarrow E$$

*is surjective. Then  $d\Phi|_{(b,x)}: T_{(b,x)}(B \times X) \rightarrow T_{\Phi(b,x)}E$  is surjective for all  $(b, x) \in B \times X$ .*

*Proof of Lemma 6.3.* The statement is local, so we can assume that  $E = \mathbb{C}^r \times X$ . Then  $\Phi(b, x) = (h(b, x), x)$  for some smooth map  $h: B \times X \rightarrow \mathbb{C}^r$ , which is linear in the  $b$  variable. Then  $\Phi$  is surjective if and only if for all  $x \in X$  the linear map  $h(\cdot, x): B \rightarrow \mathbb{C}^r$  is surjective. On the other hand

$$d\Phi = \begin{pmatrix} \partial_b h & \partial_x h \\ 0 & I \end{pmatrix}$$

and  $(\partial_b|_{(b,x)}h)(v) = h(v, x)$  because of the linearity of  $h$  in  $b$ . Therefore the surjectivity of  $h(\cdot, x)$  implies that the matrix  $\partial_b h$  has full rank, and hence it also implies the surjectivity of  $d\Phi$ . □

*Proof of Proposition 6.1.* We apply Lemma 6.2. and 6.3. Since  $\Phi$  transversal to all points, it is transversal to all submanifolds. □

Proposition 6.1 immediately implies the following algebraic version.

**Proposition 6.4.** *Let  $E \rightarrow X$  be an algebraic vector bundle,  $B$  a vector space, and  $\varphi: B \rightarrow \Gamma(E)$  a linear family of algebraic sections. Suppose that  $E$  is generated by the sections  $\varphi(b), b \in B$ , i.e.  $\Phi(b, x) := \varphi(b)(x): B \times X \rightarrow E$  is surjective, and  $Y$  is a subvariety of the total space  $E$ . Then there is an open subset  $U$  of  $B$  such that for all  $b \in U$  the section  $\varphi(b)$  is transversal to  $Y$ .*

Transversality to a variety means that  $\varphi(b)$  is transversal to all strata of  $Y$  for a complex stratification of  $Y$ . If a map  $f$  of smooth varieties is transversal to a subvariety  $Y$  in this sense then  $f^*([Y]) = [f^{-1}(Y)]$  just as if  $Y$  were smooth.

We mention that if in Proposition 6.4 we take  $B = H^0(E)$ , then we get that for a vector bundle generated by global sections we always have an open subset of the sections which is transversal to any given subvariety of the total space.

**Proposition 6.5.** *The family of sections  $s_\alpha$  (defined in (10)) generates the bundle  $\xi$ .*

*Proof.* Fix  $L^n \subseteq \mathbb{C}^N$  and  $l^1 \subseteq \mathbb{C}^N$  and let  $\psi(\beta) = \iota_L^* \beta \iota_l$  for  $\beta \in \text{Hom}(\mathbb{C}^N, (\mathbb{C}^N)^*)$ . It is enough to show that  $\psi: \text{Hom}(\mathbb{C}^N, (\mathbb{C}^N)^*) \rightarrow \text{Hom}(l, L^*)$  is surjective, which is clear since  $\psi(\beta)$  is an  $n \times 1$  submatrix of  $\beta$  in an appropriate coordinate system. □

Hence we may choose an  $s_\alpha$  transversal to  $\widehat{\Sigma}_{\mathcal{E}}$ . We have that  $V_{\mathcal{E}} = s_\alpha^{-1}(\widehat{\Sigma}_{\mathcal{E}})$  is a codimension  $l$  subvariety of  $\text{Gr}_n \mathbb{C}^N \times (\mathbb{P}^{N-1})^k$ , which represents  $[Y_{\mathcal{E}}]_{c_i \mapsto (-1)^i c_i}$  in the cohomology of  $\text{Gr}_n \mathbb{C}^N \times (\mathbb{P}^{N-1})^k$ .

Now fix a complete flag in  $\mathbb{C}^N$ , and consider the products of Schubert varieties

$$S_{\lambda, \mathbf{q}} = S_\lambda \times (S_{q_1} \times \cdots \times S_{q_k}) \subset \text{Gr}_n \mathbb{C}^N \times (\mathbb{P}^{N-1})^k.$$

Here  $\lambda$  is a partition (with number of parts  $\leq n$ ), and  $S_i \subset \mathbb{P}^{N-1}$  is a linear space of codimension  $i$ , with  $\mathbf{q} = (q_1, \dots, q_k)$ . The cohomology classes represented by the  $S_{\lambda, \mathbf{q}}$ 's form an additive basis of the cohomology group, and by the Giambelli formula of Schubert calculus we know that

$$[S_{\lambda, \mathbf{q}}] = \Delta_\lambda(-c_1, c_2, -c_3, c_4, \dots, (-1)^n c_n) \cdot \prod_{i=1}^k (-d_i)^{q_i}.$$

Recall also that this basis is self dual in the sense that  $\int [S_{\lambda, \mathbf{q}}][S_{\mu, \mathbf{w}}] = 0$  unless  $\mu = \lambda'$ , the complement of  $\lambda$  in the  $n \times (N - n)$  rectangle, and  $\mathbf{w} = \mathbf{q}'$ , i.e.,  $\mathbf{q} + \mathbf{w} = (N, \dots, N)$  (in which case, the integral is 1).

By appropriate choice of the section  $s_\alpha$  we may also assume that  $V_{\mathcal{E}}$  is transversal to the Schubert varieties  $S_{\lambda, \mathbf{q}}$ . Let  $|\lambda| + \sum q_i = \dim(\text{Gr}_n \mathbb{C}^N \times (\mathbb{P}^{N-1})^k) - l$ . Then we have that

$$\#(V_{\mathcal{E}} \cap S_{\lambda, \mathbf{q}}) = \int [V][S_{\lambda, \mathbf{q}}],$$

which is then the coefficient of  $[S_{\lambda', \mathbf{q}'}]$  when  $[Y_{\mathcal{E}}]$  is written as a linear combination of  $[S_{\mu, \mathbf{w}}]$ 's. By rephrasing we obtain the following theorem.

**Theorem 6.6.** *The coefficient of  $\Delta_{\lambda}(c_1, c_2, \dots, c_n) \prod_i (-d_i)^{q_i}$  in  $[Y_{\mathcal{E}}]$ , when the latter is written as a linear combination of classes*

$$\Delta_{\mu}(c_1, c_2, \dots, c_n) \prod_{i=1}^k (-d_i)^{w_i},$$

*is the intersection number  $\#(V_{\mathcal{E}} \cap S_{\lambda', \mathbf{q}'})$ . Hence all coefficients in this linear combination are non-negative.*

Let us now consider the special case of  $\lambda$  being the empty partition, and  $\sum q_i = l$ . We obtain that the coefficient of  $(-1)^l \prod_i d_i^{q_i}$  in  $[Y_{\mathcal{E}}]$  is the intersection number

$$\#(V \cap (\text{point} \times H_1 \times \dots \times H_k)),$$

where  $H_i$  are generic linear subspaces in  $\mathbb{P}^N$  of total codimension  $l$ . Identifying the point in  $\text{Gr}_n \mathbb{C}^N$  with  $\mathbb{C}^n$  this intersection number is the same as the definition of  $N(\mathcal{C}; \mathbf{q})$ . This proves Theorem 5.3.

Theorem 5.3 shows the geometric interpretation of the pure d coefficients of  $[Y_{\mathcal{C}}]$ , but the other extreme, the pure c coefficients are also noteworthy. By the definition of  $\text{GL}(n)$ -equivariant cohomology we have

**Theorem 6.7.** *Suppose  $\pi : E \rightarrow B$  is a rank  $n$  vector bundle with Chern classes  $c_1, \dots, c_n$ . Assume that  $\pi$  has  $k$  sections satisfying a certain transversality property. Then the cohomology class in  $B$  represented by the subvariety*

$$\{b \in B \mid s_1(b), \dots, s_k(b) \text{ form a configuration belonging to } Y_{\mathcal{E}} \subset \pi^{-1}(b)\}$$

*is equal to  $[Y_{\mathcal{E}}]^*$ .*

The transversality property can be easily phrased. Over the real numbers it is a generic property of  $k$ -tuples of sections. Thus one obtains a result on the parity of (the cohomology class represented by) the points over which  $k$  generic sections of a real projective space bundle form a given configuration  $\mathcal{C}$ .

**Remark 6.8.** Certain facts suggest some kind of relations between the  $c$  and the  $d$  variables of  $[Y_{\mathcal{E}}]$ . One of these facts is that either one can be used to calculate the degree of  $Y_{\mathcal{E}}$  – hence they can not be independent. More generally, it can be shown that  $[Y_{\mathcal{E}}]$  can be written as a polynomial of the weights  $\gamma_i - d_j$  of the representation in Section 4.2. Another fact is that for matrix Schubert varieties the pure  $c$  part determines the whole  $[Y_{\mathcal{E}}]$  (up to permutation of indexes). Since the pure  $c$  part of

$[Y_{\mathcal{C}}]$  can be presented in a more compact way in general, it would be interesting to see the relation in general. However, we may not expect that the pure  $c$  part determines the pure  $d$  part for any  $\mathcal{C}$ . For example, let  $\mathcal{C}_1$  be the configuration of 7 points on the projective plane with the collinearities 123, 145, and 167 (and otherwise general). Let  $\mathcal{C}_2$  be with the collinearities 123, 345, 567 (and otherwise general). The pure  $c$  part of the equivariant classes of both of these are  $c_1^3$ . The pure  $d$  parts are essentially different (see Theorem 4.2).

### 7. Stabilization

The interior structure of natural infinite sequences of equivariant cohomology classes of geometrically relevant varieties has remarkable connections with the theory of symmetric functions [Nak99], and iterated residue identities for hyperplane arrangements [BS06], [FR08]. In this section we make the first step towards exploring similar relations for the classes  $[Y_{\mathcal{C}}]$ , by showing the property analogous with the so-called  $d$ -stability property of Thom polynomials of contact singularities, cf. [FR08, Section 7.3]. A byproduct – important in Section 8 – is Theorem 7.4 on the vanishing of certain coefficients of  $[Y_{\mathcal{C}}]$ .

Let  $\mathcal{C}$  be a configuration of  $k$  vectors in  $\mathbb{C}^s$ , spanning  $\mathbb{C}^s$ , and let  $\text{codim}(Y_{\mathcal{C}} \subset \mathbb{C}^{s \times k}) = l$ . For  $n \geq s$  let the configuration  $\mathcal{C}^{\#n}$  be obtained from  $\mathcal{C}$  through the natural embedding of  $\mathbb{C}^s$  into  $\mathbb{C}^n$ . Hence  $\mathcal{C}^{\#n}$  is a configuration of  $k$  points in  $\mathbb{C}^n$ , spanning an  $s$  dimensional subspace. It is easy to check that

$$\text{codim}(Y_{\mathcal{C}^{\#n}} \subset \mathbb{C}^{n \times k}) = l + (k - s)(n - s).$$

Our goal is to compare the classes

$$[Y_{\mathcal{C}}] \in \mathbb{Z}[c_1, \dots, c_s, d_1, \dots, d_k] \quad \text{and} \quad [Y_{\mathcal{C}^{\#n}}] \in \mathbb{Z}[c_1, \dots, c_n, d_1, \dots, d_k]. \quad (11)$$

Clearly  $Y_{\mathcal{C}^{\#n}} \cap \mathbb{C}^{s \times k} = Y_{\mathcal{C}}$ . If this intersection was transversal, or if  $Y_{\mathcal{C}^{\#n}}$  was equal to  $Y_{\mathcal{C}}$ , then the relation between the two equivariant classes would follow from standard notions of equivariant cohomology. None of these is the case; the relation between the two polynomials does not follow from straightforward applications of equivariant cohomology notions. The relation between the two classes must involve nontrivial algebra, since it involves nontrivial geometry – consider for example the case when  $\mathcal{C}$  is the collection of 4 generic vectors in  $\mathbb{C}^2$ . In this case  $[Y_{\mathcal{C}}] = 1$  while  $[Y_{\mathcal{C}^{\#4}}]$  has a term  $2d_1 d_2 d_3 d_4$ , whose coefficient is the solution of the Schubert problem of Example 5.5.

Below we will work with the Chern roots  $\gamma_i$  of  $\text{GL}(m)$  ( $m = s$  or  $n$ ), instead of the Chern classes  $c_i$ . That is, we identify  $H^*(B\text{GL}(m)) = \mathbb{Z}[c_1, \dots, c_m]$  with the symmetric polynomials of  $\gamma_1, \dots, \gamma_m$ ; where the  $i$ 'th elementary symmetric polynomial is  $c_i$ . In our notation, hence,  $Y_{\mathcal{C}}$  may be a polynomial in  $c_i$ 's and  $d_i$ 's, or a

polynomial in  $\gamma_i$ 's and  $d_i$ 's necessarily symmetric in the  $\gamma_i$ 's. If  $S$  is an  $s$ -element subset of  $[n]$ , then  $[Y_{\mathcal{E}}](\gamma_S)$  will denote the value of  $[Y_{\mathcal{E}}]$  if we substitute the variables  $\gamma_i, i \in S$  for  $\gamma_1, \dots, \gamma_s$ .

**Theorem 7.1.** *Let  $\binom{[n]}{s}$  denote the set of  $s$ -element subsets of  $[n]$ . For such a subset  $S$ , let  $\bar{S}$  denote the complement of  $S$  in  $[n]$ . Then we have*

$$[Y_{\mathcal{E}^{\#n}}] = \sum_{S \in \binom{[n]}{s}} \frac{[Y_{\mathcal{E}}](\gamma_S) \cdot \prod_{i \in \bar{S}} \prod_{j=1}^k (\gamma_i - d_j)}{\prod_{i \in \bar{S}} \prod_{j \in S} (\gamma_i - \gamma_j)}. \tag{12}$$

*Proof.* Let  $\varepsilon^k$  denote the trivial bundle of rank  $k$ , and let  $\tau_s$  be the tautological bundle (or rank  $s$ ) over  $\text{Gr}_s \mathbb{C}^n$ . The embedding of bundles  $\tau_s \subset \varepsilon^n$  induces the embedding of bundles  $i : \text{Hom}(\varepsilon^k, \tau_s) \rightarrow \text{Hom}(\varepsilon^k, \varepsilon^n)$ . The maximal torus  $U(1)^n \times U(1)^k$  of  $G_{n,k}$  acts on the following diagram

$$\begin{array}{ccccc}
 Y_{\mathcal{E}}(\xi) \subset \text{Hom}(\varepsilon^k, \tau_s) & \xrightarrow{i} & \text{Hom}(\varepsilon^k, \varepsilon^n) & \xrightarrow{\pi_2} & \text{Hom}(\mathbb{C}^k, \mathbb{C}^n) \supset Y_{\mathcal{E}^{\#n}} \\
 & \searrow \xi & \downarrow \pi_1 & & \\
 & & \text{Gr}_s \mathbb{C}^n & & 
 \end{array}$$

where  $\pi_1$  and  $\pi_2$  are the projections of  $\text{Hom}(\varepsilon^k, \varepsilon^n) = \text{Gr}_s \mathbb{C}^n \times \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ , and  $Y_{\mathcal{E}}(\xi)$  is the collection of the  $Y_{\mathcal{E}}$ -points in each fiber of  $\xi$ . The composition  $\pi_2 \circ i$  is a birational map from  $Y_{\mathcal{E}}(\xi)$  to  $Y_{\mathcal{E}^{\#n}}$ . Therefore we can apply the fibered version of the Atiyah–Bott localization theorem, Theorem (3.8) in [BS06] (see also Proposition 5.1 of [FR08]), and we obtain the theorem.  $\square$

Another relation between the classes (11) stems from the following theorem.

**Theorem 7.2.** *Let  $n > s$ , and  $[Y_{\mathcal{E}^{\#n}}] = \sum \gamma_n^i p_i(\gamma_1, \dots, \gamma_{n-1}, d_1, \dots, d_k)$ . Then*

- (1)  $p_i = 0$  for  $i > k - s$ , and
- (2)  $p_{k-s} = k_{\mathcal{E}} \cdot [Y_{\mathcal{E}^{\#(n-1)}}]$  for an integer  $k_{\mathcal{E}}$ .

*Proof.* Apply Theorem 2.1 of [FR07].  $\square$

To explore the algebraic consequences of Theorem 7.2 we define lowering and raising operators on constant width polynomials.

**Definition 7.3.** The width of a monomial in  $\mathbb{Z}[c_1, \dots, c_n]$  is the number of factors in it. The width of a polynomial is the width of its widest term. Let  $P_w^n$  be the vector space of width  $\leq w$  polynomials in  $\mathbb{Z}[c_1, \dots, c_n]$ . The lowering operator  $L_w^n : P_w^{n+1} \rightarrow P_w^n$  is defined to be the linear extension of

$$L_w^n (c_{i_1} c_{i_2} \dots c_{i_w}) = c_{i_1-1} c_{i_2-1} \dots c_{i_w-1},$$

and  $L_w^n(c_I) = 0$  if the width of  $c_I$  is less than  $w$  ( $c_0$  is defined to be 1).  $L_w^n$  decreases the degree by  $w$ . The raising operator (increasing the degree by  $w$ )  $R_w^n : P_w^n \rightarrow P_w^{n+1}$  is defined by

$$p = \sum_{\lambda} a_{\lambda} \Delta_{\lambda}(c_1, \dots, c_n) \mapsto \sum_{\lambda} a_{\lambda} \Delta_{(\lambda_1+1, \dots, \lambda_w+1)}(c_1, \dots, c_{n+1}),$$

where  $\sum_{\lambda} a_{\lambda} \Delta_{\lambda}$  is the unique expression of  $p$  as a linear combination of Schur polynomials corresponding to partitions with  $w$  parts.

For instance we have

$$L_3^2(c_1 c_2 c_3 + 5c_3^2) = c_1 c_2,$$

and

$$R_3^2(c_1 c_2) = R_3^2(\Delta_{210}(c_1, c_2)) = \Delta_{321}(c_1, c_2, c_3) = c_1 c_2 c_3 - c_3^2.$$

We have the one-sided inverse property  $L_w^n \circ R_w^n = \text{id}$ , but not the other way around.

**Theorem 7.4.** *Let  $\mathcal{C}$  be a configuration of  $k$  vectors in  $\mathbb{C}^n$ , spanning an  $s$  dimensional subspace.*

- *The width of  $[Y_{\mathcal{C}}]^*$  is at most  $k - s$ .*
- *If  $[Y_{\mathcal{C}}]^*$  is written in the Schur basis  $\Delta_{\lambda}$ , then all occurring  $\lambda$  have at most  $k - s$  parts.*

*Proof.* If  $[Y_{\mathcal{C}}]^*$  had a term of width  $i > k - s$ , then  $L_i^n([Y_{\mathcal{C}}]^*)$  would not be 0, contradicting to Theorem 7.2 (1) (cf. [FR07, Corollary 2.5]). This proves the first statement. The second is a combinatorial rephrasing of the first one. □

**Theorem 7.5.** *Let  $n, k \geq s$ , and let us use the notations of Theorem 7.1. Let  $\lambda$  be a partition with at most  $k - s$  parts. Then we have*

$$R_{k-s}^{n-1} \circ \dots \circ R_{k-s}^{s+1} \circ R_{k-s}^s (\Delta_{\lambda}(c_1, \dots, c_s)) = \sum_{S \in \binom{[n]}{s}} \frac{\Delta_{\lambda}(\gamma_S) \cdot \prod_{i \in \bar{S}} \gamma_i^k}{\prod_{i \in \bar{S}} \prod_{j \in S} (\gamma_i - \gamma_j)}. \tag{13}$$

*Proof.* The polynomial  $\Delta_{\lambda}(c_1, \dots, c_s)$  is the equivariant class  $[Y_{\mathcal{C}}]^*$  of an appropriate matrix Schubert variety, according to Theorem 4.2. Then Theorem 7.1 gives that the right hand side is the  $\text{GL}(n)$  equivariant class of another matrix Schubert variety. Checking the indexes, and applying Theorem 4.2 again we obtain the left hand side. □

It would be interesting to find a combinatorial proof of this theorem, possibly along the line of the multivariate Lagrange interpolation formula for symmetric functions [CL96].

Finally, we have the description of the relation between the pure  $c$  parts of the equivariant classes (11).

**Theorem 7.6.** *Let  $\mathcal{C}$  be a configuration of  $k$  vectors in  $\mathbb{C}^s$ , spanning  $\mathbb{C}^s$ . Let  $n \geq s$ , and let  $\mathcal{C}^{\#n}$  be obtained from  $\mathcal{C}$  by the natural embedding  $\mathbb{C}^s \subset \mathbb{C}^n$ . Then  $[Y_{\mathcal{C}^{\#n}}]^*$  has width at most  $k - s$ , and*

$$[Y_{\mathcal{C}^{\#n}}]^* = R_{k-s}^{n-1} \circ \dots \circ R_{k-s}^{s+1} \circ R_{k-s}^s ([Y_{\mathcal{C}}]^*).$$

*Proof.* According to Theorem 7.4 we can express  $[Y_{\mathcal{C}}]^*$  as a linear combination of  $\Delta_\lambda$  polynomials with each  $\lambda$  having at most  $k - s$  parts. Let us apply the operation  $p \mapsto \sum_S \frac{p \cdot \prod_{i \in \bar{S}} \gamma_i^k}{\prod_{i \in \bar{S}} \prod_{j \in S} (\gamma_i - \gamma_j)}$  to this expression. Theorem 7.5 gives our result.  $\square$

In particular the constant  $k_{\mathcal{C}}$  above is 1. Furthermore, the pure  $c$  part of  $[Y_{\mathcal{C}}]$  determines  $[Y_{\mathcal{C}^{\#n}}]$  by adding a  $(k - s) \times (n - s)$  rectangle to each  $\lambda$  in its Schur expansion. The analogous phenomenon for equivariant classes of contact singularities is called the “finiteness of Thom series”, see [FR08]. However, finiteness of Thom series seems more the exception than the rule for contact singularities. The only known finite Thom series correspond to a trivial case (the algebras  $\mathbb{Z}[x_1, \dots, x_n]/(x_1, \dots, x_n)^2$ ), the Giambelli–Thom–Porteous formula.

Denoting the Menelaus configuration considered in a subspace of  $\mathbb{C}^n$  ( $n \geq 3$ ) by  $\mathcal{C}_M^{\#n}$  we obtain that

$$[Y_{\mathcal{C}_M^{\#n}}]^* = 3\Delta_{n-1, n-2, n-2} + 2\Delta_{n-1, n-1, n-3} + 3\Delta_{n, n-2, n-3}.$$

**Remark 7.7.** The only property of the representation of Section 4.2 used in this section was that this representation is a quiver representation. Hence the suitable rephrasing of the localization, width, and vanishing results above are valid for all quiver representations.

### 8. The calculation of equivariant classes of matrix matroid varieties

The standard straightforward methods to calculate equivariant classes of invariant subvarieties – such as the method of resolution or (Gröbner) degeneration – assume more knowledge on the ideal of the variety than we have about the ideal of matrix matroid varieties.

What we can do is list certain properties of the class  $[Y_{\mathcal{C}}]$ , and hope that a computer search will prove that there is only one element of the polynomial ring  $\mathbb{Z}[c_1, \dots, c_n, d_1, \dots, d_k]$  that satisfies all these properties. The main such property – which we will call *interpolation property* – is motivated by methods used in the theory of Thom polynomials of singularities.

**8.1. Interpolation.** For a configuration  $\mathcal{D} \in \mathbb{C}^{n \times k}$  let  $G_{\mathcal{D}}$  denote its stabilizer subgroup in  $G_{n,k}$ . The embedding  $G_{\mathcal{D}} \rightarrow G_{n,k}$  induces a map between classifying



spaces  $BG_{\mathcal{D}} \rightarrow BG_{n,k}$ , and, in turn, a homomorphism between rings of characteristic classes  $\phi_{\mathcal{D}}: H^*(BG_{n,k}) \rightarrow H^*(BG_{\mathcal{D}})$ .

**Theorem 8.1** ([FR04], Theorem 3.2). *If  $\mathcal{D} \notin Y_{\mathcal{E}}$  then  $\phi_{\mathcal{D}}([Y_{\mathcal{E}}]) = 0$ .*

Theorem 8.1 is a homogeneous interpolating condition on  $[Y_{\mathcal{E}}]$ . To obtain a non-trivial condition, however, we need to find a configuration  $\mathcal{D}$ , outside of  $Y_{\mathcal{E}}$ , with reasonably large symmetry group. Let us illustrate the usage of this theorem with an example.

**Example 8.2.** *The calculation of  $[Y_{\mathcal{E}_M}]$ .* Consider the following configuration  $\mathcal{D}_{1|2|6}$ :  $v_1, v_2$ , and  $v_6$  are three generic vectors in  $\mathbb{C}^3$ , while  $v_3 = v_4 = v_5 = 0 \in \mathbb{C}^3$ . Clearly  $\mathcal{D}_{1|2|6}$  is not in the closure of  $X_{\mathcal{E}_M}$ , since for all configurations in  $X_{\mathcal{E}_M}$  the vectors  $v_1, v_2$ , and  $v_6$  are coplanar. Therefore  $\phi_{\mathcal{D}_{1|2|6}}([Y_{\mathcal{E}_M}]) = 0$ . The stabilizer subgroup of  $\mathcal{D}_{1|2|6}$  is  $U(1)^6$  with the embedding into  $G_{3,6}$  via  $(\text{diag}(\alpha, \beta, \gamma), \text{diag}(\alpha, \beta, \delta, \eta, \theta, \gamma))$ . Hence – by abusing language and identifying general elements of a  $U(1)$  with the first Chern class of  $U(1)$  – the map  $\phi_{\mathcal{D}_{1|2|6}}: \mathbb{Z}[c_1, c_1, c_3, d_1, \dots, d_6] \rightarrow \mathbb{Z}[\alpha, \beta, \gamma, \delta, \eta, \theta]$  maps

$$\begin{aligned} c_1 &\mapsto \alpha + \beta + \gamma, & c_2 &\mapsto \alpha\beta + \alpha\gamma + \beta\gamma, & c_3 &\mapsto \alpha\beta\gamma, \\ d_1 &\mapsto \alpha, & d_2 &\mapsto \beta, & d_3 &\mapsto \delta, & d_4 &\mapsto \eta, & d_5 &\mapsto \theta, & d_6 &\mapsto \gamma. \end{aligned}$$

The vanishing of  $[Y_{\mathcal{E}_M}]$  at this map is a non-trivial interpolation property of  $[Y_{\mathcal{E}_M}]$ . In fact, one finds that the degree 4 part of the intersection of the kernels of the  $\phi_{\mathcal{D}_{1|2|6}}$ ,  $\phi_{\mathcal{D}_{1|3|5}}$ ,  $\phi_{\mathcal{D}_{2|3|4}}$ , and  $\phi_{\mathcal{D}_{4|5|6}}$  is 2-dimensional. (This is not surprising in the light of the decomposition (5).) Now let  $\mathcal{D}_{124|356}$  be the following configuration:  $v_1 = v_2 = v_4$  and  $v_3 = v_5 = v_6$  are two different nonzero vectors in  $\mathbb{C}^3$ . This configuration is not in the closure of  $X_{\mathcal{E}_M}$  because of Menelaus’ theorem. Indeed, the left hand side of (4) is  $-1$  for any configuration in  $X_{\mathcal{E}_M}$ , but it is  $\infty$  for  $\mathcal{D}_{124|356}$ . As a consequence,  $[Y_{\mathcal{E}_M}]$  must vanish at the map  $\phi_{\mathcal{D}_{124|356}}: \mathbb{Z}[c_1, c_1, c_3, d_1, \dots, d_6] \rightarrow \mathbb{Z}[\alpha, \beta, \gamma]$ ,

$$\begin{aligned} c_1 &\mapsto \alpha + \beta + \gamma, & c_2 &\mapsto \alpha\beta + \alpha\gamma + \beta\gamma, & c_3 &\mapsto \alpha\beta\gamma, \\ d_1 &\mapsto \alpha, & d_2 &\mapsto \alpha, & d_3 &\mapsto \beta, & d_4 &\mapsto \alpha, & d_5 &\mapsto \beta, & d_6 &\mapsto \beta. \end{aligned}$$

It turns out that in  $\mathbb{Z}[c_1, c_1, c_3, d_1, \dots, d_6]$  there is only a 1-dimensional set of degree 4 polynomials vanishing at the five maps  $\phi_{\mathcal{D}_{1|2|6}}$ ,  $\phi_{\mathcal{D}_{1|3|5}}$ ,  $\phi_{\mathcal{D}_{2|3|4}}$ ,  $\phi_{\mathcal{D}_{4|5|6}}$ , and  $\phi_{\mathcal{D}_{124|356}}$ . Normalization is achieved, for example, by observing that the coefficient of  $d_3^2 d_6^2$  in  $[Y_{\mathcal{E}_M}]$  has to be 1, due to Theorem 5.3.

Observe that the application of Theorem 8.1 to calculate  $[Y_{\mathcal{E}}]$  resonates to the method of determining the ideal of  $Y_{\mathcal{E}}$  discussed at the end of Section 3.2. That is, we first deal with the trivial conditions following from the closed conditions on  $X_{\mathcal{E}}$ ,

then need to work with some extra equations besides these naive ones. What makes the equivariant cohomology calculation easier is that here we do not have to have a full understanding of *all* the fake components, or *all* the extra geometry theorems of the configuration. It is enough to find *some* of these, use these to find an interpolations constraint. And it is clear when we can stop: as soon as we find enough interpolation constraints to cut down the dimension of the solution set to 1, we can be sure we found  $[Y_{\mathcal{C}}]$ .

**8.2. Calculation in practice.** The three main conditions we may use to calculate the equivariant class  $[Y_{\mathcal{C}}]$  are

- the interpolation conditions, Theorem 8.1;
- the enumerative conditions, Theorem 5.3;
- the width condition, Theorem 7.4.

The first one depends on the choice of the test configuration  $\mathcal{D}$ . The second one depends on the choice of the numbers  $\mathbf{q}$ . For certain choices of  $\mathcal{D}$  and  $\mathbf{q}$  these conditions are far from being straightforward, because we do not know whether  $\mathcal{D}$  belongs to  $Y_{\mathcal{C}}$ , or the number  $N(\mathcal{C}; \mathbf{q})$ . For some other choices, however, simple arguments answer these questions, and hence we have explicit constraints of  $[Y_{\mathcal{C}}]$ . See, for example, the calculation of  $[Y_{\mathcal{C}_M}]$  above.

It is quite possible that the interpolation conditions themselves are enough to determine the equivariant class  $[Y_{\mathcal{C}}]$  up to a scalar. For some other representations the analogous statement is a theorem, e.g. [FR04, Theorem 3.5]. However the proof there depends on a condition of the representation (called Euler condition in [FR04, Definition 3.3], closely related to the “equivariantly perfect” condition of [AB83, Section 1.]). This condition does not hold for the representation of Section 4.2.

What works in practice, is the combination of the three constraints. For all the configurations the authors considered (many more than the ones presented in this paper) there is only one polynomial of degree  $\text{codim } Y_{\mathcal{C}}$  in  $\mathbb{Z}[c_1, \dots, c_n, d_1, \dots, d_k]$  satisfying the simple straightforward constraints obtained from interpolation and enumeration, together with the width condition. We conjecture this holds for all configurations.

## 9. Hierarchy

The interpolation method highlights the importance of the hierarchy of the sets  $Y_{\mathcal{C}}$ . In fact, the effective usage of the interpolation method to calculate  $[Y_{\mathcal{C}}]$  assumes that we have another configuration  $\mathcal{D}$  such that  $\mathcal{D} \notin Y_{\mathcal{C}}$ . To indicate the non-triviality of this problem we challenge the reader with the problem of deciding whether the configuration  $\mathcal{D}_{134|256}$  is contained in  $Y_{\mathcal{C}_C}$  (for notations see Figure 2 and Example 8.2).

The hierarchy of the sets  $X_{\mathcal{C}}$  is not “normal”, in the sense that there are examples of configurations  $\mathcal{C}$  and  $\mathcal{D}$  such that certain points of  $X_{\mathcal{D}}$  are in the closure of  $X_{\mathcal{C}}$ , some others are not. A small example is  $\mathcal{C} = \mathcal{C}_M$ ,  $\mathcal{D} = 6$  points on one line. Hence, we restrict our attention to the case when  $\mathcal{D}$  is an orbit of the action in Section 4.2. In this case Theorem 8.1 yields the following: if  $\phi_{\mathcal{D}}([Y_{\mathcal{C}}]) \neq 0$  then  $Y_{\mathcal{D}} \subset Y_{\mathcal{C}}$ .

The vanishing of  $\phi_{\mathcal{D}}([Y_{\mathcal{C}}])$  has no chance of determining the adjacency of  $Y_{\mathcal{C}}$  and  $Y_{\mathcal{D}}$  if the stabilizer group  $G_{\mathcal{D}}$  is trivial (i.e. it is the kernel of the representation,  $U(1)$ ). However, when  $G_{\mathcal{D}}$  is larger, we have found no counterexample to the following conjecture.

**Conjecture 9.1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be configurations of  $k$  points in  $\mathbb{C}^n$ . Suppose that projectivizations of the non-zero vectors in  $\mathcal{D}$  form a projectively independent set (hence of cardinality  $\leq n$ ). Then

$$Y_{\mathcal{D}} \subset Y_{\mathcal{C}} \iff \phi_{\mathcal{D}}([Y_{\mathcal{C}}]) \neq 0.$$

Together with the effective algorithm of Section 8.2 computing  $[Y_{\mathcal{C}}]$ , this conjecture would serve as a computable criterion of hierarchy, cf. [FP07].

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