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On the Kazhdan–Lusztig order on cells and families

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To Nicolas Spaltenstein on the occasion of his 60th birthday

Abstract. We consider the set $\mathrm{Irr}(W)$ of (complex) irreducible characters of a finite Coxeter group W. The Kazhdan–Lusztig theory of cells gives rise to a partition of $\mathrm{Irr}(W)$ into "families" and to a natural partial order $\leqslant_{\mathcal{LR}}$ on these families. Following an idea of Spaltenstein, we show that $\leqslant_{\mathcal{LR}}$ can be characterised (and effectively computed) in terms of standard operations in the character ring of W. If, moreover, W is the Weyl group of an algebraic group G, then $\leqslant_{\mathcal{LR}}$ can be interpreted, via the Springer correspondence, in terms of the closure relation among the "special" unipotent classes of G.

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1. Introduction

Let $\mathrm{Irr}(W)$ be the set of (complex) irreducible characters of a finite Coxeter group W. There is a natural partition $\mathrm{Irr}(W) = \bigsqcup_{\mathcal{F}} \mathrm{Irr}(W \mid \mathcal{F})$ where $\mathcal{F}$ runs over the two-sided cells of W in the sense of Kazhdan–Lusztig [23]. This partition is an important ingredient in the fundamental work of Lusztig [26] on the characters of reductive groups over finite fields. Using some standard operations in the character ring of W (truncated induction from parabolic subgroups, tensoring with the sign character), Lusztig has defined another partition of $\mathrm{Irr}(W)$ into so-called "families". As shown in [26, Chap. 5] (see also [31, Chap. 23]), these two partitions turn out to be the same. The proof relies on deep results from algebraic geometry which provide certain "positivity" properties of the Kazhdan–Lusztig basis [23] of the associated Iwahori–Hecke algebra.

Now, the theory of Kazhdan–Lusztig cells gives rise not only to the partition $\mathrm{Irr}(W) = \bigsqcup_{\mathcal{F}} \mathrm{Irr}(W \mid \mathcal{F})$, but also to a natural partial order $\leqslant_{\mathcal{LR}}$ on the pieces in this partition. For example, if W is the symmetric group $\mathfrak{S}_n$, then $\mathrm{Irr}(W)$ is parametrized by the partitions of n, all families are singleton sets, and $\leqslant_{\mathcal{LR}}$ corresponds to the dominance order on partitions; see [14] and the references there. This is the prototype of a picture which applies to any finite W.

The main purpose of this paper is to obtain a better understanding of the partial order $\leqslant_{\mathcal{LR}}$. This will be relevant in a number of applications; we just mention, for example, that $\leqslant_{\mathcal{LR}}$ is a crucial ingredient in defining a "cellular structure" (in the sense of Graham–Lehrer [22]) of the associated Iwahori–Hecke algebra [15]. Our first main result will show that $\leqslant_{\mathcal{LR}}$ can be characterised in a purely elementary way in terms of standard operations in the character ring of W (induction, truncated induction, tensoring with sign), similar in spirit to Lusztig's definition of families. In particular, we obtain an efficient algorithm for computing the partial order, which can be implemented in CHEVIE [17]. We conjecture that this remains valid in the more general framework of Lusztig [25], [31] where "weights" may be attached to the generators of W. (We provide both theoretical and experimental evidence for this conjecture.)

The main inspiration for this work is a paper by Spaltenstein [36]. By pushing the ideas in [36] a little bit further, and combining them with the above characterisation of $\leqslant_{\mathcal{LR}}$, we obtain our second main result:

If W is the Weyl group of an algebraic group G, then the partial order $\leqslant_{\mathcal{LR}}$ on the families of $\mathrm{Irr}(W)$ *can be interpreted, via the Springer correspondence, in terms of the closure relation among the "special" unipotent classes of G.*

This paper is organised as follows. We recall the basic definitions on cells and families in Section 2. Here, we work in the general framework of Iwahori–Hecke algebras with unequal parameters, taking into account "weight functions" as in [25], [31]. In Definition 2.10 and Conjecture 2.12, we propose our alternative description of $\leqslant_{\mathcal{LR}}$ (in the form of an equivalence). In Section 3, we prove at least one implication in that conjectured equivalence in the general case of unequal parameters; see Proposition 3.4. This is followed by the discussion of some examples in which the reverse implication can be seen to hold by elementary methods. In Section 4, we concentrate on the equal parameter case and complete the proof of Conjecture 2.12 in that case. This allows us to discuss in Section 5 the relation with unipotent classes and the work of Spaltenstein [36].

It would be interesting to understand how our results in Section 5 are related to work of Bezrukavnikov [4, Theorem 4]. In a completely different direction, by work of Broué, Chlouveraki, Kim, Malle, Rouquier (see [8]), there is also a notion of "families" for the irreducible characters of finite complex reflection groups. It would be interesting to see if it is possible to define a partial order on these families as well. (As Jean Michel has pointed out to me, one cannot simply adopt the definitions in this paper.)

2. Kazhdan–Lusztig cells and families

Let W be a finite Coxeter group, with generating set S and corresponding length function $l\colon W \to \mathbb{Z}_{\geqslant 0}$. Let Γ be an abelian group (written additively) and $L\colon W \to \Gamma$ be a weight function, that is, we have $L(ww') = L(w)+L(w')$ whenever $w, w' \in W$ are such that $l(ww') = l(w) + l(w')$. Let $F \subseteq \mathbb{C}$ be a splitting field for W and $A = F[\Gamma]$ be the F-vector space with basis $\{v^g \mid g \in \Gamma\}$. There is a well-defined ring structure on A such that $v^g v^{g'} = v^{g+g'}$ for all $g, g' \in \Gamma$. Let $\mathbf{H} = \mathbf{H}_A(W, S, L)$ be the corresponding generic Iwahori–Hecke algebra over A with parameters $\{v_s \mid s \in S\}$ where $v_s := v^{L(s)}$ for $s \in S$. This is an associative algebra which is free as an A-module, with basis $\{T_w \mid w \in W\}$. The multiplication is given by the rule

$$T_sT_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v_s - v_s^{-1})T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. See [21], [25], [31] for further details.

We assume that there exists a total ordering $\leqslant$ of Γ which is compatible with the group structure, that is, whenever $g, g', h \in \Gamma$ are such that $g \leqslant g'$, then $g + h \leqslant g'+h$. This implies that A is an integral domain; we denote by K its field of fractions. Throughout this paper, we assume that

$$L(s) \geqslant 0 \quad \text{for all } s \in S.$$

We define $\Gamma_{\geqslant 0} = \{g \in \Gamma \mid g \geqslant 0\}$ and denote by $\mathbb{Z}[\Gamma_{\geqslant 0}] \subseteq A$ the set of all integral linear combinations of terms v^g where $g \geqslant 0$. The notations $\mathbb{Z}[\Gamma_{>0}]$, $\mathbb{Z}[\Gamma_{\leqslant 0}]$, $\mathbb{Z}[\Gamma_{<0}]$ have a similar meaning.

Example 2.1. Let $\Gamma = \mathbb{Z}$ and $\leqslant$ be the natural order. (This is the setting of Lusztig [31].) Then A is nothing but the ring of Laurent polynomials over F in the indeterminate v. We have $K = F(v)$. If, furthermore, we have $L(s) = 1$ for all $s \in S$, then we say that we are in the "equal parameter case".

Returning to the general case, let $\{C_w \mid w \in W\}$ be the Kazhdan–Lusztig basis of $\mathbf{H}$; see [23], [25], [31]. The element C_w is characterised by the property that (a) it is fixed by a certain ring involution of $\mathbf{H}$ and (b) it is congruent to T_w modulo $\sum_{y\in W} \mathbb{Z}[\Gamma_{>0}]T_y$. (This is the original convention used in [23], [25].) Let $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$ be the Kazhdan–Lusztig pre-order relations on W; for any $w \in W$, we have

$$\mathbf{H}C_w \subseteq \sum_{y\leqslant_{\mathcal{L}} w} \mathbb{Z}[\Gamma]C_y, \quad C_w\mathbf{H} \subseteq \sum_{y\leqslant_{\mathcal{R}} w} \mathbb{Z}[\Gamma]C_y, \quad \mathbf{H}C_w\mathbf{H} \subseteq \sum_{y\leqslant_{\mathcal{LR}} w} \mathbb{Z}[\Gamma]C_y.$$

Let $\sim_{\mathcal{L}}$, $\sim_{\mathcal{R}}$, $\sim_{\mathcal{LR}}$ be the associated equivalence relations on W. Thus, given $x, y \in W$, we have $x \sim_{\mathcal{L}} y$ if and only if $x \leqslant_{\mathcal{L}} y$ and $y \leqslant_{\mathcal{L}} x$. (Similarly for

$\sim_{\mathcal{R}}$ and $\sim_{\mathcal{LR}}$.) The corresponding equivalence classes are called "left cells", "right cells" and "two-sided cells", respectively. Note that all these notions depend on the weight function L and the total ordering of Γ.

Let $\mathfrak{C}$ be a left cell and set $[\mathfrak{C}]_A := \mathfrak{I}_{\mathfrak{C}}/\hat{\mathfrak{I}}_{\mathfrak{C}}$ where

$$\mathfrak{I}_{\mathfrak{C}} = A\text{-span}\{C_y \mid y \leqslant_{\mathcal{L}} w \text{ for some } w \in \mathfrak{C}\}$$
$$\hat{\mathfrak{I}}_{\mathfrak{C}} = A\text{-span}\{C_y \mid y \leqslant_{\mathcal{L}} w \text{ for some } w \in \mathfrak{C}, \text{ but } y \notin \mathfrak{C}\}.$$

Since $\mathfrak{I}_{\mathfrak{C}}$ and $\hat{\mathfrak{I}}_{\mathfrak{C}}$ are left ideals in $\mathbf{H}$, the quotient $[\mathfrak{C}]_A$ is a left $\mathbf{H}$-module with a canonical A-basis indexed by the elements of $\mathfrak{C}$. Extending scalars from A to F via the F-algebra homomorphism $\theta_1 \colon A \to F$ sending all v^g to 1 ($g \in \Gamma$), we obtain a left $F[W]$-module $[\mathfrak{C}]_1 := F \otimes_A [\mathfrak{C}]_A$. We have a direct sum decomposition of left $F[W]$-modules

$$F[W] \cong \bigoplus_{\mathfrak{C} \text{ left cell in } W} [\mathfrak{C}]_1.$$

Now let us denote by $\mathrm{Irr}(W)$ the set of irreducible representations of W over F (up to isomorphism); recall that F is assumed to be a splitting field for W. Let $E \in \mathrm{Irr}(W)$. Since we have the above direct sum decomposition, there exists a left cell $\mathfrak{C}$ such that E is a constituent of $[\mathfrak{C}]_1$; furthermore, all such left cells are contained in the same two-sided cell. This two-sided cell, therefore, only depends on E and will be denoted by $\mathcal{F}_E$. Thus, we obtain a natural surjective map

$$\mathrm{Irr}(W) \to \{\text{set of two-sided cells of } W\}, \qquad E \mapsto \mathcal{F}_E.$$

(See Lusztig [26, 5.15] for the equal parameter case; the same argument works in general.) It will be useful to introduce the following notation. Let X, Y be any subsets of W. Then we write $X \leqslant_{\mathcal{LR}} Y$ if $x \leqslant_{\mathcal{LR}} y$ for all $x \in X$ and $y \in Y$.

Definition 2.2 (Lusztig [26]). Let $E, E' \in \mathrm{Irr}(W)$. We write $E \leqslant_{\mathcal{LR}} E'$ if $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{E'}$. This defines a pre-order relation on $\mathrm{Irr}(W)$. We write $E \sim_{\mathcal{LR}} E'$ if $E \leqslant_{\mathcal{LR}} E'$ and $E' \leqslant_{\mathcal{LR}} E$ or, equivalently, if $\mathcal{F}_E = \mathcal{F}_{E'}$. Thus, we obtain a partition

$$\mathrm{Irr}(W) = \bigsqcup_{\mathcal{F} \text{ two-sided cell}} \mathrm{Irr}(W \mid \mathcal{F}),$$

where $\mathrm{Irr}(W \mid \mathcal{F})$ consists of all $E \in \mathrm{Irr}(W)$ such that $\mathcal{F}_E = \mathcal{F}$.

Remark 2.3. Let $w_0 \in W$ be the longest element and ε be the sign representation of W. If $\mathfrak{C}$ is a left cell in W, then so is $\mathfrak{C}w_0$ and we have

$$[\mathfrak{C}w_0]_1 \cong [\mathfrak{C}]_1 \otimes \varepsilon.$$

(See Lusztig [26, Lemma 5.14] and [10, Cor. 2.8].) Furthermore, multiplication by w_0 reverses the relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$; see [31, Cor. 11.7]. It follows that, for all $E, E' \in \mathrm{Irr}(W)$, we have:

(a) $\mathcal{F}_{E\otimes\varepsilon} = \mathcal{F}_E\, w_0$.

(b) $E \leqslant_{\mathcal{LR}} E'$ if and only if $E' \otimes \varepsilon \leqslant_{\mathcal{LR}} E \otimes \varepsilon$.

Thus, tensoring with ε induces an order-reversing bijection on the sets $\mathrm{Irr}(W \mid \mathcal{F})$.

In order to describe Lusztig's alternative characterisation of the sets $\mathrm{Irr}(W \mid \mathcal{F})$, we need to introduce some further notation. Recall that K is the field of fractions of $A = F[\Gamma]$. By extension of scalars, we obtain a K-algebra $\mathbf{H}_K = K \otimes_A \mathbf{H}$ which is known to be split semisimple; see [21, 9.3.5]. Furthermore, by Tits' Deformation Theorem, the irreducible representations of $\mathbf{H}_K$ (up to isomorphism) are in bijection with the irreducible representations of W; see [21, 8.1.7]. Given $E \in \mathrm{Irr}(W)$, we denote by E_v the corresponding irreducible representation of $\mathbf{H}_K$. This is uniquely characterised by the following condition:

$$\theta_1(\mathrm{trace}(T_w, E_v)) = \mathrm{trace}(w, E) \quad \text{for all } w \in W,$$

where $\theta_1 \colon A \to F$ is as above. Note also that $\mathrm{trace}(T_w, E_v) \in A$ for all $w \in W$.

Definition 2.4 (Lusztig). Given $E \in \mathrm{Irr}(W)$, we define

$$\boldsymbol{a}_E := \min\{g \in \Gamma_{\geqslant 0} \mid v^g \mathrm{trace}(T_w, E_v) \in F[\Gamma_{\geqslant 0}] \text{ for all } w \in W\}$$

Furthermore, we define numbers $c_{w,E} \in F$ by

$$\mathrm{trace}(T_w, E_v) = c_{w,E}\, v^{-\boldsymbol{a}_E} + \text{combination of terms } v^g \text{ where } g > -\boldsymbol{a}_E.$$

(In the equal parameter case, these definitions were given by Lusztig [26, (5.1.21)]. The same definitions work in general; see also [10]). The following result shows that the numbers $c_{w,E}$ can, in fact, be used to detect the two-sided cell $\mathcal{F}_E$.

Lemma 2.5 (Lusztig). *We have* $\varnothing \neq \{w \in W \mid c_{w,E} \neq 0\} \subseteq \mathcal{F}_E$ *for all* $E \in \mathrm{Irr}(W)$.

(See Lusztig [26, Lemma 5.2] for the equal parameter case; the same arguments also work in general. For more details in the general case, see [10, Prop. 4.7].)

Now let $I \subseteq S$ and consider the parabolic subgroup $W_I \subseteq W$ generated by I. Then we have a corresponding parabolic subalgebra $\mathbf{H}_I \subseteq \mathbf{H}$. By extension of scalars from A to K, we also have a subalgebra $\mathbf{H}_{K,I} = K \otimes_A \mathbf{H}_I \subseteq \mathbf{H}_K$. The above definitions (i.e., $\boldsymbol{a}_E, c_{w,E}, \ldots$) apply to the irreducible representations of W_I as well. Denote by Ind_I^S the induction of representations, either from W_I to W or from $\mathbf{H}_I$ to $\mathbf{H}$.

Lemma 2.6 (Lusztig). *Let* $M \in \mathrm{Irr}(W_I)$.

(a) *If* $E \in \mathrm{Irr}(W)$ *is a constituent of* $\mathrm{Ind}_I^S(M)$, *then* $\boldsymbol{a}_E \geqslant \boldsymbol{a}_M$.

(b) *There exists some* $E \in \mathrm{Irr}(W)$ *which is a constituent of* $\mathrm{Ind}_I^S(M)$ *and such that* $\boldsymbol{a}_E = \boldsymbol{a}_M$.

(See Lusztig [24] in the equal parameter case; the same arguments work in general. See [10, Lemma 3.5] for details.)

We now recall Lusztig's definition of families. Let $E \in \mathrm{Irr}(W)$ and $M \in \mathrm{Irr}(W_I)$. We write $M \rightsquigarrow_L E$ if E is a constituent of $\mathrm{Ind}_I^S(M)$ and $\boldsymbol{a}_E = \boldsymbol{a}_M$.

Definition 2.7 (Lusztig [26, 4.2]). The partition of $\mathrm{Irr}(W)$ into "families" is defined as follows. When $W = \{1\}$, there is only one family; it consists of the unit representation of W. Assume now that $W \neq \{1\}$ and that families have already been defined for all proper parabolic subgroups of W. Then $E, E' \in \mathrm{Irr}(W)$ are said to be in the same family for $\mathrm{Irr}(W)$ if there exists a sequence $E = E_0, E_1, \ldots, E_m = E'$ in $\mathrm{Irr}(W)$ such that, for each $i \in \{1, 2, \ldots, m\}$, the following condition is satisfied. There exists a subset $I_i \subsetneqq S$ and $M_i', M_i'' \in \mathrm{Irr}(W_{I_i})$, where M_i', M_i'' belong to the same family of $\mathrm{Irr}(W_{I_i})$, such that either

$$M_i' \rightsquigarrow_L E_{i-1} \quad \text{and} \quad M_i'' \rightsquigarrow_L E_i$$

or

$$M_i' \rightsquigarrow_L E_{i-1} \otimes \varepsilon \quad \text{and} \quad M_i'' \rightsquigarrow_L E_i \otimes \varepsilon.$$

Note that it is clear from this definition that tensoring with the sign representation permutes the families.

We can now state the following remarkable theorem. One of its applications is that it facilitates the explicit determination of the partition of $\mathrm{Irr}(W)$ in Definition 2.2; see Lusztig [26, Chap. 4].

Theorem 2.8 (Barbasch–Vogan, Lusztig [26, 5.25]). *Assume that* W *is a finite Weyl group and that we are in the equal parameter case. Let* $E, E' \in \mathrm{Irr}(W)$. *Then* $E \sim_{\mathcal{LR}} E'$ (*see Definition* 2.2) *if and only if* E, E' *belong to the same family* (*see Definition* 2.7).

Remark 2.9. The "if" part of the above result is proved by elementary methods; see [26, Chap. 5]. Our Proposition 3.4 below provides a new proof for this "if" part, which also works in the general multi-parameter case. The proof of the "only if" part in [26] relies on deep results from the theory of primitive ideals in enveloping algebras (which also explains the restriction to Weyl groups). An alternative approach is provided by [31, 23.3] and [13] where it is shown that the above theorem holds for any finite W and any weight function $L\colon W \to \Gamma$, assuming that Lusztig's conjectures **P1**–**P15** in [31, 14.2] are satisfied. This is known to be true for all finite Coxeter groups in the equal parameter case (see the comments on the proof of Theorem 4.1 below); it is also true for a number of situations involving unequal parameters. For a summary of the present state of knowledge, see [16, §5] and the references there.

Our aim is to find an alternative description of the pre-order $\leqslant_{\mathcal{LR}}$ on $\operatorname{Irr}(W)$, in the spirit of Lusztig's definition of families. The following definition is inspired by Spaltenstein [36].

Definition 2.10. We define a relation $\preceq$ on $\operatorname{Irr}(W)$ inductively as follows. If $W = \{1\}$, then $\operatorname{Irr}(W)$ only consists of the unit representation and this is related to itself. Now assume that $W \neq \{1\}$ and that $\preceq$ has already been defined for all proper parabolic subgroups of W. Let $E, E' \in \operatorname{Irr}(W)$. Then we write $E \preceq E'$ if there is a sequence $E = E_0, E_1, \ldots, E_m = E'$ in $\operatorname{Irr}(W)$ such that, for each $i \in \{1, 2, \ldots, m\}$, the following condition is satisfied. There exists a subset $I_i \subsetneqq S$ and $M_i', M_i'' \in \operatorname{Irr}(W_{I_i})$, where $M_i' \preceq M_i''$ within $\operatorname{Irr}(W_{I_i})$, such that either

$$E_{i-1} \text{ is a constituent of } \operatorname{Ind}_{I_i}^S(M_i') \quad \text{and} \quad M_i'' \rightsquigarrow_L E_i$$

or

$$E_i \otimes \varepsilon \text{ is a constituent of } \operatorname{Ind}_{I_i}^S(M_i') \quad \text{and} \quad M_i'' \rightsquigarrow_L E_{i-1} \otimes \varepsilon.$$

We note that, as in [26, 4.2], it is enough to require that, in the above definition, we have $|I_i| = |S| - 1$ for all i (that is, each W_{I_i} is a maximal parabolic subgroup).

Remark 2.11. Let $E, E' \in \operatorname{Irr}(W)$. It is clear from the above definition that we have the following implications:

(a) If E, E' belong to the same family then $E \preceq E'$ and $E' \preceq E$.

(b) If $E \preceq E'$, then we also have $E' \otimes \varepsilon \preceq E \otimes \varepsilon$.

The reverse implication in (a) does not seem to follow easily from the definitions. In Proposition 4.4, we will establish that reverse implication in the equal parameter case; the general multi-parameter case requires further work and will be dealt with in [19, Cor. 9.2].

By analogy with Theorem 2.8, we would now like to state the following:

Conjecture 2.12. *Let $E, E' \in \operatorname{Irr}(W)$. Then $E \leqslant_{\mathcal{LR}} E'$ (see Definition 2.2) if and only if $E \preceq E'$ (see Definition 2.10).*

In Section 3, we will prove the "if" part of the conjecture by a general argument (for any weight function $L: W \to \Gamma$ as above). In particular, as already announced in Remark 2.9, this will provide a new, completely elementary proof of the "if" part of Theorem 2.8. We also verify in some examples that the reverse implications hold. In Section 4, we will prove the "only if" part of the conjecture by a general argument, assuming that we are in the equal parameter case.

3. Two-sided cells and induced representations

We keep the setting of the previous section, where W is a finite Coxeter group and $L\colon W \to \Gamma$ is any weight function such that $L(s) \geqslant 0$ for all $s \in S$.

Given a subset $I \subseteq S$, let W_I be the corresponding parabolic subgroup of W and X_I be the set of distinguished left coset representatives of W_I in W. Thus, we have a bijection $X_I \times W_I \to W$, $(d, w) \mapsto dw$, where $l(dw) = l(d) + l(w)$; see §2.1 of [21]. In the following discussion, we shall make frequent use of the main result of [11], concerning the induction of cells from W_I to W.

Lemma 3.1. *Let $x \in W$ and write $x = dw$ where $d \in X_I$ and $w \in W_I$. Let $E \in \mathrm{Irr}(W)$ be a constituent of $[\mathfrak{C}]_1$ where $\mathfrak{C}$ is the left cell of W which contains x; in particular, $x \in \mathcal{F}_E$. Then there exists some $M \in \mathrm{Irr}(W_I)$ such that $w \in \mathcal{F}_M$ and E is a constituent of $\mathrm{Ind}_I^S(M)$.*

Proof. Let $\mathfrak{C}'$ be the left cell in W_I which contains w. Then, by [11, Theorem 1], we have $\mathfrak{C} \subseteq X_I\mathfrak{C}'$; furthermore, by [13, Lemma 5.2], $[\mathfrak{C}]_1$ is a direct summand of $\mathrm{Ind}_I^S([\mathfrak{C}']_1)$. Hence, since $E \in \mathrm{Irr}(W)$ is a constituent of $[\mathfrak{C}]_1$, there exists some $M \in \mathrm{Irr}(W_I)$ such that M is a constituent of $[\mathfrak{C}']_1$ and E is a constituent of $\mathrm{Ind}_I^S(M)$. Since $w \in \mathfrak{C}'$, we also have $w \in \mathcal{F}_M$, as required. □

Recall that, for any subsets X, Y of W, we write $X \leqslant_{\mathcal{LR}} Y$ if $x \leqslant_{\mathcal{LR}} y$ for all $x \in X$ and $y \in Y$.

Lemma 3.2. *Let $E \in \mathrm{Irr}(W)$ and $M \in \mathrm{Irr}(W_I)$ be such that E is a constituent of $\mathrm{Ind}_I^S(M)$. Then we have $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_M$.*

Proof. Let $\mathfrak{C}'$ be a left cell in W_I such that M is a constituent of $[\mathfrak{C}']_1$. As in the above proof, by [11, Theorem 1], we have a partition $X_I\mathfrak{C}' = \bigsqcup_{i=1}^m \mathfrak{C}_i$ where $\mathfrak{C}_1, \dots, \mathfrak{C}_m$ are left cells of W. Furthermore, by [13, Lemma 5.2], we have $\mathrm{Ind}_I^S([\mathfrak{C}']_1) = \bigoplus_{i=1}^m [\mathfrak{C}_i]_1$. Hence, since E is a constituent of $\mathrm{Ind}_I^S(M)$, there exists some i such that E is a constituent of $[\mathfrak{C}_i]_1$. Let $\mathfrak{C} := \mathfrak{C}_i$. Now note that $l(xw) = l(x) + l(w)$ for all $x \in X_I$ and $w \in W_I$. This length condition implies that $xw \leqslant_{\mathcal{L}} w$ for all $x \in X_I$ and $w \in W_I$; see [31, Theorem 6.6]. Hence, we have $w \leqslant_{\mathcal{L}} w'$ for all $w \in \mathfrak{C}$ and $w' \in \mathfrak{C}'$. Since $\mathfrak{C}' \subseteq \mathcal{F}_M$ and $\mathfrak{C} \subseteq \mathcal{F}_E$, this implies that $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_M$, as required. □

A special case of the following result appeared in [14, Lemma 3.6].

Lemma 3.3. *Let $E \in \mathrm{Irr}(W)$ and $M \in \mathrm{Irr}(W_I)$ be such that $M \rightsquigarrow_L E$. Then we have $\mathcal{F}_M \subseteq \mathcal{F}_E$.*

Proof. The algebra $\mathbf{H}$ is symmetric, with trace form $\tau\colon \mathbf{H} \to A$ given by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $1 \neq w \in W$. The sets $\{T_w \mid w \in W\}$ and $\{T_{w^{-1}} \mid w \in W\}$ form a pair of dual bases. Hence we have the following orthogonality relations:

$$\sum_{w \in W} \operatorname{trace}(T_w, E_v) \operatorname{trace}(T_{w^{-1}}, E'_v) = \begin{cases} (\dim E)\, \boldsymbol{c}_E & \text{if } E \cong E', \\ 0 & \text{otherwise;} \end{cases}$$

see [21, 8.1.8]. Here, $0 \neq \boldsymbol{c}_E \in A$ and, as observed by Lusztig, we have

$$\boldsymbol{c}_E = f_E\, v^{-2\boldsymbol{a}_E} + \text{combination of terms } v^g \text{ where } g > -2\boldsymbol{a}_E,$$

where f_E is a strictly positive real number; see [10, 3.3]. The same definitions apply, of course, to the parabolic subalgebra $\mathbf{H}_I$. Now consider the element

$$e_M := \sum_{w \in W_I} \operatorname{trace}(T_w, M_v)\, T_{w^{-1}} \in \mathbf{H}_{K,I}.$$

We shall evaluate $\operatorname{trace}(e_M, E_v)$ in two ways. On the one hand, given $E' \in \operatorname{Irr}(W)$, let us denote by $d(E', M)$ the multiplicity of E' as a constituent of $\operatorname{Ind}_I^S(M)$. By Frobenius reciprocity and the compatibility with specialisations in [21, 9.1.9], this implies that

$$\operatorname{trace}(h, E_v) = \sum_{M' \in \operatorname{Irr}(W_I)} d(E, M') \operatorname{trace}(h, M'_v) \quad \text{for all } h \in \mathbf{H}_{K,I}.$$

Using the orthogonality relations for the irreducible representations of $\mathbf{H}_{K,I}$, we conclude that

$$\begin{aligned} \operatorname{trace}(e_M, E_v) &= \sum_{M' \in \operatorname{Irr}(W_I)} d(E, M') \operatorname{trace}(e_M, M'_v) \\ &= \sum_{M' \in \operatorname{Irr}(W_I)} d(E, M') \sum_{w \in W_I} \operatorname{trace}(T_w, M_v) \operatorname{trace}(T_{w^{-1}}, M'_v) \\ &= d(E, M)\, (\dim M)\, \boldsymbol{c}_M. \end{aligned}$$

Consequently, we have

$$v^{2\boldsymbol{a}_M} \operatorname{trace}(e_M, E_v) = d(E, M)\, (\dim M)\, f_M + \text{“higher terms”},$$

where “higher terms” means an F-linear combination of terms v^g where $g \in \Gamma_{>0}$. On the other hand, recalling Definition 2.4 and taking into account our assumption $\boldsymbol{a}_M = \boldsymbol{a}_E$, we obtain

$$\begin{aligned} v^{2\boldsymbol{a}_M} \operatorname{trace}(e_M, E_v) &= \sum_{w \in W_I} \big(v^{\boldsymbol{a}_M} \operatorname{trace}(T_w, M_v)\big) \big(v^{\boldsymbol{a}_E} \operatorname{trace}(T_{w^{-1}}, E_v)\big) \\ &= \Big(\sum_{w \in W_I} c_{w,M}\, c_{w^{-1},E}\Big) + \text{“higher terms”}. \end{aligned}$$

Comparing the two expressions, we deduce that

$$\sum_{w \in W_I} c_{w,M}\, c_{w^{-1},E} = d(E,M)\,(\dim M)\, f_M. \tag{$*$}$$

Now the right hand side of $(*)$ is non-zero since $d(E,M) \neq 0$ by assumption. Hence, there exists some $w \in W_I$ such that $c_{w,M} \neq 0$ and $c_{w^{-1},E} \neq 0$. By [21, Cor. 8.2.6], we have $\operatorname{trace}(T_w, E_v) = \operatorname{trace}(T_{w^{-1}}, E_v)$. So we also have $c_{w,E} = c_{w^{-1},E} \neq 0$. By Lemma 2.5, this implies $w \in \mathcal{F}_M \cap \mathcal{F}_E$ and, hence, $\mathcal{F}_M \subseteq \mathcal{F}_E$. $\square$

Proposition 3.4. *Let $E, E' \in \operatorname{Irr}(W)$. If $E \preceq E'$, then $E \leqslant_{\mathcal{LR}} E'$. In particular, if E, E' belong to the same family, then $E \sim_{\mathcal{LR}} E'$.*

Proof. If $W = \{1\}$, there is nothing to prove. Now assume that $W \neq \{1\}$ and that the assertion has already been proved for all proper parabolic subgroups of W. It is now sufficient to consider an elementary step in Definition 2.10. That is, we can assume that there is a subset $I \subsetneqq S$ and $M', M'' \in \operatorname{Irr}(W_I)$, where $M' \preceq M''$ within $\operatorname{Irr}(W_I)$, such that one of the following two conditions holds.

(I) E is a constituent of $\operatorname{Ind}_I^S(M')$ and $M'' \rightsquigarrow_L E'$.

(II) $E' \otimes \varepsilon$ is a constituent of $\operatorname{Ind}_I^S(M')$ and $M'' \rightsquigarrow_L E \otimes \varepsilon$.

If (I) holds, then $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{M'}$ and $\mathcal{F}_{M''} \subseteq \mathcal{F}_{E'}$ by Lemmas 3.2 and 3.3. Since $M' \preceq M''$, we already know that $M' \leqslant_{\mathcal{LR}} M''$ and, hence, $\mathcal{F}_{M'} \leqslant_{\mathcal{LR}} \mathcal{F}_{M''}$ (with respect to W_I). But then we also have $\mathcal{F}_{M'} \leqslant_{\mathcal{LR}} \mathcal{F}_{M''}$ with respect to W and, hence, $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{E'}$, as required.

On the other hand, if (II) holds, then a completely similar argument shows that $\mathcal{F}_{E'\otimes\varepsilon} \leqslant_{\mathcal{LR}} \mathcal{F}_{E\otimes\varepsilon}$. But, by Remark 2.3, we have $\mathcal{F}_{E\otimes\varepsilon} = \mathcal{F}_E w_0$ and $\mathcal{F}_{E'\otimes\varepsilon} = \mathcal{F}_{E'} w_0$. Furthermore, multiplication with w_0 reverses the relation $\leqslant_{\mathcal{LR}}$. Hence, we have $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{E'}$, as required.

Finally, if E, E' belong to the same family, then Remark 2.11 immediately shows that $E \preceq E'$, $E' \preceq E$ and, hence, $E \sim_{\mathcal{LR}} E'$. $\square$

Example 3.5. Let (W, S) be of type H_4. Here, we are automatically in the equal parameter case. There are 34 irreducible representations in $\operatorname{Irr}(W)$ and they are partitioned into 13 families; see Alvis–Lusztig [2]. Using CHEVIE [17], one easily determines the relation $\preceq$. It turns out that we obtain a "linear" order such that, for all $E, E' \in \operatorname{Irr}(W)$, we have:

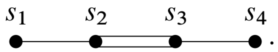
(a) $E \preceq E'$ is and only if $\boldsymbol{a}_{E'} \leqslant \boldsymbol{a}_E$.

(b) E, E' belong to the same family if and only if $\boldsymbol{a}_E = \boldsymbol{a}_{E'}$.

On the other hand, Alvis [1] has determined the two-sided cells of W; there are precisely 13 of them. Hence, by Proposition 3.4, we have $E \sim_{\mathcal{LR}} E'$ if and only if E, E' belong to the same family. Furthermore, since $\preceq$ already induces a linear order on families, it follows that $E \preceq E'$ if and only if $E \leqslant_{\mathcal{LR}} E'$. Thus, Conjecture 2.12 holds in this case.

Similar remarks apply to (W, S) of type H_3 and $I_2(m)$ (with equal or unequal parameters in the latter case): In all these cases, one easily checks that $\preceq$ is a linear order satisfying (a), (b) and, hence, Conjecture 2.12 holds. (See the summary of the relevant results on cells and families in [13, §7].)

Example 3.6. Let (W, S) be of type F_4, with generators and diagram given by

F_4 

Let $\Gamma = \mathbb{Z}$ and L be a weight function which is specified by two positive integers $a := L(s_1) = L(s_2) > 0$ and $b := L(s_3) = L(s_4) > 0$. Taking into account the symmetry of the diagram, one may assume that $a \leqslant b$. There are 25 irreducible representations in $\mathrm{Irr}(W)$. The relation $\leqslant_{\mathcal{LR}}$ on $\mathrm{Irr}(W)$ has been determined in all cases in [12]. It turns out that there are only four essentially different cases: $b = a$, $b = 2a$, $2a > b > a$ or $b > 2a$; see Table 1 in [12, p. 362].

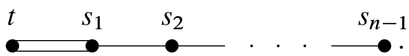
It is verified in [12] that $E \sim_{\mathcal{LR}} E'$ if and only if E, E' belong to the same family. Using CHEVIE [17], one easily determines the relation $\preceq$. By inspection, one finds that Conjecture 2.12 holds in all cases. One also finds that:

(a) If $E \preceq E'$ then $\boldsymbol{a}_{E'} \leqslant \boldsymbol{a}_E$.

(b) If $E \preceq E'$ and $\boldsymbol{a}_E = \boldsymbol{a}_{E'}$, then E, E' belong to the same family.

This example provides strong evidence for the validity of Conjecture 2.12 in the general case of unequal parameters.

Example 3.7. Let (W, S) be of type B_n, with generators and diagram given by

B_n 

We have $\mathrm{Irr}(W) = \{E^\lambda \mid \lambda \in \Lambda\}$ where Λ is the set of all pairs of partitions of total size n. For example, the unit, sign and reflection representation are labelled by $((n), \varnothing)$, $(\varnothing, (1^n))$ and $((n-1), (1))$, respectively; see [21, §5.5]. Let $\Gamma = \mathbb{Z}$. Then a weight function L is specified by two integers $b := L(t) \geqslant 0$ and $a = L(s_i) \geqslant 0$ for $1 \leqslant i \leqslant n-1$. For a conjectural description of the partial order $\leqslant_{\mathcal{LR}}$ on two-sided cells, see [3, Remark 1.2].

Here is a specific example in the case of unequal parameters, where we assume that $b > (n-1)a > 0$. This is the "asymptotic" case originally studied by Bonnafé and Iancu [6], [5]. By Proposition 3.4 and [18, Prop. 5.4], we have

$$E^\lambda \preceq E^\mu \implies E^\lambda \leqslant_{\mathcal{LR}} E^\mu \implies \lambda \trianglelefteq \mu$$

where $\trianglelefteq$ denotes the dominance order on pairs of partitions. In order to prove the reverse implications, it will be enough to show that $\lambda \trianglelefteq \mu \Rightarrow E^\lambda \preceq E^\mu$. Thus, we are reduced to a purely combinatorial problem. This, and a full description of $\preceq$ for all choices of the parameters a, b, will be discussed in [19].

4. The equal parameter case

Throughout this section, we assume that $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$. Our aim is to show that, in this setting, Conjecture 2.12 holds. For this purpose, we have to rely on some deep properties of the relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$ which are stated in Theorem 4.1 below. These in turn are established by using certain "positivity" properties of the Kazhdan–Lusztig basis of $\mathbf{H}$ which are only available in the equal parameter case; see Lusztig [31, Chap. 16] and the references there (as far as finite Weyl groups are concerned) and DuCloux [9] (as far as types H_3, H_4, $I_2(m)$ are concerned).

Theorem 4.1. *In the equal parameter case, the following hold.*

(a) (Lusztig [31]) *If $E, E' \in \mathrm{Irr}(W)$ are such that $E \leqslant_{\mathcal{LR}} E'$, then $\boldsymbol{a}_{E'} \leqslant \boldsymbol{a}_E$. In particular, if $E \sim_{\mathcal{LR}} E'$, then $\boldsymbol{a}_E = \boldsymbol{a}_{E'}$.*

(b) (Lusztig [31]) *If $E, E' \in \mathrm{Irr}(W)$ are such that $E \leqslant_{\mathcal{LR}} E'$ and $\boldsymbol{a}_{E'} = \boldsymbol{a}_E$, then $E \sim_{\mathcal{LR}} E'$.*

(c) (Lusztig–Xi [34]) *Let $x, y \in W$ be such that $x \leqslant_{\mathcal{LR}} y$. Then there exists some $z \in W$ such that $x \leqslant_{\mathcal{L}} z$ and $z \sim_{\mathcal{R}} y$.*

Comments on the proof. Using the "positivity" properties mentioned above, Lusztig shows in [31, Chap. 16] that the conjectural properties **P1–P15** in [31, 14.2] hold for $\mathbf{H}$. Then (a) and (b) are a combination of **P4**, **P11** and [31, Prop. 20.6]. The statement in (c) is due to Lusztig–Xi [34, §3]. Note that, in [34], this result is stated for affine Weyl groups; but the same proof works when W is finite. Indeed, besides general properties of the relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, $\leqslant_{\mathcal{LR}}$, the ingredients needed in the proof are listed in [34, 2.2, 2.3, 2.5]. Now, the references for these properties cover also the case of finite Coxeter groups; the above-mentioned "positivity" properties are required here, too. An additional reference for [34, 2.2 (h)] (which is attributed to Springer, unpublished) is provided by [38, 1.3]. $\square$

Remark 4.2. By Lusztig's conjectures in [31, 14.2], one can expect that (a) and (b) remain valid in the general case of unequal parameters. The proof of (c) seems to require more than just using the conjectural properties **P1–P15** in [31, 14.2]. It is not clear (at least not to me) if one can expect (c) to hold in the general case of unequal parameters.

As a first application of Theorem 4.1 (a), we obtain the following converse to Lemma 3.3.

Lemma 4.3. *Let $I \subseteq S$. Let $E \in \mathrm{Irr}(W)$ and $M \in \mathrm{Irr}(W_I)$ be such that $\mathcal{F}_M \subseteq \mathcal{F}_E$ and E is a constituent of $\mathrm{Ind}_I^S(M)$. Then $M \rightsquigarrow_L E$.*

Proof. By Lemma 2.6 (b), there exists some $E' \in \operatorname{Irr}(W)$ which is a constituent of $\operatorname{Ind}_I^S(M)$ and such that $\boldsymbol{a}_{E'} = \boldsymbol{a}_M$. By Lemma 3.3, we have $\mathcal{F}_M \subseteq \mathcal{F}_{E'}$. Thus, we have $\mathcal{F}_M \subseteq \mathcal{F}_E \cap \mathcal{F}_{E'}$ and so $\mathcal{F}_E = \mathcal{F}_{E'}$. Using Theorem 4.1 (a), we conclude that $\boldsymbol{a}_E = \boldsymbol{a}_{E'} = \boldsymbol{a}_M$, as required. $\square$

Next recall from Remark 2.11 that, if $E, E' \in \operatorname{Irr}(W)$ belong to the same family, then $E \preceq E'$ and $E' \preceq E$. Now we can also prove the reverse implication.

Proposition 4.4. *Let $E, E' \in \operatorname{Irr}(W)$ be such that $E \preceq E'$. Then $\boldsymbol{a}_{E'} \leqslant \boldsymbol{a}_E$. Furthermore, if $E \preceq E'$ and $E' \preceq E$, then $\boldsymbol{a}_E = \boldsymbol{a}_{E'}$ and E, E' belong to the same family of* $\operatorname{Irr}(W)$.

Proof. By Proposition 3.4, we have $E \leqslant_{\mathcal{LR}} E'$. So Theorem 4.1 (a) implies that $\boldsymbol{a}_{E'} \leqslant \boldsymbol{a}_E$. Now assume that $E \preceq E'$ and $E' \preceq E$. Then, clearly, $\boldsymbol{a}_E = \boldsymbol{a}_{E'}$.

We now show by an inductive argument that, if $E \preceq E'$ and $\boldsymbol{a}_E = \boldsymbol{a}_{E'}$, then E, E' belong to the same family. If $W = \{1\}$, there is nothing to prove. Now assume that $W \neq \{1\}$ and that the assertion has already been proved for all proper parabolic subgroups of W. As in the proof of Proposition 3.4, it is sufficient to consider an elementary step in Definition 2.10. That is, we can assume that there is a subset $I \subsetneqq S$ and $M', M'' \in \operatorname{Irr}(W_I)$, where $M' \preceq M''$ within $\operatorname{Irr}(W_I)$, such that one of the following two conditions holds.

(I) E is a constituent of $\operatorname{Ind}_I^S(M')$ and $M'' \rightsquigarrow_L E'$.

(II) $E' \otimes \varepsilon$ is a constituent of $\operatorname{Ind}_I^S(M')$ and $M'' \rightsquigarrow_L E \otimes \varepsilon$.

First of all, since $M' \preceq M''$, we already know that $\boldsymbol{a}_{M''} \leqslant \boldsymbol{a}_{M'}$.

Now, if (I) holds, then $\boldsymbol{a}_E \geqslant \boldsymbol{a}_{M'} \geqslant \boldsymbol{a}_{M''} = \boldsymbol{a}_{E'}$. Since $\boldsymbol{a}_E = \boldsymbol{a}_{E'}$, we conclude that $\boldsymbol{a}_{M'} = \boldsymbol{a}_{M''}$. Hence, by induction, M', M'' belong to the same family of $\operatorname{Irr}(W_I)$. Furthermore, since $\boldsymbol{a}_E = \boldsymbol{a}_{M'}$, we have $M' \rightsquigarrow_L E$. Thus, the first set of conditions in Definition 2.7 is satisfied and so E, E' belong to the same family of $\operatorname{Irr}(W)$.

On the other hand, if (II) holds, then $\boldsymbol{a}_{E'\otimes\varepsilon} \geqslant \boldsymbol{a}_{M'} \geqslant \boldsymbol{a}_{M''} = \boldsymbol{a}_{E\otimes\varepsilon}$. Assume, if possible, that $\boldsymbol{a}_{E'\otimes\varepsilon} > \boldsymbol{a}_{E\otimes\varepsilon}$. Then $E \otimes \varepsilon \not\leqslant_{\mathcal{LR}} E' \otimes \varepsilon$ by Theorem 4.1 (a). Consequently, we also have $E \not\leqslant_{\mathcal{LR}} E'$ by Remark 2.3. Since $E \leqslant_{\mathcal{LR}} E'$ and $\boldsymbol{a}_E = \boldsymbol{a}_{E'}$, this contradicts Theorem 4.1 (b). Hence, we must have $\boldsymbol{a}_{E'\otimes\varepsilon} = \boldsymbol{a}_{E\otimes\varepsilon}$. Now we can argue as above and conclude that the second set of conditions in Definition 2.7 is satisfied. Hence, E, E' belong to the same family of $\operatorname{Irr}(W)$. $\square$

(Note that the above proof only requires (a) and (b) in Theorem 4.1.)

Remark 4.5. In [19, Cor. 9.1] we will show that Proposition 4.4 remains valid in the general multi-parameter case. The proof relies on a case-by-case argument and a detailed study of the relation $\preceq$ in type B_n.

Besides the above-mentioned "positivity" properties, another distinguished feature of the equal parameter case is the existence of "special" irreducible representations. (As discussed in [12, Example 4.11], one cannot expect the existence of representations with similar properties in the general case of unequal parameters.) Given $E \in \mathrm{Irr}(W)$, let $\boldsymbol{b}_E$ be the smallest $i \geqslant 0$ such that E is a constituent of the i-th symmetric power of the natural reflection representation of W. It is an empirical observation that we always have $\boldsymbol{a}_E \leqslant \boldsymbol{b}_E$; following Lusztig [24], we say that E is "special" if $\boldsymbol{a}_E = \boldsymbol{b}_E$. Let

$$\mathcal{S}(W) := \{E \in \mathrm{Irr}(W) \mid E \text{ special}\}.$$

Theorem 4.6 (Lusztig [26, 4.14]). *Each family of* $\mathrm{Irr}(W)$ *(see Definition 2.7) contains a unique* $E \in \mathcal{S}(W)$.

(See also [21, §6.5] where non-crystallographic Coxeter groups are included from the outset in the discussion.)

Theorem 4.7 (Lusztig [26, 5.25]). *Let* $\mathfrak{C}$ *be a left cell and let* $E \in \mathcal{S}(W)$. *If* $\mathfrak{C} \subseteq \mathcal{F}_E$, *then* E *occurs with multiplicity* 1 *in* $[\mathfrak{C}]_1$.

(Alternative proofs are provided by [28], [13]; these references also cover the cases where W is of type H_3, H_4 or $I_2(m)$.)

Remark 4.8. Let $I \subsetneqq S$ and let $\mathcal{S}(W_I)$ denote the set of all $M \in \mathrm{Irr}(W_I)$ which are special (with respect to W_I). Let $M \in \mathcal{S}(W_I)$. Then it is known (see [24]) that there is a unique $E \in \mathcal{S}(W)$ such that $\boldsymbol{a}_E = \boldsymbol{a}_M$ and $\mathrm{Ind}_I^S(M)$ equals E plus a sum of irreducible representations $E' \in \mathrm{Irr}(W)$ such that $\boldsymbol{a}_{E'} > \boldsymbol{a}_E$; in particular, we have $M \rightsquigarrow_L E$. Let us write $E = j_I^S(M)$ in this case.

We define $\mathcal{S}^\circ(W)$ to be the set of all $j_I^S(M)$ where $I \subsetneqq S$ and $M \in \mathcal{S}(W_I)$. With this definition, we can now state the following result of Spaltenstein which will be a further key ingredient in our argument.

Lemma 4.9 (Cf. Spaltenstein [36]). *Let* $E \in \mathcal{S}(W)$ *be such that* $E \notin \mathcal{S}^\circ(W)$. *Then* $\boldsymbol{a}_{E\otimes\varepsilon} < \boldsymbol{a}_E$.

Proof. By standard reduction arguments, it is enough to prove this in the case where (W, S) is irreducible. If W is of type H_3, H_4 or $I_2(m)$, the assertion is easily checked by an explicit computation and CHEVIE [17]. One could also check the assertion for finite Weyl groups in this way, using the explicit knowledge of $\mathcal{S}(W)$ and of the invariants $\boldsymbol{a}_E$ from [24]. However, a related verification has already been done by Spaltenstein [36, §5]. Thus, all we need to do is to see how the setting in [36, §5] translates to our setting here.

So now assume that W is a finite Weyl group. Let G be a simple algebraic group (over $\mathbb{C}$ or over $\overline{\mathbb{F}}_p$ where p is a large prime) with Weyl group W. Using the Springer correspondence (see [37], [27]), we can naturally associate with every $E \in \mathrm{Irr}(W)$ a pair consisting of a unipotent class of G, which we denote by O_E, and a G-equivariant irreducible local system on O_E. By [26, 13.1.1], we have

$$\dim \mathcal{B}_u = \boldsymbol{a}_E \quad \text{for } E \in \mathcal{S}(W),$$

where $\mathcal{B}_u$ denotes the variety of Borel subgroups containing an element $u \in O_E$.

Now Spaltenstein [36, §5] has shown that, if $E \in \mathcal{S}(W)$ and $E \notin \mathcal{S}^\circ(W)$, then O_E is strictly contained in the Zariski closure of $O_{\overline{E}}$ where $\overline{E}$ is the unique special representation of W in the same family as $E \otimes \varepsilon$. In particular, we have $\dim \mathcal{B}_{\bar{u}} < \dim \mathcal{B}_u$ where $u \in O_E$ and $\bar{u} \in O_{\overline{E}}$. Hence, we also have $\boldsymbol{a}_{\overline{E}} < \boldsymbol{a}_E$. Finally, by Proposition 3.4 and Theorem 4.1 (a), we have $\boldsymbol{a}_{\overline{E}} = \boldsymbol{a}_{E \otimes \varepsilon}$. □

Given a two-sided cell $\mathcal{F}$ in W, we denote by $\boldsymbol{a}(\mathcal{F})$ the common value of $\boldsymbol{a}_E$ where $E \in \mathrm{Irr}(W)$ is such that $\mathcal{F}_E = \mathcal{F}$; see Theorem 4.1 (a). With this convention, we can now state the following version of Lemma 4.9 which does not refer to "special" representations in $\mathrm{Irr}(W)$. (One may conjecture that this remains true in the general case of unequal parameters.)

Corollary 4.10. *Let $\mathcal{F}$ be a two-sided cell in W such that $\mathcal{F} \cap W_I = \varnothing$ for all proper subsets $I \subsetneqq S$. Then $\boldsymbol{a}(\mathcal{F} w_0) < \boldsymbol{a}(\mathcal{F})$.*

Proof. By Proposition 3.4 and Theorem 4.6, there exists some $E \in \mathcal{S}(W)$ such that $\mathcal{F}_E = \mathcal{F}$. Assume, if possible, that there exists some $I \subsetneqq S$ and $M \in \mathcal{S}(W_I)$ such that $E = j_I^S(M)$. In particular, this would mean that E is a constituent of $\mathrm{Ind}_I^S(M)$ and $\boldsymbol{a}_M = \boldsymbol{a}_E$. Hence, by Lemma 3.3, we would have $\mathcal{F}_M \subseteq \mathcal{F}_E = \mathcal{F}$ and so $\mathcal{F} \cap W_I \neq \varnothing$, a contradiction. Thus, we have $E \notin \mathcal{S}^\circ(W)$. Now Lemma 4.9 implies that $\boldsymbol{a}_{E\otimes\varepsilon} < \boldsymbol{a}_E$.

By Remark 2.3, we have $\mathcal{F}_{E\otimes\varepsilon} = \mathcal{F}_E w_0$. Hence, we have $\boldsymbol{a}_E = \boldsymbol{a}(\mathcal{F}_E)$ and $\boldsymbol{a}_{E\otimes\varepsilon} = \boldsymbol{a}(\mathcal{F}_E w_0)$. This yields $\boldsymbol{a}(\mathcal{F} w_0) < \boldsymbol{a}(\mathcal{F})$, as required. □

Theorem 4.11. *Recall our standing assumption that we are in the equal parameter case. Then Conjecture 2.12 holds.*

Proof. The "if" part is already proved in Proposition 3.4. To prove the "only if" part, we use an inductive argument. If $W = \{1\}$, there is nothing to prove. Now assume that $W \neq \{1\}$ and that the "only if" part has already been proved for all proper parabolic subgroups W. Let $E, E' \in \mathrm{Irr}(W)$ be such that $E \leqslant_{\mathcal{LR}} E'$. We must show that $E \preceq E'$. Since $E \leqslant_{\mathcal{LR}} E'$, we have $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{E'}$. We claim that one of the following two conditions is satisfied:

(I) $\mathcal{F}_{E'} \cap W_I \neq \varnothing$ for some $I \subsetneqq S$.

(II) $\mathcal{F}_E w_0 \cap W_I \neq \varnothing$ for some $I \subsetneqq S$.

To prove this, we use an argument due to Spaltenstein [36]. Assume, if possible, that $\mathcal{F}_{E'} \cap W_I = \varnothing$ and $\mathcal{F}_E w_0 \cap W_I = \varnothing$ for all $I \subsetneqq S$. By Corollary 4.10, this implies that $\boldsymbol{a}(\mathcal{F}_{E'} w_0) < \boldsymbol{a}(\mathcal{F}_{E'})$ and $\boldsymbol{a}(\mathcal{F}_E) < \boldsymbol{a}(\mathcal{F}_E w_0)$. Furthermore, since $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{E'}$, we have $\boldsymbol{a}(\mathcal{F}_{E'}) \leqslant \boldsymbol{a}(\mathcal{F}_E)$ by Theorem 4.1 (a). Thus, we conclude that $\boldsymbol{a}(\mathcal{F}_{E'} w_0) < \boldsymbol{a}(\mathcal{F}_E w_0)$. On the other hand, since $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{E'}$, we also have $\mathcal{F}_{E'} w_0 \leqslant_{\mathcal{LR}} \mathcal{F}_E w_0$ (see Remark 2.3). So, Theorem 4.1 (a) implies that $\boldsymbol{a}(\mathcal{F}_E w_0) \leqslant \boldsymbol{a}(\mathcal{F}_{E'} w_0)$, and we have reached a contradiction. Thus, (I) or (II) holds, as claimed.

Now let us first assume that (I) holds. Let E_0 be the unique special representation in the same family as E and E_0' be the unique special representation in the same family as E'; see Theorem 4.6. Then $E \preceq E_0$ and $E_0' \preceq E'$ by Remark 2.11 (a). Hence, it will be enough to show that $E_0 \preceq E_0'$. Note that, by Proposition 3.4, we have $\mathcal{F}_E = \mathcal{F}_{E_0}$ and $\mathcal{F}_{E'} = \mathcal{F}_{E_0'}$.

Let $y \in \mathcal{F}_{E'} \cap W_I$. Then we claim that there exists some $x \in \mathcal{F}_E$ such that $x \leqslant_{\mathcal{L}} y$. This is seen as follows. Recall from Remark 2.3 that multiplication by the longest element $w_0 \in W$ reverses the relations $\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$ and $\leqslant_{\mathcal{LR}}$. Now take any element $x' \in \mathcal{F}_E$. Since $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{E'}$, we have $x' \leqslant_{\mathcal{LR}} y$. Then $yw_0 \leqslant_{\mathcal{LR}} x'w_0$ and so, by Theorem 4.1 (c), there exists some $z \in W$ such that $yw_0 \leqslant_{\mathcal{L}} z$ and $z \sim_{\mathcal{R}} x'w_0$. In particular, $z \in \mathcal{F}_E w_0$ and so $x := zw_0 \in \mathcal{F}_E$. Since $yw_0 \leqslant_{\mathcal{L}} z = xw_0$, we now deduce that $x \leqslant_{\mathcal{L}} y$, as required.

Let us write $x = dw$ where $d \in X_I$ and $w \in W_I$, as in Lemma 3.1. Thus, $x = dw \leqslant_{\mathcal{L}} y$ where $y \in W_I$. Then, by relation (†) in [11, §4], we have $w \leqslant_{\mathcal{LR},I} y$ where the subscript I indicates that this relation is with respect to W_I.

Let $\mathfrak{C}$ be the left cell in W which contains x. Then E_0 is a constituent of $[\mathfrak{C}]_1$; see Theorem 4.7. By Lemma 3.1, there exists some $M \in \mathrm{Irr}(W_I)$ such that $w \in \mathcal{F}_M$ and E_0 is a constituent of $\mathrm{Ind}_I^S(M)$. Similarly, let $\mathfrak{C}'$ be the left cell in W which contains y; now E_0' is a constituent of $[\mathfrak{C}']_1$. Again, there exists some $M' \in \mathrm{Irr}(W_I)$ such that $y \in \mathcal{F}_{M'}$ and E_0' is a constituent of $\mathrm{Ind}_I^S(M')$. Furthermore, since $y \in \mathcal{F}_{M'} \cap \mathcal{F}_{E_0'}$, we must have $\mathcal{F}_{M'} \subseteq \mathcal{F}_{E_0'}$. So we can now conclude that $M' \rightsquigarrow_L E_0'$; see Lemma 4.3. Since $w \leqslant_{\mathcal{LR},I} y$, we have $\mathcal{F}_M \leqslant_{\mathcal{LR},I} \mathcal{F}_{M'}$ and so $M \leqslant_{\mathcal{LR},I} M'$. By our inductive hypothesis, we deduce that $M \preceq M'$ within $\mathrm{Irr}(W_I)$. Thus, the first set of conditions in Definition 2.10 is satisfied. Hence, we have $E_0 \preceq E_0'$ and so $E \preceq E'$. This completes the proof in the case where (I) holds.

Finally, assume that (II) holds. Then we can argue as follows. By Remark 2.3, we have $\mathcal{F}_{E\otimes\varepsilon} = \mathcal{F}_E w_0$ and $\mathcal{F}_{E'\otimes\varepsilon} = \mathcal{F}_{E'} w_0$. In particular, (II) is equivalent to $\mathcal{F}_{E\otimes\varepsilon} \cap W_I \neq \varnothing$. Furthermore, since $\mathcal{F}_E \leqslant_{\mathcal{LR}} \mathcal{F}_{E'}$, we have $\mathcal{F}_{E'\otimes\varepsilon} \leqslant_{\mathcal{LR}} \mathcal{F}_{E\otimes\varepsilon}$. We can now apply the same argument as above and conclude that $E' \otimes \varepsilon \preceq E \otimes \varepsilon$. Then Remark 2.11 (b) shows that we also have $E \preceq E'$, as required. $\square$

5. Unipotent classes and two-sided cells

We continue to assume that we are in the equal parameter case. In addition, we now assume that W is the Weyl group of a connected reductive algebraic group G (over $\mathbb{C}$ or over $\overline{\mathbb{F}}_p$ where p is a large prime). By the Springer correspondence (see [37], [27]), we can naturally associate with every $E \in \mathrm{Irr}(W)$ a pair consisting of a unipotent class of G, which we denote by O_E, and a G-equivariant irreducible local system on O_E. Thus, we obtain a map

$$\mathrm{Irr}(W) \to \{\text{set of unipotent classes in } G\}, \quad E \mapsto O_E.$$

(The local system on O_E will not play a role for our purposes here.)

Definition 5.1 (Lusztig). A unipotent class O of G is called "special" if $O = O_E$ where $E \in \mathcal{S}(W)$. The map $E \mapsto O_E$ gives a bijection between $\mathcal{S}(W)$ and the set of special unipotent classes in G.

Remark 5.2. Let $\mathcal{F}$ be a two-sided cell in W and consider the collection of unipotent classes

$$\mathcal{C}(\mathcal{F}) := \{O_E \mid E \in \mathrm{Irr}(W) \text{ such that } \mathcal{F}_E = \mathcal{F}\}.$$

By Theorems 2.8 and 4.6, there exists a unique $E_0 \in \mathcal{S}(W)$ such that $\mathcal{F}_{E_0} = \mathcal{F}$; in particular, $O_{E_0} \in \mathcal{C}(\mathcal{F})$. Then it is known that

$$O \subseteq \bar{O}_{E_0} \quad \text{for all } O \in \mathcal{C}(\mathcal{F});$$

see [20, Prop. 2.2]. (Here, and below, $\bar{X}$ denotes the Zariski closure in G for any subset $X \subseteq G$.) Thus, the special unipotent class O_{E_0} can be characterized as the unique unipotent class in $\mathcal{C}(\mathcal{F})$ which is maximal with respect to the Zariski closure relation.

Let $\mathcal{U}_G$ be the unipotent variety of G. Let O be a special unipotent class. The corresponding "special piece" in $\mathcal{U}_G$ is defined to be the set of all elements in $\bar{O}$ which are not contained in $\bar{O}'$ where O' is any special unipotent class such that $\bar{O}' \subsetneqq \bar{O}$. By Spaltenstein [35] and Lusztig [30], the special pieces form a partition of $\mathcal{U}_G$. Note that every special piece is a union of a special unipotent class (which is open dense in the special piece) and of a certain number (possibly zero) of non-special unipotent classes.

We can now associate with every two-sided cell in W a special piece in $\mathcal{U}_G$, as follows. Let $\mathcal{F}$ be a two-sided cell in W. As already noted above, there exists a unique $E_0 \in \mathcal{S}(W)$ such that $\mathcal{F}_{E_0} = \mathcal{F}$. Let O_{E_0} be the corresponding special unipotent class and $\mathcal{O}_{\mathcal{F}}$ be the unique special piece in $\mathcal{U}_G$ containing O_{E_0}. Thus, we obtain a canonical bijection (see also Lusztig [30, Theorem 0.2]):

$$\{\text{set of two-sided cells of } W\} \xrightarrow{\text{1-1}} \{\text{set of special pieces in } \mathcal{U}_G\}, \quad \mathcal{F} \mapsto \mathcal{O}_{\mathcal{F}}.$$

As remarked in [32, §14], this map is part of Lusztig's bijection [29] between the set of two-sided cells in an associated affine Weyl group and the set of *all* unipotent classes of G.

Corollary 5.6 below gives an interpretation of the order relation $\leqslant_{\mathcal{LR}}$ on the two-sided cells of W in terms of the closure relation among the special pieces in $\mathcal{U}_G$. This will heavily rely on Theorem 4.11 and on the following result.

Theorem 5.3 (Spaltenstein [35], [36]). *Let $E, E' \in \mathcal{S}(W)$. Then we have*

$$E \preceq_s E' \iff O_E \subseteq \bar{O}_{E'} \iff O_{\bar{E}'} \subseteq \bar{O}_{\bar{E}}.$$

Here, we have used the following notation. Given $E \in \mathcal{S}(W)$, we denote by $\bar{E} \in \mathcal{S}(W)$ the unique special representation in the same family as $E \otimes \varepsilon$. (Thus, we obtain an involution $E \mapsto \bar{E}$ on $\mathcal{S}(W)$.) Furthermore, the relation $\preceq_s$ on $\mathcal{S}(W)$ is defined inductively as follows. If $W = \{1\}$, then $\mathcal{S}(W)$ only consists of the unit representation and this is related to itself. Now assume that $W \neq \{1\}$ and that $\preceq_s$ has already been defined for all proper parabolic subgroups of W. Let $E, E' \in \mathcal{S}(W)$. Then we write $E \preceq_s E'$ if there exists a subset $I \subsetneqq S$ and $M', M'' \in \mathcal{S}(W_I)$, where $M' \preceq_s M''$ within $\mathcal{S}(W_I)$, such that either

$$E \text{ is a constituent of } \operatorname{Ind}_I^S(M') \quad \text{and} \quad M'' \rightsquigarrow_L E'$$

or

$$\bar{E}' \text{ is a constituent of } \operatorname{Ind}_I^S(M') \quad \text{and} \quad M'' \rightsquigarrow_L \bar{E}.$$

Note the formal similarity in the definitions of $\preceq_s$ and the relation $\preceq$ considered in Section 2. More precisely, we have:

Lemma 5.4. *Let $E, E' \in \mathcal{S}(W)$ be such that $E \preceq_s E'$. Then we also have $E \preceq E'$.*

Proof. We proceed by an inductive argument. If $W = \{1\}$, there is a nothing to prove. Now assume that $W = \{1\}$ and that the assertion has already been proved for all proper parabolic subgroups of W. By the definition of $\preceq_s$, there exists a subset $I \subsetneqq S$ and $M', M'' \in \mathcal{S}(W_I)$, where $M' \preceq_s M''$ within $\mathcal{S}(W_I)$, such that one of the following conditions is satisfied:

(I) E is a constituent of $\operatorname{Ind}_I^S(M')$ and $M'' \rightsquigarrow_L E'$.

(II) $\bar{E}'$ is a constituent of $\operatorname{Ind}_I^S(M')$ and $M'' \rightsquigarrow_L \bar{E}$.

By our inductive hypothesis, we already know that $M' \preceq M''$ within $\operatorname{Irr}(W_I)$. Consequently, if (I) holds, then the first set of conditions in Definition 2.10 is satisfied and so $E \preceq E'$. Now assume that (II) holds. Then we obtain that $\bar{E}' \preceq \bar{E}$. By the definition of $\bar{E}$, $\bar{E}'$ and Remark 2.11 (a), we have $\bar{E} \preceq E \otimes \varepsilon$, $E' \otimes \varepsilon \preceq \bar{E}'$ and so $E' \otimes \varepsilon \preceq E \otimes \varepsilon$. Hence, Remark 2.11 (b) implies that $E \preceq E'$, as required. $\square$

Lemma 5.5. *Let $P \subseteq G$ be a parabolic subgroup of G, with unipotent radical U_P and Levi complement L such that L has Weyl group $W_I \subseteq W$ where $I \subseteq S$. Let $E \in \mathrm{Irr}(W)$ and O_E be the corresponding unipotent class in G; let $M \in \mathrm{Irr}(W_I)$ and O_M be the corresponding unipotent class in L.*

(a) *Assume that E is a constituent of $\mathrm{Ind}_I^S(M)$. Then $O_E \cap U_P O_M \neq \varnothing$.*

(b) *Assume that E is special and $M \rightsquigarrow_L E$. Then M is special and $U_P O_M \subseteq \bar{O}_E$.*

Proof. (a) Springer's restriction formula [37, Theorem 4.4] (see also Lusztig [27, Theorem 8.3]) expresses the multiplicity of E as a constituent of $\mathrm{Ind}_I^S(M)$ in geometric terms, using the variety

$$\mathfrak{X}_{u,u'}(P) := \{x \in G \mid x^{-1}ux \in u'U_P\}, \quad \text{where } u \in O_E \text{ and } u' \in O_M.$$

In particular, the assumption that E is a constituent of $\mathrm{Ind}_I^S(M)$ implies that $\mathfrak{X}_{u,u'}(P)$ must be non-empty. Thus, we have $O_E \cap U_P O_M \neq \varnothing$, as required.

(b) We check that O_E is induced from O_M in the sense of Lusztig–Spaltenstein [33]. To begin with, since E is special, the unipotent class O_E has property (B) in [33, §3]; see the remark at the end of [24, §2], or [21, Theorem 6.5.13 (c)]. On the other hand, since $M \rightsquigarrow_L E$, the representation M must also be special. (This follows, for example, from [21, §5.2 and §6.5].) In particular, property (B) holds for O_M, too. Then [33, Theorem 3.5] shows that O_E is induced from O_M, that is, O_E is the unique unipotent class in G such that $O_E \cap U_P O_M$ is dense in $U_P O_M$. Hence, $U_P O_M$ must be contained in the closure of O_E, as desired. $\square$

We can now state the promised geometric interpretation of $\leqslant_{\mathcal{LR}}$.

Corollary 5.6. *Let $\mathcal{F}$, $\mathcal{F}'$ be two-sided cells in W. Then we have $\mathcal{F} \leqslant_{\mathcal{LR}} \mathcal{F}'$ if and only if $\mathcal{O}_{\mathcal{F}} \subseteq \bar{\mathcal{O}}_{\mathcal{F}'}$.*

Proof. First assume that $\mathcal{F} \leqslant_{\mathcal{LR}} \mathcal{F}'$. The following argument for proving $\mathcal{O}_{\mathcal{F}} \subseteq \bar{\mathcal{O}}_{\mathcal{F}'}$ is inspired by the discussion in [36, §2]. If $W = \{1\}$, there is nothing to prove. Now assume that $W \neq \{1\}$ and that the assertion has already been proved for all proper parabolic subgroups of W. As in the proof of Theorem 4.11, one of the following two conditions must be satisfied:

(I) $\mathcal{F}' \cap W_I \neq \varnothing$ for some $I \subsetneqq S$.

(II) $\mathcal{F} w_0 \cap W_I \neq \varnothing$ for some $I \subsetneqq S$.

Assume first that (I) holds. Let $E, E' \in \mathcal{S}(W)$ be such that $\mathcal{F} = \mathcal{F}_E$ and $\mathcal{F}' = \mathcal{F}_{E'}$. Then we must show that $O_E \subseteq \bar{O}_{E'}$. As in the proof of Theorem 4.11, since E, E' are special and $E \leqslant_{\mathcal{LR}} E'$, there exist $M, M' \in \mathrm{Irr}(W_I)$, where $M \leqslant_{\mathcal{LR},I} M'$ with respect to W_I, such that E is a constituent of $\mathrm{Ind}_I^S(M)$ and $M' \rightsquigarrow_L E'$. Now let $P \subseteq G$ be a parabolic subgroup of G, with unipotent radical U_P and Levi complement

L such that L has Weyl group W_I. Applying Lemma 5.5, we conclude that M' is special and that we have the following relations among the associated unipotent classes:

$$O_E \cap U_P O_M \neq \varnothing \quad \text{and} \quad U_P O_{M'} \subseteq \bar{O}_{E'}. \tag{$*$}$$

Let $M_0 \in \mathcal{S}(W_I)$ be the unique special representation in the same family as M (with respect to W_I). Then $M_0 \sim_{\mathcal{LR},I} M$ (see Proposition 3.4) and so $M_0 \leqslant_{\mathcal{LR},I} M'$. Hence, applying our inductive hypothesis, we can conclude that $O_{M_0} \subseteq \bar{O}_{M'}$ (within L). Furthermore, since M, M_0 belong to the same family, we have $O_M \subseteq \bar{O}_{M_0}$; see Remark 5.2. Thus, we have reached the conclusion that $O_M \subseteq \bar{O}_{M'}$. This certainly implies that $U_P O_M$ is contained in the closure of $U_P O_{M'}$. Combining this with ($*$), it follows that $O_E \subseteq \bar{O}_{E'}$, as required.

Now assume that (II) holds. Then the same argument shows that $O_{\bar{E}'} \subseteq \bar{O}_{\bar{E}}$; note that, by Proposition 3.4 and Remark 2.3, we have $\mathcal{F}_{\bar{E}_0} = \mathcal{F}_{E_0 \otimes \varepsilon} = \mathcal{F}_{E_0} w_0$ for every $E_0 \in \mathcal{S}(W)$. But, by the second equivalence in Theorem 5.3, we then also have that $O_E \subseteq \bar{O}_{E'}$, as required.

Conversely, assume that $\mathcal{O}_{\mathcal{F}} \subseteq \bar{\mathcal{O}}_{\mathcal{F}'}$. Let again $E, E' \in \mathcal{S}(W)$ be such that $\mathcal{F} = \mathcal{F}_E$ and $\mathcal{F}' = \mathcal{F}_{E'}$. Then the assumption certainly implies that $O_E \subseteq \bar{O}_{E'}$. So the first equivalence in Theorem 5.3 shows that $E \preceq_s E'$. By Lemma 5.4 and Proposition 3.4, this implies $E \preceq E'$ and $E \leqslant_{\mathcal{LR}} E'$, as required. □

Remark 5.7. The closure relation among the special unipotent classes in G, and the order-reversing bijection $O_E \mapsto O_{\bar{E}}$ ($E \in \mathcal{S}(W)$), are explicitly known; see Carter [7, §13.2], Spaltenstein [35]. Hence, by the above result, we also have an explicit description of the partial order $\leqslant_{\mathcal{LR}}$ on the families of $\mathrm{Irr}(W)$.

On the other hand, the advantage of Theorem 4.11 is that it provides a purely elementary description of $\leqslant_{\mathcal{LR}}$ in terms of the relation $\preceq$, independently of the theory of algebraic groups. Moreover, the equivalence between $\leqslant_{\mathcal{LR}}$ and $\preceq$ applies to more general situations where no geometric interpretation is available; see the examples in Section 3.

Note added in proof. After the submission of this paper, I learned that a version of Corollary 5.6 already appeared as Proposition 2.23 in an article by Barbasch and Vogan, *Ann. of Math.* **121** (1985), 41–110. However, the details of the proof of the "if" part are omitted there, and the proof of the "only if" part is different from the one given here. In our proof of Corollary 5.6, the results of Spaltenstein [36] play an essential role in establishing the equivalence.

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