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Some groups of mapping classes not realized by diffeomorphisms

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Abstract. Let Σ be a closed surface of genus $g \geq 2$ and $z \in \Sigma$ a marked point. We prove that the subgroup of the mapping class group $\mathrm{Map}(\Sigma, z)$ corresponding to the fundamental group $\pi_1(\Sigma, z)$ of the closed surface does not lift to the group of diffeomorphisms of Σ fixing z. As a corollary, we show that the Atiyah–Kodaira surface bundles admit no invariant flat connection, and obtain another proof of Morita's non-lifting theorem.

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1. Introduction

Given a closed orientable surface Σ and a finite, possibly empty, set $\boldsymbol{z} \subset \Sigma$ of marked points, consider the group

$$\mathrm{Diff}_+(\Sigma, \boldsymbol{z}) = \{f \in \mathrm{Diff}_+(\Sigma) \mid f(\boldsymbol{z}) = \boldsymbol{z}\}$$

of orientation-preserving diffeomorphisms of Σ which map the set of marked points to itself. (When $\boldsymbol{z}$ is empty we drop it from our notation.) We denote by $\mathrm{Diff}_0(\Sigma, \boldsymbol{z})$ the normal subgroup of $\mathrm{Diff}_+(\Sigma, \boldsymbol{z})$ consisting of those diffeomorphisms which are isotopic to the identity via an isotopy which fixes the set $\boldsymbol{z}$. The mapping class group is the quotient group

$$\mathrm{Map}(\Sigma, \boldsymbol{z}) = \mathrm{Diff}_+(\Sigma, \boldsymbol{z}) / \mathrm{Diff}_0(\Sigma, \boldsymbol{z}).$$

In [16], Morita proved that if Σ has genus at least 18 and the set of punctures is empty, then the exact sequence

$$0 \to \mathrm{Diff}_0(\Sigma) \to \mathrm{Diff}_+(\Sigma) \to \mathrm{Map}(\Sigma) \to 0$$

does not split. The bound was later improved to genus at least 5 by Morita ([17], Theorem 4.21). Recently Franks–Handel [6] have extended this result so that it holds for genus at least 3. Cantat–Cerveau [3] have proved that finite index subgroups of the mapping class group do not lift to the group of analytic diffeomorphisms. A much more powerful result is due to Marković [12] and Marković–Šarić [13], who have proved that for genus at least 2, the mapping class group does not even lift to the group of homeomorphisms. The proofs of at least some of these results apply also to the case with marked points.

Given a subgroup $\Gamma \hookrightarrow \mathrm{Map}(\Sigma, z)$, the *realization problem* asks whether Γ lifts to $\mathrm{Diff}_+(\Sigma, z)$. This has been the focus of much interest for various classes of subgroups over the years since Nielsen first raised the question. Affirmative answers were given for cyclic groups by Nielsen [18], for finite groups by Kerckhoff [9], and for abelian groups by Birman–Lubotzky–McCarthy [2]. In this paper, we exhibit rather small subgroups of $\mathrm{Map}(\Sigma, z)$ that do not lift to $\mathrm{Diff}_+(\Sigma, z)$. Specifically, in the case of a surface of genus at least 2 with a single marked point we prove:

Theorem 1.1. *Let Σ be a closed surface of genus $g \geq 2$ and $z \in \Sigma$ a marked point. No finite index subgroup of the point-pushing subgroup $\pi_1(\Sigma, z) \subset \mathrm{Map}(\Sigma, z)$ lifts to $\mathrm{Diff}_+(\Sigma, z)$.*

The point-pushing subgroup fits into the Birman exact sequence

$$1 \to \pi_1(\Sigma, z) \xrightarrow{F} \mathrm{Map}(\Sigma, z) \to \mathrm{Map}(\Sigma) \to 1 \tag{1.1}$$

as long as $g \geq 2$. Observe that if (Σ, z) is a torus with a single marked point, then the mapping class group does in fact lift to $\mathrm{Diff}_+(\Sigma, z)$.

We sketch now the proof of Theorem 1.1. Seeking a contradiction, assume that there is a homomorphism Φ such that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{Diff}_+(\Sigma, z) \arrow[d] \\ \pi_1(\Sigma, z) \arrow[ur, dashed, "\Phi"] \arrow[r, hook, "F"] & \mathrm{Map}(\Sigma, z) \end{tikzcd}$$

where F is the inclusion from (1.1). The homomorphism Φ yields an action of $\pi_1(\Sigma, z)$ on Σ by diffeomorphisms fixing z and hence a representation of $\pi_1(\Sigma, z)$ in $\mathrm{GL}^+(T_z\Sigma)$. By Milnor's inequality this representation has Euler-number bounded in absolute value by $g - 1$. On the other hand, we compute that the Euler-number must be $2 - 2g$; this contradiction gives Theorem 1.1.

Combining Theorem 1.1 with some topological constructions, we show that the centralizers of most finite order elements of $\mathrm{Map}(\Sigma)$ do not lift to $\mathrm{Diff}_+(\Sigma)$. Concretely, we construct a subgroup of $\mathrm{Map}(\Sigma)$ isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \pi_1(S, z)$ for some closed surface S that does not lift to $\mathrm{Diff}_+(\Sigma)$. This relies on the existence of finite order elements and thus does not apply to finite index subgroups of $\mathrm{Map}(\Sigma)$. Using Theorem 1.1 and this construction, we derive the following version of Morita's theorem:

Theorem 1.2 (Morita's non-lifting theorem). *Let (Σ, z) be a surface of genus g with $|z| = k$ marked points. Assume either that $g \geq 6$ or that $g \geq 2$ and $k \geq 1$. Then the exact sequence*

$$0 \to \mathrm{Diff}_0(\Sigma, z) \to \mathrm{Diff}_+(\Sigma, z) \to \mathrm{Map}(\Sigma, z) \to 0 \tag{1.2}$$

does not split. In fact, if $g \geq 2$ and $k \geq 1$ then no finite index subgroup of $\mathrm{Map}(\Sigma, z)$ *lifts to* $\mathrm{Diff}_+(\Sigma, z)$.

Morita originally proved his theorem by finding a surface bundle over an 6-dimensional manifold with a cohomological obstruction to the existence of a flat connection. (All connections are taken to be smooth.) The theorem of Earle–Eells [4] on the contractibility of $\mathrm{Diff}_0(\Sigma)$ implies that a Σ-bundle over a base B admits a flat connection if and only if the topological monodromy representation $\pi_1(B) \to \mathrm{Map}(\Sigma)$ can be lifted to a map $\pi_1(B) \to \mathrm{Diff}_+(\Sigma)$. In particular, if the sequence (1.2) split, then every surface bundle would admit a flat connection, so Morita's theorem follows from his example.

In contrast, for surface bundles over surfaces, Kotschick–Morita [11] proved that every surface bundle admits a flat connection after "stabilization"; in particular, there can be no cohomological obstruction to flatness in this case. This raised the open problem of finding a surface bundle over a surface that does not admit a flat connection. The details of the proof of Theorem 1.2 give a partial solution to this problem. In the case of a punctured surface, Theorem 1.1 gives a surface group isomorphic to $\pi_1(\Sigma, z)$ inside $\mathrm{Map}(\Sigma, z)$ that does not lift to $\mathrm{Diff}_+(\Sigma, z)$. This yields a surface bundle with a distinguished section, with base space a closed surface, which admits no flat connection such that the distinguished section is parallel. (In fact, this bundle is just the trivial bundle $\Sigma \times \Sigma$, and the distinguished section is the diagonal.) We believe that this is the first such surface group inside a punctured mapping class group known. In the case of a closed surface, the construction described above corresponds to a topological construction of Kodaira and Atiyah, and we conclude (see remarks preceding the proof for definitions):

Theorem 1.3. *When $k \geq 3$, the Atiyah–Kodaira bundle $\Sigma \to M_k \to S'$ admits no flat connection invariant under the order-k deck transformation $\mathcal{T} : M_k \to M_k$.*

However, the full question remains open in the case when the surface is closed.

Question. *Does there exist a closed surface bundle over a surface that admits no flat connection?*

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2. A few facts about Euler-numbers

Let Σ be a closed surface of genus g and let $\tilde{\Sigma} \to \Sigma$ be its universal cover. Choose base points $z \in \Sigma$ and $\tilde{z} \in \tilde{\Sigma}$ projecting to z. The choice of base points yields an identification between the fundamental group $\pi_1(\Sigma, z)$ and the deck-transformation group of the cover $\tilde{\Sigma} \to \Sigma$. Before going any further, let us remark that the composition $\gamma \star \eta$ of two elements $\gamma, \eta \in \pi_1(\Sigma, z)$ is obtained by first running γ and then η. By construction, the universal cover $\tilde{\Sigma}$ consists of homotopy classes rel endpoints of continuous paths in Σ beginning at z. Here we can identify $\tilde{z}$ with, for instance, the homotopy class of the constant path. The fundamental group $\pi_1(\Sigma, z)$ acts on $\tilde{\Sigma}$ by precomposition, meaning that we first run a path representing the element in the fundamental group and then a path representing the element in $\tilde{\Sigma}$. In particular, the obtained action of $\pi_1(\Sigma, z) \curvearrowright \tilde{\Sigma}$, the so-called action by deck-transformations, is a left action.

Assume now that $\rho\colon \pi_1(\Sigma, z) \to \mathrm{Homeo}^+(\mathbb{S}^1)$ is an action of the fundamental group of Σ on the circle. Let E_ρ be the quotient of $\tilde{\Sigma} \times \mathbb{S}^1$ under the action

$$\pi_1(\Sigma, z) \curvearrowright (\tilde{\Sigma} \times \mathbb{S}^1), \quad (\gamma, (x, \theta)) \mapsto (\gamma x, \rho(\gamma)\theta).$$

The projection of $\tilde{\Sigma} \times \mathbb{S}^1$ onto the first factor is $\pi_1(\Sigma)$-equivariant and has fiber $\mathbb{S}^1$; this descends to give E_ρ the structure of a circle bundle over Σ. The trivial connection on $\tilde{\Sigma} \times \mathbb{S}^1$ induces a flat connection on E_ρ. Conversely, every flat circle bundle over Σ is obtained in this way.

The *Euler-number* $e(E_\rho) \in \mathbb{Z}$ of the bundle $E_\rho \to \Sigma$ is the obstruction for the bundle E_ρ to admit a section, or equivalently, for the action ρ to lift to an action on the universal cover $\mathbb{R}$ of $\mathbb{S}^1$.

Milnor–Wood inequality. *Assume that E_ρ is a flat orientable circle bundle over a closed surface Σ of genus g. Then $\left|e(E_\rho)\right| \leq 2g - 2$.*

It should be observed that there are flat circle bundles with Euler-number $2 - 2g$. For instance, endowing Σ with a hyperbolic metric, we can identify the universal cover $\tilde{\Sigma}$ with the hyperbolic plane. The action of $\pi_1(\Sigma, z)$ on $\mathbb{H}^2$ extends to an action on the circle at infinity $\partial_\infty \mathbb{H}^2$. The associated flat circle bundle is isomorphic to the unit tangent bundle of Σ and hence has Euler-number equal to the Euler characteristic $\chi(\Sigma) = 2 - 2g$. We record this fact for further reference (see Appendix C of [15]):

Lemma 2.1. *Let Σ be a closed orientable hyperbolic surface of genus g and identify $\pi_1(\Sigma, z)$ with the corresponding group of deck-transformations of $\mathbb{H}^2$. The circle bundle corresponding to the induced action of $\pi_1(\Sigma, z)$ on $\partial_\infty \mathbb{H}^2 = \mathbb{S}^1$ has Euler-number $2 - 2g$.* $\square$

We point out that Goldman [7] proved a converse to this lemma: if $\rho\colon \pi_1(\Sigma, z) \to \mathrm{PSL}_2\,\mathbb{R}$ has $\left|e(E_\rho)\right| = 2g-2$, then ρ is an isomorphism onto a discrete subgroup of $\mathrm{PSL}_2\,\mathbb{R}$ and thus comes from a hyperbolic metric on Σ as in the lemma.

Other examples of circle bundles over Σ can be constructed as follows. A linear action $\rho\colon \pi_1(\Sigma, z) \to \mathrm{GL}_2^+\,\mathbb{R}$ of $\pi_1(\Sigma, z)$ on $\mathbb{R}^2$ induces an action on the space of directions $P_+\mathbb{R}^2 = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_+$ of $\mathbb{R}^2$. The latter can be identified with the circle and hence the same construction as above yields a circle bundle E_ρ. A circle bundle E_ρ arising in this way is called a *flat linear circle bundle*. The linear action ρ induces a different circle bundle $\widehat{E}_\rho$ via the induced projective action on the projective line $P\mathbb{R}^2 = (\mathbb{R}^2 \setminus \{0\})/(\mathbb{R} \setminus \{0\})$, which can also be identified with the circle. By construction there is a two-to-one fiberwise covering $E_\rho \to \widehat{E}_\rho$. In particular, $e(\widehat{E}_\rho) = 2e(E_\rho)$. We have then:

Milnor's inequality. *Assume that E_ρ is a flat linear orientable circle bundle over a closed surface Σ of genus g. Then $\left|e(E_\rho)\right| \leq g - 1$.*

In [14], Milnor proved that if a $\mathrm{GL}_2^+\,\mathbb{R}$-bundle over a closed surface of genus g admits a flat symmetric connection, then its Euler-number is bounded in absolute value by $g - 1$. This is equivalent to Milnor's inequality above. Later, Wood [19] extended Milnor's work to prove the Milnor–Wood inequality.

For a general oriented circle bundle $S^1 \to E \to B$, the Euler class is a characteristic class $e(E) \in H^2(B)$. When the base space is a surface, we identify this with the Euler-number by the identification $H^2(\Sigma) = \mathbb{Z}$. We will use the same symbol for the Euler-number and Euler class; it should be clear from context what is meant.

3. Surfaces with one puncture

Let Σ be a closed surface of genus g and $z \in \Sigma$ a marked point, and define the group $\mathcal{G}(\Sigma, z)$ to consist of those orientation-preserving homeomorphisms f of Σ which fix z so that f and f^{-1} are differentiable at z. In this section we prove the following generalization of Theorem 1.1:

Proposition 3.1. *Let Σ be a closed surface of genus $g \geq 2$ and $z \in \Sigma$ a marked point. If $\Gamma \subset \pi_1(\Sigma, z)$ is a finite index subgroup, then the inclusion of Γ into* $\mathrm{Map}(\Sigma, z)$ *under the homomorphism F from* (1.1) *does not lift to $\mathcal{G}(\Sigma, z)$.*

Observe that since $\mathrm{Diff}_+(\Sigma, z)$ is a subgroup of $\mathcal{G}(\Sigma, z)$, Theorem 1.1 follows directly from Proposition 3.1. Although Proposition 3.1 applies only to punctured surfaces, we will upgrade it in Section 4 to prove Theorem 1.2 for closed surfaces.

Before going any further we describe the homomorphism

$$F\colon \pi_1(\Sigma, z) \hookrightarrow \mathrm{Map}(\Sigma, z)$$

from (1.1) in detail. Given $\gamma \in \pi_1(\Sigma, z)$, let $\vec{\gamma}\colon [0,1] \to \Sigma$ be a loop in the corresponding homotopy class. The map $t \mapsto \vec{\gamma}(1-t)$ can be interpreted as an isotopy from the identity Id_z to itself. By the theorem on extension of isotopies we obtain an isotopy $f_t\colon \Sigma \to \Sigma$ with $f_0 = \mathrm{Id}_\Sigma$ and $f_t(z) = \vec{\gamma}(1-t)$. Birman proved that the element $F_\gamma \in \mathrm{Map}(\Sigma, z)$ corresponding to $f_1 \in \mathrm{Diff}_+(\Sigma, z)$ depends only on the element $\gamma \in \pi_1(\Sigma, z)$. Observing that

$$F_{\gamma \star \eta} = F_\gamma \circ F_\eta$$

we have that $F\colon \pi_1(\Sigma, z) \to \mathrm{Map}(\Sigma, z)$ is a homomorphism.

Starting now the proof of Proposition 3.1, assume that there is a homomorphism

$$\Phi\colon \pi_1(\Sigma, z) \to \mathcal{G}(\Sigma, z)$$

such that for each $\gamma \in \pi_1(\Sigma, z)$ the homeomorphism Φ_γ represents the mapping class $F_\gamma \in \mathrm{Map}(\Sigma, z)$. Endowing Σ with a hyperbolic metric we identify its universal cover with $\mathbb{H}^2$; choose a point $\tilde{z}$ covering z. We obtain then a homomorphism

$$\widetilde{\Phi}\colon \pi_1(\Sigma, z) \to \mathcal{G}(\mathbb{H}^2, \tilde{z})$$

mapping γ to the unique lift of Φ_γ which fixes $\tilde{z}$. Here $\mathcal{G}(\mathbb{H}^2, \tilde{z})$ is the group of homeomorphisms of $\mathbb{H}^2$ fixing $\tilde{z}$ which are differentiable at $\tilde{z}$ with inverse differentiable at $\tilde{z}$.

Lemma 3.2. *The homeomorphism $\widetilde{\Phi}_\gamma\colon \mathbb{H}^2 \to \mathbb{H}^2$ extends to a homeomorphism of the closed disk $\overline{\mathbb{H}}^2 = \mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$. Moreover, the restriction of $\widetilde{\Phi}_\gamma$ to $\partial_\infty \mathbb{H}^2$ coincides with the action of γ as a deck-transformation.*

Lemma 3.2 is probably well known to experts and non-experts alike. However, here is a proof:

Proof. We start by observing that the action Φ can be lifted in a different way. By construction, if we forget the marked point, the homeomorphism Φ_γ is homotopic to the identity. If f_t is such a homotopy with $f_0 = \mathrm{Id}_\Sigma$ and $f_1 = \Phi_\gamma$, let $\hat{f}_t$ be the unique lift of f_t to $\mathbb{H}^2$ with $\hat{f}_0 = \mathrm{Id}_{\mathbb{H}^2}$. We obtain a new lift $\widehat{\Phi}_\gamma = \hat{f}_1$ of Φ_γ. It follows directly from the construction of the homomorphism F and from the fact that Φ_γ represents $F(\gamma)$ that

$$\widehat{\Phi}_\gamma(\tilde{z}) = \gamma^{-1}\tilde{z}$$

where we have identified $\gamma \in \pi_1(\Sigma, z)$ with the corresponding deck-transformation. In particular, the two lifts $\widehat{\Phi}_\gamma$ and $\widetilde{\Phi}_\gamma$ differ by the deck-transformation γ, meaning that

$$\gamma \circ \widehat{\Phi}_\gamma = \widetilde{\Phi}_\gamma. \tag{3.1}$$

By construction, the lift $\widehat{\Phi}_\gamma$ moves every point in $\mathbb{H}^2$ a uniformly bounded distance from itself. In particular $\widehat{\Phi}$ extends continuously to the identity map on the boundary $\partial_\infty \mathbb{H}^2$ of the hyperbolic plane. The claim follows from this fact and (3.1). □

We come now to the meat of the proof of Theorem 1.2. Recall that $\overline{\mathbb{H}}^2$ is the union of $\mathbb{H}^2$ with the circle at infinity $\partial_\infty \mathbb{H}^2$. The half-open annulus $\overline{\mathbb{H}}^2 \setminus \tilde{z}$ can be compactified in a canonical way by attaching to the open end the space of directions $P_+ T_{\tilde{z}} \mathbb{H}^2 = (T_{\tilde{z}} \mathbb{H}^2 \setminus \{0\})/\mathbb{R}_+$ of the tangent space at $\tilde{z}$. Let $\mathcal{A}$ be the so-obtained closed annulus. By Lemma 3.2, the action of $\pi_1(\Sigma, z)$ via $\widetilde{\Phi}$ induces an action on $\overline{\mathbb{H}}^2 \setminus \{\tilde{z}\}$. Moreover, the assumption that $\widetilde{\Phi}_\gamma$ is differentiable at $\tilde{z}$ for all $\gamma \in \pi_1(\Sigma, z)$ implies that this action extends to an action on $\mathcal{A}$ which restricts to $\partial\mathcal{A}$ as follows.

- On the component $\partial_1 \mathcal{A}$ corresponding to $\partial_\infty \mathbb{H}^2$ the action of $\pi_1(\Sigma, z)$ the action is equal to the one induced by the deck-transformation group by Lemma 3.2.
- On the component $\partial_2 \mathcal{A}$ corresponding to the space of directions of $T_{\tilde{z}} \mathbb{H}^2$, the action is induced by the representation

$$\pi_1(\Sigma, z) \to \mathrm{GL}^+(T_{\tilde{z}} \mathbb{H}^2), \quad \gamma \mapsto d\widehat{\Phi}_\gamma|_{\tilde{z}}.$$

In particular, it follows from Lemma 2.1 that the circle bundle E_1 over Σ induced by the action on $\partial_1 \mathcal{A}$ has Euler-number

$$e(E_1) = 2 - 2g.$$

Similarly, it follows from Milnor's inequality that the circle bundle E_2 over Σ induced by the action on $\partial_2 \mathcal{A}$ satisfies

$$|e(E_2)| = g - 1.$$

But since the annulus bundle $\mathcal{A}$ admits a fiberwise deformation retract onto E_1 and also onto E_2, these bundles have the same Euler-number

$$e(E_1) = e(\mathcal{A}) = e(E_2).$$

This contradiction shows that the image of $\pi_1(\Sigma, z)$ under F does not lift to $\mathcal{G}(\Sigma, z)$. The same argument applies to finite index subgroups; this concludes the proof of Proposition 3.1. □

As mentioned above, Theorem 1.1 follows directly from Proposition 3.1.

An alternate perspective on Proposition 3.1. In the remainder of this section, we sketch an alternate perspective on the above proof in the language of surface bundles. This perspective will be used in the remarks following the proof of Theorem 1.2 and in the proof of Theorems 1.3 and 4.3.

The previous section considered the flat linear circle bundle $E_{d\Phi} \to \Sigma$, which *a priori* depends on the lift Φ of F; however, the isomorphism type of $E_{d\Phi}$ as a topological circle bundle does not depend on Φ. In fact, this circle bundle can be defined without reference to any lift, as we describe below.

The theorem of Earle–Eells, extended to punctured surfaces by Earle–Schatz [5], gives a one-to-one correspondence between Σ-bundles with distinguished section over a base B (up to isomorphism) and their monodromy representation $\pi_1(B) \to \mathrm{Map}(\Sigma, z)$ (up to conjugacy). The "vertical Euler class" of a Σ-bundle with distinguished section is a characteristic class defined as follows. Given such a bundle $\Sigma \to E \xrightarrow{\pi} B$ with section $\sigma\colon B \to E$, the vectors tangent to the fibers span a 2-dimensional subbundle $T\pi \leq TE$. Passing to the space of directions and restricting to the section σ induces a circle bundle $UT\pi|_\sigma \to B$. The vertical Euler class is defined to be the Euler class $e(UT\pi|_\sigma) \in H^2(B)$ of this circle bundle. This class is discussed in many references, including [16]. We will need only the following property.

Fact. *If the monodromy $r\colon \pi_1(B) \to \mathrm{Map}(\Sigma, z)$ of a Σ-bundle with section lifts to $\rho\colon \pi_1(B) \to \mathcal{G}(\Sigma, z)$, yielding as above the flat linear circle bundle $E_{d\rho} \to B$, then $E_{d\rho}$ is isomorphic to $UT\pi|_\sigma$ as a circle bundle.*

To apply this fact to the map $F\colon \pi_1(\Sigma, z) \to \mathrm{Map}(\Sigma, z)$, we must identify the Σ-bundle with section over Σ whose monodromy is F. It is easy to check that the desired bundle is the product bundle $p_1\colon \Sigma \times \Sigma \to \Sigma$, with section given by the diagonal $\Delta\colon \Sigma \to \Sigma \times \Sigma$.

Along the diagonal, we can identify the tangent space $T_{(p,p)}(\Sigma \times \Sigma)$ with $T_p\Sigma \times T_p\Sigma$. Under this identification, $Tp_1 = \ker dp_1$ consists of vectors of the form $(0, v) \in T_p\Sigma \times T_p\Sigma$. Mapping $(0, v) \mapsto (v, v)$ gives an isomorphism between $Tp_1|_\Delta$ and $T\Delta$, the subbundle spanned by vectors tangent to the diagonal. It follows that $e(UTp_1|_\Delta) = e(UT\Delta) = 2 - 2g$. By Milnor's inequality, this bundle is not isomorphic to any flat linear circle bundle. Thus the fact above implies that no lift $\Phi\colon \pi_1(\Sigma, z) \to \mathcal{G}(\Sigma, z)$ exists.

For a finite index subgroup of $\pi_1(\Sigma, z)$ corresponding to the cover $p\colon \Sigma' \to \Sigma$, the same argument applies to the bundle $\Sigma' \times \Sigma \to \Sigma$, with section given by the graph of p.

4. The proof of Theorem 1.2

In this section we deduce Theorem 1.2 from Proposition 3.1, but before doing so we need some notation.

Theorem 1.2. *Let* $(\Sigma, \boldsymbol{z})$ *be a surface of genus* g *with* k *marked points. Assume that either* $g \geq 6$ *or that* $g \geq 2$ *and* $k \geq 1$*. Then the exact sequence*

$$0 \to \mathrm{Diff}_0(\Sigma, \boldsymbol{z}) \to \mathrm{Diff}_+(\Sigma, \boldsymbol{z}) \to \mathrm{Map}(\Sigma, \boldsymbol{z}) \to 0$$

does not split. In fact, if $g \geq 2$ *and* $k \geq 1$ *then no finite index subgroup of* $\mathrm{Map}(\Sigma, \boldsymbol{z})$ *lifts to* $\mathrm{Diff}_+(\Sigma, \boldsymbol{z})$.

Given a surface as in Theorem 1.2, let $\mathcal{G}(\Sigma, \boldsymbol{z})$ be the group of those orientation-preserving homeomorphisms f of Σ which fix the marked points $\boldsymbol{z}$ pointwise so that f and f^{-1} are differentiable at each $z \in \boldsymbol{z}$. If $\mathcal{G}_0(\Sigma, \boldsymbol{z})$ denotes the normal subgroup of $\mathcal{G}(\Sigma, \boldsymbol{z})$ consisting of those elements which are isotopic to the identity relative to the set $\boldsymbol{z}$ then the quotient group

$$\mathrm{PMap}(\Sigma, \boldsymbol{z}) = \mathcal{G}(\Sigma, \boldsymbol{z}) / \mathcal{G}_0(\Sigma, \boldsymbol{z})$$

is the *pure mapping class group*, a finite index subgroup of the mapping class group $\mathrm{Map}(\Sigma, \boldsymbol{z})$. We could equivalently define $\mathrm{PMap}(\Sigma, \boldsymbol{z})$ using diffeomorphisms instead of $\mathcal{G}(\Sigma, \boldsymbol{z})$.

We can now start with the proof of Theorem 1.2. We will divide the proof into cases depending on the genus g and number of marked points k in $(\Sigma, \boldsymbol{z})$; the proof for each case will depend upon the previous one.

Case 1. $g \geq 2$ *and* $k = 1$. Since the group $\mathrm{Diff}_+(\Sigma, z)$ is a subgroup of $\mathcal{G}(\Sigma, z)$, the claim follows directly from Proposition 3.1. □

Case 2. $g \geq 2$ *and* $k \geq 2$. Consider the configuration space

$$\mathcal{C}_k(\Sigma) = \big\{(x_1, \ldots, x_k) \in \Sigma^k \mid x_i \neq x_j \text{ if } i \neq j\big\}$$

of ordered k-tuples of pairwise distinct points in the closed surface Σ. We can consider $\mathcal{C}_k(\Sigma)$ as a fiber bundle over Σ via the following projection:

$$p_1 \colon \mathcal{C}_k(\Sigma) \to \Sigma, \quad p_1 \colon (x_1, \ldots, x_k) \mapsto x_1$$

In particular, we obtain a homomorphism

$$\pi_1(p_1) \colon \pi_1(\mathcal{C}_k(\Sigma), (z_1, \ldots, z_k)) \to \pi_1(\Sigma, z_1).$$

We claim that $\pi_1(p_1)$ has a right inverse:

Lemma 4.1. *There is a homomorphism*

$$\eta\colon \pi_1(\Sigma, z_1) \to \pi_1(\mathcal{C}_k(\Sigma), (z_1, \dots, z_k))$$

with $\pi_1(p_1) \circ \eta = \mathrm{Id}$.

Proof. It suffices to construct a section $\Sigma \to \mathcal{C}_k(\Sigma)$ of the fiber bundle $p_1 \colon \mathcal{C}_k(\Sigma) \to \Sigma$. In order to construct such a section, it suffices to find maps $\alpha_i \colon \Sigma \to \Sigma$ for $i = 2, \dots, k$, each without fixed points and satisfying $\alpha_i(z_1) = z_i$ and $\alpha_i(x) \neq \alpha_j(x)$ for $i \neq j$. Given such α_i, let $\sigma\colon \Sigma \to \Sigma^k$ be the map given by $\sigma(x) = (x, \alpha_2(x), \dots, \alpha_k(x))$. By construction, the image of σ is contained in $\mathcal{C}_k(\Sigma)$. On the other hand, $p_1 \circ \sigma = \mathrm{Id}$; in other words, σ is the desired section.

To find such maps, let $T \subset \Sigma$ be a compact subsurface homeomorphic to a torus with one boundary component and which contains all the points $z_1, \dots, z_k$. Let C be a homotopically essential simple closed curve in $T \setminus \partial T$ with $z_i \in C$ for $i = 1, \dots, k$; let also $\mathbb{T}$ be the closed torus obtained by collapsing the boundary of T to a point. Equivalently, $\mathbb{T}$ is obtained by collapsing $\Sigma \setminus (T \setminus \partial T)$ to a point; this gives a map $\Sigma \to \mathbb{T}$. We can now identify C with a factor of $\mathbb{T} \approx \mathbb{S}^1 \times \mathbb{S}^1$, giving in particular a projection $\mathbb{T} \to C$. Composing with the map $\Sigma \to \mathbb{T}$ above, we obtain a retraction $a\colon \Sigma \to C$ which fixes each point in C. Fixing a parametrization of C, let α_i be the composition

$$\alpha_i\colon \quad \Sigma \xrightarrow{a} C \xrightarrow{r_i} C \hookrightarrow \Sigma$$

where the middle map $r_i \colon C \to C$ is the rotation taking z_1 to z_i. Since the image of each α_i is C, any fixed point of α_i must lie in C; since α_i acts by a nontrivial rotation on C, α_i has no fixed points. Similarly, since each α_i is the composition of a with a different rotation, we have $\alpha_i(x) \neq \alpha_j(x)$ for $i \neq j$, as desired. $\square$

Order now the points $z_1, \dots, z_k$ in $\boldsymbol{z}$ and let $\vec{z}$ be the so-obtained point in $\mathcal{C}_k(\Sigma)$. Recall that $\mathrm{PMap}(\Sigma, \boldsymbol{z})$ is the pure mapping class group of $(\Sigma, \boldsymbol{z})$, i.e. the subgroup of the mapping class group consisting of mapping classes whose representatives in $\mathrm{Diff}_+(\Sigma)$ fix each one of the marked points. Forgetting all the marked points, and forgetting all the marked points but z_1, we obtain the following versions of the Birman exact sequence (1.1):

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathcal{C}_k(\Sigma), \vec{z}) \arrow[r] \arrow[d, shift left, "\pi_1(p)"] & \mathrm{PMap}(\Sigma, \boldsymbol{z}) \arrow[r] \arrow[d] & \mathrm{Map}(\Sigma) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_1(\Sigma, z_1) \arrow[r] \arrow[u, bend left, "\eta"] & \mathrm{Map}(\Sigma, z_1) \arrow[r] & \mathrm{Map}(\Sigma) \arrow[r] & 1.
\end{tikzcd}$$

Here η is the homomorphism provided by Lemma 4.1.

Assume now that G is a finite index subgroup in $\mathrm{Map}(\Sigma, \boldsymbol{z})$ which lifts to $\mathrm{Diff}_+(\Sigma, \boldsymbol{z})$. Intersecting with the point-pushing subgroup $\pi_1(\mathcal{C}_k(\Sigma), \vec{z})$, we obtain a finite index subgroup of $\pi_1(\mathcal{C}_k(\Sigma, \vec{z}))$ which lifts to $\mathrm{Diff}_+(\Sigma, \boldsymbol{z})$. Composing with the section η provided by Lemma 4.1, we obtain a lift of a finite index subgroup $\Gamma < \pi_1(\Sigma, z_1)$ to $\mathrm{Diff}_+(\Sigma, \boldsymbol{z})$. Since $\mathrm{Diff}_+(\Sigma, \boldsymbol{z})$ is a subgroup of $\mathrm{Diff}_+(\Sigma, z_1)$ and hence of $\mathcal{G}(\Sigma, z_1)$, this contradicts Proposition 3.1. This concludes the proof of Case 2. □

Remark. Before going further, observe that we have actually proved that, under the assumptions of Case 2, no finite index subgroup of $\mathrm{Map}(\Sigma, \boldsymbol{z})$ lifts to $\mathcal{G}(\Sigma, \boldsymbol{z})$.

Case 3. $g \geq 6$ *and* $k = 0$. In this case we will prove that the centralizer of a certain finite order element $T \in \mathrm{Map}(\Sigma)$ does not lift to $\mathrm{Diff}_+(\Sigma)$. We have significant freedom in our choice of T; we require only that the order of T be at least 3, and that the quotient $\Sigma/\langle T\rangle$ have genus at least 2. The first step is to verify that such finite order elements exist for all Σ. Though in the proof we work with an order 3 automorphism τ, any number $k \geq 3$ would work just as well; see the remark following the proof of Lemma 4.2 to see why it is necessary that τ have order at least 3.

Fact. *If* $g \geq 6$*, then there is a diffeomorphism* $\tau\colon \Sigma \to \Sigma$ *of order* 3 *with at least* 2 *fixed points so that the quotient* $\Sigma/\langle\tau\rangle$ *has genus* $h \geq 2$*.*

There are many different ways to find such a finite-order diffeomorphism. One uniform way is to begin with a degree 3 cyclic branched cover of the sphere branched at $g-4$ points. By the Hurwitz formula, the resulting surface has genus $g-6$. Now add three genus 2 handles symmetrically, so they are permuted freely by the order 3 deck transformation; in the quotient this corresponds to adding a single genus 2 handle to the sphere. The result is a genus g surface Σ with an order 3 automorphism τ so that the quotient $\Sigma/\langle\tau\rangle$ has genus 2.

Let $\tau\colon \Sigma \to \Sigma$ be the diffeomorphism provided by the fact above, $T \in \mathrm{Map}(\Sigma)$ the corresponding mapping class, and

$$C(T) = \{f \in \mathrm{Map}(\Sigma) \mid f \circ T = T \circ f\}$$

its centralizer. We claim that $C(T)$ does not lift to $\mathrm{Diff}_+(\Sigma)$. Seeking a contradiction, assume that such a lifting

$$\Psi\colon C(T) \to \mathrm{Diff}_+(\Sigma)$$

exists. By definition, the diffeomorphism $\Psi(T)$ has order 3 and is isotopic to τ. In particular, both diffeomorphisms are conjugate and we may assume without loss of generality that $\Psi(T) = \tau$, so that the image of Ψ is contained in the centralizer $C(\tau) < \mathrm{Diff}_+(\Sigma)$.

Remark. The authors did not find a reference for this fact, so we give a short argument here. Each of τ and $\tau' = \Psi(T)$ is an isometry of some hyperbolic structure X and X'

on Σ, respectively. Identifying the universal cover of X and X' with the hyperbolic plane, we obtain that the groups G generated by all lifts of τ and G' generated by all lifts of τ' are Fuchsian groups. In fact, the assumption that τ is isotopic to τ' implies that G and G' are isomorphic. Satz IV.10 in Zieschang–Vogt–Coldewey [20] implies that the actions of G and G' are conjugate. This yields a conjugation between τ and τ'. Before moving on, we observe that a second and slightly more sophisticated proof follows from the fact that the fixed point set of the mapping class T in Teichmüller space is totally geodesic with respect to the Teichmüller metric, and thus *a fortiori* connected.

By construction, the quotient surface $S = \Sigma/\langle\tau\rangle$ has genus $h \geq 2$. Let now $z_1, \dots, z_k \in S$ be the projection to S of the fixed points of τ and set $\boldsymbol{z} = \{z_1, \dots, z_k\}$. Every $f \in \mathrm{Diff}_+(\Sigma)$ which commutes with τ induces a homeomorphism of $(S, \boldsymbol{z})$. This gives a homomorphism

$$\alpha\colon C(\tau) \to \mathrm{Homeo}(S, \boldsymbol{z})$$

whose kernel is the cyclic group generated by τ. Let $C(\tau, \boldsymbol{z})$ be the finite index subgroup of $C(\tau)$ consisting of those diffeomorphisms which commute with τ and fix each of its fixed points. The key fact, and the reason we require τ to have order 3, is the following lemma:

Lemma 4.2. *The image of $C(\tau, \boldsymbol{z})$ under α is contained in $\mathcal{G}(S, \boldsymbol{z})$.*

Proof. It is well known that there is a conformal structure on Σ such that τ is biholomorphic. In particular, if x is one of the fixed points of τ we can find coordinates ζ around x such that $\tau(\zeta) = \omega \cdot \zeta$ where ω is a primitive third root of unity. Every differentiable $f : \Sigma \to \Sigma$ which fixes x and commutes with τ has differential

$$df_x : T_x\Sigma \to T_x\Sigma$$

satisfying $df_x \cdot \omega = \omega \cdot df_x$. Since ω has order 3, the elements 1 and ω span $\mathbb{C}$ as a real vector space. Since df_x commutes with multiplication by each, df_x is complex differentiable. This implies that the induced map $S \to S$ is also differentiable at the projection of x. This concludes the proof of the lemma. Note that we could not have concluded that df_x is complex differentiable if ω instead had order 2, since any linear map commutes with -1. □

By composing with Ψ, we obtain an action

$$C(T) \xrightarrow{\Psi} C(\tau) \xrightarrow{\alpha} \mathrm{Homeo}(S, \boldsymbol{z})$$

of $C(T)$ on $(S, \boldsymbol{z})$. Since $\langle\tau\rangle$ is the kernel of α, this descends to an action

$$C(T)/\langle T\rangle \to \mathrm{Homeo}(S, \boldsymbol{z}).$$

As in the construction of α, we can identify $C(T)/\langle T\rangle$ with a certain subgroup of $\mathrm{Map}(S, z)$. A mapping class in $\mathrm{Map}(S, z)$ lifts to the branched cover Σ exactly if it preserves up to conjugacy the subgroup of $\pi_1(S \setminus z)$ determining the cover $\Sigma \setminus z \to S \setminus z$. Since this subgroup has finite index in $\pi_1(S \setminus z)$, its stabilizer has finite index in $\mathrm{Map}(S, z)$. Among these, $C(T)/\langle T\rangle$ is identified with the finite index subgroup consisting of those mapping classes whose lift to Σ commutes with T. Let Γ be the intersection of $C(T)/\langle T\rangle$ with $\mathrm{PMap}(S, z)$; note that Γ has finite index in $\mathrm{Map}(S, z)$.

We consider the restriction of the action $C(T)/\langle T\rangle \to \mathrm{Homeo}(S, z)$ above to the subgroup Γ. Since Γ is contained in $\mathrm{PMap}(S, z)$, the image under Ψ of any lift will be contained in $C(\tau, z)$. Lemma 4.2 implies that the action $\Gamma \to \mathrm{Homeo}(S, z)$ has image contained in $\mathcal{G}(S, z)$. Thus we have a lift of the finite index subgroup $\Gamma < \mathrm{Map}(S, z)$ to $\mathcal{G}(S, z)$, contradicting the remark following the proof of Case 2. This contradiction completes the proof of Case 3, and thus concludes the proof of Theorem 1.2. $\square$

For a minimal example of a non-lifting subgroup, consider the intersection of $\Gamma \subset \mathrm{Map}(S, z)$ with the surface group $\eta(\pi_1(S, z_1))$; this gives a surface group inside $\mathrm{Map}(S, z)$ whose preimage in $C(T)$ does not lift to $\mathrm{Diff}_+(\Sigma)$. This preimage is a central extension of a surface group by the cyclic group $\langle T\rangle$; by possibly passing to an index 3 subgroup, we may assume this extension is trivial, yielding a subgroup of $\mathrm{Map}(\Sigma)$ isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \pi_1(S', z)$ which does not lift to $\mathrm{Diff}_+(\Sigma)$.

Observations on the proof of Theorem 1.2. In this section, we give an informal discussion interpreting the above proof in terms of surface bundles. We then use this perspective to give two observations, Theorems 1.3 and 4.3 below.

As discussed in the introduction, Case 1 above is equivalent to the statement that not every surface bundle with section admits a flat connection so that the section is parallel. This was proved in Proposition 3.1 by exhibiting the product bundle $\Sigma \times \Sigma$ with section given by the diagonal Δ.

The content of Lemma 4.1 in Case 2 is then that this bundle admits k disjoint sections, one of which is the diagonal. The proof given above was chosen because it requires no conditions on the genus g of Σ. In the special case when $k|(g-1)$, another construction is as follows. Let $\sigma\colon \Sigma \to \Sigma$ generate a free action of $\mathbb{Z}/k\mathbb{Z}$ on Σ; then the graphs $\Delta = \Gamma_{\mathrm{id}}, \Gamma_\sigma, \Gamma_{\sigma^2}, \dots, \Gamma_{\sigma^{k-1}}$ give k disjoint sections of $\Sigma \times \Sigma$.

Fiberwise branched covers. In Case 3, we exploit the connection between $\mathrm{Map}(\Sigma)$ and $\mathrm{Map}(S, z)$, where $S = \Sigma/\langle\tau\rangle$ and z is the image of the fixed points of τ. For surface bundles, this corresponds to passing to a fiberwise branched cover, as follows; we allow the order of τ to be any $k \geq 3$. If $S \to E \to B$ is a surface bundle with n disjoint sections $\sigma_1, \dots, \sigma_n\colon B \to E$, the union of the sections gives a (disconnected) codimension 2 subspace of E. Depending on the bundle and sections, E may admit a cyclic branched cover $\widetilde{E} \to E$ of order k, branched over the sections σ_i; in this case $\widetilde{E}$ becomes a Σ-bundle $\Sigma \to \widetilde{E} \to B$. The action of τ on Σ then corresponds to the order-k automorphism $\mathcal{T}\colon \widetilde{E} \to \widetilde{E}$ generating the deck transformations of the branched cover $\widetilde{E} \to E$. The observation above that $C(T)/\langle T\rangle$ has finite index in $\mathrm{Map}(S, z)$ becomes here the following fact: even if E does not admit such a branched cover, there is always some finite cover $B' \to B$ so that the pullback bundle $S \to E' \to B'$ admits a cyclic branched cover, branched over the preimages in E' of the sections σ_i.

Combining this construction with the choice of sections $\Gamma_{\sigma^i} \subset \Sigma \times \Sigma$ recovers the classical example of Kodaira [10] and Atiyah [1] . Their surface bundle is constructed as follows: start with a surface S admitting a free action of $\mathbb{Z}/k\mathbb{Z}$ generated by σ. The bundle $S \times S \to S$ does not admit a branched cover branched over the union of the sections Γ_{σ^i}. However, taking $\pi\colon S' \to S$ to be the cover corresponding to the kernel of $\pi_1(S) \to H_1(S) \to H_1(S; \mathbb{Z}/k\mathbb{Z})$, the pullback $S' \times S \to S'$ does admit a branched cover $M_k \to S' \times S$ of order k, branched over the union of the sections $\Gamma_{\sigma^i \circ \pi}$. Composing with the projection $S' \times S \to S'$ gives a bundle $\Sigma \to M_k \to S'$, where the fiber Σ is a branched cover of the original fiber S of order k, branched over k points. (Note that the manifold M_k fibers over a surface in two different ways; the fibering considered here is that of the original authors.)

Aside from the choice of sections, these steps correspond exactly to the considerations above, and so the results of Case 3 apply identically to this case, giving the following theorem:

Theorem 1.3. *When $k \geq 3$, the Atiyah–Kodaira bundle $\Sigma \to M_k \to S'$ admits no flat connection invariant under the order-k deck transformation $\mathcal{T}\colon M_k \to M_k$.*

The surface group $\pi_1(S', z) \subset \mathrm{Map}(\Sigma)$ singled out in the previous section is the monodromy of this surface bundle. We remark that by returning to the choice of sections considered in Case 3, the same theorem is obtained for the surface bundles constructed by González-Díez and Harvey in [8].

We now sketch a description of Morita's m-construction; this is a generalization of the construction of Kodaira and Atiyah, used by Morita in [16] to give the original proof of Morita's theorem. Roughly, the m-construction begins with a surface bundle over a manifold of dimension n satisfying certain conditions, then modifies it by pulling back along covers of the base, covers and branched covers of the fiber, and the bundle projection itself; the result is another surface bundle whose base has dimension $n + 2$.

More precisely, given an admissible surface bundle $s \to E \to B$, first pull back to the total space to obtain a bundle over E with fiber s; this bundle naturally admits a "diagonal" section. Possibly passing to a finite cover of the base, we may take a fiberwise cover, obtaining a new bundle with fiber S, where $S \to s$ is a cover with deck transformation group $\mathbb{Z}/m\mathbb{Z}$. As discussed above, combining the "diagonal" section with this $\mathbb{Z}/m\mathbb{Z}$-action yields m disjoint sections of this S-bundle. Again possibly passing to a finite cover of the base, we may take a fiberwise branched cover, yielding a bundle $\Sigma \to \widetilde{E} \to E'$, where $\Sigma \to S$ is a cyclic branched cover of order m branched at m points. Note that the deck transformation $\mathcal{T}\colon \widetilde{E} \to \widetilde{E}$ of this cyclic branched cover has order m.

Fixing a single fiber of the original bundle $s \to E \to B$ and following through this construction, we see that the preimage of this fiber in $\widetilde{E}$ gives an Atiyah–Kodaira bundle $\Sigma \to M_m \to S'$ inside $\Sigma \to \widetilde{E} \to E'$. Thus we have the following consequence of Theorem 1.3.

Theorem 4.3. *When $m \geq 3$, given any admissible bundle $s \to E \to B$, the Σ-bundle $\Sigma \to \widetilde{E} \to E'$ resulting from Morita's m-construction admits no flat connection invariant under the order-m deck transformation $\mathcal{T}\colon \widetilde{E} \to \widetilde{E}$.*

For comparison, the corresponding form of Morita's theorem is as follows.

Theorem 4.4 (Morita's Theorem). *There exists a bundle $s \to E^6 \to B^4$ so that the Σ-bundle $\Sigma \to \widetilde{E}^8 \to E'^6$ resulting from Morita's m-construction admits no flat connection.*

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